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Research article

Modeling and analysis of demand-supply dynamics with a collectability factor using delay differential equations in economic growth via the Caputo operator

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Abstract: In this paper, to investigate the dynamic interplay between supply and demand, with a focus on collectability, a novel mathematical model was introduced via conformable operator. This model considers the possibility that operating expenses or a lack of raw materials causes a manufacturing delay than the supply of goods instantly matching demand. This maturation (delay) is represented by the delay factor (τ). Stability analysis revolves around the equilibrium point other than zero. Chaotic behavior emerges through Hopf bifurcation at the critical delay parameter value. If this delay parameter is even slightly perturbed, oscillatory limit cycles can be induced in the market dynamics, leading to equilibrium with brisk market expansion, frequent recessions, and sudden collapses. We conducted sensitivity and directional analysis on a number of factors while also examining the stability and duration of the Hopf bifurcation. Numerical findings were validated using MATLAB. Additionally, the Caputo operator was used to examine the fractional of demand and supply dynamics. Importantly, we assumed a pivotal role in advancing fair labor practices and fostering economic growth on a national scale.

Keywords: production; demand; apply stability; delay; Hopf bifurcation; Caputo
Mathematics Subject Classification: 34Axx, 34Dxx, 34Fxx

1. Introduction

Analysing and predicting demand-supply dynamics is critical for businesses and governments alike in today's challenging financial scene. The capacity to precisely predict and analyze these dynamics enables decision-makers to optimize production, distribution, and resource allocation, resulting in increased operational efficiency and customer satisfaction. Traditional models, on the other hand, frequently miss a vital factor—collectability—which plays a significant role in determining the effectiveness of demand fulfillment. The concept of the market depends on the demand-supply dynamics. The common commodities like houses observed by Dorofeenko et al. [1], and gasoline by Aleksandrov et al. [2] and Plante [3] exhibit the true dynamics of demand and supply. Gasoline prices drop due to a huge surplus supply [3]. People are affected by the prices of houses and gasoline; thus, it is a must to develop better mathematical models related to the dynamics of demand and supply. Many mathematical tools have been developed in the past to study this concept of demand and supply examined by Heo [4], and Weinrich [5]. In this analysis, the principles of a dynamical system are utilized to examine the intricate dynamics of demand and supply. This approach expands upon the traditional Marshall model, offering a more comprehensive understanding. We focus on a specific item within the global market to investigate the interplay between demand and supply forces. The concept of aggregate demand and aggregate supply is not considered by Karlan et al. [6]. In competitive market product, characteristics are standardized, and buyers and sellers cannot affect the price. Firms can enter or leave the market without any barriers examined by Stone [7]. According to the law of demand, the quantity demanded decreases as the price increases, keeping all the other factors unchanged. According to the law of supply, supply quantity rises as prices rise while all other parameters remain constant. According to Alfred Marshall, the equilibrium price is determined by the intersection of the supply and demand curves (Figure 1). Market is assumed to operate at this price. However, in reality, the factors that are kept constant in the definitions of demand and supply curves are called determinants [7]. The fundamental changes that take place in these factors have a considerable impact on the dynamics of demand and supply, having enormous impacts. It leads to the deviation of the price from the equilibrium point. As per the law of demand and supply, the demand curve and the supply curve are static and are independent of time. It is assumed that the amount of demand and the amount of supply remains stagnant over a period of time. In the proposed demand-supply model, the amounts are dependent on time that is the represent the numbers at that specific moment, not over time; these models are more realistic.

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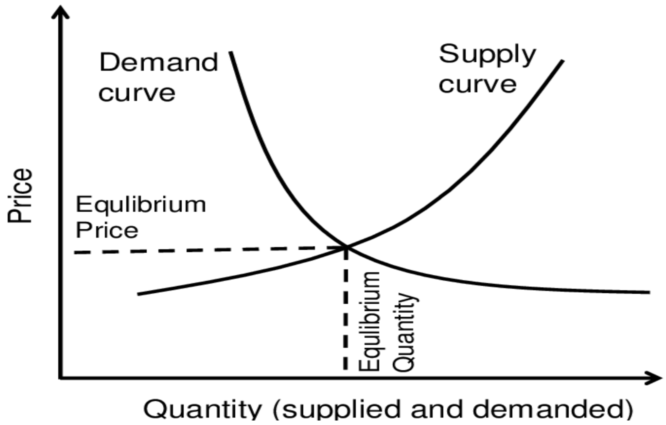

Figure 1. Market equilibrium occurs when the demand curve and supply curve intersect, indicating a state of balance in the market.

Shananin et al. analyze the impact of consumer financing on home economics in Russia during the COVID-19 epidemic [8]. Ramsey-type models were examined by Gimaltdinov as optimum control issues [9]. Mazloumfard and Glantz studied bank profits under conditions of monopolistic competition and the impact of tax pressure [10]. Tadmon and Njike worked on Okun's law and method for calculating the minimal reservation wage in terms of model parameters [11]. Arabob et al. studied the order of derivatives being reduced from units, causing a delay in the fluctuation of financial assets [12]. Wang et al. explore the dynamics of bank data using a competitive model with encouraging results [13]. Selyutin and Rudenko observed that Savings and capital are two categories of banking services, together with loans and deposits. Holdings and retainage are both types of investments [14]. Comes studied a three-level Lotka-Volterra (TLVR) model, a two-way financial exchange from the Parent Bank to the Subsidiaries Bank, and the reverse is taken into account [15]. Marasco et al. examined the Fokker Planck Kolmogorov stochastic equation solution utilized in the TLVR model to evaluate the equilibrium of the banking sector. It is possible to build custom Lotka-Volterra models for n-level banking [16]. Ruan examined and carefully analyzed the exponential characteristic equation's zeros [17]. By utilizing delay differential equations, demand-supply dynamics with collectability factors are achieved. However, in this case, the process and its signs are unquestionably different. With the help of technological developments, many real world problems have been symbolized using mathematical models [32−39].

We start by providing some background knowledge on mathematical models. The development and redesigned mathematical model are used to undertake a comprehensive numerical examination of the dynamics of bank capital. Additionally, a delay differential equation is used to generalize the model. The impact of the DDE order on the dynamics of banking capital is proven based on the numerical analysis. We conclude by summarizing the findings.

2. Mathematical model

The average price of a product over the global market at time t is denoted as $P(t)$. The complete amount of demand of the product in the global market is denoted as $D(t)$. The complete amount of supply of products in a global market is denoted as $S(t)$. However, the supply of the product is always not immediate, but gets delayed due to the availability of raw material, transportation, complications of production process, etc. This maturation time or production time is incorporated as delay parameters in the term defining amount of supply $S(t)$. The threshold prices of demand and supply are denoted as $P_d > 0$ and $P_s > 0$, respectively. A product is assumed to be collectable when its price becomes very high, such as when an investor is increasingly more likely to buy a stock when it exceeds its fundamental worth. This is denoted by collectability factor $F_d > 0$ and does not depend on the variables (P, D, S). The cost of other commodities, resources, taxes etc. related to supply is denoted as $F_d > 0$. Both F_d and F_s are independent of the three variables (P, D, S). This entire demand-supply dynamic is mathematically represented by the following system of first order non-linear delay differential equations:

$$DP = \alpha[R - S(t - \tau)] \tag{1}$$

$$DD = \beta(P_d - P)[1 - \beta_1(P_d - P)^2] + F_d \tag{2}$$

$$DS = -\gamma(P_s - P) + \delta[R - S(t - \tau)] + F_s \tag{3}$$

where $\alpha, \beta, \beta_1, \gamma, \delta$ are all positive parameters.

2.1. Boundedness of the system

From Eqs (1)−(3).
Let W=P+D+S

$$\frac{dW(t)}{dt} = \frac{dP(t)}{dt} + \frac{dD(t)}{dt} + \frac{dS(t)}{dt}.$$

Additoinally, $\varphi = \min(\alpha, \beta, \gamma, \delta, \beta_1, P_d, P_S)$

$$\frac{dW(t)}{dt} \le \varphi(P + D + S) + F_d + F_S$$

$$0 \le \varphi(P + D + S) + F_d + F_S \to 0, 0 \le (P + D + S) \le \frac{F_d + F_S}{\varphi}.$$

As t→ ∞, and applying comparison theorem

$$0 \le (P + D + S) \le \frac{F_d + F_S}{\varphi}.$$

Hence, a three-dimensional space contains all of the equations in the systems (1)−(3).
X= $[(P, D, S)\ \varepsilon\ R_+^3 : 0 \le (P + D + S) \le \frac{F_d + F_S}{\varphi}]$ as t→ ∞, for all positive initial value {P(0) > 0, D(0) > 0, S(0), $S(t - \tau) \sim S$= Const. $\forall\ t\ \varepsilon[-\tau, 0]\}\ \varepsilon\ D \subset R_+^3$, where

$$\varphi = \min(\alpha, \beta, \gamma, \delta, \beta_1, P_d, P_S).$$

2.2. Positively of dynamics

Hence, a three-dimensional space contains all of the equations in the systems (1)−(3). $X = \left[(P, D, S)\varepsilon\ R_+^3 : 0 \le (P + D + S) \le \frac{F_d + F_S}{\varphi}\right] as\ t \to \infty$, for all positive initial value $D(0) > 0, S(0), S(t - \tau) \sim S$= Const. $\forall\ t\ \varepsilon[-\tau, 0]\}\ \varepsilon\ D \subset R_+^3$, Where

$$\varphi = \min(\alpha, \beta, \gamma, \delta, \beta_1, P_d, P_S).$$

From (3)

$$\frac{dS}{dt} \geq -\delta S$$
$$\frac{dS}{S} \geq -\delta dt$$
$$S \geq -e^{\delta t}.$$

And similarly, we can calculate for P and D.

2.3. Equilibrium point

From Eq (1)

$$\frac{dP^*}{dt} = 0 \Rightarrow \alpha[D^* - S^*]$$
$$\alpha[D^* - S^*] = 0$$
$$D^* - S^* = 0$$
$$D^* = S^*.$$

From Eq (3)

$$P^* = P_S - \frac{F_S}{r}.$$

2.4. Cycle of period

The dynamic behavior for equilibrium points $E^*(P^*, D^*, S^*)$ of the model given by (1)–(3) is analyses:

$$\frac{dP^*}{dt} = \alpha[D^* - S^*(t-\tau)] \tag{4}$$

$$\frac{dD^*}{dt} = \beta(P_d - P^*)[1 - \beta_1(P_d - P^*)^2] + F_d \tag{5}$$

$$\frac{dS^*}{dt} = -\gamma(P_s - P^*) + \delta[D - S^*(t-\tau)] + F_s. \tag{6}$$

The exponential characteristic equation about equilibrium E^* is given by:

$$\lambda^3 + (\alpha\beta\beta_1(P_d^2 + P^2 - 2P_dP) + 2\alpha\beta\beta_1(P_d - P) - \alpha\beta)\lambda + e^{-\lambda\tau}(\delta\lambda^2 - \alpha\gamma\lambda + (\alpha\beta\delta - \delta\alpha\beta\beta_1(P_d^2 + P^2 - 2P_dP) - \alpha\beta\delta + \alpha\delta\beta\beta_1(P_d^2 + P^2 - 2P_dP) = 0$$

$$\lambda^3 + a_1\lambda + e^{-\lambda\tau}(b_1\lambda^2 - b_2\lambda + b_3) = 0. \tag{7}$$

Where $a_1 = \alpha\beta\beta_1(P_d^2 + P^2 - 2P_dP) + 2\alpha\beta\beta_1(P_d - P) - \alpha\beta$

$$b_1 = \delta, b_2 = \alpha\gamma, b_3 = (\alpha\beta\delta - \delta\alpha\beta\beta_1(P_d^2 + P^2 - 2P_dP) - \alpha\beta\delta + \alpha\delta\beta\beta_1(P_d^2 + P^2 - 2P_dP).$$

All parameters are a_1, b_1, b_2, b_3 positive.
Equation (7) have a solution Iff $\lambda = \mathrm{i}\omega$

$$(\mathrm{i}\omega)^3 + a_1(\mathrm{i}\omega) + e^{-\mathrm{i}\omega\tau}(b_1(\mathrm{i}\omega)^2 - b_2(\mathrm{i}\omega) + b_3) = 0. \tag{8}$$

Separating real and imaginary part

$$-\omega^3 = b_2\omega\sin\tau\omega - (b_1\omega^2 + b_3)\cos\tau\omega \tag{9}$$

$$a_1\omega = b_2\cos\tau\omega + (b_1\omega^2 + b_3)\sin\tau\omega. \tag{10}$$

Squaring and adding (9) and (10), we get

$$\omega^6 - b_1^2\omega^4 + (a_1^2 - b^2 - 2b_1b_3)\omega^2 + b_3^2 = 0. \tag{11}$$

Put $b_1^2 = a, (a_1^2 - b^2 - 2b_1b_3) = b, b_3^2 = c$ and $\omega^2 = z$, we get

$$z^3 - az^2 + bz + c = 0. \tag{12}$$

Equation (12) has at least one real positive root if $c < 0$.

Suppose $k(z) = z^3 - az^2 + bz + c$

$$k(0) = c < 0, \lim_{z\to\infty} k(z) = \infty, \exists\, z_0 \in (0, \infty)$$

$k(z_0) = 0$, if

Equation (12) has positive at least one positive root iff $c \geq 0$.

$$A = a^2 - 3b \geq 0$$

$$k(z) = z^3 - az^2 + bz + c$$

$$k'(z) = 3z^2 - 2az + b$$

$$k'(z) = 0 \to 3z^2 - 2az + b = 0 \tag{13}$$

$$z_{1,2} = \frac{2a \mp \sqrt{4a^2 - 12b}}{6} = \frac{a \mp \sqrt{A}}{3}. \tag{14}$$

Equation (13) has doesn't any real root if $A < 0, k(z)$ is monotone increasing fuxn in z.

$k(z) = c \geq 0$, Eq (11) has no positive root.

Clearly if $A \geq 0$, then $z_1 = \frac{a+\sqrt{A}}{3}$ is least possible of $k(z)$.

If $c \geq 0$ then Eq (12) is absolute if $z_1 > 0$ and $k(z_1) > 0$.

Suppose that each of two $z_1 \leq 0$ or $z_1 > 0$ and $k(z_1) > 0$.

If $z_1 \leq 0, \sin k(z)$ is increasing for $z \geq z_1$ and $k(0) = c \geq 0$.

Consequently, it can be deduced that k(z) does not possess any positive real zeros. If $z_1 > 0$ and $k(z_1) > 0, \sin z_2 = \frac{-a+\sqrt{A}}{3}$ is superlative value.

It follows that $k(z_1) \leq k(z_2)$ $k(0) = c \geq 0$, $k(z)$ has no positive root.

Lemma 1.

The variable denoted as z_1 is determined by the Eq (14).

(1) If $c < 0$, there exists at least one positive real root in Eq (12).

(2) If $c \geq 0$ and $A = a^2 - 3b < 0$, If the condition holds, Eq (12) does not possess any positive roots.

(3) If $c \geq 0$, then Eq (11) has positive root if $z_1 > 0$ and $k(z_1) \leq 0$.

Let's assume that Eq (12) possesses a positive root. Without loss of generality (WLOG), we can consider three positive roots, namely z_1, z_2, z_3. Consequently, Eq (11) will also have three positive roots:

$$\omega_1 = \sqrt{z_1},\ \omega_2 = \sqrt{z_2}, \omega_3 = \sqrt{z_3}.$$

From Eq (10)

$$\sin \omega\tau = \frac{b_1\omega - \omega^3}{d}$$

$$\tau = \frac{1}{\omega}\left[\sin^{-1}\frac{(b_1\omega - \omega^3)}{d} + 2(l-1)\pi\right] : l = 1,2,3$$

$$\tau_m^{(l)} = \frac{1}{\omega m}\left[\sin^{-1}\frac{(b_1\omega - \omega^3 k)}{d} + 2(l-1)\pi\right] : m = 1,2,3, \quad \mathrm{l} = 0,1,2 \dots \dots$$

As a result, the roots of Eq (11) would consist of a pair of purely imaginary numbers. When $\tau = \tau_m^{(l)}, m = 1,2,3; l = 0,1,2 \dots ..$

$$\lim_{j\to\infty} \tau_m^{(l)} = \infty, \qquad \mathrm{m} = 1,2,3,4$$

$$\tau_0 = \tau_{m_0}^{(l_0)} = \min_{1\le m\le 3, l\ge 1}[\tau_m^{(l)}], \omega_0 = \omega_{m_0}, y_0 = y_{m_0}. \tag{15}$$

Lemma 2. Suppose that $a_1 \ge 0, (c_1 + d),\ b_1(c_1 - d) > 0$.

(1) If $c \ge 0$ and $A = a^2 - 3b < 0$, then all the roots of Eq (7) will have a negative real part for all $\tau \ge 0$.

(2) If $c < 0\ or\ c \ge 0, z_1 > 0$ and $k(z_1) \le 0$, then all the roots of Eq (7) will have a negative real part for all values of τ in the interval $\tau\, \epsilon (0, \tau_0)$.

Proof. When τ is equal to zero, Eq (7) transforms into

$$\lambda^3 + (a_1 + a_2)\lambda^2 + (b_1 + b_2)\lambda + (c_1 + c_2) = 0. \tag{16}$$

According to Routh-Hurwitz's Criteria:

The condition for all roots of Eq (8) to have a negative real part is if and only if

$$(c_1 + c_2) \ge 0, (a_1 + a_2)(b_1 + b_2) - (c_1 + c_2) > 0.$$

If $c \ge 0$ and $A = a^2 - 3b < 0$.

Lemma 1 (2) show that Eq (7) has no roots with zero real part $\forall\, \tau \ge 0$.

When $c < 0\ or\ c \ge 0, z_1 > 0$ and $k(z_1) \le 0$.

Lemma 1 (1) and (2) implies when $\tau \ne \tau_m^{(l)}, m = 1,2,3, l \ge 1$, Eq (7) does not possess any roots with a zero real part, and the minimum value of τ for which Eq (7) exhibits purely imaginary roots.

3. Numerical example

We can obtain insight into the system's behavior by analyzing the numerical results. We can see how changes in parameters or beginning circumstances alter the dynamics of demand and supply. Furthermore, we may investigate the implications of the collectability factor on the supply process and its impact on overall system behavior:

$$\alpha = 4, \beta = 0.1, P_d = 10, \beta_1 = 0.01, F_D = 1, \gamma = 0.01, P_S = 10, \delta = 2, F_S = 1.$$

4. Result and discussion

Figure 2 reflects that the supply is delayed below a critical value of $\tau < 3.6999$. The demand-supply surplus initial shows fluctuations for these perturbations but becomes stable in the long run. Figure 3 shows the period three market attractor. Figure 4 has same observation of the period three market, which is supported by the phase diagram. Figure 5 shows that whenever $\tau \ge 3.6999$, the hopf-bifurcation is observed, which means the entire dynamic will be involved into near ending cycles of pick growth recession and flash crash. Figure 6 shows that the attractor will never be at a stable equilibrium for the perturbation $\tau \ge 3.6999$. Figure 7 explained about the stability of (P, D, S) when the parametric value of β_1 varies.

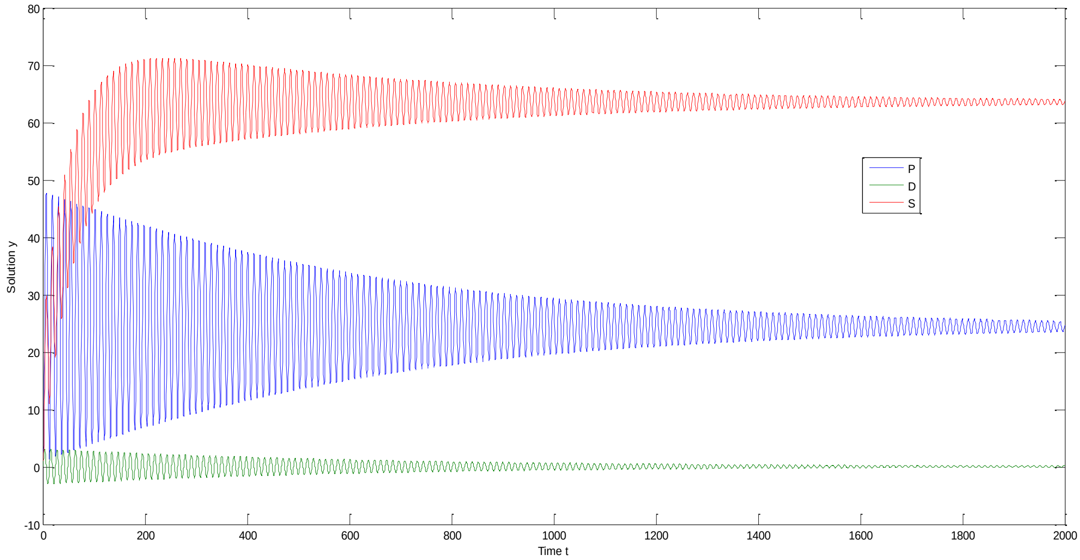

Figure 2. Asymptotic behavior of a demand-supply dynamic model at τ<3.6999.

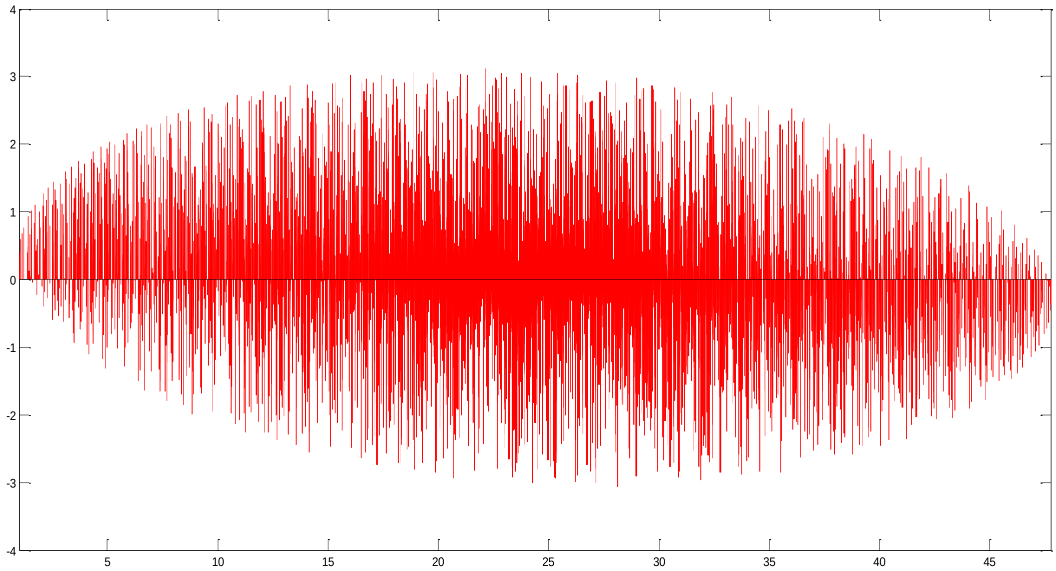

Figure 3. Bar phase diagram of a demand-supply dynamic model at $\tau < 3.6999$.

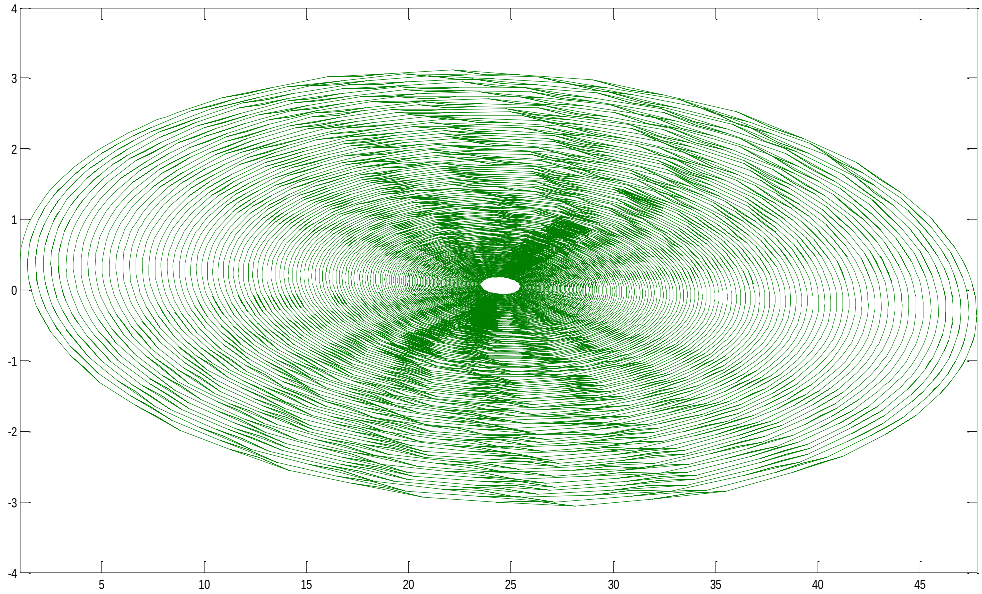

Figure 4. Phase diagram of a demand-supply dynamic model at $\tau < 3.6999$.

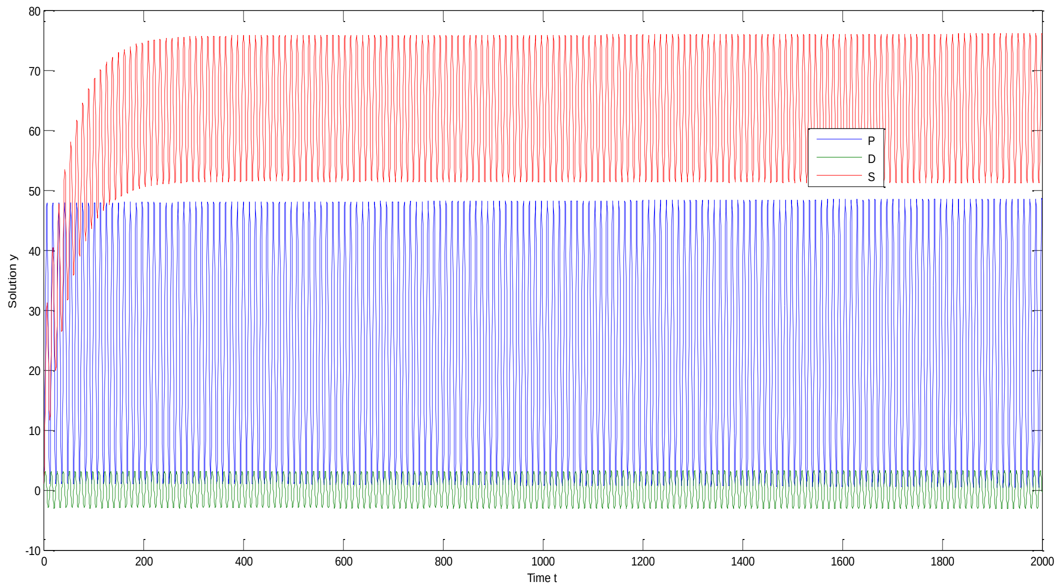

Figure 5. Hopf-bifurcation of a demand-supply dynamic model at $\tau \geq 3.6999$.

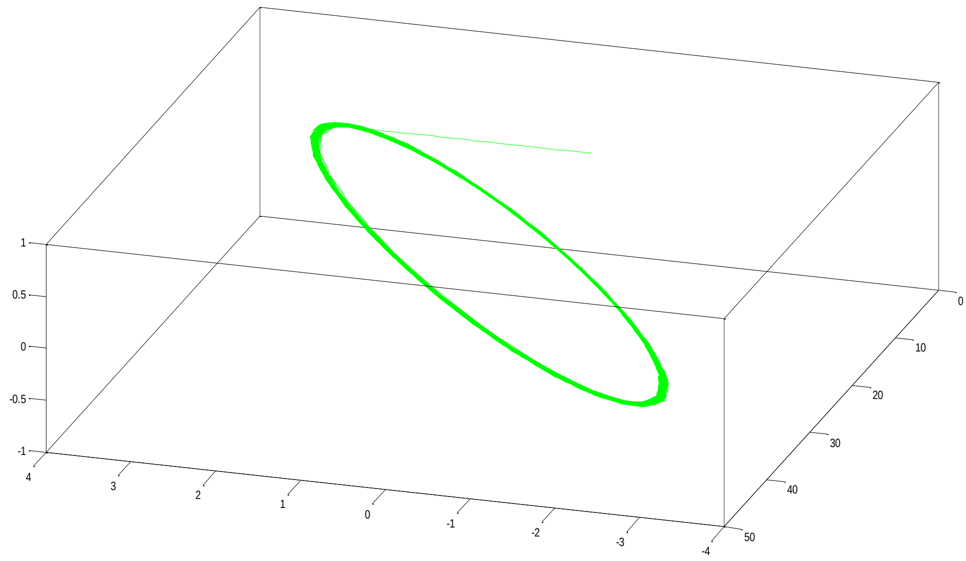

Figure 6. Phase diagram of a demand-supply dynamic model when $\tau \geq 3.6999$.

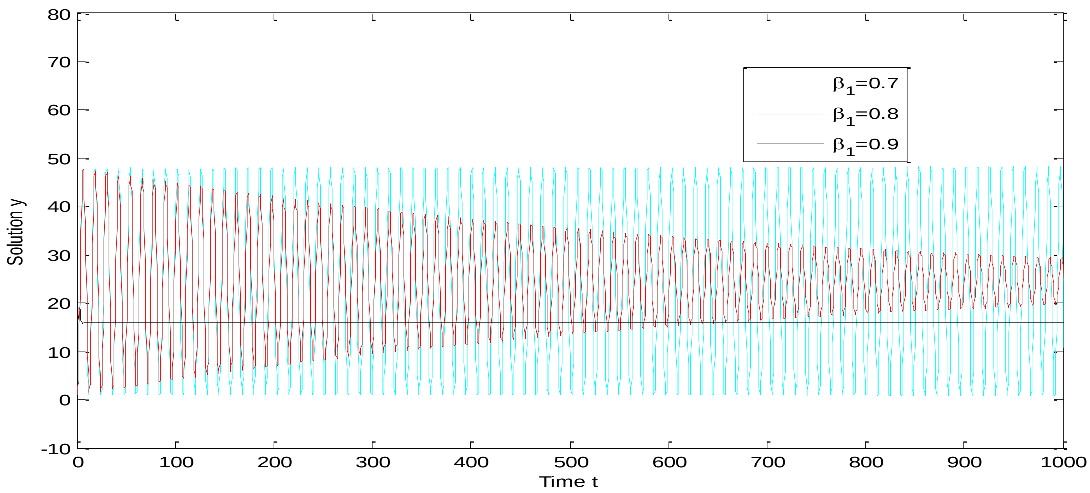

Figure 7. Sensitivity analysis (P, D, S) with respect to β_1.

5. Direction and stability of cyclic period

A collection of continuous functions showing a bifurcation phenomenon at the critical value of the positive steady state is created. The bifurcating periodic solutions' course, stability, and periodicity are emphasized. Using the basic approach and several reductions proposed by Hassard et al. [18], a particular formula will be created to investigate the properties of the Hopf-bifurcation at the complex level. The obtained equation will effectively allow the computation of the Hopf-bifurcation characteristics.

Let $V_1 = P - P^*, V_2 = D - D^*, V_3 = S - S^*$, and the system is altered by $t \to \frac{t}{\tau}$, to normalize the τ into

$$\frac{dV_1}{dt} = \alpha V_2 + \alpha D^* - \alpha V_3(t-1) - \alpha S^*(t-1)$$

$$\frac{dV_2}{dt} = \beta P_d - 2\beta\beta_1 P_d^2 + 6\beta\beta_1 P_d(V_1 + P^*) + 2\beta\beta_1 P_d V_1 - 4\beta\beta_1 P_d V_1 P^*$$
$$-2\beta\beta_1 P^{*2} - \beta\beta_1 V_1 P_d^2 - \beta\beta_1 (V_1 + P^*)^3 - \beta\beta_1 P^* P_d^2$$
$$\frac{dV_3}{dt} = -\gamma P_S + \gamma V_1 + \gamma P^* + \delta V_2 + \delta D^* - \delta V_3(t-1) - \delta S^*(t-1). \tag{17}$$

In this stage, we can handle the equation $A = A\big((-1,0), R_+^3\big)$. Without loss of generality, we represent he the critical value τ_l by τ_0. Let $\tau = \tau_0 + \upsilon$, where $\upsilon = 0$ corresponds to the hopf-bifurcation value of the system (13). For simplicity in notation, we rewrite (17) as:

$$\upsilon'(t) = E_\upsilon(\upsilon_t) + F(\upsilon, \upsilon_t). \tag{18}$$

Where $\upsilon(t) = (V_1(t), V_2(t), V_3(t))^T \epsilon R^3, \upsilon_t(\phi) \epsilon A$, and defined by $\upsilon_t(\phi) = \upsilon_t(t + \phi)$& $E_\upsilon: A \to R, F: R \times D \to R$ is given as:

$$E_\upsilon \chi = (\tau_0 + \upsilon)\begin{bmatrix} 0 & \alpha & 0 \\ b & 0 & 0 \\ \gamma & \delta & 0 \end{bmatrix}\begin{bmatrix} \chi_1(0) \\ \chi_2(0) \\ \chi_3(0) \end{bmatrix} + (\tau_0 + \upsilon)\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & \delta \end{bmatrix}\begin{bmatrix} \chi_1(-1) \\ \chi_2(-1) \\ \chi_3(-1) \end{bmatrix}.$$

Where $b = \beta\beta_1(6 + P_d(2 - P_d) - 3V_1(V_1 + 2P^*) - P^*(4 + 3P^*)$.

And $F(\upsilon, \phi) = (\tau_0 + \upsilon)\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}$.

Where,

$$F_1 = \alpha\chi_1^2(0) - (1 - D^*)\chi_1(0)\chi_3(0) + P^*\chi_2(0)\chi_3(0) + S^*\chi_1(0)\chi_2(0) + \chi_1(0)\chi_2(0)\chi_3(0)$$
$$F_2 = -\chi_2^2(0) - (1 - P^*)\chi_2(0)\chi_3(0) + D^*\chi_1(0)\chi_3(0) + S^*\chi_1(0)\chi_2(0) + \chi_1(0)\chi_2(0)\chi_3(0)$$
$$F_3 = \gamma\chi_3^2(0) + \delta\chi_1(-1)\chi_3(0) + \delta\chi_2(-1)\chi_3(0)$$
$$\chi(\phi) = \big(\chi_1(\phi), \chi_2(\phi), \chi_3(\phi)\big)^T \epsilon A(-1,0), R).$$

Using the Riesz Representation theorem, $\exists\ \zeta(\phi, \upsilon)$ of the bounded variation for $\phi \epsilon [-1,0)$, s.t.

$$E_\upsilon \chi = \int_{-1}^{0} d\zeta(\phi, 0)\phi(0)\ for\ \phi \epsilon A.$$

We can choose

$$\zeta(\phi, \upsilon) = (\tau_0 + \upsilon)\begin{bmatrix} 0 & \alpha & 0 \\ b & 0 & 0 \\ \gamma & \delta & 0 \end{bmatrix}\varsigma(\phi) + (\tau_0 + \upsilon)\begin{bmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & \delta \end{bmatrix}\varsigma(\phi + 1).$$

Here δ is the direct delta function. For $\chi \epsilon$ A$([-1,0], R_+^3)$, and calculate

$$\mathcal{R}(\upsilon)\chi = \begin{cases} \dfrac{d\chi(\phi)}{d\phi}, & \phi \in [-1,0) \\ \displaystyle\int_{-1}^{0} d\,\zeta(\phi, 0)\chi(\phi), & \phi = 0. \end{cases}\quad and\ R(\upsilon)\chi = \begin{cases} 0, & \phi \in [-1,0) \\ m(\upsilon, \chi) & \phi = 0. \end{cases}$$

Subsequently, the system (17) holds and equivalence to

$$v'(t) = \mathcal{R}(v)\chi + R(v)v_t, for \tag{19}$$

$$\eta \in A^1([-1,0], {R_+}^3)$$

$$\mathcal{R}^*\eta(s) = \begin{cases} -\frac{d\eta(s)}{ds}, & s \in [-1,0) \\ \int_{-1}^{0} d\,\zeta^T(-t,0)\eta(-t), & s = 0. \end{cases}.$$

Furthermore, the bilinear inner product

$$< \eta(s), \chi(\phi) > = \overline{\eta(0)}\chi(0) - \int_{-1}^{0}\int_{\varrho=\phi}^{\theta} \overline{\eta}(\varrho - \phi)d\zeta(\phi)\chi(\varrho)\,d\varrho. \tag{20}$$

$\mathcal{R}^*$ and $\mathcal{R} = \mathcal{R}(0)$ and the operators $\mathcal{R}^*$ and $i\omega_0$ represent the algebraic entities, with $i\omega_0$ being the eigenvalues of $\mathcal{R}(0)$. Consequently, these values serve as coefficient of $\mathcal{R}^*$. Consider the eigen vector $r(\phi) = r(0)e^{i\omega_0\phi}$ associated with eigen value $i\omega_0$. It follows that $\mathcal{R}(0) = i\omega_0 r(\phi)$. When $\phi = 0$, we arrive at the expression

$$\left[i\omega_0 L - \int_{-1}^{0} d\zeta(\phi)e^{i\omega_0\phi}\right] r(0) = 0, \text{ which option } r(0) = (1, \sigma_1, \rho_1)^T$$

$$\sigma_1 = \frac{(P^* - P^*D^*)D^*S^* + (D^* - P^*D^*)(i\omega_0 + \alpha P^*)}{P^*S^*(D^* - P^*D^*) - (P^* - P^*D^*)(i\omega_0 + bD^*)}$$

$$\rho_1 = \frac{P^*D^*S^{*2} - (i\omega_0 + \alpha P^*)(i\omega_0 + bD^*)}{P^*S^*(D^* - P^*D^*) - (P^* - P^*D^*)(i\omega_0 + bD^*)}.$$

Similarly, it can be calculated that $r^*(s) = M(1, \sigma_2, \rho_2)e^{i\omega_0\tau_0 s}$ is the eigen value of $\mathcal{R}^*$ corresponding to $-i\omega_0$, where

$$\sigma_2 = \frac{(P^* - P^*D^*)D^*S^* + (D^* - P^*D^*)(\alpha P^* - i\omega_0)}{P^*S^*(D^* - P^*D^*) - (P^* - P^*D^*)(bD^* - i\omega_0)}$$

$$\rho_2 = \frac{P^*D^*S^{*2} - (i\omega_0 + \alpha P^*)(bD^* - i\omega_0)}{P^*S^*(D^* - P^*D^*) - (P^* - P^*D^*)(bD^* - i\omega_0)}.$$

In order to assure $< r^*(s), r(\phi) > = 1$, and we calculate the value of M, from Eq (15),

$$< r^*(s), r(\phi) >$$

$$= \overline{M}(1, \overline{\sigma_2}, \overline{\rho_2},)(1, \sigma_1, \rho_1)^T - \int_{-1}^{0}\int_{\varrho=\phi}^{\phi} \overline{M}(1, \quad \overline{\sigma_2}, \overline{\rho_2},)e^{-i\omega_0\tau_0(\varrho=\phi)}d\zeta(\phi)(1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0}\,d\zeta$$

$$= \overline{M}\left\{1 + \sigma_1\overline{\sigma_2} + \overline{\rho_1\rho_2} - \int_{-1}^{0}(1, \overline{\sigma_2}, \overline{\rho_2},)\,\phi e^{i\omega_0\tau_0\phi}(1, \sigma_1, \rho_1)^T\right\}\overline{M}\{1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} + \tau_0\overline{\sigma_2}S^*(\mathcal{B}\rho_1 - g\sigma_1)e^{i\omega_0\tau_0}\}.$$

Hence, we choose

$$\overline{M} = \frac{1}{(1 + \sigma_1\overline{\sigma_2} + \rho_1\overline{\rho_2} + \tau_0\overline{\sigma_2}S^*(\mathcal{B}\rho_1 - g\sigma_1)e^{i\omega_0\tau_0})}$$

$$s.t. < r^*(s), r(\phi) > = 1, < r^*(s), \overline{r(\phi)} \; > = 0.$$

We've used the approach suggested by Hassard et al. [18] to figure out the specifics of the central surface. To achieve this, we used the same notations that were described in their work, A_0 at $\upsilon = 0$. In particular, we designate the result of Eq (14) as υ_t,when $\upsilon = 0$.

$$\mathcal{H}(t) =< r^*(s), \upsilon_t(\phi) >, W(t,\phi) = \upsilon_t(\phi) - 2Re\big(\mathcal{H}(t) r(\phi)\big). \tag{21}$$

The center manifold theory A_0

$$W(t,\phi) = W\left((r(t), \overline{r(t)}, \phi\right).$$

Where $W(r,\overline{r},\phi) = W_{20}(\phi)\frac{\mathcal{H}^2}{2} + W_{11}(\phi)\mathcal{H}\overline{\mathcal{H}} + W_{02}(\phi)\frac{\overline{\mathcal{H}}^2}{2} + \cdots$.

Let $r \; and \; \bar{r}$ represent the local coordinates associated with the center manifold A_0, which is aligned with the directions of r^* and $\overline{r^*}$, respectively. The parameter W is considered positive when the solution υ_t is positive. It is important to concentrate only on real solutions. For the specific of the Eq (15), $\upsilon_t \epsilon A_0$, when $\upsilon = 0$,

$$\begin{aligned}
&\mathcal{H}'(t) = i\omega_0\tau_0\mathcal{H} + < \bar{r}(\phi), F(0, W(\mathcal{H}, \overline{\mathcal{H}}, \phi) + 2Re(\mathcal{H}(t), r(\phi)) > \\
&= i\omega_0\tau_0\mathcal{H} + \overline{r^*}\,(0) F(0, W(\mathcal{H}, \overline{\mathcal{H}}, 0) + 2Re(\mathcal{H}(t) r(t) \\
&\cong i\omega_0\tau_0\mathcal{H} + \bar{r}^*(0) F_0(\mathcal{H}, \overline{\mathcal{H}}).
\end{aligned}$$

On rewriting this equation again:

$$\mathcal{H}'(t) \cong i\omega_0\tau_0\mathcal{H}(t) + c(\mathcal{H}, \overline{\mathcal{H}}). \tag{22}$$

Where $c(\mathcal{H}, \overline{\mathcal{H}}) = \bar{r}^*(0) F_0(\mathcal{H}, \overline{\mathcal{H}})$

$$c(\mathcal{H}, \overline{\mathcal{H}}) = c_{20}(\phi)\frac{\mathcal{H}^2}{2} + c_{11}(\phi)\mathcal{H}\overline{\mathcal{H}} + c_{21}(\phi)\frac{\mathcal{H}^2\overline{\mathcal{H}}}{2} + \cdots$$

Observing as $\upsilon_t(\phi) = (V_{1t}, V_{2t}, V_{3t}) = W(t,\phi) + \mathcal{H} r(\phi) + \overline{\mathcal{H}\overline{r(\phi)}}$
And $r(0) = M(1, \sigma_1, \rho_1)^T e^{i\omega_0\tau_0\phi}$, we have

$$V_{1t}(0) = \mathcal{H} + \overline{\mathcal{H}} + W_{20}^1(0)\frac{\mathcal{H}^2}{2} + W_{11}^1(0)\mathcal{H}\overline{\mathcal{H}} + W_{02}^1(0)\frac{\overline{\mathcal{H}}^2}{2} + \cdots$$

$$V_{2t}(0) = \sigma_1\mathcal{H} + \overline{\sigma_1\mathcal{H}} + W_{20}^2(0)\frac{\mathcal{H}^2}{2} + W_{11}^2(0)\mathcal{H}\overline{\mathcal{H}} + W_{02}^2(0)\frac{\overline{\mathcal{H}}^2}{2} + \cdots$$

$$V_{3t}(0) = \rho_{11}\mathcal{H} + \overline{\rho_{11}\mathcal{H}} + W_{20}^3(0)\frac{\mathcal{H}^2}{2} + W_{11}^3(0)\mathcal{H}\overline{\mathcal{H}} + W_{02}^3(0)\frac{\overline{\mathcal{H}}^2}{2} + \cdots$$

$$V_{1t}(-1) = \mathcal{H}e^{-i\omega_0\tau_0} + \overline{\mathcal{H}}e^{i\omega_0\tau_0} + W_{20}^1(-1)\frac{\mathcal{H}^2}{2} + W_{11}^1(-1)\mathcal{H}\overline{\mathcal{H}} + W_{02}^1(-1)\frac{\overline{\mathcal{H}}^2}{2} + \cdots$$

$$V_{2t}(-1) = \sigma_1\mathcal{H}e^{-i\omega_0\tau_0} + \overline{\sigma_1\mathcal{H}}e^{i\omega_0\tau_0} + W_{20}^2(-1)\frac{\mathcal{H}^2}{2} + W_{11}^2(-1)\mathcal{H}\overline{\mathcal{H}} + W_{02}^2(-1)\frac{\overline{\mathcal{H}}^2}{2} + \cdots$$

Consequently, by comparing coefficients with Eq (21), the following can be derived:

$$\begin{aligned}
c_{20} = -2\tau_0\overline{M}[\alpha + (1-D^*)\rho_1 - \sigma_1(P^*\rho_1 + S^*) + \overline{\sigma_1}(b\sigma_1^2 + (1-P^*)\sigma_1\rho_1)z - \sigma_1 S^* - \rho_1 D^*) \\
+ \overline{\sigma_2}\,\sigma_1(d\rho_1 - \delta e^{-i\omega_0\tau_0} - f\sigma_1 e^{-i\omega_0\tau_0}
\end{aligned}$$

$$c_{11} == -2\tau_0\bar{M}[+(1-D^*)Re\{\rho_1\} - P^*Re\{\overline{\rho_1}\sigma_1\} + S^*Re\{\rho_1\}$$
$$+\overline{\sigma_2}\,(\sigma_1\overline{\rho_1} - +(1-P^*)Re\{\sigma_1\overline{\rho_1}\} - D^*Re\{\overline{\rho_1}\} - S^*Re\{\sigma_1\})$$
$$+\overline{\rho_2}\big(d\rho_1\overline{\rho_1} - \delta Re\{\rho_1 e^{i\omega_0\tau_0}\} - fRe\{\overline{\sigma_1}\rho_1 e^{i\omega_0\tau_0}\}\big)]$$

$$c_{02} = -2\tau_0\bar{M}\big[\alpha + (1-D^*)\overline{\rho_1} - \overline{\sigma_1}(P^*\overline{\rho_1} + S^*) + \overline{\sigma_1}(b\overline{\sigma_1^2} + (1-P^*)\sigma_1\overline{\rho_1} - \overline{\sigma_1}S^* - \overline{\rho_1}D^*\big)$$
$$+\overline{\rho_1\rho_2}(d\overline{\rho_1} - \delta e^{i\omega_0\tau_0} - f\overline{\sigma_1}e^{-i\omega_0\tau_0})]$$

$$c_{21} = -2\tau_0\bar{M}[\alpha\big(W_{20}^1(0) + 2W_{11}^1(0)\big) + (1-D^*)\left(\tfrac{1}{2}W_{20}^1(0)\overline{\rho_1} + W_{11}^1(0)\rho_1 + \tfrac{1}{2}W_{20}^3(0) + W_{11}^3(0)\right) - (2Re(\sigma_1\rho_1) - P^*\left(\tfrac{1}{2}W_{20}^2(0)\overline{\rho_1} + \tfrac{1}{2}W_{20}^3(0)\overline{\rho_1} + W_{11}^1(0)\rho_1 + W_{11}^3(0)\rho_1\right) - S^*(\tfrac{1}{2}W_{20}^2(0) + \tfrac{1}{2}W_{20}^1(0)\overline{\sigma_1} + W_{11}^1(0)\sigma_1 + \overline{\sigma_2}\Big(-(W_{20}^2(0)\overline{\sigma_1} + 2W_{11}^2(0)\sigma_1) + (1-P^*)\left(\tfrac{1}{2}W_{20}^2(0)\overline{\sigma_1} + W_{11}^1(0)\rho_1 + W_{11}^3(0)\overline{\sigma_1}\right) - (2Re\{\overline{\rho_1}\sigma_1\} + \rho_1\sigma_1) - D^*\left(\tfrac{1}{2}W_{20}^1(0)\overline{\rho_1} + \tfrac{1}{2}W_{20}^3(0) + W_{11}^1(0)\rho_1 + W_{11}^3(0)\right) - S^*\left(\tfrac{1}{2}W_{20}^2(0) + W_{20}^1(0)\overline{\sigma_1} + W_{11}^2(0) + W_{11}^1(0)\sigma_1\right) + \overline{\rho_2}\Big(\delta(W_{20}^3(0)\overline{\rho_1} + 2W_{20}^3(0)\rho_1) - \delta(W_{20}^1(-1)\overline{\rho_1}) + W_{11}^1(-1)\rho_1 + \tfrac{1}{2}W_{20}^3(0)e^{i\omega_0\tau_0} + W_{11}^3(0)e^{-i\omega_0\tau_0}\Big) - \delta\left(\tfrac{1}{2}W_{20}^1(-1)\overline{\rho_1} + W_{11}^2(-1)\rho_1 + \tfrac{1}{2}W_{20}^3(0)\overline{\sigma_1}e^{i\omega_0\tau_0} + W_{11}^3(0)\sigma_1 e^{-i\omega_0\tau_0}\right)\Big)].$$

To determine the value of c_{21}, it is important to compute $W_{20}(\phi)$ & $W_{11}(\phi)$ as per Eqs (19) and (21) respectively:

$$W' = v_t - \mathcal{H}'r - \overline{\mathcal{H}'}r = \begin{cases}\mathcal{R}W - 2Re\{\bar{r}^*(0)F_0 r(\phi)\}, & \phi\epsilon[-1,0) \\ \mathcal{R}W - 2Re\{\bar{r}^*(0)F_0 r(0)\} + F_0, & \phi = 0\end{cases}.$$

Let

$$W' = \mathcal{R}W + K. \tag{23}$$

Where,

$$K(\mathcal{H},\bar{\mathcal{H}},\phi) = K_{20}(\phi)\frac{\mathcal{H}^2}{2} + K_{11}(\phi)\mathcal{H}\bar{\mathcal{H}} + K_{02}(\phi)\frac{\bar{\mathcal{H}}^2}{2} + K_{21}(\phi)\frac{\bar{\mathcal{H}}^2\bar{\mathcal{H}}}{2}\ldots \tag{24}$$

Alternatively, when considering A_0 in the vicinity of the origin,

$$W' = W_{\mathcal{H}}\mathcal{H}' + W_{\mathcal{H}}\bar{\mathcal{H}}.$$

By expanding the above series and calculating the coefficients, we obtain the following results:

$$[\mathcal{R} - 2i\omega_0 L]W_{20}(\phi) = -K_{20}(\phi), \mathcal{R}W_{11}(\phi) = -K_{11}(\phi). \tag{25}$$

From Eq (18), we know $\phi\epsilon[-1,0)$,

$$K(\mathcal{H},\bar{\mathcal{H}},\phi) = -\bar{r^*}(0)\overline{F_0}r(\phi) - \bar{r^*}(0)\overline{F_0 r(\phi)} = -cr(\phi) - \bar{c}\overline{r(\phi)}.$$

Comparing the coefficient with Eq (20), for $\phi\epsilon[-1,0)$ we get

$$K_{20}(\phi) = -c_{20}r(\phi) - \overline{c_{02}}\,\overline{r(\phi)}$$

$$K_{11}(\phi) = -c_{11}r(\phi) - \overline{c_{11}}\,\overline{r(\phi)}.$$

From Eqs (23) and (25) and definition of $\mathcal{R}$, we get

$$W_{20}(\phi) = 2i\omega_0\tau_0 W_{20}(\phi) + c_{20}(\phi) + \overline{c_{02}}\,\overline{r(\phi)}.$$

Further examine the $W_{20}(\phi)$:

$$W_{20}(\phi) = \frac{ic_{20}}{\omega_0\tau_0} r(0)e^{i\omega_0\tau_0\phi} + \frac{i\overline{c_{20}}}{3\omega_0\tau_0} r(0)e^{-i\omega_0\tau_0\phi} + A_1 e^{2i\omega_0\tau_0\phi}.$$

Similarly,

$$W_{11}(\phi) = \frac{ic_{11}}{\omega_0\tau_0} r(0)e^{i\omega_0\tau_0\phi} + \frac{i\overline{c_{11}}}{\omega_0\tau_0} \bar{r}\,(0)e^{-i\omega_0\tau_0\phi} + A_2.$$

Here A_1 & A_2 repreent three-dimensional vectors, and their values can be computed by substituting $\phi = 0$ in K. Indeed, as a matter of fact

$$K(\mathcal{H}, \bar{\mathcal{H}}, \phi) = 2Re\{\bar{r}^*(0)F_0 r(0)\} + F_0.$$

We get,

$$K_{20}(\phi) = -c_{20}r(\phi) - \overline{c_{02}}\,\overline{r(\phi)} + F_{H^2}$$

$$K_{11}(\phi) = -c_{11}r(\phi) - \overline{c_{11}}\,\overline{r(\phi)} + F_{\mathcal{H}\bar{\mathcal{H}}}.$$

Where $F_0 = F_{H^2}\frac{\mathcal{H}^2}{2} + F_{\mathcal{H}\bar{\mathcal{H}}}\mathcal{H}\bar{\mathcal{H}} + \cdots$

With the def. of $\mathcal{R}$,

$$\int_{-1}^{0} d\,\zeta(\phi)W_{20}(\phi) = 2i\omega_0\tau_0 W_{20}(0) + c_{20}r(0) + \overline{c_{02}}\,\overline{r(0)} - F_{H^2}.$$

Furthermore,

$$\int_{-1}^{0} d\,\zeta(\phi,)W_{11}(\phi) = c_{11}r(0) + \overline{c_{11}}\,\overline{r(0)} - F_{\mathcal{H}\bar{\mathcal{H}}}$$

$$\left[i\omega_0\tau_0 L - \int_{-1}^{0} e^{i\omega_0\tau_0\phi} d\zeta(\phi)\right] r(0) = 0 \; and \; \left[-i\omega_0\tau_0 L - \int_{-1}^{0} e^{-i\omega_0\tau_0\phi} d\zeta(\phi)\right]\overline{r(0)} = 0.$$

This shows

$$\left[2i\omega_0\tau_0 L - \int_{-1}^{0} e^{2i\omega_0\tau_0\phi} d\zeta(\phi)\right] A_1 = F_{H^2} \; and - \left[\int_{-1}^{0} d\zeta(\phi)\right] A_2 = F_{\mathcal{H}\bar{\mathcal{H}}}.$$

Hence,

$$\begin{bmatrix} 2i\omega_0 + \alpha P^* & -P^*S^* & P^* - P^*D^* \\ -D^*S^* & 2i\omega_0 + bD^* & D^* - P^*D^* \\ \delta S^* e^{-2i\omega_0\tau_0} & -\delta S^* e^{-2i\omega_0\tau_0} & (2i\omega_0 + \delta S^*) \end{bmatrix} A_1$$

$$= -2\begin{bmatrix} \alpha + (1 - D^*)\rho_1 - \sigma_1(P^*\rho_1 + S^*) \\ b\sigma_1^* + (1 - P^*)\sigma_1\rho_1 - \sigma_1 S^* - \rho_2 D^* \\ \rho_1(\delta\rho_1 - \delta e^{-i\omega_0\tau_0} - \delta\sigma_1 e^{-i\omega_0\tau_0} \end{bmatrix}$$

$$\begin{bmatrix} \alpha P^* & -P^*S^* & P^* - P^*D^* \\ -D^*S^* & -D^* & D^* - P^*D^* \\ \delta S^* & -\delta S^* & \delta S^* \end{bmatrix} A_2$$

$$= -2\begin{bmatrix} \alpha + (1 - D^*)Re\{\rho_1\} - P^*Re\{\overline{\rho_1}\sigma_1\} + S^*Re\{\sigma_1\}) \\ -\overline{\sigma_1}\sigma_1 + (1 - P^*)Re\{\overline{\rho_1}\sigma_1\} - D^*Re\{\overline{\rho_1}\} - S^*Re\{\sigma_1\} \\ \delta\rho_1\overline{\rho_1} - \delta Re\{\rho_1\} - \delta Re\{\overline{\rho_1}\sigma_1\}e^{i\omega_0\tau_0} \end{bmatrix}.$$

Thus, parameter c_{21} is demonstrate. Upon analyzing the information provided above, each c_{ij} can be evaluated using the parameters, leading to the computation of the following values:

$$\begin{gathered} J_1(0) = \frac{i}{2\omega_0\tau_0}\left(c_{11}c_{20} - 2|c_{11}|^2 - \frac{|c_{02}|^2}{3}\right) + \frac{c_{21}}{2}, \\ \kappa_2 = -\frac{Re\{J_1(0)\}}{Re(\lambda'(\tau_0))}, \\ \iota_2 = 2Re\{J_1(0)\}, \\ T_2 = -\frac{\mathrm{Im}\{J_1(0)\} + \kappa_2 Im\{\lambda'(\tau_0)\}}{\omega_0\tau_0} \end{gathered}. \tag{26}$$

Theorem 1. The value of κ_2 dictates the direction of the Hopf-bifurcation. The Hopf-bifurcation is in a supercritical state when $\kappa_2 > 0$ and a subcritical state when $\kappa_2 < 0$. As a result, for $\tau > \tau_0$ and $\tau < \tau_0$, there are bifurcating periodic solutions, respectively. The value of ι_2 determines how stable the bifurcating solutions are. If ι_2 and unstable if $\iota_2 > 0$, bifurcating periodic solutions are orbitally asymptotically stable. The value of T_2 affects how long the bifurcating periodic solutions last. If $T_2 > 0$ ($T_2 < 0$), the period grows and vice versa.

6. Sensitivity analysis

To determine the coefficient of generalized sensitivity, Rihan [19], and Thomaseth and Cobelli [20] employ the "direct method". This direct approach involves making the assumption of fixed parameters and subsequently computing the sensitivity coefficient $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta_1}, \boldsymbol{\gamma}, \boldsymbol{\delta})$ through utilization of the sensitivity equation along with the initial solution of Eqs (1)–(3). By focusing on a specific criterion, say γ1, the partial derivatives of the solution $(\boldsymbol{P}, \boldsymbol{D}, \boldsymbol{S})$ with respect to $\boldsymbol{\beta_1}$ yield the criteria for the sensitivity Eqs (27)–(29), as presented below:

$$\frac{dS_1}{dt} = \alpha[S_2 - S_3(t-\tau)] \tag{27}$$

$$\frac{dS_2}{dt} = [-\beta - \beta\beta_1(6P_dP - P_d^2 + 3P^2)]S_1 \tag{28}$$

$$\frac{dS_3}{dt} = -\gamma S_1 + \delta S_2 - \delta S_3(t-\tau). \tag{29}$$

Where $S_1 = \frac{\partial P}{\partial \beta_1}, S_2 = \frac{\partial D}{\partial \beta_1}, S_3 = \frac{\partial S}{\partial \beta_1}$.

By evaluating Eqs (27)–(29) with respect to parameter β_1, we were also able to investigate the effects of the systems (1)–(3) on the variables (P, D, S). We kept all other factors constant while performing a sensitivity analysis for the variables (P, D, S) pertaining to β_1. It is interesting that the system remained stable throughout this time, even though β_1 varied between 0.7 and 0.9.

Similarly, we can calculate the sensitivity analysis $(\alpha, \beta, \gamma, \delta)$.

7. Model with fractional operators

The same model is also studied using different fractional derivative operators. In addition, to guarantee that the left and right sides of the ensuing fractional model have the same dimension $(time)^{-\eta}$, all the parameters having the dimension $(time)^{-1}$ are replaced by powers of v, while the other parameters remain unchanged. There is significance associated with various operators of fractional derivatives. Riemann-Liouville proposed the definition with singular kernel [21], which was the first and most widely accepted definition in the field of fractional calculus. The definition of the derivative of fractional order with the non-singular kernel was presented by Caputo [21]. Since this definition is based on the idea of power law, it is inapplicable to issues when the fading memory process is present. Caputo and Fabricio's formulation [22] represent the next advancement in this field, as it addresses processes that display fading memory processes with exponential decay and the Delta-Dirac characteristic. Last, the definition of fractional derivatives and integrals that can easily handle the process by exhibiting a passage from fading memory to power law have been introduced by Atangana [23,26–31]. The proposed fractional mathematical model in the context of Caputo and C-F fractional operators is now examined here.

7.1. Model in Caputo fractional derivative framework

The demand-supply dynamic is mathematically represented by the following system of first order non-linear delay differential equation with conformal operators; where $\alpha, \beta, \gamma, \delta, \beta_1$ are all positive parameters and $0 < \varepsilon \leq 1$. Consider the model in Caputo fractional derivative framework as follows:

$$\mathscr{b}_{\zeta_t^{\eta}} P = \alpha^{\eta}[D - S], P(0) = P_0 \tag{30}$$

$$\mathscr{b}_{\zeta_t^{\eta}} D = \beta^{\eta}(P_d - P)\left[1 - \beta_1^{\eta}(P_d - P)^2\right] + F_d, D(0) = D_0 \tag{31}$$

$$\mathscr{b}_{\zeta_t^{\eta}} S = -\gamma^{\eta}(P_s - P) + \delta^{\eta}[D - S] + F_s, S(0) = S_0. \tag{32}$$

Here, $'\mathscr{b}_{\zeta_t^{\eta}}'$ denotes the Caputo derivative of fractional order $'\eta'$ and for calculating the fractional derivatives, it is assumed that $S(t - \tau) \approx S(t)$.

Definition 1. Let $'f'$ be an integrable function on $'R'$. If there exists a number $0 < \eta < 1$, then the fractional Caputo derivative with order $'\eta'$ is given by:

$$\mathscr{b}_{\zeta_t^{\eta}} f(t) = \frac{1}{\omega(1-\eta)} \int_0^t \frac{1}{(t-\varphi)^{\eta}} \frac{d}{d\varphi} f(\varphi) d\varphi \tag{33}$$

where $'\mathscr{b}_{\zeta_t^{\eta}}'$ denotes the Caputo derivative of fractional order η.

Theorem 2. The Cauchy problem with the Caputo derivative allows a unique solution [24] if the following two conditions hold for two positive constants $\mathcal{L}$ and $\bar{\mathcal{L}}$:

(I) Lipchitz condition: $\left|g\left(t, v_1(t)\right) - g\left(t, v_2(t)\right)\right| \leq \mathcal{L}|v_1 - v_2|$ for all $v_1, v_2 \in R$ and for all $t \in [t_0, T]$.

(II) Linear growth condition: $|g(t, v)|^2 \leq \bar{\mathcal{L}}(1 + |v|^2)$.

7.2. Model in the Caputo fractional derivative framework

From (30)–(32) to find the numerical solution [26], consider

$$^{\mathscr{b}}_{0}\zeta_t^{\eta} \psi(t) = k\left(t, \psi(t)\right), t \geq 0, \psi(0) = \psi_0. \tag{34}$$

Using the fundamental theorem, we can rewrite the above equation as:

$$\psi(t) = \psi(0) + \frac{1}{\omega(\eta)} \int_0^t (t - \varphi)^{\eta - 1} k(\psi, \varphi) d\varphi. \tag{35}$$

At $t = t_{p+1}$, we have

$$\psi\left(t_{p+1}\right) = \psi(0) + \frac{1}{\omega(\eta)} \int_0^{t_{p+1}} \left(t_{p+1} - \varphi\right)^{\eta - 1} k(\psi, \varphi) d\varphi. \tag{36}$$

At $t = t_p$, we have

$$\psi(t_p) = \psi(0) + \frac{1}{\omega(\eta)} \int_0^{t_p} (t_p - \varphi)^{\eta-1} k(\psi, \varphi) d\varphi. \tag{37}$$

From the above two equations, we have

$$\psi(t_{p+1}) - \psi(t_p) = \frac{1}{\omega(\eta)} \left[\int_0^{t_{p+1}} (t_{p+1} - \varphi)^{\eta-1} k(\psi, \varphi) d\varphi - \int_0^{t_p} (t_p - \varphi)^{\eta-1} k(\psi, \varphi) d\varphi \right]. \tag{38}$$

The application of the handled scheme with Lagrange polynomial interpolation gives the following numerical iteration formula.

$$P_{p+1} = \frac{\eta h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_1(t_m, P_m)[(p-m+1)^\eta (p-m+2+2\eta) - (p-m)^\eta (p-m+2+2\eta)] - \frac{h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_1(t_{m-1}, P_{m-1})[(p-m+1)^{\eta+1} - (p-m)^\eta (p-m+1+\eta)]. \tag{39}$$

Where,

$$f_1(t, P) = \alpha^\eta [D - S] \tag{40}$$

$$D_{p+1} = \frac{\eta h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_2(t_m, D_m)[(p-m+1)^\eta (p-m+2+2\eta) - (p-m)^\eta (p-m+2+2\eta)] - \frac{h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_2(t_{m-1}, D_{m-1})[(p-m+1)^{\eta+1} - (p-m)^\eta (p-m+1+\eta)]. \tag{41}$$

Where,

$$f_2(t, D) = \beta^\eta (P_d - P)\left[1 - \beta_1^\eta (P_d - P)^2\right] + F_d. \tag{42}$$

$$S_{p+1} = \frac{\eta h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_3(t_m, S_m)[(p-m+1)^\eta (p-m+2+2\eta) - (p-m)^\eta (p-m+2+2\eta)] - \frac{h^\eta}{\omega(\eta+2)} \sum_{m=0}^{r} f_3(t_{m-1}, S_{m-1})[(p-m+1)^{\eta+1} - (p-m)^\eta (p-m+1+\eta)] \tag{43}$$

where $f_3(t, S) = -\gamma^\eta (P_s - P) + \delta^\eta [D - S] + F_s$.

8. Conclusions

We present a mathematical model that describes the dynamics of demand and supply, which genialized the Marshall model and collectability factor. Collectability is seen more often in reality, and many stocks are overvalued. According to the Marshall model, market equilibrium is the only global attractor. When the delay parameter crosses are below the critical value $\tau < 3.6999$, the entire demand-supply surplus initial shows fluctuations for these perturbations and when delay parameter cross the critical value ($\tau \geq 3.6999$), market enters the danger zone i.e., small perturbations lead to regular market fluctuations around market equilibrium, recession, large growth cycles, and flash crash. The caputo operator is used to examine the frcational part of the demand and supply dynamics. The recently proposed model includes a wider variety of market phenomena connected to supply and demand dynamics. Further, the stability of fluctuations around market equilibrium, recession, large growth cycles, and flash crash is also examined. Thus, this model offers insights into the intricacies of real-world markets and their directional analysis to various disturbances by integrating the collectability component and examining the influence of the delay parameter.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no conflict of interest.

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