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Research article

Some new Young type inequalities

Yonghui Ren*

School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China

* **Correspondence:** Email: yonghuiren1992@163.com.

Abstract: In this paper, we gave some generalized Young type inequalities due to Zuo and Li [J. Math. Inequal., 16 (2022), 1169–1178], and we also presented a new Young type inequality. As applications, we obtained some operator inequalities and matrix versions inequalities including the Hilbert-Schmidt norm and trace.

Keywords: Young's inequality; Kantorovich constant; operator inequalities; Hilbert-Schmidt norm; trace
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1. Introduction

Let $(\mathcal{H}, \langle\cdot,\cdot\rangle)$ be a complex Hilbert space and let $\mathbb{B}(\mathcal{H})$ denote the algebra of all bounded linear operators acting on $\mathcal{H}$. A self-adjoint operator A is said to be positive if $\langle Ax, x\rangle \geq 0$ for all $x \in \mathcal{H}$, while it is said to be strictly positive if A is positive and invertible. As usual, we say that $A > B$ when $A - B > 0$ and $A \geq B$ when $A - B \geq 0$, respectively. Moreover, $\mathbb{M}_n$ denotes the sets of all $n \times n$ complex matrices. The unitarily invariance of the $|||\cdot|||$ on $\mathbb{M}_n$ means that $|||UAV||| = |||A|||$ for any $A \in \mathbb{M}_n$ and all unitary matrices $U, V \in \mathbb{M}_n$. For $A = [a_{ij}] \in \mathbb{M}_n$, the Hilbert-Schmidt (or Frobenius) norm and the trace norm of A are defined by

$$\|A\|_2 = \sqrt{\sum_{j=1}^{n} s_j^2(A)}$$

and

$$\|A\|_1 = \sum_{j=1}^{n} s_j(A),$$

respectively, where $s_j(A)$ are the singular values of A, that is, the eigenvalues of the positive semi-definite matrix

$$|A| = (A^*A)^{\frac{1}{2}},$$

and arranged in a nonincreasing order. It is well known that $\|\cdot\|_2$ is unitarily invariant. In addition, we defined

$$A\nabla_s B = (1-s)A + sB$$

and

$$A\sharp_s B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^s A^{\frac{1}{2}}$$

for $s \in [0,1]$, denoted by $A\nabla B$ and $A\sharp B$ when $s = \frac{1}{2}$, respectively. Similarly, we define the weighted means by

$$a\nabla_s b = (1-s)a + sb$$

and

$$a\sharp_s b = a^{1-s}b^s$$

for $a, b > 0$ and $s \in [0,1]$.

The famous Young's inequality states that the convex combination of two positive numbers is greater than or equal to the product of powers of these numbers with corresponding coefficients as exponents, which reads

$$a\sharp_s b \le a\nabla_s b \tag{1}$$

for $a, b > 0$ and $0 \le s \le 1$, with equation if and only if $a = b$. It extends the classical arithmetic-geometric means inequality. In 2015, Alzer et al. [1] showed the following refinements and reverses of Eq (1)

$$\left(\frac{s}{\tau}\right)^{\lambda} \le \frac{(a\nabla_s b)^{\lambda} - (a\sharp_s b)^{\lambda}}{(a\nabla_\tau b)^{\lambda} - (a\sharp_\tau b)^{\lambda}} \le \left(\frac{1-s}{1-\tau}\right)^{\lambda} \tag{2}$$

for $a, b > 0$, $0 < s \le \tau < 1$ and $\lambda \ge 1$. In recent years, there have been a large number of works directly inspired by the Alzer-Fonseca-Kovačec inequality, see [2–7] et al. Moreover, letting $\lambda = 1$, the following results obtained by Zhao and Wu [8] can be regarded as some further refinements of (2) when $s = \frac{1}{2}$ and $\tau = \frac{1}{2}$, respectively,

(i) If $0 \le s \le \frac{1}{2}$, then

$$sa + (1-s)b \le a^s b^{1-s} + (1-s)(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt[4]{ab} - \sqrt{a})^2; \tag{3}$$

(ii) If $\frac{1}{2} \le s \le 1$, then

$$sa + (1-s)b \le a^s b^{1-s} + s(\sqrt{a} - \sqrt{b})^2 - r_0(\sqrt[4]{ab} - \sqrt{b})^2; \tag{4}$$

where $a, b > 0$, $r_0 = \min\{2r, 1-2r\}$, $r = \min\{s, 1-s\}$ for $s \in [0,1]$. Sababheh and Moslehian [9] show a nice multiple-term refinements of (3) and (4). Interested readers could refer to [10–14] and references therein for some other results about Young's inequality.

Unless otherwise specified, we will default to $a, b > 0$ and $0 \le s \le 1$ in the rest of this article for our convenience.

There are some results related to Young's inequality mentioned above. Ghazanfari et al. [15] presented an inequality

$$(1 - s^2 + s^3)a + (1-s^2)b \le s^{s-2}a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2. \tag{5}$$

It is easy to verify that both sides in the inequality (5) are greater than or equal to the corresponding sides in the inequalities (3) and (4), respectively. This indicates that the inequality (5) is a new Young type inequality.

In 2020, Ren [4] showed a generalization of the inequality (5),

$$(1 - s^{N+1} + s^{N+2})a + (1 - s^2)b \leq s^{sN-(N+1)}a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2 \tag{6}$$

for $N \in \mathbb{N}^+$. Later, Yang and Li [16] gave a more generalized inequality than (6): If $N_1, N_2 \in \mathbb{N}^+$, then

$$(1 - s^{N_1+1} + s^{N_1+2})a + (1 - s^{N_2+2})b \leq s^{s(N_1-N_2)-(N_1+1)}a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2. \tag{7}$$

Very recently, Zuo and Li [17] got a new generalization of inequality (5): If $N \in \mathbb{N}^+$, then

$$(1 - s^{N+1} + s^{N+2})a + (1 - s^{N+1})b \leq s^{s-(N+1)}a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2. \tag{8}$$

In addition, the Kantorovich constant and the Specht's ratio are defined by

$$K(h, 2) = \frac{(h + 1)^2}{4h} \quad \text{for} \quad h > 0$$

and

$$S(h) = \begin{cases} \dfrac{h^{\frac{1}{h-1}}}{e \log\left(h^{\frac{1}{h-1}}\right)}, & \text{if } h \in (0, 1) \cup (1, \infty), \\ 1, & \text{if } h = 1. \end{cases}$$

$K(h, 2)$ and $S(h)$ have some common properties, for example:

(i) $K(1, 2) = S(1) = 1$;

(ii) $K(h, 2)$ and $S(h)$ are decreasing on $h \in (0, 1)$ and increasing on $h \in (1, +\infty)$;

(iii) $K(h, 2) \geq 1$ and $S(h) \geq 1$.

Zuo et al. [18] and Furuichi [19] showed

$$S(h^r)a\sharp_s b \leq K(h, 2)^r a\sharp_s b \leq a\nabla_s b \tag{9}$$

for $r = \min\{s, 1 - s\}$ and $h = \frac{a}{b} > 0$.

In this paper, we try to give some generalizations of (8), and we also present a new generalization of Young type inequality, then we refine these inequalities with the Kantorovich constant. As applications, we obtain some operator inequalities, Hilbert-Schmidt norm inequalities and trace inequalities.

2. Main results

First, we show a generalization of the inequality (8).

Theorem 1. *Let $N_1, N_2 \in \mathbb{N}^+$ and $1 \geq s \geq 0$, $a, b > 0$. Then, we have*

$$(1 - s^{N_1+1} + s^{N_1+2})a + (1 - s^{N_1+N_2+1})b \leq s^{s(1-N_2)-(N_1+1)}a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2. \tag{10}$$

Proof. By computations, we can get

$$\begin{aligned}
&s^{s(1-N_2)-(N_1+1)}a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2 - (1 - s^{N_1+1} + s^{N_1+2})a - (1 - s^{N_1+N_2+1})b \\
&= s^{s(1-N_2)-(N_1+1)}a^s b^{1-s} - 2\sqrt{ab} + (1-s)s^{N_1+1}a + ss^{N_1+N_2}b \\
&\geq s^{s(1-N_2)-(N_1+1)}a^s b^{1-s} - 2\sqrt{ab} + (s^{N_1+1}a)^{(1-s)}(s^{N_1+N_2}b)^s \quad \text{(by (1))} \\
&= (s^{\frac{s(1-N_2)-(N_1+1)}{2}}a^{\frac{s}{2}}b^{\frac{1-s}{2}} - s^{-\frac{s(1-N_2)-(N_1+1)}{2}}a^{\frac{1-s}{2}}b^{\frac{s}{2}})^2 \\
&\geq 0.
\end{aligned}$$

□

We now explain that Theorem 1 is a new generalization of the Young type inequality. In fact, comparing inequality (10) with (7), it is not difficult to find that

$$(1 - s^{N_1+N_2+1})b \geq (1 - s^{N_2+2})b$$

and

$$s^{s(1-N_2)-(N_1+1)} \geq s^{s(N_1-N_2)-(N_1+1)}$$

for $N_1, N_2 \in \mathbb{N}$, $1 \geq s \geq 0$, $b > 0$.

Remark 1. *Taking $N_2 = 0$ in Theorem 1, we get (8).*

Next, we show another generalization of (8), which can be regarded as a complement of Theorem 1.

Theorem 2. *Let $N_1, N_2 \in \mathbb{N}^+$ and $1 \geq s \geq 0$, $a, b > 0$. Then, we have*

$$(1 - s^{N_1+1} + s^{N_1+2})a + (1 - s^{N_1-N_2+2})b \leq s^{sN_2-(N_1+1)}a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2.$$

Proof. By computations, we can get

$$\begin{aligned}
&s^{sN_2-(N_1+1)}a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2 - (1 - s^{N_1+1} + s^{N_1+2})a - (1 - s^{N_1-N_2+2})b \\
&= s^{sN_2-(N_1+1)}a^s b^{1-s} - 2\sqrt{ab} + (1-s)s^{N_1+1}a + ss^{N_1-N_2+1}b \\
&\geq s^{sN_2-(N_1+1)}a^s b^{1-s} - 2\sqrt{ab} + (s^{N_1+1}a)^{(1-s)}(s^{N_1-N_2+1}b)^s \quad \text{(by (1))} \\
&= (s^{\frac{sN_2-(N_1+1)}{2}}a^{\frac{s}{2}}b^{\frac{1-s}{2}} - s^{-\frac{sN_2-(N_1+1)}{2}}a^{\frac{1-s}{2}}b^{\frac{s}{2}})^2 \\
&\geq 0.
\end{aligned}$$

□

Comparing Theorem 2 with Theorem 1, it is easy to see that

$$(1 - s^{N_1+N_2+1})b \geq (1 - s^{N_1-N_2+2})b$$

and

$$s^{s(1-N_2)-(N_1+1)} \geq s^{sN_2-(N_1+1)}$$

for $N_2 \in \mathbb{N}^+$, $1 \geq s \geq 0$, $b > 0$. Therefore, Theorem 2 is a new generalization of the Young type inequality.

Remark 2. *Taking $N_2 = 1$ in Theorem 2, we get (8).*

Next, we show a new generalization of Young type inequality.

Theorem 3. *Let $N_1, N_2 \in \mathbb{N}$, $1 \geq s \geq 0$ and $a, b > 0$. Then, we have*

$$(1 - s^{N_1} + s^{N_1+1})a + (1 - s^{N_2+1})b \leq s^{s(N_1-N_2)-N_1} a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2.$$

Proof. By computations, we obtain

$$\begin{aligned}
&(\sqrt{a} - \sqrt{b})^2 + s^{s(N_1-N_2)-N_1} a^s b^{1-s} - (1 - s^{N_1} + s^{N_1+1})a - (1 - s^{N_2+1})b \\
&= (1 - s)(s^{N_1} a) + s(s^{N_2} b) - 2\sqrt{ab} + s^{s(N_1-N_2)-N_1} a^s b^{1-s} \\
&\geq (s^{N_1} a)^{1-s}(s^{N_2} b)^s - 2\sqrt{ab} + s^{s(N_1-N_2)-N_1} a^s b^{1-s} \quad \text{(by (1))} \\
&= \left(s^{\frac{s(N_2-N_1)+N_1}{2}} a^{\frac{1-s}{2}} b^{\frac{s}{2}} - s^{\frac{s(N_1-N_2)-N_1}{2}} a^{\frac{s}{2}} b^{\frac{1-s}{2}}\right)^2 \\
&\geq 0.
\end{aligned}$$

□

Letting $N_2 = 0$ in Theorem 3, we get

$$(1 - s^N + s^{N+1})a + (1 - s)b \leq s^{sN-N} a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2 \tag{11}$$

for $N \in \mathbb{N}$, $1 \geq s \geq 0$ and $a, b > 0$. As we can see that both sides in (11) are greater than or equal to the corresponding sides in the inequalities (3) and (4), respectively, this indicates that Theorem 3 is a new generalization of the Young type inequality.

Following the ideas of Theorem 2, we can get a new Young type inequality by replacing N_2 with $1 - N_2$ in Theorem 3. However, we omit it to avoid repetition of the article.

We next improve Theorems 1–Theorem 3 with the Kantorovich constant by (9).

Theorem 4. *Let $N_1, N_2 \in \mathbb{N}^+$ and $1 \geq s \geq 0$, $a, b > 0$,*

$$h = \frac{s^{1-N_2} a}{b}$$

and

$$r = \min\{s, 1 - s\}.$$

We have

$$(1 - s^{N_1+1} + s^{N_1+2})a + (1 - s^{N_1+N_2+1})b \leq K(h, 2)^{-r} s^{s(1-N_2)-(N_1+1)} a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2. \tag{12}$$

Proof. Compute

$$\begin{aligned}
&K(h, 2)^{-r} s^{s(1-N_2)-(N_1+1)} a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2 - (1 - s^{N_1+1} + s^{N_1+2})a - (1 - s^{N_1+N_2+1})b \\
&= K(h, 2)^{-r} s^{s(1-N_2)-(N_1+1)} a^s b^{1-s} - 2\sqrt{ab} + (1 - s)s^{N_1+1} a + s s^{N_1+N_2} b \\
&\geq K(h, 2)^{-r} s^{s(1-N_2)-(N_1+1)} a^s b^{1-s} - 2\sqrt{ab} + K(h, 2)^r (s^{N_1+1} a)^{(1-s)} (s^{N_1+N_2} b)^s \\
&= \left(K(h, 2)^{-\frac{r}{2}} s^{\frac{s(1-N_2)-(N_1+1)}{2}} a^{\frac{s}{2}} b^{\frac{1-s}{2}} - K(h, 2)^{\frac{r}{2}} s^{-\frac{s(1-N_2)-(N_1+1)}{2}} a^{\frac{1-s}{2}} b^{\frac{s}{2}}\right)^2 \\
&\geq 0.
\end{aligned}$$

□

Theorem 5. *Let* $N_1, N_2 \in \mathbb{N}^+$, $1 \geq s \geq 0$, $a, b > 0$ *and* $r = \min\{s, 1-s\}$.
(i) If

$$h = \frac{s^{N_2} a}{b},$$

then, we have

$$(1 - s^{N_1+1} + s^{N_1+2})a + (1 - s^{N_1-N_2+2})b \leq K(h,2)^{-r} s^{sN_2-(N_1+1)} a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2.$$

(ii) If

$$h = \frac{s^{N_1-N_2} a}{b},$$

then, we have

$$(1 - s^{N_1} + s^{N_1+1})a + (1 - s^{N_2+1})b \leq K(h,2)^{-r} s^{s(N_1-N_2)-N_1} a^s b^{1-s} + (\sqrt{a} - \sqrt{b})^2.$$

Proof. Using the same technique as in Theorem 4, we complete the proof of Theorem 5. □

Replacing a by a^2 and b by b^2 in Theorems 4 and 5, respectively, we get the following corollary:

Corollary 1. *Let* $N_1, N_2 \in \mathbb{N}^+$, $a, b > 0$ *and* $r = \min\{s, 1-s\}$ *for* $1 \geq s \geq 0$.
(i) If

$$h = \frac{s^{1-N_2} a^2}{b^2},$$

then, we have

$$(1 - s^{N_1+1} + s^{N_1+2})a^2 + (1 - s^{N_1+N_2+1})b^2 \leq K(h,2)^{-r} s^{s(1-N_2)-(N_1+1)} (a^s b^{1-s})^2 + (a-b)^2.$$

(ii) If

$$h = \frac{s^{N_2} a^2}{b^2},$$

then, we have

$$(1 - s^{N_1+1} + s^{N_1+2})a^2 + (1 - s^{N_1-N_2+2})b^2 \leq K(h,2)^{-r} s^{sN_2-(N_1+1)} (a^s b^{1-s})^2 + (a-b)^2.$$

(iii) If

$$h = \frac{s^{N_1-N_2} a^2}{b^2},$$

then, we have

$$(1 - s^{N_1} + s^{N_1+1})a^2 + (1 - s^{N_2+1})b^2 \leq K(h,2)^{-r} s^{s(N_1-N_2)-N_1} (a^s b^{1-s})^2 + (a-b)^2.$$

Based on the scalars results mentioned above, we next present some operator inequalities, Hilbert-Schmidt norm inequalities and trace inequalities as promised.

Lemma 6. *[20] Let* $X \in \mathbb{M}_n$ *be self-adjoint and* f *and* g *be continuous real functions such that* $f(t) \geq g(t)$ *for all* $t \in Sp(X)$ *(the spectrum of* X*). Then* $f(X) \geq g(X)$.

Theorem 7. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be strictly positive operators and let positive real numbers m, m' and M, M' satisfy either of the following conditions:*
(i) $0 < s^{1-N_2} mI \le A \le s^{1-N_2} m'I < M'I \le B \le MI$;
(ii) $0 < s^{N_2-1} mI \le B \le s^{N_2-1} m'I < M'I \le A \le MI$.
Then, we have

$$(1 - s^{N_1+1} + s^{N_1+2})B + (1 - s^{N_1+N_2+1})A \le K(h', 2)^{-r} s^{s(1-N_2)-(N_1+1)} A\sharp_s B + 2(A\nabla B - A\sharp B),$$

where

$$h' = \frac{M'}{s^{1-N_2} m'}, \quad r = \min\{s, 1 - s\}$$

and $N_1, N_2 \in \mathbb{N}^+$.

Proof. Let $b = 1$ in (12), we have

$$(1 - s^{N_1+1} + s^{N_1+2})a + (1 - s^{N_1+N_2+1}) \le K(s^{1-N_2} a, 2)^{-r} s^{s(1-N_2)-(N_1+1)} a^s + (a + 1 - 2\sqrt{a}\,). \tag{13}$$

Under the conditions (i), we have

$$I \le h'I = \frac{M'}{s^{1-N_2} m'} I \le X = A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \le hI = \frac{M}{s^{1-N_2} m} I,$$

and then

$$Sp(X) \subseteq [h', h] \subseteq (1, +\infty).$$

By Lemma 6 and (13), we obtain

$$(1 - s^{N_1+1} + s^{N_1+2})X + (1 - s^{N_1+N_2+1})I \le \max_{h' \le x \le h} K(x, 2)^{-r} s^{s(1-N_2)-(N_1+1)} X^s + (X + I - 2X^{\frac{1}{2}}).$$

Since the Kantorovich constant $K(h, 2)$ is an increasing function on $h \in (1, +\infty)$, then

$$\begin{aligned}&(1 - s^{N_1+1} + s^{N_1+2})A^{-\frac{1}{2}} B A^{-\frac{1}{2}} + (1 - s^{N_1+N_2+1})I \\ &\le K(h', 2)^{-r} s^{s(1-N_2)-(N_1+1)} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^s + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} + I - 2(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}}.\end{aligned} \tag{14}$$

Under the conditions (ii), we have

$$\frac{1}{h} I = \frac{s^{N_2-1} m}{M} I \le X = A^{-\frac{1}{2}} B A^{-\frac{1}{2}} \le \frac{1}{h'} I = \frac{s^{N_2-1} m'}{M'} I \le I,$$

and then

$$Sp(X) \subseteq [\frac{1}{h}, \frac{1}{h'}] \subseteq (0, 1).$$

Since the Kantorovich constant $K(h, 2)$ is a decreasing function on $h \in (0, 1)$, we can similarly get

$$\begin{aligned}&(1 - s^{N_1+1} + s^{N_1+2})A^{-\frac{1}{2}} B A^{-\frac{1}{2}} + (1 - s^{N_1+N_2+1})I \\ &\le K(\frac{1}{h'}, 2)^{-r} s^{s(1-N_2)-(N_1+1)} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^s + A^{-\frac{1}{2}} B A^{-\frac{1}{2}} + I - 2(A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^{\frac{1}{2}}.\end{aligned} \tag{15}$$

In fact, inequality (14) is equal to (15) with the property

$$K(t, 2) = K(\frac{1}{t}, 2).$$

Multiplying $A^{\frac{1}{2}}$ from the both sides to the inequalities (14) or (15), we get the required results. □

Theorem 8. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be strictly positive operators and let positive real numbers m, m' and M, M' satisfy either of the following conditions*
(i) $0 < s^{N_2}mI \leq A \leq s^{N_2}m'I < M'I \leq B \leq MI$;
(ii) $0 < s^{-N_2}mI \leq B \leq s^{-N_2}m'I < M'I \leq A \leq MI$;
Then, we have

$$(1 - s^{N_1+1} + s^{N_1+2})B + (1 - s^{N_1-N_2+2})A \leq K(h', 2)^{-r} s^{sN_2-(N_1+1)} A\sharp_s B + 2(A\nabla B - A\sharp B),$$

where

$$h' = \frac{M'}{s^{N_2}m'}, \quad r = \min\{s, 1 - s\}$$

and $N_1, N_2 \in \mathbb{N}^+$.

Proof. Using the same method as in Theorem 7, we can get Theorem 8 by Theorem 5 (i). □

Theorem 9. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be strictly positive operators and let positive real numbers m, m' and M, M' satisfy either of the following conditions*
(i) $0 < s^{N_1-N_2}mI \leq A \leq s^{N_1-N_2}m'I < M'I \leq B \leq MI$;
(ii) $0 < s^{N_2-N_1}mI \leq B \leq s^{N_2-N_1}m'I < M'I \leq A \leq MI$;
Then, we have

$$(1 - s^{N_1} + s^{N_1+1})B + (1 - s^{N_2+1})A \leq K(h', 2)^{-r} s^{s(N_1-N_2)-N_1} A\sharp_s B + 2(A\nabla B - A\sharp B).$$

where

$$h' = \frac{M'}{s^{N_1-N_2}m'}, \quad r = \min\{s, 1 - s\}$$

and $N_1, N_2 \in \mathbb{N}^+$.

Proof. Using the same method as in Theorem 7, we can obtain Theorem 9 by Theorem 5 (ii). □

Theorem 10. *Let $A, B \in \mathbb{M}_n$ be positive definite and $X \in \mathbb{M}_n$. Then, we have*

$$(1 - s^{N_1+1} + s^{N_1+2})\|A\|_2^2 + (1 - s^{N_1+N_2+1})\|B\|_2^2 \leq K^{-r} s^{s(1-N_2)-(N_1+1)} \|A^s X B^{1-s}\|_2^2 + \|AX - XB\|_2^2,$$

where $N_1, N_2 \in \mathbb{N}^+$, $r = \min\{s, 1 - s\}$ for $1 \geq s \geq 0$ and

$$K = \min\{K(\frac{s^{1-N_2}\lambda_i^2}{\mu_l^2}, 2), 1 \leq i, l \leq n\}$$

for λ_i, μ_l are eigenvalues of A, B, respectively.

Proof. Since $A, B \in \mathbb{M}_n$ are positive definite, by the spectral theorem, there exist unitary matrices $U, V \in \mathbb{M}_n$ such that

$$A = U\Lambda_1 U^*, B = V\Lambda_2 V^*,$$

where

$$\Lambda_1 = \text{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n), \quad \Lambda_2 = \text{diag}(\mu_1, \mu_2, \cdots, \mu_n), \quad \lambda_i, \mu_i > 0, i = 1, 2, \cdots, n.$$

Let

$$Y = U^*XV = [y_{il}],$$

then,

$$\|AX - XB\|_2^2 = \left\|U[(\lambda_i - \mu_l)y_{il}]V^*\right\|_2^2 = \sum_{i,l=1}^{n}(\lambda_i - \mu_l)^2|y_{il}|^2.$$

Similarly, we have

$$\|A^sXB^{1-s}\|_2^2 = \sum_{i,l=1}^{n}(\lambda_i^s\mu_l^{1-s})^2|y_{il}|^2$$

and

$$\|A\|_2^2 = \sum_{i,l=1}^{n}\lambda_i^2|y_{il}|^2.$$

By the unitarily invariance of the Hilbert-Schmidt norm and Corollary 1 (i), we get the following results

$$\begin{aligned}
&(1 - s^{N_1+1} + s^{N_1+2})\|A\|_2^2 + (1 - s^{N_1+N_2+1})\|B\|_2^2 \\
&= (1 - s^{N_1+1} + s^{N_1+2})\sum_{i,l=1}^{n}\lambda_i^2|y_{il}|^2 + (1 - s^{N_1+N_2+1})\sum_{i,l=1}^{n}\mu_l^2|y_{il}|^2 \\
&= \sum_{i,l=1}^{n}\Big[(1 - s^{N_1+1} + s^{N_1+2})\lambda_i^2 + (1 - s^{N_1+N_2+1})\mu_l^2\Big]|y_{il}|^2 \\
&\le \sum_{i,l=1}^{n}\Big[K^{-r}s^{s(1-N_2)-(N_1+1)}(\lambda_i^s\mu_l^{1-s})^2 + (\lambda_i - \mu_l)^2\Big]|y_{il}|^2 \\
&= K^{-r}s^{s(1-N_2)-(N_1+1)}\|A^sXB^{1-s}\|_2^2 + \|AX - XB\|_2^2.
\end{aligned}$$

□

Theorem 11. *Let $A, B \in \mathbb{M}_n$ be positive definite and $X \in \mathbb{M}_n$. Then, we have*

$$(1 - s^{N_1+1} + s^{N_1+2})\|A\|_2^2 + (1 - s^{N_1-N_2+2})\|B\|_2^2 \le K^{-r}s^{sN_2-(N_1+1)}\|A^sXB^{1-s}\|_2^2 + \|AX - XB\|_2^2,$$

where $N_1, N_2 \in \mathbb{N}^+$, $r = \min\{s, 1-s\}$ for $1 \ge s \ge 0$ and

$$K = \min\{K(\frac{s^{N_2}\lambda_i^2}{\mu_l^2}, 2), 1 \le i, l \le n\}$$

for λ_i, μ_l are eigenvalues of A, B, respectively.

Proof. We get Theorem 11 by Theorem 10 and Corollary 1 (ii). □

Theorem 12. *Let $A, B \in \mathbb{M}_n$ be positive definite and $X \in \mathbb{M}_n$. Then, we have*

$$(1 - s^{N_1} + s^{N_1+1})\|A\|_2^2 + (1 - s^{N_2+1})\|B\|_2^2 \le K^{-r}s^{s(N_1-N_2)-N_1}\|A^sXB^{1-s}\|_2^2 + \|AX - XB\|_2^2,$$

where $N_1, N_2 \in \mathbb{N}^+$, $r = \min\{s, 1-s\}$ for $1 \ge s \ge 0$ and

$$K = \min\{K(\frac{s^{N_1-N_2}\lambda_i^2}{\mu_l^2}, 2), 1 \le i, l \le n\}$$

for λ_i, μ_l are eigenvalues of A, B, respectively.

Proof. We get Theorem 12 by Theorem 10 and Corollary 1 (iii). □

Lemma 13. *[21] Let $A, B \in \mathbb{M}_n$, then,*

$$\sum_{j=1}^{n} s_j(AB) \le \sum_{j=1}^{n} s_j(A)s_j(B).$$

Theorem 14. *Let $A, B \in \mathbb{M}_n$ be positive definite and $N_1, N_2 \in \mathbb{N}^+$, $r = \min\{s, 1-s\}$ for $1 \ge s \ge 0$ and*

$$K = \min\{K(\frac{s^{1-N_2}s_j(A)}{s_j(B)}, 2), 1 \le j \le n\}.$$

Thus, we have

$$\mathrm{tr}[(1 - s^{N_1+1} + s^{N_1+2})A + (1 - s^{N_1+N_2+1})B] \le K^{-r}s^{s(1-N_2)-(N_1+1)}\|A^s\|_2\|B^{1-s}\|_2 + \|A\|_1 + \|B\|_1 - 2\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|_1.$$

Proof. By Theorem 4, Lemma 13 and the famous Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\mathrm{tr}[(1 - s^{N_1+1} + s^{N_1+2})A + (1 - s^{N_1+N_2+1})B] \\
&= \sum_{j=1}^{n}((1 - s^{N_1+1} + s^{N_1+2})s_j(A) + (1 - s^{N_1+N_2+1})s_j(B)) \\
&\le \sum_{j=1}^{n}\Big[\max K\Big(\frac{s^{1-N_2}s_j(A)}{s_j(B)}, 2\Big)^{-r} s^{s(1-N_2)-(N_1+1)}s_j^s(A)s_j^{1-s}(B) + (\sqrt{s_j(A)} - \sqrt{s_j(B)})^2\Big] \\
&= K^{-r}s^{s(1-N_2)-(N_1+1)}\sum_{j=1}^{n}s_j(A^s)s_j(B^{1-s}) + \sum_{j=1}^{n}s_j(A) + \sum_{j=1}^{n}s_j(B) - 2\sum_{j=1}^{n}\sqrt{s_j(A)s_j(B)} \\
&\le K^{-r}s^{s(1-N_2)-(N_1+1)}\Big(\sum_{j=1}^{n}s_j^2(A^s)\Big)^{\frac{1}{2}}\Big(\sum_{j=1}^{n}s_j^2(B^{1-s})\Big)^{\frac{1}{2}} + \sum_{j=1}^{n}s_j(A) + \sum_{j=1}^{n}s_j(B) - 2\sum_{j=1}^{n}s_j(A^{\frac{1}{2}}B^{\frac{1}{2}}) \\
&= K^{-r}s^{s(1-N_2)-(N_1+1)}\|A^s\|_2\|B^{1-s}\|_2 + \|A\|_1 + \|B\|_1 - 2\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|_1.
\end{aligned}$$

□

Theorem 15. *Let $A, B \in \mathbb{M}_n$ be positive definite and $N_1, N_2 \in \mathbb{N}^+$, $r = \min\{s, 1-s\}$ for $1 \ge s \ge 0$ and*

$$K = \min\{K(\frac{s^{N_2}s_j(A)}{s_j(B)}, 2), 1 \le j \le n\}.$$

We have

$$\mathrm{tr}[(1 - s^{N_1+1} + s^{N_1+2})A + (1 - s^{N_1-N_2+2})B] \le K^{-r}s^{sN_2-(N_1+1)}\|A^s\|_2\|B^{1-s}\|_2 + \|A\|_1 + \|B\|_1 - 2\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|_1.$$

Proof. Following the line of Theorem 14, we can get the Theorem 15 by Theorem 5 (i). □

Theorem 16. *Let $A, B \in \mathbb{M}_n$ be positive definite and $N_1, N_2 \in \mathbb{N}^+$, $r = \min\{s, 1-s\}$ for $1 \ge s \ge 0$ and*

$$K = \min\{K(\frac{s^{N_1-N_2}s_j(A)}{s_j(B)}, 2), 1 \le j \le n\}.$$

We have

$$\mathrm{tr}[(1 - s^{N_1} + s^{N_1+1})A + (1 - s^{N_2+1})B] \le K^{-r}s^{s(N_1-N_2)-N_1}\|A^s\|_2\|B^{1-s}\|_2 + \|A\|_1 + \|B\|_1 - 2\|A^{\frac{1}{2}}B^{\frac{1}{2}}\|_1.$$

Proof. Following the line of Theorem 14, we can get the Theorem 16 by Theorem 5 (ii). □

3. Conclusions

In this paper, we give two generalized Young type inequalities of Zuo and Li [17], and we also present a new Young type inequality by comparing with the results obtained by Zhao and Wu [8]. As applications, we obtain some inequalities including operator, Hilbert-Schmidt norm and trace using our scalars results.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares that he has no competing interest.

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