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Research article

Novel categories of supra soft continuous maps via new soft operators

Alaa M. Abd El-latif[1,2] **and Mesfer H. Alqahtani**[3,*]

[1] Mathematics Department, College of Sciences and Arts, Northern Border University, Rafha 91911, Saudi Arabia

[2] Mathematics Department, Faculty of Education, Ain Shams University, Roxy, 11341, Cairo, Egypt

[3] Department of Mathematics, University College of Umluj, University of Tabuk, Tabuk 48322, Saudi Arabia

* **Correspondence:** Email: m_halqahtani@ut.edu.sa.

Abstract: In this paper we continue presenting new types of soft operators for supra soft topological spaces (or SSTSs). Specifically, we investigate more interesting properties and relationships between the supra soft somewhere dense interior (or SS-sd-interior) operator, the SS-sd-closure operator, the SS-sd-cluster operator, and the SS-sd-boundary operator. We prove that the SS-sd-interior operator, SS-sd-boundary operator, and SS-sd-exterior operator form a partition for the absolute soft set. Furthermore, we apply the notion of SS-sd-sets to soft continuity. In addition, we use the SS-sd-interior operator and the SS-sd-closure operator to provide equivalent conditions and many characterizations for SS-sd-continuous, SS-sd-irresolute, SS-sd-open, SS-sd-closed, and SS-sd-homeomorphism maps. Examples include the following: The soft mapping is an SS-sd-homeomorphism if, and only if it is both SS-sd-continuous and an SS-sd-closed if, and only if, the soft mapping in addition to its inverse is SS-sd-continuous. Moreover, a bijective soft mapping is SS-sd-open if, and only if, it is SS-sd-closed. Furthermore, we provide many examples and counterexamples to show our results, which are extensions of previous studies. A diagram summarizing our results is also introduced.

Keywords: supra soft sd-operators; supra soft sd-interior points; supra soft sd-continuous maps; supra soft sd-homeomorphism maps; applications
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1. Introduction

In [1, 2] the definition of somewhere dense sets was introduced in general topology to contain most of all old classes of generalized open sets. Al-Shami [3] investigated more characterizations for this class. Mashhour et al. [4] introduced the definition of supra topological spaces. Kozae et al. [5]

applied these spaces to digital spaces in 2016. Al-shami and Alshammari [6] presented new rough set models and operators by using the supra spaces. Al-shami [7] successfully applied the notion of somewhere dense sets to improve the approximations and accuracy measure of a rough set, which has been extended in [8].

Molodtsov [9] introduced the concept of soft sets in 1999 to deal with uncertainties. Maji et al. [10] investigated a deep framework for the theory of soft sets. Ahmad and Kharal [11] defined the notion of soft continuity in 2011, which was later investigated in [12,13]. Some classes of soft functions defined by soft open sets modulo soft sets of the first category have been presented in [14].

Shabir and Naz [15], and Çagman et al. [16] presented the concepts of the soft topological spaces (or STSs) in 2011. After that, the researchers introduced several kinds of soft open sets and forms of soft continuity, named soft $g\beta$-closed sets [17], soft semi-open sets and soft semi irresolute soft functions [18, 19], soft b-open sets and soft b-continuous functions [20, 21], and decompositions of several types of soft continuity [22].

The definitions of soft ideal and soft local functions were introduced in [23]. After that, various authors [24, 25] used the definition of soft semi-open sets to introduce the notion of soft semi local functions. Abd El-latif [26] used the soft ideal to define new soft ideal rough topological spaces. Many weaker classes of soft open sets have been generalized by applying the soft ideal notion [27–32]. Later, novel versions of soft separation axioms [33, 34], soft connectedness [35], and soft semi-compactness [36] based on soft ideals were presented.

Al-Shami [37] introduced the concept of soft sd-sets as a generalization to several old kinds of weaker forms of soft open sets. He and his co-authors [38] used this new notion to define different kinds of soft functions. Ameen et al. [39] investigated these new types of soft continuity for soft Baire spaces. Asaad et al. [40] introduced the notion of soft open functions and soft Baire spaces based on soft sd-sets.

El-Shafei and Al-Shami [41] studied the connectedness by using a soft sd-closure operator. Al Ghour, in 2023, introduced new weaker versions of soft continuity, named soft c-continuity and soft nearly c-continuity [42], as well as soft semi ω-open functions [43].

El-Sheikh and Abd El-latif [44] defined the concept of SSTSs. Later, more investigation and topological properties were applied to SSTSs, particularly, supra slc-sets (continuity) [45], SS-δ_i-open sets and applications to soft continuity [46], SS-regular-closed sets (continuity) [47], SS-b-open soft sets (continuity) [48], SS-generalized closed sets (based on soft ideals) [49,50], and SSTSs defined by separation axioms [51,52].

Abd El-latif [53] introduced the notions of SS-sd-sets and SS-sc-sets. He presented some kinds of SS-operators, named the SS-sd-cluster (respectively, closure, interior) operator. In addition, he studied the relationships among them.

In this manuscript, we continue to investigate more characterizations of SS-operators as based on SS-sd-sets. In addition to introducing the SS-sd-operators, we study the relationships between them in detail. Since the null soft set is not SS-sd-set, we redefine these operators after excluding it. We provide many examples to confirm our findings; we also investigate some results and relationships from the literature. Furthermore, we present the notions of SS-sd-continuous maps, SS-sd-irresolute maps, SS-sd-open maps, SS-sd-closed maps, and SS-sd-homeomorphism maps. We study many of their equivalent conditions. A comparison of the corresponding results of previous studies is discussed and summarized in Figure 1.

2. Preliminaries

In this manuscrit, we follow the results and terminologies mentioned in [11, 13, 15, 44, 48, 53]. Let (U, μ, Θ) be an SSTS; the collections of SS- (respectively, semi-, regular-, β-, α-, pre-, b-) continuous maps will denoted by SS- (respectively, semi, regular-, β-, α-, pre-, b-) cts.

Definition 2.1. *[9] A pair (K, Θ), denoted by K_Θ, over the initial universe U and the set of parameters denoted by Θ, is called a soft set, which is a parameterized family of subsets of the universe U. i.e., $K_\Theta = \{K(\theta) : \theta \in \Theta,\ K : \Theta \to P(U)\}$. The family of all soft sets will be denoted by $S(U)_\Theta$.*

If $K(\theta) = \varphi$ (respectively, $K(\gamma) = U$) for all $\theta \in \Theta$, then (K, Θ) is called a null (respectively, an absolute) soft set and will be denoted by $\tilde{\varphi}$ (respectively, $\tilde{U}$).

Definition 2.2. *[15] Let $\tau \subseteq S(U)_\Theta$ be called a soft topology on U if τ contains $\tilde{U}$ and $\tilde{\varphi}$ and is closed under arbitrary soft unions and finite soft intersections. The triplet (U, τ, Θ) is called an STS over U. Also, the elements of τ are called soft open sets, and their soft complements are called soft closed sets.*

Definition 2.3. *[15, 54] Let (U, τ, Θ) be an STS and $(K, \Theta) \in S(U)_\Theta$. The soft closure of (K, Θ), denoted by $cl(K, \Theta)$, is the intersection of all soft closed supersets of (K, Θ). Also, the soft interior of (K, Θ), denoted by $int(G, \Theta)$, is the union of all soft open subsets of (K, Θ).*

Definition 2.4. *[15, 18] The soft set $(G, \Theta) \in S(U)_\Theta$ is called a soft point in $\tilde{U}$, denoted by s_θ, if there exist $s \in U$ and $\theta \in \Theta$ such that $G(\theta) = \{s\}$ and $G(\theta') = \varphi$ for each $\theta' \in \Theta - \{\theta\}$. Also, $s_\theta \tilde{\in} (F, \Theta)$ if for the element $\theta \in \Theta$, $G(\theta) \subseteq F(\theta)$.*

Theorem 2.5. *[11] For the soft function $\psi_{sd} : (U_1, \tau_1, \Theta_1) \to (U_2, \tau_2, \Theta_2)$, the following statements hold.*

(1) $\psi_{sd}^{-1}((N^{\tilde{c}}, \Theta_2)) = (\psi_{sd}^{-1}(N, \Theta_2))^{\tilde{c}}\ \forall\ (N, \Theta_2) \in S(U_2)_{\Theta_2}$.

(2) $\psi_{sd}(\psi_{sd}^{-1}((N, \Theta_2))) \tilde{\subseteq} (N, \Theta_2)\ \forall\ (N, \Theta_2) \in S(U_2)_{\Theta_2}$. *The equality holds if ψ_{sd} is surjective.*

(3) $(M, \Theta_1) \tilde{\subseteq} \psi_{sd}^{-1}(\psi_{sd}((M, \Theta_1)))\ \forall\ (M, \Theta_1) \in S(U_1)_{\Theta_1}$. *The equality holds if ψ_{sd} is injective.*

(4) $\psi_{sd}(\tilde{U}_1) \tilde{\subseteq} \tilde{U}_2$. *The equality holds if ψ_{sd} is surjective.*

Definition 2.6. *[44] The collection $\mu \subseteq S(U)_\Theta$ is called an SSTS on U if μ contains $\tilde{U}$ and $\tilde{\varphi}$ and is closed under arbitrary soft unions.*
The elements of μ are called SS-open sets and their soft complements are called SS-closed sets.
Also, the SS-interior of a soft subset (K, Θ) of $\tilde{U}$, denoted by $int^s(K, \Theta)$, is the soft union of all SS-open subsets of (K, Θ).
Moreover, the SS-closure of (K, Θ), denoted by $cl^s(K, \Theta)$, is the soft intersection of all SS-supersets of (K, Θ).
Furthermore, the SS-boundary of (K, Θ), denoted by $b^s(K, \Theta)$, where $b^s(K, \Theta) = cl^s(K, \Theta) - int^s(K, \Theta)$.

Definition 2.7. *[44] Let (U, τ, Θ) be an STS and (U, μ, Θ) be an SSTS . We say that μ is an SSTS associated with τ if $\tau \subset \mu$.*

Definition 2.8. *[44] A soft function $\psi_{sd} : (U_1, \tau_1, \Theta_1) \to (U_2, \tau_2, \Theta_2)$ with μ_1 as an associated SSTS with τ_1 is said to be an SS-cts if $\psi_{sd}^{-1}(G, \Theta_2) \in \mu_1\ \forall\ (G, \Theta_2) \in \tau_2$.*

Definition 2.9. *[53] Let* (U,μ,Θ) *be an SSTS and* $(K,\Theta)\in S(U)_\Theta$*; then,* (K,Θ) *is called an SS-sd-set if there exists* $\tilde{\varphi}\neq(O,\Theta)\in\mu$ *such that*

$$(O,\Theta)\tilde{\subseteq}cl^s[(O,\Theta)\tilde{\cap}(K,\Theta)].$$

The soft complement of an SS-sd-set is said to be an SS-sc-set. The family of all SS-sd-sets (respectively, SS-sc-sets) will denoted by $SD(U)_\Theta$ *(respectively,* $SC(U)_\Theta$*).*

Theorem 2.10. *[53] Let* (U,μ,Θ) *be an SSTS and* $(G,\Theta)\in S(U)_\Theta$*; then,* $(K,\Theta)\in SD(U)_\Theta$ *if and only if* $int^s(cl^s(K,\Theta))\neq\tilde{\varphi}$.

Theorem 2.11. *[53] Let* (U,μ,Θ) *be an SSTS and* $(G,\Theta)\in S(U)_\Theta$*; then,* $(K,\Theta)\in SC(U)_\Theta$ *if and only if there exists a proper SS-closed subset* (H,Θ) *of* $\tilde{U}$ *such that* $int^s(K,\Theta)\tilde{\subseteq}(H,\Theta)$.

Corollary 2.12. *[53] Every soft subset (superset) of an SS-sc-set (SS-sd-set) is an SS-sc-set (SS-sd-set).*

Proposition 2.13. *[53] A soft subset* (L,Θ) *of an SSTS* (U,μ,Θ) *is either an SS-sd-set or SS-sc-set.*

3. On soft operators based on supra soft sd-sets

In this section, we extend more properties for the SS-sd-interior operator, SS-sd-closure operator and SS-sd-cluster operator [53]. In addition, we present the SS-sd-boundary operator. Moreover, the relationships between these operators are studied and validated through the use of many examples and counterexamples. Furthermore, we prove that for any soft subset (A,Θ) of an SSTS (U,μ,Θ), the class $\{b^s_{sd}(A,\Theta), int^s_{sd}(A,\Theta), ext^s_{sd}(A,\Theta)\}$ forms a partition of $\tilde{U}$. Also, the null soft set is not SS-sd-set which leads to introduce another sufficient definitions for the above-mentioned operators, which were designed to maintain the systematic relations between different kinds of SS-cts (respectively, open, closed) maps.

Definition 3.1. *[53] The SS-sd-interior points of a soft subset* (G,Θ) *of an SSTS* (U,μ,Θ)*, denoted by* $int^s_{sd}(G,\Theta)$*, is the largest SS-sd-subsets of* (G,Θ)*. Also, the SS-sd-closure points of a soft subset* (H,Θ) *of an SSTS* (U,μ,Θ)*, denoted by* $cl^s_{sd}(H,\Theta)$*, is the smallest SS-sc-superset of* (H,Θ)*.*

Corollary 3.2. *For an SS-closed subset* (Y,Θ) *of an SSTS* (U,μ,Θ) *in which* $b^s(Y,\Theta)\in SD(U)_\Theta$*, we have that* $(Y,\Theta)\in SD(U)_\Theta$*.*

Proof. Let $(Y,\Theta)\in\mu^c$. Then,

$$b^s(Y,\Theta)=cl^s(Y,\Theta)-int^s(Y,\Theta)=(Y,\Theta)-int^s(Y,\Theta)\tilde{\subseteq}(Y,\Theta).$$

Since $b^s(Y,\Theta)\in SD(U)_\Theta$, $(Y,\Theta)\in SD(U)_\Theta$ according to Corollary 2.12.

Proposition 3.3. *For the SS-sd-interior (closure) operators* $int^s_{sd}, cl^s_{sd}: S(U)_\Theta \longrightarrow S(U)_\Theta$ *we have the following:*

(1) $int^s_{sd}(G,\Theta)=\begin{cases}\tilde{\varphi}, & (G,\Theta) \text{ is an SS-sc-set only,}\\ (G,\Theta), & \text{otherwise.}\end{cases}$

(2) $cl^s_{sd}(G,\Theta)=\begin{cases}\tilde{U}, & (G,\Theta) \text{ is an SS-sd-set only,}\\ (G,\Theta), & \text{otherwise.}\end{cases}$

Proof.

(1) Suppose conversely, that $int^s_{sd}(G,\Theta) \neq \tilde{\varphi}$, whereas (G,Θ) is an SS-sc-set only. It follows that, for any soft point $s_\theta \tilde{\in} (G,\Theta)$ there exists $(D,\Theta) \in SD(U)_\Theta$ such that $s_\theta \tilde{\in} (D,\Theta) \tilde{\subseteq} (G,\Theta)$. Given Corollary 2.12, $(G,\Theta) \in SD(U)_\Theta$, which contradicts Proposition 2.13. Hence, $int^s_{sd}(G,\Theta) = \tilde{\varphi}$. Otherwise, $int^s_{sd}(G,\Theta) = (G,\Theta)$.

(2) It is similar to the proof of (1).

Theorem 3.4. *[53] For a soft subset (T,Θ) of an SSTS (U,μ,Θ), we have the following:*

(1) $cl^s_{sd}(T^{\tilde{c}},\Theta) = [int^s_{sd}(T,\Theta)]^{\tilde{c}}$ *and* $int^s_{sd}(T^{\tilde{c}},\Theta) = [cl^s_{sd}(T,\Theta)]^{\tilde{c}}$.

(2) $cl^s_{sd}(H,\Theta) \tilde{\subseteq} cl^s(H,\Theta)$.

(3) $int^s(K,\Theta) \tilde{\subseteq} int^s_{sd}(K,\Theta)$.

Corollary 3.5. *For a soft subset (G,Θ) of an SSTS (U,μ,Θ), if $cl^s_{sd}(G,\Theta) = \tilde{U}$, then $[int^s_{sd}(G,\Theta)]^{\tilde{c}} \neq \tilde{U}$.*

Proof. Suppose that $cl^s_{sd}(G,\Theta) = \tilde{U}$. According to Proposition 3.3 (2), (G,Θ) is an SS-sd-set only. Hence, $(G^{\tilde{c}},\Theta)$ is an SS-sc-set only. Therefore, $[int^s_{sd}(G,\Theta)]^{\tilde{c}} = cl^s_{sd}(G^{\tilde{c}},\Theta) = (G^{\tilde{c}},\Theta) \neq \tilde{U}$ according to Theorem 3.4 (1).

Proposition 3.6. *Let (U,μ,Θ) be an SSTS and (P,Θ) and (Q,Θ) be an SS-sd-sets only. Then, the following holds.*

(1) $(P,\Theta) \tilde{\cap} (Q,\Theta) \neq \tilde{\varphi}$.

(2) $cl^s_{sd}(P,\Theta) \tilde{\cup} cl^s_{sd}(Q,\Theta) = cl^s_{sd}[(P,\Theta) \tilde{\cup} (Q,\Theta)]$.

Proof.

(1) Assume conversely, that $(P,\Theta),(Q,\Theta)$ are disjoint SS-sd-sets only. It follows that, $(P,\Theta) \tilde{\subseteq} (Q^{\tilde{c}},\Theta)$ and $(Q^{\tilde{c}},\Theta)$ is an SS-sc-set only. According to Proposition 2.13, (P,Θ) is an SS-sc-set only, which is a contradiction. Thus, $(P,\Theta) \tilde{\cap} (Q,\Theta) \neq \tilde{\varphi}$.

(2) Since (P,Θ) and (Q,Θ) are SS-sd-sets only, $cl^s_{sd}(P,\Theta) = cl^s_{sd}(Q,\Theta) = \tilde{U}$ according to Proposition 3.3. Hence,

$$\tilde{U} = cl^s_{sd}(P,\Theta) \tilde{\cup} cl^s_{sd}(Q,\Theta) \tilde{\supseteq} cl^s_{sd}[(P,\Theta) \tilde{\cup} (Q,\Theta)].$$

However, we have

$$cl^s_{sd}[(P,\Theta) \tilde{\cup} (Q,\Theta)] \tilde{\supseteq} cl^s_{sd}(P,\Theta) \tilde{\cup} cl^s_{sd}(Q,\Theta) \text{ from [53, Theorem 4.15 (6)].}$$

Therefore,

$$cl^s_{sd}(P,\Theta) \tilde{\cup} cl^s_{sd}(Q,\Theta) = cl^s_{sd}[(P,\Theta) \tilde{\cup} (Q,\Theta)].$$

Corollary 3.7. *[53] If $(N,\Theta) \in SD(U)_\Theta$ and $(M,\Theta) \in S(U)_\Theta$ such that $(N,\Theta) \tilde{\cap} (M,\Theta) = \tilde{\varphi}$, then $(N,\Theta) \tilde{\cap} cl^s_{sd}(M,\Theta) = \tilde{\varphi}$.*

Proposition 3.8. *Let (U, μ, Θ) be an SSTS and (P, Θ) and (Q, Θ) be SS-sd-sets. Then, (P, Θ) and (Q, Θ) are disjoint $\Leftrightarrow cl^s_{sd}(P, \Theta)\tilde{\cap} cl^s_{sd}(Q, \Theta) = \tilde{\varphi}$.*

Proof. If $cl^s_{sd}(P, \Theta)\tilde{\cap} cl^s_{sd}(Q, \Theta) = \tilde{\varphi}$, then it is clear that (P, Θ) and (Q, Θ) are disjoint. Conversely, assume that (P, Θ) and (Q, Θ) are disjoint SS-sd-sets. It follows that, (P, Θ) and (Q, Θ) are proper soft sets. According to Corollary 3.7,

$$(P, \Theta)\tilde{\cap} cl^s_{sd}(Q, \Theta) = \tilde{\varphi} \text{ and } (Q, \Theta)\tilde{\cap} cl^s_{sd}(P, \Theta) = \tilde{\varphi}.$$

According to Proposition 3.3 (2),

$$(P, \Theta) \text{ and } (Q, \Theta) \text{ are SS-sc-sets, and hence } cl^s_{sd}(P, \Theta) = (P, \Theta) \text{ and } cl^s_{sd}(Q, \Theta) = (Q, \Theta).$$

Therefore,

$$cl^s_{sd}(P, \Theta)\tilde{\cap} cl^s_{sd}(Q, \Theta) = \tilde{\varphi}.$$

Proposition 3.9. *For a soft subset (P, Θ) of an SSTS (U, μ, Θ), we have that $int^s_{sd}(cl^s_{sd}(P, \Theta)) = cl^s_{sd}(int^s_{sd}(P, \Theta))$.*

Proof. We have the following according to Proposition 3.3:

$$cl^s_{sd}(int^s_{sd}(P, \Theta)) = \begin{cases} (P, \Theta), & (P, \Theta) \text{ is both an SS-sd-set and SS-sc-set,} \\ \tilde{U}, & (P, \Theta) \text{ is an SS-sd-set only,} \\ \tilde{\varphi}, & (P, \Theta) \text{ is an SS-sc-set only.} \end{cases}$$

$= int^s_{sd}(cl^s_{sd}(P, \Theta))$.

Definition 3.10. *[53] A soft point s_θ is said to be an SS-sd-cluster point of a soft subset (C, Θ) of an SSTS (U, μ, Θ) if for each SS-sd-set (G, Θ) that contains s_θ,*

$$[(C, \Theta) \setminus s_\theta]\tilde{\cap}(G, \Theta) \neq \tilde{\varphi}.$$

The set of all SS-sd-cluster points of (C, Θ), denoted by $d^s_{sd}(C, \Theta)$, is called an SS-sd-derived set.

Theorem 3.11. *[53] Let (U, μ, Θ) be an SSTS and $(C, \Theta), (D, \Theta) \in S(U)_\Theta$; then, the following holds:*

(1) *$d^s_{sd}(C, \Theta)\tilde{\subseteq}(C, \Theta) \Leftrightarrow (C, \Theta)$ is a proper SS-sc-set.*

(2) *If $(C, \Theta)\tilde{\subseteq}(D, \Theta)$, then $d^s_{sd}(C, \Theta)\tilde{\subseteq} d^s_{sd}(D, \Theta)$.*

Theorem 3.12. *For a soft subset (A, Θ) of an SSTS (U, μ, Θ); if $(A, \Theta)\tilde{\cup} d^s_{sd}(A, \Theta) \neq \tilde{U}$, then $(A, \Theta)\tilde{\cup} d^s_{sd}(A, \Theta)$ is an SS-sc-set.*

Proof. According to Theorem 3.11 (1), we have to prove that $d^s_{sd}[(A, \Theta)\tilde{\cup} d^s_{sd}(A, \Theta)]\tilde{\subseteq}(A, \Theta) \tilde{\cup} d^s_{sd}(A, \Theta)$. So, assume that $s_\theta \tilde{\notin}(A, \Theta)\tilde{\cup} d^s_{sd}(A, \Theta)$, and hence $s_\theta \tilde{\notin}(A, \Theta)$ and $s_\theta \tilde{\notin} d^s_{sd}(A, \Theta)$. Hence,

$$[(A, \Theta) \setminus s_\theta]\tilde{\cap}(G, \Theta) = \tilde{\varphi} \text{ for some SS-sd-set } (G, \Theta) \text{ containing } s_\theta.$$

Since $s_\theta \tilde{\notin}(A, \Theta)$, $(A, \Theta)\tilde{\cap}(G, \Theta) = \tilde{\varphi}$. According to Corollary 3.7,

$$cl^s_{sd}(A, \Theta)\tilde{\cap}(G, \Theta) = \tilde{\varphi}.$$

Since $d^s_{sd}(A, \Theta)\tilde{\subseteq} cl^s_{sd}(A, \Theta)$, $d^s_{sd}(A, \Theta)\tilde{\cap}(G, \Theta) = \tilde{\varphi}$. Thus,

$$[(A,\Theta)\tilde{\cap}(G,\Theta)]\tilde{\cup}[d^s_{sd}(A,\Theta)\tilde{\cap}(G,\Theta)] = (G,\Theta)\tilde{\cap}[(A,\Theta)\tilde{\cup}d^s_{sd}(A,\Theta)] = \tilde{\varphi}.$$

It follows that,

$$s_\theta \tilde{\notin} d^s_{sd}[(A,\Theta)\tilde{\cup}d^s_{sd}(A,\Theta)].$$

Therefore, $(A,\Theta)\tilde{\cup}d^s_{sd}(A,\Theta)$ is an SS-sc-set.

Corollary 3.13. *For a soft subset (A,Θ) of an SSTS (U,μ,Θ), $cl^s_{sd}(A,\Theta) = (A,\Theta)\tilde{\cup}d^s_{sd}(A,\Theta)$.*

Proof. It follows from Theorem 3.12.

Definition 3.14. *Let (U,μ,Θ) be an SSTS and $(A,\Theta) \in S(U)_\Theta$. Then, $s_\theta \tilde{\in} S(U)_\Theta$ is called an SS-sd-boundary point of (F,Θ) if*

$$s_\theta \tilde{\in} [cl^s_{sd}(F,\Theta) - int^s_{sd}(F,\Theta)].$$

The set of all SS-sd-boundary points of (F,Θ) is called an SS-sd-boundary set of (F,Θ), and it is denoted by $b^s_{sd}(F,\Theta)$. Also, the SS-sd-exterior of (F,Θ) is denoted by $ext^s_{sd}(F,\Theta)$, and $ext^s_{sd}(F,\Theta) = int^s_{sd}(F^{\tilde{c}},\Theta)$.

Theorem 3.15. *Let (U,μ,Θ) be an SSTS and $(A,\Theta) \in S(U)_\Theta$; then, the following holds:*

(1) $b^s_{sd}(A,\Theta) = cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}} = cl^s_{sd}(A,\Theta)\tilde{\cap}cl^s_{sd}(A^{\tilde{c}},\Theta) = [int^s_{sd}(A,\Theta)\tilde{\cup}ext^s_{sd}(A,\Theta)]^{\tilde{c}}$.

(2) $b^s_{sd}(A,\Theta) = b^s_{sd}(A^{\tilde{c}},\Theta)$.

Proof.

(1) $[int^s_{sd}(A,\Theta)\tilde{\cup}ext^s_{sd}(A,\Theta)]^{\tilde{c}} = [int^s_{sd}(A,\Theta)]^{\tilde{c}}\tilde{\cap}[int^s_{sd}(A^{\tilde{c}},\Theta)]^{\tilde{c}}$
$= cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}}$ from Theorem 3.4 (1)
$= cl^s_{sd}(A,\Theta)\tilde{\cap}cl^s_{sd}(A^{\tilde{c}},\Theta)$
$= cl^s_{sd}(A,\Theta) - int^s_{sd}(A,\Theta)$
$= b^s_{sd}(A,\Theta)$.

(2) $b^s_{sd}(A^{\tilde{c}},\Theta) = cl^s_{sd}(A^{\tilde{c}},\Theta)\tilde{\cap}[int^s_{sd}(A^{\tilde{c}},\Theta)]^{\tilde{c}} = [int^s_{sd}(A,\Theta)]^{\tilde{c}}\tilde{\cap}cl^s_{sd}(A,\Theta) = b^s_{sd}(A,\Theta)$.

Theorem 3.16. *Let (U,μ,Θ) be an SSTS and $(F,\Theta) \in S(U)_\Theta$; then, the following holds:*

(1) $cl^s_{sd}(F,\Theta) = int^s_{sd}(F,\Theta)\tilde{\cup}b^s_{sd}(F,\Theta)$.

(2) $int^s_{sd}(F,\Theta) = (F,\Theta) - b^s_{sd}(F,\Theta)$.

Proof.

(1) $int^s_{sd}(F,\Theta)\tilde{\cup}b^s_{sd}(F,\Theta) = int^s_{sd}(F,\Theta)\tilde{\cup}[cl^s_{sd}(F,\Theta)\tilde{\cap}[int^s_{sd}(F,\Theta)]^{\tilde{c}}]$ from Theorem 3.15 (1)
$= [int^s_{sd}(F,\Theta)\tilde{\cup}cl^s_{sd}(F,\Theta)]\tilde{\cap}[int^s_{sd}(F,\Theta)\tilde{\cup}[int^s_{sd}(F,\Theta)]^{\tilde{c}}]$
$= cl^s_{sd}(F,\Theta)\tilde{\cap}\tilde{U}$
$= cl^s_{sd}(F,\Theta)$.

(2) $(F,\Theta) - b^s_{sd}(F,\Theta) = (F,\Theta)\tilde{\cap}[cl^s_{sd}(F,\Theta)\tilde{\cap}[int^s_{sd}(F,\Theta)]^{\tilde{c}}]^{\tilde{c}}$
$= (F,\Theta)\tilde{\cap}[[cl^s_{sd}(F,\Theta)]^{\tilde{c}}\tilde{\cup}[int^s_{sd}(F,\Theta)]]$
$= [(F,\Theta)\tilde{\cap}[cl^s_{sd}(F,\Theta)]^{\tilde{c}}]\tilde{\cup}[(F,\Theta)\tilde{\cap}int^s_{sd}(F,\Theta)]$
$= \tilde{\varphi}\tilde{\cup}int^s_{sd}(F,\Theta)$
$= int^s_{sd}(F,\Theta)$.

Proposition 3.17. *For a soft subset (A,Θ) of an SSTS (U,μ,Θ), the class $\{b^s_{sd}(A,\Theta), int^s_{sd}(A,\Theta), ext^s_{sd}(A,\Theta)\}$ forms a partition of $\tilde{U}$.*

Proof. $b^s_{sd}(A,\Theta)\tilde{\cup}int^s_{sd}(A,\Theta)\tilde{\cup}ext^s_{sd}(A,\Theta) = [cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}}]\tilde{\cup}int^s_{sd}(A,\Theta)\tilde{\cup}[cl^s_{sd}(A,\Theta)]^{\tilde{c}} = \tilde{U}$. Also, $b^s_{sd}(A,\Theta)\tilde{\cap}int^s_{sd}(A,\Theta)\tilde{\cap}ext^s_{sd}(A,\Theta) = [cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}}]\tilde{\cap}int^s_{sd}(A,\Theta)\tilde{\cap}[cl^s_{sd}(A,\Theta)]^{\tilde{c}} = \tilde{\varphi}$.

Proposition 3.18. *Let (U,μ,Θ) be an SSTS and $(A,\Theta)\in S(U)_\Theta$; then, the following holds:*

(1) $b^s_{sd}[int^s_{sd}(A,\Theta)]\tilde{\subseteq}b^s_{sd}(A,\Theta)$.

(2) $b^s_{sd}[cl^s_{sd}(A,\Theta)]\tilde{\subseteq}b^s_{sd}(A,\Theta)$.

Proof.

(1) $b^s_{sd}[int^s_{sd}(A,\Theta)] = cl^s_{sd}(int^s_{sd}(A,\Theta))\tilde{\cap}[int^s_{sd}(int^s_{sd}(A,\Theta))]^{\tilde{c}}\tilde{\subseteq}cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}} = b^s_{sd}(A,\Theta)$.

(2) $b^s_{sd}[cl^s_{sd}(A,\Theta)]cl^s_{sd}(cl^s_{sd}(A,\Theta))\tilde{\cap}[int^s_{sd}(cl^s_{sd}(A,\Theta))]^{\tilde{c}} = cl^s_{sd}(A,\Theta)\tilde{\cap}cl^s_{sd}[cl^s_{sd}(A,\Theta)]^{\tilde{c}}$

$$= cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}} = b^s_{sd}(A,\Theta).$$

Remark 3.19. *The reverse inclusions of Proposition 3.18 are not satisfied as shown in the next example.*

Example 3.20. *Assume that $U=\{s_1,s_2,s_3\}$. Let $\Theta=\{\theta_1,\theta_2\}$ be the set of parameters. Let $(M_i,\Theta), i=1,2,3,4$ be soft sets over the universe U, where*
$M_1(\theta_1)=\{s_1,s_2\}$, $M_1(\theta_2)=\{s_1,s_3\}$,
$M_2(\theta_1)=\{s_1\}$, $M_2(\theta_2)=\varphi$,
$M_3(\theta_1)=\{s_2\}$, $M_3(\theta_2)=\{s_1,s_3\}$,
$M_4(\theta_1)=\{s_1\}$, $M_4(\theta_2)=\{s_1\}$,
then $\mu=\{\tilde{U},\tilde{\varphi},(M_i,\Theta), i=1,2,3,4\}$ defines an SSTS on U. For the soft sets (A,Θ) and (B,Θ), where:
$A(\theta_1)=\varphi$, $A(\theta_2)=\{s_2\}$,
$B(\theta_1)=\{s_1,s_2\}$, $B(\theta_2)=U$, *we have the following:*

(1) $b^s_{sd}(A,\Theta)=(A,\Theta)\tilde{\nsubseteq}b^s_{sd}[int^s_{sd}(A,\Theta)]=\tilde{\varphi}$.

(2) $b^s_{sd}(B,\Theta)=\{(\theta_1,\{s_3\}),(\theta_2,\varphi)\}\tilde{\nsubseteq}b^s_{sd}[cl^s_{sd}(A,\Theta)]=\tilde{\varphi}$.

Proposition 3.21. *Let (U,μ,Θ) be an SSTS and $(A,\Theta)\in S(U)_\Theta$; then, the following holds:*

(1) *(A,Θ) is a non-null SS-sd-set if, and only if $b^s_{sd}(A,\Theta)\tilde{\cap}(A,\Theta)=\tilde{\varphi}$.*

(2) *(A,Θ) is a proper SS-sc-set if, and only if $b^s_{sd}(A,\Theta)\tilde{\subseteq}(A,\Theta)$.*

(3) *(A,Θ) is both an SS-sd-set and SS-sc-set if, and only if $b^s_{sd}(A,\Theta)=\tilde{\varphi}$.*

Proof.

(1) " $\Rightarrow$ " Let (A,Θ) be a non-null SS-sd-set. It follows that, $int^s_{sd}(A,\Theta)=(A,\Theta)$ according to Proposition 3.3 (1). Hence,

$$b^s_{sd}(A,\Theta)\tilde{\cap}(A,\Theta) = [cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}}]\tilde{\cap}(A,\Theta) = [cl^s_{sd}(A,\Theta)\tilde{\cap}[(A,\Theta)]^{\tilde{c}}]\tilde{\cap}(A,\Theta) = \tilde{\varphi}.$$

" $\Leftarrow$ " Consider $b^s_{sd}(A,\Theta)\tilde{\cap}(A,\Theta)=\tilde{\varphi}$, then $b^s_{sd}(A,\Theta)\tilde{\cap}(A,\Theta)=\tilde{\varphi}$.

Therefore,

$$[cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}}]\tilde{\cap}(A,\Theta) = [int^s_{sd}(A,\Theta)]^{\tilde{c}}]\tilde{\cap}(A,\Theta) = \tilde{\varphi}.$$

It follows that,

$$(A,\Theta)\tilde{\subseteq}int^s_{sd}(A,\Theta). \text{ However, we have } int^s_{sd}(A,\Theta)\tilde{\subseteq}(A,\Theta).$$

Thus, $int^s_{sd}(A,\Theta) = (A,\Theta)$; so, (A,Θ) is a non-null SS-sd-set according to Proposition 3.3 (1).

(2) " $\Rightarrow$ " Let (A,Θ) be a proper SS-sc-set, then $cl^s_{sd}(A,\Theta) = (A,\Theta)$ according to Proposition 3.3 (2). Hence,

$$b^s_{sd}(A,\Theta) = cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}} = (A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}}\tilde{\subseteq}(A,\Theta).$$

" $\Leftarrow$ " Consider that $b^s_{sd}(A,\Theta)\tilde{\subseteq}(A,\Theta)$. It follows that,

$$cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}}\tilde{\subseteq}(A,\Theta).$$

Hence, $cl^s_{sd}(A,\Theta)\tilde{\subseteq}(A,\Theta)$. However, we have that $(A,\Theta)\tilde{\subseteq}cl^s_{sd}(A,\Theta)$. Therefore, (A,Θ) is a proper SS-sc-set according to Proposition 3.3 (2).

(3) " $\Rightarrow$ " Let (A,Θ) be both an SS-sd-set and SS-sc-set. It follows that,

$$int^s_{sd}(A,\Theta) = (A,\Theta) = cl^s_{sd}(A,\Theta), \text{ which follows}$$

$$b^s_{sd}(A,\Theta) = cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}} = (A,\Theta)\tilde{\cap}(A^{\tilde{c}},\Theta) = \tilde{\varphi}.$$

" $\Leftarrow$ " Assume that $b^s_{sd}(A,\Theta) = \tilde{\varphi}$; then, $cl^s_{sd}(A,\Theta)\tilde{\cap}[int^s_{sd}(A,\Theta)]^{\tilde{c}} = \tilde{\varphi}$. Hence, $cl^s_{sd}(A,\Theta)\tilde{\subseteq}int^s_{sd}(A,\Theta)$. However, we have that $int^s_{sd}(A,\Theta)\tilde{\subseteq}cl^s_{sd}(A,\Theta)$. Thus, $int^s_{sd}(A,\Theta) = cl^s_{sd}(A,\Theta)$. Since $int^s_{sd}(A,\Theta)\tilde{\subseteq}(A,\Theta)$, $cl^s_{sd}(A,\Theta)\tilde{\subseteq}(A,\Theta)$. Therefore,

$$cl^s_{sd}(A,\Theta) = (A,\Theta). \tag{3.1}$$

Also, since $(A,\Theta)\tilde{\subseteq}cl^s_{sd}(A,\Theta)$, $(A,\Theta)\tilde{\subseteq}int^s_{sd}(A,\Theta)$. Therefore,

$$int^s_{sd}(A,\Theta) = (A,\Theta). \tag{3.2}$$

Given Proposition 3.3 and Eqs (3.1) and (3.2), (A,Θ) is both an SS-sd-set and SS-sc-set.

Proposition 3.22. *Let (U,μ,Θ) be an SSTS and $(C,\Theta),(D,\Theta)\in S(U)_\Theta$; then, the following holds:*

(1) $b^s_{sd}[(C,\Theta)\tilde{\cap}(D,\Theta)]\tilde{\subseteq}b^s_{sd}(C,\Theta)\tilde{\cup}b^s_{sd}(D,\Theta)$.

(2) $b^s_{sd}[(C,\Theta)\tilde{\cup}(D,\Theta)]\tilde{\subseteq}b^s_{sd}(C,\Theta)\tilde{\cup}b^s_{sd}(D,\Theta)$.

Proof.

(1) Assume conversely, that $s_\theta\tilde{\notin}b^s_{sd}(C,\Theta)\tilde{\cup}b^s_{sd}(D,\Theta)$. It follows that, $s_\theta\tilde{\notin}b^s_{sd}(C,\Theta)$ and $s_\theta\tilde{\notin}b^s_{sd}(D,\Theta)$; thus, $s_\theta\tilde{\notin}cl^s_{sd}(C,\Theta)\tilde{\cap}[int^s_{sd}(C,\Theta)]^{\tilde{c}}$ and $s_\theta\tilde{\notin}cl^s_{sd}(D,\Theta)\tilde{\cap}[int^s_{sd}(D,\Theta)]^{\tilde{c}}$. Hence, $s_\theta\tilde{\notin}cl^s_{sd}(C,\Theta)\tilde{\cap}cl^s_{sd}(D,\Theta)$ and $s_\theta\tilde{\notin}[int^s_{sd}(C,\Theta)]^{\tilde{c}}\tilde{\cup}[int^s_{sd}(D,\Theta)]^{\tilde{c}}$. Therefore, $s_\theta\tilde{\notin}cl^s_{sd}[(C,\Theta)\tilde{\cap}(D,\Theta)]$ and $s_\theta\tilde{\notin}int^s_{sd}[(C,\Theta)\tilde{\cap}(D,\Theta)]^{\tilde{c}}$. Thus, $s_\theta\tilde{\notin}b^s_{sd}[(C,\Theta)\tilde{\cap}(D,\Theta)]$.

(2) It follows by a similar method to (1).

Remark 3.23. *The reverse inclusions of Proposition 3.22 are not satisfied in general, as shown in the next example.*

Example 3.24. *In Example 3.20, for the soft sets* (C, Θ) *and* (D, Θ)*, where*
$C(\theta_1) = \{s_3\}, \quad C(\theta_2) = \{s_1, s_2\},$
$D(\theta_1) = \{s_1, s_2\}, \quad D(\theta_2) = U$*, we have the following:*

(1) $b^s_{sd}(C, \Theta)\tilde{\cup}b^s_{sd}(D, \Theta) = \{(\theta_1, \{s_3\}), (\theta_2, \{s_2\})\}\tilde{\nsubseteq}b^s_{sd}[(C, \Theta)\tilde{\cap}(D, \Theta)] = \tilde{\varphi}$.

(2) $b^s_{sd}(C, \Theta)\tilde{\cup}b^s_{sd}(D, \Theta) = \{(\theta_1, \{s_3\}), (\theta_2, \{s_2\})\}\tilde{\nsubseteq}b^s_{sd}[(C, \Theta)\tilde{\cup}(D, \Theta)] = b^s_{sd}(\tilde{U}) = \tilde{\varphi}$.

Proposition 3.25. *For a soft subset* (T, Θ) *of an SSTS* (U, μ, Θ) *we have the following:*

$$b^s_{sd}(T, \Theta) = \begin{cases} (T, \Theta), & (T, \Theta) \text{ is an SS-sc-set only,} \\ (T^{\tilde{c}}, \Theta), & (T, \Theta) \text{ is an SS-sd-set only,} \\ \tilde{\varphi}, & (T, \Theta) \text{ is both an SS-sd-set and SS-sc-set.} \end{cases}$$

Proof. Clear by following Proposition 3.3.

Note 3.26. *Depending on Proposition 3.25, we notice that for any soft subset* (T, Θ) *of an SSTS* (U, μ, Θ)*, we have that either* $b^s_{sd}(T, \Theta)\tilde{\subseteq}(T, \Theta)$ *or* $b^s_{sd}(T, \Theta)\tilde{\subseteq}(T^{\tilde{c}}, \Theta)$*.*

Theorem 3.27. *For a soft subset* (G, Θ) *of an SSTS* (U, μ, Θ) *we have that* $b^s_{sd}(G, \Theta)$ *is an SS-sc-set.*

Proof. To prove this result for $b^s_{sd}(A, \Theta) = cl^s_{sd}(A, \Theta)\tilde{\cap}cl^s_{sd}(A^{\tilde{c}}, \Theta)$, we have the following cases:
Case I: If $cl^s_{sd}(A, \Theta) = \tilde{U}$, then $b^s_{sd}(A, \Theta) = cl^s_{sd}(A^{\tilde{c}}, \Theta)$ is an SS-sc-set.
Case II: If $cl^s_{sd}(A^{\tilde{c}}, \Theta) = \tilde{U}$, then $b^s_{sd}(A, \Theta) = cl^s_{sd}(A, \Theta)$ is an SS-sc-set.
Case III: If $cl^s_{sd}(A, \Theta) \neq \tilde{U}$ and $cl^s_{sd}(A^{\tilde{c}}, \Theta) \neq \tilde{U}$, then $b^s_{sd}(A, \Theta) = cl^s_{sd}(A, \Theta)\tilde{\cap}cl^s_{sd}(A^{\tilde{c}}, \Theta)$ is an SS-sc-set.

4. Applications of supra soft sd-sets for soft continuity

Herein, we apply the notion of SS-sd-sets to soft continuity. Specifically, we introduce the definition of SS-sd-cts maps as an extension to most of the previous types of weaker forms of such notions. A diagram to illustrate the relationships among our new class and other previous soft continuity notions is explored in Figure 1. In addition, many interesting properties and conditions equivalent to this concept are discussed. Moreover, we define the SS-sd-irresolute maps. We prove that the compositions of SS-sd-irresolute maps and SS-sd-cts maps are also SS-sd-cts. Finally, many important examples are provided to show the effectiveness and efficiency of the proposed method and compared to others.

Definition 4.1. *A soft function* $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ *with* μ_1 *as an associated SSTS with* τ_1 *is said to be an SS-sd-cts if either* $\psi^{-1}_{sd}(G, \Theta_2) = \tilde{\varphi}$ *or* $\psi^{-1}_{sd}(G, \Theta_2) \in SD(U_1)_{\Theta_1}$ *for each* $(G, \Theta_2) \in \tau_2$*.*

Note 4.2. *In Definition 4.1, the probability of null soft set will be eliminated if we considered* ψ_{sd} *as a surjective soft function.*

Theorem 4.3. *[53] Every SS- (respectively, semi- ,* α*-, b-, pre-, regular-,* β*-) open set is SS-sd-set.*

Proposition 4.4. *[48] Every SS- (respectively, semi-,* α*-, b-, pre-, regular-) cts function is SS-*β*-cts.*

The authors of [44] proved that the collection of SS-β-cts functions is a wider class of SS-continuity, as shown in Proposition 4.4. In the next theorem, we shall prove that the collection of SS-sd-cts functions is even wider.

Theorem 4.5. *Every SS- (respectively, semi-, α-, b-, pre-, regular-, β-) cts function is SS-sd-cts.*

Proof. It follows from Theorem 4.3.

Remark 4.6. *In general, the converse of Theorem 4.5 is not true, as shown in the next example.*

Example 4.7. *Let $U_1 = \{m_1, m_2, m_3, m_4\}$, $U_2 = \{n_1, n_2, n_3, n_4\}$, $\Theta_1 = \{j_1, j_2\}$ and $\Theta_2 = \{k_1, k_2\}$.*

$$Define\ s : U_1 \to U_2\ and\ d : \Theta_1 \to \Theta_2\ as\ follows :$$

$$s(m_1) = n_4,\ s(m_2) = n_3,\ s(m_3) = n_1,\ s(m_4) = n_2,\ d(j_1) = k_1,\ d(j_2) = k_2.$$

$$Let\ \tau_1 = \{\tilde{U}_1, \tilde{\varphi}, (E_2, \Theta_1)\}\ be\ an\ STS\ over\ U_1\ and$$

$$\mu_1 = \{\tilde{U}_1, \tilde{\varphi}, (E_1, \Theta_1), (E_2, \Theta_1), (E_3, \Theta_1), (E_4, \Theta_1), (E_5, \Theta_1), (E_6, \Theta_1), (E_7, \Theta_1)\}$$

be an associated SSTS with τ_1, where

$$E_1(j_1) = \{m_1\}, \quad E_1(j_2) = \varphi.$$

$$E_2(j_1) = \{m_1, m_2\}, \quad E_2(j_2) = \{m_1\}.$$

$$E_3(j_1) = \{m_1, m_2\}, \quad E_3(j_2) = \{m_3, m_4\}.$$

$$E_4(j_1) = \{m_3, m_4\}, \quad E_4(j_2) = \{m_1, m_2\}.$$

$$E_5(j_1) = \{m_1, m_3, m_4\}, \quad E_5(j_2) = \{m_1, m_2\}.$$

$$E_6(j_1) = U, \quad E_6(j_2) = \{m_1, m_2\}.$$

$$E_7(j_1) = \{m_1, m_2\}, \quad E_7(j_2) = \{m_1, m_3, m_4\}.$$

$$Let\ \tau_2 = \{\tilde{U}_2, \tilde{\varphi}, (W_1, \Theta_2)\}\ be\ a\ STS\ over\ U_2\ where,$$

$$W_1(k_1) = \{n_1, n_2, n_3\}, \quad W_1(k_2) = U_2.$$

Then,

$$\psi_{sd}^{-1}((W_1, \Theta_2)) = \{(j_1, \{m_2, m_3, m_4\}), (j_2, U_1)\}$$

is an SS-sd-set, but it is not SS-β-open. Hence, ψ_{sd} is an SS-sd-cts, but it is not SS-β-cts.

Corollary 4.8. *It follows from Theorem 4.5 and [48, Corollary 6.1] that we have the following implications for an SSTS (U, μ, Θ), which are not reversible.*

$$\begin{array}{ccccccccccc}
\text{SS-regular-cts} & \longrightarrow & \text{SS-cts} & \longrightarrow & \text{SS-}\alpha\text{-cts} & \longrightarrow & \text{SS-semi-cts} & \longrightarrow & \text{SS-}\beta\text{-cts} & \longrightarrow & \text{SS-sd-cts} \\
 & & & & \downarrow & & \searrow & & \nearrow & & \\
 & & & & \text{SS-pre-cts} & \longrightarrow & & \text{SS-b-cts} & & &
\end{array}$$

Figure 1. The relationships among the class of SS-sd-cts functions and other previous such notions.

Theorem 4.9. *Let $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ be a soft function with μ_1 as an associated SSTS with τ_1; then, the following are equivalent:*

(1) *ψ_{sd} is SS-sd-cts.*

(2) *For each $(E, \Theta_2) \in \tau_2^c$, either $\psi_{sd}^{-1}(E, \Theta_2) \in SC(U_1)_{\Theta_1}$ or $\psi_{sd}^{-1}(E, \Theta_2) = \tilde{U}_1$.*

(3) *$cl_{sd}^s(\psi_{sd}^{-1}(E, \Theta_2)) \tilde{\subseteq} \psi_{sd}^{-1}(cl(E, \Theta_2))\ \forall\ (E, \Theta_2) \tilde{\subseteq} \tilde{U}_2$.*

(4) *$\psi_{sd}(cl_{sd}^s(G, \Theta_1)) \tilde{\subseteq} cl(\psi_{sd}(G, \Theta_1))\ \forall\ (G, \Theta_1) \tilde{\subseteq} \tilde{U}_1$.*

(5) *$\psi_{sd}^{-1}(int(E, \Theta_2)) \tilde{\subseteq} int_{sd}^s(\psi_{sd}^{-1}(E, \Theta_2))\ \forall\ (E, \Theta_2) \tilde{\subseteq} \tilde{U}_2$.*

Proof.

(1) $\Rightarrow$ (2) Let $(E, \Theta_2) \in \tau_2^c$; then, $(E^{\tilde{c}}, \Theta_2) \in \tau_2$. Hence, either $\psi_{sd}^{-1}(E^{\tilde{c}}, \Theta_2) = [\psi_{sd}^{-1}(E, \Theta_2)]^{\tilde{c}} \in SD(U_1)_{\Theta_1}$ or $[\psi_{sd}^{-1}(E, \Theta_2)]^{\tilde{c}} = \tilde{\varphi}$ given (1). It follows that,

$$either\ \psi_{sd}^{-1}(E, \Theta_2) \in SC(U_1)_{\Theta_1}\ or\ \psi_{sd}^{-1}(E, \Theta_2) = \tilde{U}_1.$$

(2) $\Rightarrow$ (3) Let $(E, \Theta_2) \tilde{\subseteq} \tilde{U}_2$. Since $cl(E, \Theta_2) \in \tau_2^c$, given (2)

$$either\ \psi_{sd}^{-1}(cl(E, \Theta_2)) = \tilde{U}_1\ and\ \ we\ get\ the\ proof,$$
$$or\ \psi_{sd}^{-1}(E, \Theta_2) \in SC(U_1)_{\Theta_1},\ which\ leads\ to$$
$$cl_{sd}^s(\psi_{sd}^{-1}(E, \Theta_2)) \tilde{\subseteq} cl_{sd}^s(\psi_{sd}^{-1}(cl(E, \Theta_2))) = \psi_{sd}^{-1}(cl(E, \Theta_2)).$$

Thus, the proof is obtained.

(3) $\Rightarrow$ (4) Let $(G, \Theta_1) \tilde{\subseteq} \tilde{U}_1$. Given that $\psi_{sd}(G, \Theta_1) \tilde{\subseteq} \tilde{U}_2$, and applying (3), we get

$$cl_{sd}^s(\psi_{sd}^{-1}(\psi_{sd}(G, \Theta_1))) \tilde{\subseteq} \psi_{sd}^{-1}(cl(\psi_{sd}(G, \Theta_1))).$$

It follows that,

$$\psi_{sd}[cl_{sd}^s(\psi_{sd}^{-1}(\psi_{sd}(G, \Theta_1)))] \tilde{\subseteq} \psi_{sd}[\psi_{sd}^{-1}(cl(\psi_{sd}(G, \Theta_1)))] \tilde{\subseteq} cl(\psi_{sd}(G, \Theta_1)),\ from\ Theorem\ 2.5\ (2).$$

Hence,

$$\psi_{sd}(cl_{sd}^s(G, \Theta_1)) \tilde{\subseteq} cl(\psi_{sd}(G, \Theta_1)),\ from\ Theorem\ 2.5\ (3).$$

(4) $\Rightarrow$ (5) Let $(E, \Theta_2) \tilde{\subseteq} \tilde{U}_2$. It follows that, $\psi_{sd}^{-1}(E^{\tilde{c}}, \Theta_2) \tilde{\subseteq} \tilde{U}_1$. Applying (4), we get that

$$\psi_{sd}[cl_{sd}^s[\psi_{sd}^{-1}(E^{\tilde{c}}, \Theta_2)]] \tilde{\subseteq} cl(\psi_{sd}[\psi_{sd}^{-1}(E^{\tilde{c}}, \Theta_2)]) \tilde{\subseteq} cl(E^{\tilde{c}}, \Theta_2) = [int(E, \Theta_2)]^{\tilde{c}}\ from\ Theorem\ 3.4.$$

Hence,

$$\psi_{sd}^{-1}[\psi_{sd}(cl_{sd}^s[\psi_{sd}^{-1}(E^{\tilde{c}}, \Theta_2)])] \tilde{\subseteq} \psi_{sd}^{-1}[[int(E, \Theta_2)]^{\tilde{c}}] = [\psi_{sd}^{-1}(int(E, \Theta_2))]^{\tilde{c}}.$$

Therefore,

$$cl_{sd}^s[(\psi_{sd}^{-1}(E, \Theta_2))]^{\tilde{c}} \tilde{\subseteq} [\psi_{sd}^{-1}(int(E, \Theta_2))]^{\tilde{c}},\ from\ Theorem\ 2.5\ (3).$$

Thus,

$$\psi_{sd}^{-1}(int(E, \Theta_2)) \tilde{\subseteq} [cl_{sd}^s[(\psi_{sd}^{-1}(E, \Theta_2))]^{\tilde{c}}]^{\tilde{c}} = int_{sd}^s(\psi_{sd}^{-1}(E, \Theta_2)).$$

(5) $\Rightarrow$ (1) Let $(E,\Theta_2)\in\tau_2$; then, $(E,\Theta_2)=int(E,\Theta_2)$. It follows that,

$$\psi_{sd}^{-1}(E,\Theta_2)\tilde{\subseteq} int_{sd}^{s}(\psi_{sd}^{-1}(E,\Theta_2)),\ from\ (5).$$

Alternatively,

$$int_{sd}^{s}(\psi_{sd}^{-1}(E,\Theta_2))\tilde{\subseteq}\psi_{sd}^{-1}(E,\Theta_2).$$

Therefore,

$$either\ \psi_{sd}^{-1}(E,\Theta_2)=\tilde{\varphi}\ or\ \psi_{sd}^{-1}(E,\Theta_2)\in SD(U_1)_{\Theta_1}\ according\ to\ Proposition\ 3.3\ (1).$$

Thus, ψ_{sd} is an SS-sd-cts.

Definition 4.10. *A soft function $\psi_{sd}:(U_1,\tau_1,\Theta_1)\to(U_2,\tau_2,\Theta_2)$ with μ_1,μ_2 associated SSTSs with τ_1,τ_2, respectively, is said to be an SS-sd-irresolute if either $\psi_{sd}^{-1}(G,\Theta_2)=\tilde{\varphi}$ or $\psi_{sd}^{-1}(G,\Theta_2)\in SD(U_1)_{\Theta_1}$ for each $(G,\Theta_2)\in SD(U_2)_{\Theta_2}$.*

Theorem 4.11. *Every SS-sd-irresolute function is an SS-sd-cts.*

Proof. It is immediately obvious from Theorem 4.3.

Remark 4.12. *In general, the converse of Theorem 4.11 is not true, as shown in the next example.*

Example 4.13. *In Example 4.7, consider that*

$$\mu_2=\{\tilde{U}_2,\tilde{\varphi},(W_1,\Theta_2),(W_2,\Theta_2),(W_3,\Theta_2),(W_4,\Theta_2),(W_5,\Theta_2)\}$$

is an associated SSTS with τ_2, where

$$W_1(k_1)=\{n_1,n_2,n_3\},\quad W_1(k_2)=U_2.$$

$$W_2(k_1)=\{n_3\},\quad W_2(k_2)=\varphi.$$

$$W_3(k_1)=\{n_1,n_3\},\quad W_3(k_2)=\varphi.$$

$$W_4(k_1)=\{n_1\},\quad W_4(k_2)=\{n_1\}.$$

$$W_5(k_1)=\{n_1,n_3\},\quad W_5(k_2)=\{n_1\}.$$

Hence, ψ_{sd} is an SS-sd-cts, as shown in Example 4.7. In addition, for the soft set (W_2,Θ_2), we have that $(W_2,\Theta_2)\in SD(U_2)_{\Theta_2}$, where $\psi_{sd}^{-1}(W_2,\Theta_2)=\{(j_1,\{m_2\}),(j_2,\varphi)\}$. One can notice that neither $\psi_{sd}^{-1}(W_2,\Theta_2)=\tilde{\varphi}$ nor $\psi_{sd}^{-1}(W_2,\Theta_2)\in SD(U_1)_{\Theta_1}$. Therefore, ψ_{sd} is not SS-sd-irresolute.

Theorem 4.14. *Let $\psi_{sd}:(U_1,\tau_1,\Theta_1)\to(U_2,\tau_2,\Theta_2)$ be an SS-sd-irresolute with μ_1,μ_2 associated SSTSs with τ_1,τ_2, respectively, and $\rho_{sd}:(U_2,\tau_2,\Theta_2)\to(U_3,\tau_3,\Theta_3)$ be an SS-sd-cts with μ_3 as an associated SSTS with τ_3; then, the soft composition $\rho_{sd}o\psi_{sd}:(U_1,\tau_1,\Theta_1)\to(U_3,\tau_3,\Theta_3)$ is an SS-sd-cts.*

Proof. Let $(Q,\Theta_3)\in\tau_3$. Since ρ_{sd} is an SS-sd-cts, $\rho_{sd}^{-1}(Q,\Theta_3)\in SD(U_2)_{\Theta_2}$. Since ψ_{sd} is SS-sd-irresolute, $[\rho_{sd}o\psi_{sd}]^{-1}(Q,\Theta_3)=\psi_{sd}^{-1}[\rho_{sd}^{-1}(Q,\Theta_3)]\in SD(U_1)_{\Theta_1}$. Hence, $\rho_{sd}o\psi_{sd}$ is an SS-sd-cts.

Corollary 4.15. *The soft composition of two SS-sd-irresolute functions is also SS-sd-irresolute.*

Proof. It is straightforward from Theorem 4.14.

Theorem 4.16. *Let $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ be a soft function with μ_1, μ_2 associated SSTSs with τ_1, τ_2, respectively, then the following statements are equivalent:*

(1) *ψ_{sd} is SS-sd-irresolute.*

(2) *For each $(L, \Theta_2) \in SC(U_2)_{\Theta_2}$, either $\psi_{sd}^{-1}(L, \Theta_2) \in SC(U_1)_{\Theta_1}$ or $\psi_{sd}^{-1}(L, \Theta_2) = \tilde{U}_1$.*

(3) *$cl^s_{sd}(\psi_{sd}^{-1}(L, \Theta_2)) \tilde{\subseteq} \psi_{sd}^{-1}(cl^s_{sd}(L, \Theta_2))\ \forall\ (L, \Theta_2) \tilde{\subseteq} \tilde{U}_2$.*

(4) *$\psi_{sd}(cl^s_{sd}(M, \Theta_1)) \tilde{\subseteq} cl^s_{sd}(\psi_{sd}(M, \Theta_1))\ \forall\ (M, \Theta_1) \tilde{\subseteq} \tilde{U}_1$.*

(5) *$\psi_{sd}^{-1}(int^s_{sd}(L, \Theta_2)) \tilde{\subseteq} int^s_{sd}(\psi_{sd}^{-1}(L, \Theta_2))\ \forall\ (L, \Theta_2) \tilde{\subseteq} \tilde{U}_2$.*

Proof. It follows by a similar manner to the proof of Theorem 4.9.

Theorem 4.17. *A soft function $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ with μ_1, μ_2 associated SSTSs with τ_1, τ_2, respectively, is SS-sd-irresolute if*

$$cl^s(\psi_{sd}^{-1}(Y, \Theta_2)) \tilde{\subseteq} \psi_{sd}^{-1}(cl^s_{sd}(Y, \Theta_2))\ \forall\ (Y, \Theta_2) \tilde{\subseteq} \tilde{U}_2.$$

Proof. Assume that $(Y, \Theta_2) \tilde{\subseteq} \tilde{U}_2$. Since $cl^s_{sd}(Y, \Theta) \tilde{\subseteq} cl^s(Y, \Theta)$ from Theorem 3.4 (2), $cl^s_{sd}(\psi_{sd}^{-1}(Y, \Theta_2)) \tilde{\subseteq} cl^s(\psi_{sd}^{-1}(Y, \Theta_2)) \tilde{\subseteq} \psi_{sd}^{-1}(cl^s_{sd}(Y, \Theta_2))$ considering the given condition. Hence, ψ_{sd} is SS-sd-irresolute according to Theorem 4.16 (3).

Remark 4.18. *In general, the converse of Theorem 4.17 is not true, as shown in the next example.*

Example 4.19. *In Example 4.7, consider that*

$$\tau_1 = \{\tilde{U}_1, \tilde{\varphi}, (X_1, \Theta_1)\} \text{ be a } STS \text{ over } U_1 \text{ and}$$

$$\mu_1 = \{\tilde{U}_1, \tilde{\varphi}, (X_1, \Theta_1), (X_2, \Theta_1), (X_3, \Theta_1)\}$$

is an associated SSTS with τ_1, where

$$X_1(j_1) = \{m_1, m_2\}, \quad X_1(j_2) = \{m_1\}.$$

$$X_2(j_1) = \{m_1, m_2\}, \quad X_2(j_2) = \{m_2\}.$$

$$X_3(j_1) = \{m_1, m_2\}, \quad X_3(j_2) = \{m_1, m_2\}.$$

$$\text{Let } \tau_2 = \{\tilde{U}_2, \tilde{\varphi}, (W_1, \Theta_2)\} \text{ be an } STS \text{ over } U_2 \text{ and}$$

$$\mu_2 = \{\tilde{U}_2, \tilde{\varphi}, (W_1, \Theta_2), (W_2, \Theta_2), (W_3, \Theta_2), (W_4, \Theta_2), (W_5, \Theta_2)\}$$

be an associated SSTS with τ_2, where

$$W_1(k_1) = \{n_1, n_2, n_3\}, \quad W_1(k_2) = U_2.$$

$$W_2(k_1) = \{n_3\}, \quad W_2(k_2) = \varphi.$$

$$W_3(k_1) = \{n_1, n_3\}, \quad W_3(k_2) = \varphi.$$

$$W_4(k_1) = \{n_1\}, \quad W_4(k_2) = \{n_1\}.$$

$$W_5(k_1) = \{n_1, n_3\}, \quad W_5(k_2) = \{n_1\}.$$

Hence, ψ_{sd} is an SS-sd-irresolute, whereas for the soft set (W_2, Θ_2), we have that

$$\psi_{sd}^{-1}(cl_{sd}^{s}(W_2, \Theta_2)) = \{(j_1, \{m_2\}), (j_2, \varphi)\} \text{ and } cl^{s}(\psi_{sd}^{-1}(Y, \Theta_2)) = \tilde{U}_1.$$

$$Therefore,\ \tilde{U}_1 = cl^{s}(\psi_{sd}^{-1}(Y, \Theta_2)) \tilde{\nsubseteq} \psi_{sd}^{-1}(cl_{sd}^{s}(Y, \Theta_2)) = \{(j_1, \{m_2\}), (j_2, \varphi)\}.$$

Theorem 4.20. *A soft function $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ with μ_1, μ_2 associated SSTSs with τ_1, τ_2, respectively, is SS-sd-irresolute if one of the following conditions is satisfied:*

(1) $\psi_{sd}(cl^{s}(G, \Theta_1)) \tilde{\subseteq} cl_{sd}^{s}(\psi_{sd}(G, \Theta_1))\ \forall\ (G, \Theta_1) \tilde{\subseteq} \tilde{U}_1$.

(2) $\psi_{sd}^{-1}(int_{sd}^{s}(Y, \Theta_2)) \tilde{\subseteq} int^{s}(\psi_{sd}^{-1}(Y, \Theta_2))\ \forall\ (Y, \Theta_2) \tilde{\subseteq} \tilde{U}_2$.

Proof. If condition (1) is satisfied, then

$$\psi_{sd}(cl^{s}(G, \Theta_1)) \tilde{\subseteq} cl_{sd}^{s}(\psi_{sd}(G, \Theta_1))\ \forall\ (G, \Theta_1) \tilde{\subseteq} \tilde{U}_1.$$

Since $cl_{sd}^{s}(Y, \Theta) \tilde{\subseteq} cl^{s}(Y, \Theta)$ from Theorem 3.4 (2),

$$\psi_{sd}(cl_{sd}^{s}(G, \Theta_1)) \tilde{\subseteq} \psi_{sd}(cl^{s}(G, \Theta_1)) \tilde{\subseteq} cl_{sd}^{s}(\psi_{sd}(G, \Theta_1)).$$

Therefore, ψ_{sd} is SS-sd-irresolute according to Theorem 4.16 (4).
If condition (2) is satisfied, then $\psi_{sd}^{-1}(int_{sd}^{s}(Y, \Theta_2)) \tilde{\subseteq} int^{s}(\psi_{sd}^{-1}(Y, \Theta_2))\ \forall\ (Y, \Theta_2) \tilde{\subseteq} \tilde{U}_2$. Since $int^{s}(Y, \Theta) \tilde{\subseteq} int_{sd}(Y, \Theta_2)$,

$$\psi_{sd}^{-1}(int_{sd}^{s}(Y, \Theta_2)) \tilde{\subseteq} int^{s}(\psi_{sd}^{-1}(Y, \Theta_2)) \tilde{\subseteq} int_{sd}(\psi_{sd}^{-1}(Y, \Theta_2)).$$

Hence, ψ_{sd} is SS-sd-irresolute according to Theorem 4.16 (5).

Remark 4.21. *In general, the converse of Theorem 4.20 is not true, as shown in the next examples.*

Examples 4.22. *In Example 4.19, the following is applied:*

(1) *For the soft set $(T, \Theta_1) = \{(j_1, \{m_1, m_2\}), (j_2, \{m_3\})\}$, we have that*

$$cl_{sd}^{s}(\psi_{sd}(T, \Theta_1)) = \{(k_1, \{n_3, n_4\}), (k_2, \{n_1\})\} \text{ and } \psi_{sd}(cl^{s}(T, \Theta_1)) = \tilde{U}_1.$$

$$Therefore,\ \tilde{U}_1 = \psi_{sd}(cl^{s}(T, \Theta_1)) \tilde{\nsubseteq} cl_{sd}^{s}(\psi_{sd}(T, \Theta_1)) = \{(k_1, \{n_3, n_4\}), (k_2, \{n_1\})\}.$$

(2) *For the soft set $(C, \Theta_2) = \{(k_1, \{n_2, n_3, n_4\}), (j_2, U_2)\}$, we have that*

$$int^{s}(\psi_{sd}^{-1}(C, \Theta_2)) = \{(j_1, \{m_1, m_2\}), (j_2, \{m_1, m_2\})\} \text{ and}$$

$$\psi_{sd}^{-1}(int_{sd}^{s}(C, \Theta_2)) = \{(j_1, \{m_1, m_2, m_3\}), (j_2, U_2)\}.$$

Therefore,

$$\psi_{sd}^{-1}(int_{sd}^{s}(C, \Theta_2)) \tilde{\nsubseteq} int^{s}(\psi_{sd}^{-1}(C, \Theta_2)).$$

5. Supra soft sd-homeomorphism mappings

This section is devoted to introduce a new approach for SS-maps, named SS-sd-open maps, SS-sd-closed maps, and SS-sd-homeomorphism maps. Moreover, we clearly demonstrate their equivalent properties, with the support of examples.

Definition 5.1. *A soft mapping $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ with μ_2 as an associated SSTS with τ_2 is said to be as follows:*

(1) *SS-sd-open if $\psi_{sd}(G, \Theta_1) \in SD(U_2)_{\Theta_2}$ for each non-null soft open subset (G, Θ_1) of $\tilde{U}_1$.*

(2) *SS-sd-closed if either $\psi_{sd}(H, \Theta_1) \in SC(U_2)_{\Theta_2}$ or $\psi_{sd}(H, \Theta_1) = \tilde{U}_2$, $\forall\ (H, \Theta_1) \in \tau_1^c$.*

Theorem 5.2. *Let $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ be a soft mapping with μ_2 as an associated SSTS with τ_2 and $(G, \Theta_1)\tilde{\subseteq}\tilde{U}_1$; then,*

$$\psi_{sd} \text{ is an SS-sd-open if, and only if } \psi_{sd}(int(G, \Theta_1))\tilde{\subseteq} int^s_{sd}[\psi_{sd}(G, \Theta_1)].$$

Proof. " $\Rightarrow$ " Let ψ_{sd} be an SS-sd-open map and $(G, \Theta_1)\tilde{\subseteq}\tilde{U}_1$. So, either $int(G, \Theta_1) = \tilde{\varphi}$ or $int(G, \Theta_1) \neq \tilde{\varphi}$.
If $int(G, \Theta_1) = \tilde{\varphi}$, then the result is obtained.
If $int(G, \Theta_1) \neq \tilde{\varphi}$, then $\psi_{sd}(int(G, \Theta_1)) \in SD(U_2)_{\Theta_2}$.
Since $int(G, \Theta_1)\tilde{\subseteq}(G, \Theta_1)$, $\psi_{sd}(int(G, \Theta_1))\tilde{\subseteq}\psi_{sd}((G, \Theta_1))$. It follows that,

$$int^s_{sd}[\psi_{sd}(int(G, \Theta_1))] = \psi_{sd}(int(G, \Theta_1))\tilde{\subseteq} int^s_{sd}[\psi_{sd}((G, \Theta_1))].$$

" $\Leftarrow$ " Suppose that (G, Θ_1) is a non-null soft open subset of $\tilde{U}_1$. It follows that,

$$\psi_{sd}(int(G, \Theta_1)) = \psi_{sd}(G, \Theta_1)\tilde{\subseteq} int^s_{sd}[\psi_{sd}(G, \Theta_1)] \text{ according to the assumption.}$$

However, we have

$$int^s_{sd}[\psi_{sd}(G, \Theta_1)]\tilde{\subseteq}\psi_{sd}(G, \Theta_1).$$

Therefore,

$$int^s_{sd}[\psi_{sd}(G, \Theta_1)] = \psi_{sd}(G, \Theta_1).$$

Thus,

$$(G, \Theta_1) \in SD(U_1)_{\Theta_1}\text{; hence, } \psi_{sd} \text{ is SS-sd-open.}$$

Proposition 5.3. *Let $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ be a soft mapping with μ_2 as an associated SSTS with τ_2 and $(G, \Theta_1)\tilde{\subseteq}\tilde{U}_1$; then,*

$$\psi_{sd} \text{ is an SS-sd-closed if, and only if } cl^s_{sd}[\psi_{sd}(G, \Theta_1)]\tilde{\subseteq}\psi_{sd}(cl(G, \Theta_1)).$$

Proof. " $\Rightarrow$ " Suppose that ψ_{sd} is an SS-sd-closed map and $(G, \Theta_1)\tilde{\subseteq}\tilde{U}_1$. It follows that, either $\psi_{sd}(cl(G, \Theta_1)) \in SC(U_2)_{\Theta_2}$ or $\psi_{sd}(G, \Theta_1) = \tilde{U}_2$, where $cl(G, \Theta_1) \in \tau_1^c$. In fact, both cases leads to $cl^s_{sd}[\psi_{sd}(G, \Theta_1)]\tilde{\subseteq}\psi_{sd}(cl(G, \Theta_1))$.
" $\Leftarrow$ " Let $(G, \Theta_1) \in \tau_1^c$. Then, $cl(G, \Theta_1) = (G, \Theta_1)$. By assumption,

$$\psi_{sd}(G, \Theta_1)\tilde{\subseteq} cl^s_{sd}[\psi_{sd}(G, \Theta_1)]\tilde{\subseteq}\psi_{sd}(cl(G, \Theta_1)) = \psi_{sd}(G, \Theta_1).$$

Hence,

$$cl_{sd}^{s}[\psi_{sd}(G,\Theta_1)] = \psi_{sd}(G,\Theta_1).$$

Therefore,

$$\psi_{sd}(G,\Theta_1) \in SC(U_2)_{\Theta_2}\text{; hence, } \psi_{sd} \text{ is an SS-sd-closed.}$$

Proposition 5.4. *Let $\psi_{sd} : (U_1,\tau_1,\Theta_1) \rightarrow (U_2,\tau_2,\Theta_2)$ be a bijective soft mapping with μ_2 as an associated SSTS with τ_2; then, ψ_{sd} is an SS-sd-open if, and only if it is SS-sd-closed.*

Proof. " $\Rightarrow$ " Let $(G,\Theta_1) \in \tau_1^c$; then, $(G^{\tilde{c}},\Theta_1) \in \tau_1$. Since ψ_{sd} is bijective and SS-sd-open,

$$\text{either } [\psi_{sd}(G,\Theta_1)]^{\tilde{c}} = \psi_{sd}(G^{\tilde{c}},\Theta_1) \in SD(U_1)_{\Theta_1} \text{ or } [\psi_{sd}(G,\Theta_1)]^{\tilde{c}} = \tilde{\varphi}.$$

It follows that,

$$\text{either } \psi_{sd}(G,\Theta_1) \in SC(U_2)_{\Theta_2} \text{ or } \psi_{sd}(G,\Theta_1) = \tilde{U}_2.$$

Therefore, ψ_{sd} is SS-sd-closed.
" $\Leftarrow$ " It follows by a similar argument.

Remark 5.5. *The proof of Proposition 5.4 cannot be obtained in general without the bijectivity condition, as shown in the next example.*

Example 5.6. *Let $U_1 = \{m_1,m_2,m_3,m_4\}$, $U_2 = \{n_1,n_2,n_3,n_4\}$, $\Theta_1 = \{j_1, j_2\}$ and $\Theta_2 = \{k_1,k_2\}$.*

$$Define\ s : U_1 \rightarrow U_2\ and\ d : \Theta_1 \rightarrow \Theta_2\ as\ follows:$$

$$s(m_1) = n_1,\ s(m_2) = n_1,\ s(m_3) = n_1,\ s(m_4) = n_1,\ d(j_1) = k_1,\ d(j_2) = k_2.$$

$$Let\ \tau_1 = \{\tilde{U},\tilde{\varphi},(A,\Theta_1)\}\ be\ an\ STS\ over\ U_1,\ where$$

$$A(j_1) = \{m_3\}, \quad A(j_2) = \varphi.$$

$$Let\ \tau_2 = \{\tilde{U}_2,\tilde{\varphi},(B_1,\Theta_2)\}\ be\ an\ STS\ over\ U_2\ and$$

$$\mu_2 = \{\tilde{U}_2,\tilde{\varphi},(B_1,\Theta_2),(B_2,\Theta_2),(B_3,\Theta_2),(B_4,\Theta_2)\}$$

be an associated SSTS with τ_2, where

$$B_1(k_1) = \{n_1,n_2,n_3\}, \quad B_1(k_2) = U_2.$$

$$B_2(k_1) = \{n_1,n_3\}, \quad B_2(k_2) = \varphi.$$

$$B_3(k_1) = \{n_1\}, \quad B_3(k_2) = \{n_1\}.$$

$$B_4(k_1) = \{n_1,n_3\}, \quad B_4(k_2) = \{n_1\}.$$

Then,

$$\psi_{sd}(A,\Theta_1) = \{(k_1,\{n_1\}),(k_2,\varphi)\}$$

is an SS-sd-subset of $\tilde{U}_2$. On the other hand, we have that

$$\psi_{sd}(A^{\tilde{c}},\Theta_1)] = \{(k_1,\{n_1\}),(k_2,\{n_1\})\}$$

is not SS-sc-subset of $\tilde{U}_2$. It follows that, ψ_{sd} is an SS-sd-open, but it is not SS-sd-closed because it is not bijective.

Definition 5.7. *A bijective soft mapping $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ with μ_2 as an associated SSTS with τ_2 is said to be an SS-sd-homeomorphism if it is an SS-sd-cts and SS-sd-open.*

Theorem 5.8. *For a bijective soft mapping $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ with μ_1, μ_2 associated SSTSs with τ_1, τ_2, respectively. The following statements are equivalent:*

(1) *ψ_{sd} is SS-sd-homeomorphism.*

(2) *ψ_{sd} and ψ_{sd}^{-1} are each an SS-sd-cts.*

(3) *ψ_{sd} is both an SS-sd-closed and SS-sd-cts.*

Proof. It immediately follows from Proposition 5.4 and Definition 5.7.

Theorem 5.9. *A bijective soft mapping $\psi_{sd} : (U_1, \tau_1, \Theta_1) \rightarrow (U_2, \tau_2, \Theta_2)$ with μ_1, μ_2 associated SSTSs with τ_1, τ_2, respectively, is an SS-sd-homeomorphism if one of the following conditions is satisfied:*

(1) *$\psi_{sd}(cl^s_{sd}(G, \Theta_1)) \tilde{\subseteq} cl(\psi_{sd}(G, \Theta_1))$ and $cl^s_{sd}(\psi_{sd}(G, \Theta_1)) \tilde{\subseteq} \psi_{sd}(cl^s(G, \Theta_1))$, $\forall\ (G, \Theta_1) \tilde{\subseteq} \tilde{U}_1$.*

(2) *$\psi_{sd}(int(G, \Theta_1)) \tilde{\subseteq} int^s_{sd}(\psi_{sd}(G, \Theta_1))$, $\forall\ (G, \Theta_1) \tilde{\subseteq} \tilde{U}_1$ and $\psi_{sd}^{-1}(int(Z, \Theta_2)) \tilde{\subseteq} int^s_{sd}(\psi_{sd}^{-1}(Z, \Theta_2))$, $\forall\ (Z, \Theta_2) \tilde{\subseteq} \tilde{U}_2$.*

Proof. If condition (1) is satisfied, then $\psi_{sd}(cl^s_{sd}(G, \Theta_1)) \tilde{\subseteq} cl(\psi_{sd}(G, \Theta_1))$. From Theorem 4.9 (4), ψ_{sd} is SS-sd-cts. Also, since $cl^s_{sd}(\psi_{sd}(G, \Theta_1)) \tilde{\subseteq} \psi_{sd}(cl^s(G, \Theta_1))$, ψ_{sd} is an SS-sd-closed according to Proposition 5.3. Hence, ψ_{sd} is an SS-sd-homeomorphism according to Theorem 5.8.
If condition (2) is satisfied, then from Theorem 4.9 (5), Theorem 5.2 and Definition 5.7, the reader can obtain that ψ_{sd} is an SS-sd-homeomorphism.

6. Conclusions and upcoming applications

Recently, the generalizations of topological structures and weaker forms of sets have became easier to obtain for applications [5–8, 26, 55]. This gave us the motivation to introduce more generalizations to such types of weaker forms of sets. We aimed to investigate more properties of the SS-sd-operators [53]. Specifically, we studied the relations between them before and after excluding the null soft set. In addition, we introduced the SS-sd-boundary operator and discussed many of its properties. Another aim of this paper, was to introduce a wider class of SS-maps by using the SS-sd-sets and the SS-sc-sets. Finally, the SS-sd-homeomorphism maps were introduced as maps with the following characteristics: Bijective, SS-sd-cts, and SS-sd-open.

Our upcoming project is to generalize the aforementioned notions as based on the soft ideal [50]. Moreover, through the use of the above-mentioned approaches, more topological properties such as separation axioms and connectedness, will be introduced and our future work will be in this direction. Finally, the improvement of the accuracy measures for subsets in information systems will be considered by using the introduced generalizations.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflict of interest regarding the publication of this paper.

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