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Research article

Analysis of a nonlinear problem involving discrete and proportional delay with application to Houseflies model

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Abstract: This manuscript established a comprehensive analysis of a general class of fractional order delay differential equations with Caputo-Fabrizio fractional derivative (CFFD). Functional analysis was used to examine the existence and uniqueness of the suggested class and to generate sufficient requirements for Ulam-Hyers (UH) type stability. Further, a numerical method based on Lagrange interpolation is used to compute approximate solution. Then, some applications in physical dynamics including a houseflies model and a Cauchy type problem were discussed to illustrate the established analysis with graphical illustrations.

Keywords: CFFD; existence theory; UH stability; numerical results
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1. Introduction

The development of fractional calculus (FC) has gotten a lot of interest recently in order to study differentiation and integration to non-integer order. This area of FC has significant applications in various field of science and technology, where real-world problems can be represented as mathematical equations. Fractional differentiation and integration research is intrinsically multidisciplinary and finds application in a wide range of fields, including complex systems, medical imaging, continuum mechanics, elasticity, signal analysis, quantum mechanics, bioengineering, biomedicine, financial systems, social systems, pollution control, turbulence, population growth and dispersal, landscape evolution, and pollution control (see [1]). The utilization of real order derivatives has been found to be very useful in many practical applications as compared to integer order derivatives. For instance, when modelling anomalous diffusion phenomena, the fractal structure more accurately captures the actual conditions of the medium, such as in the case of reservoirs where determining an Euclidian structure is challenging by nature. In addition, FC provides novel mathematical tools for modeling physical and biological processes. For more sophisticated applications, we refer to [2].

In addition, the aforementioned field is applicable in a wide range of other scientific subfields, such as physics [3, 4], biological sciences [5, 6], chemical sciences [7], and a variety of other fields [8]. While integer order was being developed, the idea of any arbitrary order derivative was also being developed. It has not been deemed the answer to problems that occur in the actual world due to the fact that it is quite complicated. In later years, as technology continued to evolve and new definitions of complex functions were developed, the field received the attention it deserved. The idea has currently been effectively used to address a wide range of complex scientific problems. Numerous real-world issues have been resolved by using fractional derivatives instead of integer order. One illustration of these problems is the state-space model for lithium-ion batteries with non-integer order derivatives (see [9]). The Caputo fractional order derivative model of Zika virus transmission dynamics [10], current empiricism, and classical science are all examples of fractional-order modeling of electric circuits [11].

It is worth mentioning that "fractional derivative" does not have a single widely accepted definition. Numerous definitions, such as the Riemann-Liouville (R-L), Caputo definitions [12], conformable fractional derivative [13], Atangana-Baleanu defintion [14], etc, can be found in the literature. These ideas have recently been used as the foundation for breaking down numerous mathematical problems [15–17]. Problems related to material heterogeneities can not be explained using R-L or Caputo concepts [18]. To overcome this difficulty, authors in [19] defined another definition of arbitrary order derivative with non-singular kernel called Caputo-Fabrizio definition. Some properties of the new concepts have been studied by Losada and Nieto in [18]. Various problems have been investigated for existence and uniqueness by the applications of mathematical analysis using the Caputo-Fabrizio fractional derivative (CFFD), we refer to [20] and [21] for properties and study of boundary value problems using CFFD.

It is also very important to note that delay-type problems have received a lot of attention from researchers working in fractional calculus. This is a significant subfield in the context of this sector. Many scholars have devoted a tremendous amount of time to finding solutions to delay-type problems (see [22, 23]). Additionally, a delay can be introduced to a dynamical problem to account for the time it takes for a disease to manifest its symptoms. As an example, we incorporate a delay in a fractional-order epidemic model. As we have already said, models utilizing ordinary fractional order derivatives need a lot of storage space and result in inefficient operation. As a result, researchers have developed a short-memory fractional modeling process in addition to short-memory fractional derivatives in order to overcome the problem. You can get qualitative analysis of delay differential equations in [24] and [25].

Here it is worth mentioning that equations involve two delay terms appear in various neurological models. For instance Kitching in 1977 highlighted the necessity to take into account a number of time delay elements in the life cycle of the Australian blowfly Lucila cuprina while estimating its population. Therefore, it is interesting to mention that two delays terms problems have increasingly been used in modelling various process in physiological, medical models (see [26]). Inspired from the above discussion, we are going to investigate a general class of aforementioned equation under CFFD. The proposed problem involves discrete and proportional delay terms. To the best of our knowledge, such problems have not been studied under the mentioned fractional operator. The suggested problem is

$$\begin{cases} {}^{CF}\mathcal{D}_{0^+}^{\delta}\mathrm{U}(\mathrm{z}) = \Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda \mathrm{z}), \mathrm{U}(\mathrm{z}-\tau)), z \in [0, \mathbf{T}], \\ \mathrm{U}(0) = \mathrm{U}_0 + \Psi(\mathrm{U}), \\ \mathrm{U}(\mathrm{z}) = \Phi(\mathrm{z}), \mathrm{z} \in [-\tau, 0), \end{cases} \tag{1.1}$$

where $0 < \delta \leq 1$, $0 < \lambda < 1$, $\Phi \in C[I \times R^3, R]$, $\Psi \in C[R, R]$, and $\Phi : [-\tau, 0) \to R$, is a continuous history function.

Here, we remark that delay terms appear in differential equations with advanced arguments in mathematical models of various real world problems. For instance, the economic models, influence and processing are subject to natural delays. Researchers have contributed to the theory and stability analysis of delay differential equations. However, the problems involving mixed delay terms of discrete and proportional type are rarely considered under the advanced arguments of fractional calculus. Particularly, the considered problem with mixed delay terms has not been studied by using the non-singular type Caputo-Fabrizio derivative. Also, the numerical study of such problems has previously not been undertaken. Hence that work partially fills that gap. A comprehensive analysis containing existence theory, and stability analysis will be established. To that end, we will use tools of nonlinear analysis.

Stability theory in mathematics deals with the stability of dynamical system trajectories and solutions to differential equations under slight changes in initial circumstances. In the investigation of dynamical systems, stability means that the trajectories do not change too much under small perturbations. Stability is also an important consequence from a numerical and optimization point of view. Stability theory can be established by using various concepts like Lyapunov stability, Mittag-Leffler stability, exponential stability, UH stability. In this article, we study Hyers Ulam stability for our considered problem. During a talk in 1940, Ulam posed a question on the stability of group homomorphisms which then served as the inspiration for the stability problem of functional equations. In the setting of Banach spaces and additive mappings, Hyers provided a partial affirmative response to Ulam's query. This was a major first step towards additional solutions in this field [27]. Later on, the concept was explored by Rassias for various problems of functional analysis (see [28]). The aforementioned concept has been extensively investigated for various problems of fractional order differential equations. Here, we refer to [29–31]. Recently researchers have studied the mentioned aspects for some applicable problems in [32] and [33]. Also, stability due to Hyers and Ulam has been attracted the attention of researchers working in the area of mathematical models. It has also been studied in relation to dynamical problems as well, for instance see [34] and [35] respectively.

Additionally, mathematical models with nonlocal operators of differentiations and integrations have also been studied. The mentioned operators have the ability to capture the crossover dynamics of real-word phenomena more efficiently. Here, for interesting applications, we refer to [36]. Computing the exact or analytical solution to nonlinear problems with fractional order derivatives is a quiet difficult task. Therefore, researchers usually use various numerical methods. For instance, authors [37] have applied a randomized Euler scheme to find the numerical solutions for irregular delay problems of ordinary differential equations. In the same way, authors [38] studied the existence, uniqueness, and approximation of solutions to Carathéodory delay differential equations. A survey was conducted on the numerical solution of fractional differential equations (see [39]). In addition, authors [40] presented numerical methods for delay differential equations. Also numerical results for the area devoted to nonsingular type differential operators is curial importance for researchers, and significant contribution has been done (see [41]). Keeping all the mentioned points in mind, we will investigate the qualitative theory of existence uniqueness, stability analysis and numerical investigations for the considered problem. To make our results applicable and novel, the results are testified by the famous house houseflies model and a Cauchy type dynamical problem. The concerned problems are special examples of the famous logistic equations which has numerous applications in ecology as well as in biological and physical disciplines.

2. Preliminaries

Let, $\mathbf{J} = [0, \mathbf{T}]$, and $\Omega = C(\mathbf{J}, R)$. Then, for any $\mathrm{U} \in \Omega$ the supremum norm $\|\cdot\|$ on Ω is defined as follows

$$\|\mathrm{U}\| = \sup_{\mathrm{z}\in\mathbf{J}} |\mathrm{U}(\mathrm{z})|.$$

Thus, Ω is a Banach space with the above norm defined on it.

Definition 2.1. [19] For, $\delta \in (0, 1]$, the CFFD of $\mathrm{U}(\mathrm{z}) \in \Omega$ can be described as

$$ {}^{CF}\mathcal{D}^{\delta}_{0^+}\mathrm{U}(\mathrm{z}) = \frac{M(\delta)}{(1-\delta)} \int_0^{\mathrm{z}} \exp\left(-\frac{\delta}{1-\delta}(\mathrm{z}-\xi)^{\delta-1}\right)\mathrm{U}'(\xi)d\xi, \tag{2.1}$$

where, $M(\delta)$ is a normalization function.

Definition 2.2. [19] For, $\delta \in (0, 1]$ the Caputo-Fabrizio fractional integral for $\mathrm{U}(\mathrm{z}) \in \Omega$ can be described as

$$ {}^{CF}\mathbf{I}^{\delta}_{0^+}\mathrm{U}(\mathrm{z}) = \frac{1-\delta}{M(\delta)}\mathrm{U}(\mathrm{z}) + \frac{\delta}{M(\delta)} \int_0^{\mathrm{z}} \mathrm{U}(\xi)d\xi. \tag{2.2}$$

Lemma 2.3. *[20] Let $x \in L(\mathbf{J})$, where $x \to 0$ at $z \to 0$, the solution of*

$$\begin{aligned} {}^{CF}\mathcal{D}^{\delta}_{0^+}\mathrm{U}(\mathrm{z}) &= x(\mathrm{z}), \ \textit{with} \ \ \delta \in (0, 1], \\ \mathrm{U}(0) &= \mathrm{U}_0 \end{aligned}$$

is given by

$$\mathrm{U}(\mathrm{z}) = \mathrm{U}_0 + \frac{1-\delta}{M(\delta)}x(\mathrm{z}) + \frac{\delta}{M(\delta)} \int_0^{\mathrm{z}} x(\xi)d\xi.$$

3. Existence theory

In this part of our manuscript, we are going to study the problem (1.1) for existence theory.

Lemma 3.1. *Let $\Phi \in C(\mathbf{J} \times \mathcal{R}^3)$. Then, the solution of problem*

$$\begin{cases} {}^{CF}\mathcal{D}_{0^+}^{\delta}\mathrm{U}(\mathrm{z}) = \Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda \mathrm{z}), \mathrm{U}(\mathrm{z}-\tau)), z \in \mathbf{J}, \\ \mathrm{U}(0) = \mathrm{U}_0 + \Psi(\mathrm{U}), \\ \mathrm{U}(\mathrm{z}) = \Phi(\mathrm{z}), \mathrm{z} \in [-\tau, 0), \end{cases} \tag{3.1}$$

is given by

$$\mathrm{U}(\mathrm{z}) = \begin{cases} \mathrm{U}_0 + \Psi(\mathrm{U}) + \dfrac{1-\delta}{M(\delta)}\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda \mathrm{z}), \mathrm{U}(\mathrm{z}-\tau)) \\ +\dfrac{\delta}{M(\delta)}\displaystyle\int_0^{\mathrm{z}} \Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi-\tau))d\xi, z \in \mathbf{J}. \end{cases} \tag{3.2}$$

Proof. Using Lemma 2.3, we have

$$\mathrm{U}(\mathrm{z}) = \begin{cases} \mathrm{U}_0 + \Psi(\mathrm{U}) + \dfrac{1-\delta}{M(\delta)}\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda \mathrm{z}), \mathrm{U}(\mathrm{z}-\tau)) \\ +\dfrac{\delta}{M(\delta)}\displaystyle\int_0^{\mathrm{z}} \Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi-\tau))d\xi, \; z \in \mathbf{J}. \end{cases} \tag{3.3}$$

□

To demonstrate the existence of solutions to problem (1.1). Let Ω be the space of continuous function, define operator $\mathbf{Z} : \Omega \to \Omega$ by:

$$\mathbf{Z}\mathrm{U}(\mathrm{z}) = \begin{cases} \mathrm{U}_0 + \Psi(\mathrm{U}) + \dfrac{1-\delta}{M(\delta)}\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda \mathrm{z}), \mathrm{U}(\mathrm{z}-\tau)) \\ +\dfrac{\delta}{M(\delta)}\displaystyle\int_0^{\mathrm{z}} \Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi-\tau))d\xi, \; z \in \mathbf{J}. \end{cases} \tag{3.4}$$

Divide the above operator (3.4) into two sub operators as follows:

$$\mathbf{Z_1}\mathrm{U}(\mathrm{z}) = \mathrm{U}_0 + \Psi(\mathrm{U}) + \frac{1-\delta}{M(\delta)}\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda \mathrm{z}), \mathrm{U}(\mathrm{z}-\tau)), \tag{3.5}$$

and

$$\mathbf{Z_2}\mathrm{U}(\mathrm{z}) = \frac{\delta}{M(\delta)}\int_0^{\mathrm{z}} \Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi-\tau))d\xi. \tag{3.6}$$

From Eqs (3.5) and (3.6), $\mathbf{Z}(\mathrm{U})$ can be written as:

$$\mathbf{Z}(\mathrm{U}) = \mathbf{Z_1}(\mathrm{U}) + \mathbf{Z_2}(\mathrm{U}). \tag{3.7}$$

For further analysis, we need the following assumptions:

($\mathbf{A_1}$) For continuous function $\Psi(\mathrm{U})$, and $\mathbb{G} > 0$,

$$|\Psi(\mathrm{U}) - \Psi(\mathrm{V})| \leq \mathbb{G}|\mathrm{U} - \mathrm{V}|.$$

($\mathbf{A_2}$) For continuous $\Phi(\mathrm{z}, \mathrm{U}, \mathrm{V}, w)$ and $\mathcal{D}_1 > 0$, $\mathcal{D}_2 > 0$, $\mathcal{D}_3$, and $\mathcal{D}_4$,

$$|\Phi(\mathrm{z}, U, V, w)| \leq \mathcal{D}_1|\mathrm{U}| + \mathcal{D}_2|\mathrm{V}| + \mathcal{D}_3|w| + \mathcal{D}_4.$$

$(\mathbf{A_3})$ For $\mathbb{C}_1,\ \mathbb{C}_2,\ \mathbb{C}_3 > 0$,

$$|\Phi(\mathrm{z}, \mathrm{U}, \mathrm{V}, w) - \Phi(\mathrm{z}, \widehat{\mathrm{U}}, \widehat{\mathrm{V}}, \widehat{w})| \leq \mathbb{C}_1|\mathrm{U} - \widehat{\mathrm{U}}| + \mathbb{C}_2|\mathrm{V} - \widehat{\mathrm{V}}| + \mathbb{C}_3|w - \widehat{w}|.$$

Theorem 3.2. *Under the assumptions* $(\mathbf{A_2}, \mathbf{A_3})$*, and if*

$$\Pi = \left(\mathbb{G} + \frac{1-\delta}{M(\delta)}[\mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_3]\right) < 1$$

holds, then the proposed problem has at least one solution .

Proof. **Step1** : We need to show that the operator $\mathbf{Z_1}$ is a contraction.

Let $\mathscr{D} = \{\mathrm{U} \in B : \ \|\mathrm{U}\| \leq r\} \subset B$ be the closed, convex, and bounded subset. Then, obviously $\mathbf{Z_1}$ is continuous. Let $\mathrm{U}, \mathrm{V} \in \mathscr{D}$, consider

$$\begin{aligned}
\|\mathbf{Z_1}(\mathrm{U}) - \mathbf{Z_1}(\mathrm{V})\| &= \sup_{\mathrm{z}\in\mathbf{J}}\left\{\left|\mathrm{U}_0 + \Psi(\mathrm{U}) + \frac{(1-\delta)\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda\mathrm{z}), \mathrm{U}(\mathrm{z}-\tau))}{M(\delta)}\right.\right.\\
&\quad \left.\left. - \left(\mathrm{U}_0 + \Psi(\mathrm{V}) + \frac{(1-\delta)\Phi(\mathrm{z}, \mathrm{V}(\mathrm{z}), \mathrm{V}(\lambda\mathrm{z}), \mathrm{V}(\mathrm{z}-\tau))}{M(\delta)}\right)\right|\right\}\\
&\leq \sup_{\mathrm{z}\in\mathbf{J}}|\Psi(\mathrm{U}) - \Psi(\mathrm{V})| + \frac{1-\delta}{M(\delta)}\sup_{\mathrm{z}\in\mathbf{J}}\Big\{|\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda\mathrm{z}), \mathrm{U}(\mathrm{z}-\tau))\\
&\quad -\Phi(\mathrm{z}, \mathrm{V}(\mathrm{z}), \mathrm{V}(\lambda\mathrm{z}), \mathrm{V}(\mathrm{z}-\tau))|\Big\}\\
&\leq \mathbb{G}\|\mathrm{U} - \mathrm{V}\| + \frac{1-\delta}{M(\delta)}\sup_{\mathrm{z}\in\mathbf{J}}[\mathbb{C}_1|\mathrm{U}(\mathrm{z}) - \mathrm{V}(\mathrm{z})| + \mathbb{C}_2|\mathrm{U}(\lambda\mathrm{z}) - \mathrm{V}(\lambda\mathrm{z})|\\
&\quad +\mathbb{C}_3|\mathrm{U}(\mathrm{z}-\tau) - \mathrm{V}(\mathrm{z}-\tau)|]\\
&\leq \left(\mathbb{G} + \frac{1-\delta}{M(\delta)}[\mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_3]\right)\|\mathrm{U} - \mathrm{V}\|\\
&\leq \Pi\|\mathrm{U} - \mathrm{V}\|.
\end{aligned}$$

Hence, $\mathbf{Z_1}$ is a contraction.

Step2 : Next, to show that $\mathbf{Z}_2$ is equi-continuous, take $\mathrm{U} \in \mathscr{D}$ and consider

$$\begin{aligned}
\|\mathbf{Z_2}(\mathrm{U})\| &= \sup_{\mathrm{z}\in\mathbf{J}}\left|\frac{\delta}{M(\delta)}\int_0^{\mathrm{z}}\Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi-\tau))d\xi\right|\\
&\leq \left(\frac{\delta T}{M(\delta)}\right)[\mathcal{D}_1 + \mathcal{D}_2 + \mathcal{D}_3]r\\
&< \infty.
\end{aligned}$$

Thus, $\mathbf{Z_2}(\mathrm{U})$ is bounded. As Φ is continuous, and therefore, $\mathbf{Z_2}(\mathrm{U})$ is also continuous. Further, let $\mathrm{z}_1 < \mathrm{z}_2 \in \mathbf{J}$. Then

$$\begin{aligned}
\|\mathbf{Z_2}\mathrm{U}(\mathrm{z}_2) - \mathbf{Z_2}\mathrm{U}(\mathrm{z}_1)\| &= \sup_{\mathrm{z}_1,\mathrm{z}_2\in\mathbf{J}}\left\{\left|\frac{\delta}{M(\delta)}\int_0^{\mathrm{z}_2}\Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi-\tau))d\xi\right.\right.\\
&\quad \left.\left. -\frac{\delta}{M(\delta)}\int_0^{\mathrm{z}_1}\Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi-\tau))d\xi\right|\right\}
\end{aligned}$$

$$\leq \frac{\delta[\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3]r}{M(\delta)}\Big(\int_0^{\mathrm{z}_2}d\xi-\int_0^{\mathrm{z}_1}d\xi\Big)$$
$$\leq \frac{\delta[\mathcal{D}_1+\mathcal{D}_2+\mathcal{D}_3]r}{M(\delta)}(\mathrm{z}_2-\mathrm{z}_1)\to 0,\ \text{as}\ \mathrm{z}_1\to\mathrm{z}_2.$$

Thus, $\mathbf{Z_2}$ is equi-continuous, and bounded. Hence, in view of Arzelá-Ascoli and Schauder fixed point theorems, problem (1.1) has at least one solution . □

Theorem 3.3. *Under assumptions* $(\mathbf{A_1})$, $(\mathbf{A_3})$, *and if the condition*

$$\Lambda=\Big[\mathbb{G}+\frac{(\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3)}{M(\delta)}(1-\delta[1-\mathbf{T}])\Big]<1$$

holds, then the problem (1.1) *has a unique solution.*

Proof. Let $\mathrm{U},\widehat{\mathrm{U}}\in\Omega$, and consider

$$\begin{aligned}\|\mathbf{Z}(\mathrm{U})-\mathbf{Z}(\widehat{\mathrm{U}})\|=&\sup_{\mathrm{z}\in[0,\mathbf{T}]}\bigg\{\bigg(\bigg|\mathrm{U}_0+\Psi(\mathrm{U})+\frac{(1-\delta)\Phi(\mathrm{z},\mathrm{U}(\mathrm{z}),\mathrm{U}(\lambda\mathrm{z}),\mathrm{U}(\mathrm{z}-\tau))}{M(\delta)}\\&+\frac{\delta}{M(\delta)}\int_0^{\mathrm{z}}\Phi(\xi,\mathrm{U}(\xi),\mathrm{U}(\lambda\xi),\mathrm{U}(\xi-\tau))d\xi\bigg)\\&-\bigg(\mathrm{U}_0+\Psi(\widehat{\mathrm{U}})+\frac{(1-\delta)\Phi(\mathrm{z},\widehat{\mathrm{U}}(\mathrm{z}),\widehat{\mathrm{U}}(\lambda\mathrm{z}),\widehat{\mathrm{U}}(\mathrm{z}-\tau))}{M(\delta)}\\&+\frac{\delta}{M(\delta)}\int_0^{\mathrm{z}}\Phi(\xi,\widehat{\mathrm{U}}(\xi),\widehat{\mathrm{U}}(\lambda\xi),\widehat{\mathrm{U}}(\xi-\tau))d\xi\bigg)\bigg|\bigg\}\\\leq&\sup_{\mathrm{z}\in\mathbf{J}}\bigg\{|\Psi(\mathrm{U})-\Psi(\widehat{\mathrm{U}})|+\frac{1-\delta}{M(\delta)}\bigg|\Phi(\mathrm{z},\mathrm{U}(\mathrm{z}),\mathrm{U}(\lambda\mathrm{z}),\mathrm{U}(\mathrm{z}-\tau))\\&-\Phi(\mathrm{z},\widehat{\mathrm{U}}(\mathrm{z}),\widehat{\mathrm{U}}(\lambda\mathrm{z}),\widehat{\mathrm{U}}(\mathrm{z}-\tau))\bigg|\\&+\frac{\delta}{M(\delta)}\int_0^{\mathrm{z}}\bigg|\Phi(\xi,\mathrm{U}(\xi),\mathrm{U}(\lambda\xi),\mathrm{U}(\xi-\tau))\\&-\Phi(\xi,\widehat{\mathrm{U}}(\xi),\widehat{\mathrm{U}}(\lambda\xi),\widehat{\mathrm{U}}(\xi-\tau))\bigg|d\xi\bigg\}\\\leq&\sup_{\mathrm{z}\in\mathbf{J}}\bigg\{\mathbb{G}|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|+\frac{1-\delta}{M(\delta)}[\mathbb{C}_1|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|+\mathbb{C}_2|\mathrm{U}(\lambda\mathrm{z})-\widehat{\mathrm{U}}(\lambda\mathrm{z})|\\&+\mathbb{C}_3|\mathrm{U}(\mathrm{z}-\tau)-\widehat{\mathrm{U}}(\mathrm{z}-\tau)|]+\frac{\delta}{M(\delta)}\int_0^{\mathrm{z}}[\mathbb{C}_1|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|\\&+\mathbb{C}_2|\mathrm{U}(\lambda\mathrm{z})-\widehat{\mathrm{U}}(\lambda\mathrm{z})|+\mathbb{C}_3|\mathrm{U}(\mathrm{z}-\tau)-\widehat{\mathrm{U}}(\mathrm{z}-\tau)|]d\xi\bigg\},\end{aligned}$$

which further yields that

$$\begin{aligned}\|\mathbf{Z}(\mathrm{U})-\mathbf{Z}(\widehat{\mathrm{U}})\|\leq&\sup_{\mathrm{z}\in\mathbf{J}}\mathbb{G}|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|+\sup_{\mathrm{z}\in\mathbf{J}}\bigg\{\frac{1-\delta}{M(\delta)}[\mathbb{C}_1|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|+\mathbb{C}_2|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|\\&+\mathbb{C}_3|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|]+\frac{\delta}{M(\delta)}\int_0^{\mathrm{z}}[\mathbb{C}_1|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|\\&+\mathbb{C}_2|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}\mathrm{z})|+\mathbb{C}_3|\mathrm{U}(\mathrm{z})-\widehat{\mathrm{U}}(\mathrm{z})|]d\xi\bigg\}.\end{aligned}$$

Hence, we have

$$\begin{aligned}||\mathbf{Z}(\mathrm{U}) - \mathbf{Z}(\widehat{\mathrm{U}})|| &\leq \left[\mathbb{G} + \frac{(\mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_3)}{M(\delta)}(1 - \delta[1 - \mathbf{T}])\right]||\mathrm{U} - \widehat{\mathrm{U}}|| \\ &\leq \Lambda||\mathrm{U} - \widehat{\mathrm{U}}||.\end{aligned}$$

In view of Banach contraction theorem, $\mathbf{Z}$ is a contraction operator, and therefore has a unique fixed point. Therefore, problem (1.1) has a unique solution. □

Remark 3.1. It is remarkable that all Lipschitz continuous functions on a bounded set are also Hölder continuous. Therefore, If in ($\mathbf{A}_3$) instead of Lipschitz continuity we consider Hölder continuity, the results of Theorems 3.2, and 3.3 also hold.

4. Stability analysis

This section is devoted to establishing stability results for the considered problem. We contend that the UH stability idea is important for practical issues in economics, biology, and numerical analysis. The interesting feature of stability is that researching a UH stable system does not require reaching the exact solution, which is typically challenging or time-consuming. According to UH stability, there is a close approximate solution to exact solution of the problem. Because in many mathematical models of economics, biological and physical problems which are nonlinear and we do not know the exact solution, therefore, we need to find best approximate or numerical solution. For more details theory, and applications of UH stability analysis, we refer to [27], [29], and [30].

Consider the problem

$$\begin{cases} {}^{CF}{}_0\mathcal{D}_z^{\delta}\mathrm{U}(z) = \Phi(z, \mathrm{U}(z), \mathrm{U}(\lambda z), \mathrm{U}(z - \tau)) + h(z),\ z \in \mathbf{J}, \\ \mathrm{U}(0) = \mathrm{U}_0 + \Psi(\mathrm{U}), \\ \mathrm{U}(z) = \Phi(z), z \in [-\tau, 0). \end{cases} \tag{4.1}$$

Here, $h \in \Omega$ such that for $\epsilon > 0$, $|h(z)| \leq \epsilon$. Then, Eq (4.1) has the solution

$$\mathrm{U}(z) = \begin{cases} \mathrm{U}_0 + \Psi(\mathrm{U}) + \dfrac{(1 - \delta)[\Phi(z, \mathrm{U}(z), \mathrm{U}(\lambda z), \mathrm{U}(z - \tau)) + h(z)]}{M(\delta)} \\ + \delta \displaystyle\int_0^z \frac{[\Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi - \tau)) + h(\xi)]}{M(\delta)} d\xi, z \in \mathbf{J}. \end{cases} \tag{4.2}$$

In operator form using Theorem 3.3, Eq (4.2) can be shortened to

$$\mathbf{Z}(\mathrm{U}(z)) = \mathrm{U}(z) + h(z),\ z \in \mathbf{J}. \tag{4.3}$$

From Eq (4.2), one has using Eq (4.1)

$$\begin{aligned}|\mathbf{Z}(\mathrm{U}(z)) - \mathrm{U}(z)| &\leq \left[\frac{1 - \delta(1 - T)}{M(\delta)}\right]\epsilon \\ &\leq \Pi\epsilon,\end{aligned}$$

where, $\Pi = \frac{1-\delta(1-T)}{M(\delta)}$.

Theorem 4.1. *Equation* (3.1) *is UH and generalized UH stable if*

$$\Lambda = \left[\mathbb{G} + \frac{(\mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_3)}{M(\delta)}(1 - \delta[1 - \mathbf{T}])\right] < 1$$

holds.

Proof. Let $\widehat{\mathrm{U}},\ \mathrm{U} \in \Omega$ represent any and unique solution respectively of Eq (3.1). Then,

$$\begin{aligned}
&\|\mathrm{U} - \widehat{\mathrm{U}}\| = \sup \mathrm{z} \in \mathbf{J}|\mathrm{U}(\mathrm{z} - \mathbf{Z}(\widehat{\mathrm{U}}(\mathrm{z}))| \\
&\leq \sup \mathrm{z} \in \mathbf{J}|\mathrm{U}(\mathrm{z}) - \mathbf{Z}(\mathrm{U}(\mathrm{z}))| + \sup \mathrm{z} \in \mathbf{J}|\mathbf{Z}(\mathrm{U}(\mathrm{z})) - \mathbf{Z}(\widehat{\mathrm{U}}(\mathrm{z}))| \\
&\leq \Pi\epsilon + \Lambda\|\mathrm{U} - \widehat{\mathrm{U}}\| \\
&\leq \frac{\Pi\epsilon}{1 - \Lambda}.
\end{aligned} \tag{4.4}$$

Therefore, Eq (3.1) is UH and generalized UH stable. Similarly, we can deduce other kinds of UH stability in the same way. □

5. Numerical approximation

Usually to find an analytical or exact solution for nonlinear problems with fractional order derivative is a difficult job. Therefore, we need some sophisticated numerical tools to compute the approximate solutions for the problem under consideration. Since, problem (1.1) is nonlinear and its analytical or exact solution is difficult to compute, we establish a numerical algorithm to compute the approximate solution. Following the numerical method [41], a numerical scheme is established based on the interpolation process. The equivalent integral form of Eq (1.1) is given by

$$\begin{aligned}
\mathrm{U}(\mathrm{z}) &= \mathrm{U}_0 + \Psi(\mathrm{U}) + \frac{1-\delta}{M(\delta)}\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda\mathrm{z}), \mathrm{U}(\mathrm{z} - \tau)) \\
&\quad + \frac{\delta}{M(\delta)}\int_0^{\mathrm{z}} \Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi - \tau))d\xi.
\end{aligned} \tag{5.1}$$

At $\mathrm{z} = \mathrm{z}_{m+1}$, Eq (5.1) can be written as

$$\begin{aligned}
\mathrm{U}(\mathrm{z}_{m+1}) &= \mathrm{U}_0 + \Psi(\mathrm{U}_m) + \frac{1-\delta}{M(\delta)}\Phi(\mathrm{z}_m, \mathrm{U}(\mathrm{z}_m), \mathrm{U}(\lambda\mathrm{z}_m), \mathrm{U}(\mathrm{z}_m - \tau)) \\
&\quad + \frac{\delta}{M(\delta)}\int_0^{\mathrm{z}_{m+1}} \Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi - \tau))d\xi \\
&= \mathrm{U}_0 + \Psi(\mathrm{U}_m) + \frac{1-\delta}{M(\delta)}\Phi(\mathrm{z}_m, \mathrm{U}(\mathrm{z}_m), \mathrm{U}(\lambda\mathrm{z}_m), \mathrm{U}(\mathrm{z}_m - \tau)) \\
&\quad + \frac{\delta}{M(\delta)}\sum_{j=0}^{m+1}\int_{\mathrm{z}_j}^{\mathrm{z}_{j+1}} \Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi - \tau))d\xi.
\end{aligned} \tag{5.2}$$

Now, using interpolation with equally spaced arguments and approximating the function

$$\Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi - \tau))$$

in terms of Lagrange polynomials, one has

$$\begin{aligned}\Phi(\xi, \mathrm{U}(\xi), \mathrm{U}(\lambda\xi), \mathrm{U}(\xi-\tau)) &= \frac{\Phi(\mathrm{z}_j, \mathrm{U}(\mathrm{z}_j), \mathrm{U}(\lambda \mathrm{z}_j), \mathrm{U}(\mathrm{z}_j-\tau))}{h}(\xi - t_{j-1}) \\ &\quad -\frac{\Phi(\mathrm{z}_{j-1}, \mathrm{U}(\mathrm{z}_{j-1}), \mathrm{U}(\lambda \mathrm{z}_{j-1}), \mathrm{U}(\mathrm{z}_{j-1}-\tau))}{h}(\xi - t_j).\end{aligned} \tag{5.3}$$

Putting Eq (5.3) in Eq (5.2), we have

$$\begin{aligned}\mathrm{U}(\mathrm{z}_{m+1}) &= \mathrm{U}_0 + \Psi(\mathrm{U}_m) + \frac{1-\delta}{M(\delta)}\Phi(\mathrm{z}_m, \mathrm{U}(\mathrm{z}_m), \mathrm{U}(\lambda \mathrm{z}_m), \mathrm{U}(\mathrm{z}_m-\tau)) \\ &\quad +\frac{\delta}{M(\delta)}\sum_{j=0}^{m+1}\int_{\mathrm{z}_j}^{\mathrm{z}_{j+1}}\Big[\frac{\Phi(\mathrm{z}_j, \mathrm{U}(\mathrm{z}_j), \mathrm{U}(\lambda \mathrm{z}_j), \mathrm{U}(\mathrm{z}_j-\tau))}{h}(\xi - t_{j-1}) \\ &\quad -\frac{\Phi(\mathrm{z}_{j-1}, \mathrm{U}(\mathrm{z}_{j-1}), \mathrm{U}(\lambda \mathrm{z}_{j-1}), \mathrm{U}(\mathrm{z}_{j-1}-\tau))}{h}(\xi - t_j)\Big]d\xi \\ &= \mathrm{U}_0 + \Psi(\mathrm{U}_m) + \frac{1-\delta}{M(\delta)}\Phi(\mathrm{z}_m, \mathrm{U}(\mathrm{z}_m), \mathrm{U}(\lambda \mathrm{z}_m), \mathrm{U}(\mathrm{z}_m-\tau)) \\ &\quad +\frac{\delta}{M(\delta)h}\sum_{j=0}^{m+1}\Big[\Phi(\mathrm{z}_j, \mathrm{U}(\mathrm{z}_j), \mathrm{U}(\lambda \mathrm{z}_j), \mathrm{U}(\mathrm{z}_j-\tau))\int_{\mathrm{z}_j}^{\mathrm{z}_{j+1}}\int_{\mathrm{z}_j}^{\mathrm{z}_{j+1}}(\xi - t_{j-1})d\xi \\ &\quad -\Phi(\mathrm{z}_{j-1}, \mathrm{U}(\mathrm{z}_{j-1}), \mathrm{U}(\lambda \mathrm{z}_{j-1}), \mathrm{U}(\mathrm{z}_{j-1}-\tau))\int_{\mathrm{z}_j}^{\mathrm{z}_{j+1}}(\xi - t_j)d\xi\Big] \\ &= \mathrm{U}_0 + \Psi(\mathrm{U}_m) + \frac{1-\delta}{M(\delta)}\Phi(\mathrm{z}_m, \mathrm{U}(\mathrm{z}_m), \mathrm{U}(\lambda \mathrm{z}_m), \mathrm{U}(\mathrm{z}_m-\tau)) \\ &\quad +\frac{3\delta h}{2M(\delta)}\Phi(\mathrm{z}_m, \mathrm{U}(\mathrm{z}_m), \mathrm{U}(\lambda \mathrm{z}_m), \mathrm{U}(\mathrm{z}_m-\tau)) - \frac{\delta h}{2M(\delta)}\Phi(\mathrm{z}_{m-1}, \mathrm{U}(\mathrm{z}_{m-1}), \mathrm{U}(\lambda \mathrm{z}_{m-1}), \mathrm{U}(\mathrm{z}_{m-1}-\tau)) \\ &= \mathrm{U}_0 + \Psi(\mathrm{U}_m) + \Big[\frac{1-\delta}{M(\delta)} + \frac{3h\delta}{2M(\delta)}\Big]\Phi(\mathrm{z}_m, \mathrm{U}(\mathrm{z}_m), \mathrm{U}(\lambda \mathrm{z}_m), \mathrm{U}(\mathrm{z}_m-\tau)) \\ &\quad -\Big[\frac{1-\delta}{M(\delta)} + \frac{h\delta}{2M(\delta)}\Big]\Phi(\mathrm{z}_{m-1}, \mathrm{U}(\mathrm{z}_{m-1}), \mathrm{U}(\lambda \mathrm{z}_{m-1}), \mathrm{U}(\mathrm{z}_{m-1}-\tau))\Big].\end{aligned} \tag{5.4}$$

The formula (5.4) will be utilized to simulate our results.

6. Example

Example 6.1. *Consider a Cauchy type problem with nonlocal initial condition as*

$$\begin{cases} {}^{CF}\mathscr{D}_{0^+}^{\delta}\mathrm{U}(\mathrm{z}) = \dfrac{\mathrm{z}^2}{10^2} + \dfrac{1}{60}\cos(|\mathrm{U}(\mathrm{z})|) + \dfrac{\sin(|\mathrm{U}(0.5\mathrm{z})|)}{60} + \dfrac{1}{60}\mathrm{U}(\mathrm{z}-0.5), \\ \mathrm{U}(0) = 1 + \dfrac{\sin|\mathrm{U}|}{20}, \\ \mathrm{U}(\mathrm{z}) = 0, \mathrm{z} \in [-0.5, 0). \end{cases} \tag{6.1}$$

Here, $\mathrm{z} \in [0,1] = \mathbf{J}$*,* $\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(0.5\mathrm{z}), \mathrm{U}(\mathrm{z}-0.5)) = \frac{\mathrm{z}^2}{10^2} + \frac{1}{60}\cos(|\mathrm{U}(\mathrm{z})|) + \frac{\sin(|\mathrm{U}(\lambda \mathrm{z})|)}{60} + \frac{1}{60}\mathrm{U}(\mathrm{z}-0.5)$*, and* $\Psi(\mathrm{U}) = \frac{\mathrm{U}(\mathrm{z})}{20}$*. Now,*

$$
\begin{aligned}
&|\Phi(\mathrm{z},\mathrm{U}(\mathrm{z}),\mathrm{U}(\lambda\mathrm{z}),\mathrm{U}(\mathrm{z}-\tau))-\Phi(\mathrm{z},\widehat{\mathrm{U}}(\mathrm{z}),\widehat{\mathrm{U}}(\lambda\mathrm{z}),\widehat{\mathrm{U}}(\mathrm{z}-0.5))|\\
\leq\ & \frac{1}{60}\left|\mathrm{U}-\widehat{\mathrm{U}}\right|+\frac{1}{60}\left|\mathrm{U}-\widehat{\mathrm{U}}\right|+\frac{1}{60}\left|\mathrm{U}-\widehat{\mathrm{U}}\right|\\
=\ & \frac{1}{20}\left|\mathrm{U}-\widehat{\mathrm{U}}\right|.
\end{aligned}
$$

$$
\begin{aligned}
\|\Phi(U)\| &= \sup_{\mathrm{z}\in\mathbf{J}}\left\{\left|\frac{\mathrm{z}^2}{10^2}+\frac{1}{60}\cos(|\mathrm{U}(\mathrm{z})|)+\frac{\sin(|\mathrm{U}(\lambda\mathrm{z})|)}{60}+\frac{1}{60}\mathrm{U}(\mathrm{z}-0.5)\right|\right\}\\
&\leq \sup_{\mathrm{z}\in\mathbf{J}}\left\{\frac{\mathrm{z}^2}{10^2}+\frac{1}{60}|\mathrm{U}(\mathrm{z})|+\frac{1}{60}|\mathrm{U}(\lambda\mathrm{z})|+\frac{1}{60}|\mathrm{U}(\mathrm{z}-0.5)|\right\}\\
&\leq \frac{1}{20}\|\mathrm{U}\|+\frac{1}{100}.
\end{aligned}
$$

Hence, Ψ *satisfies assumption* $(\mathbf{A_1})$ *with* $\mathbb{G}=\frac{1}{20}$ *and also satisfies assumptions* $(\mathbf{A_2},\mathbf{A_3})$ *with constants* $\mathcal{D}_1=\frac{1}{60}=\mathbb{C}_1,\mathcal{D}_2=\frac{1}{60}=\mathbb{C}_2,\mathcal{D}_3=\frac{1}{60}=\mathbb{C}_3$, *and* $\mathcal{D}_4=\frac{1}{100}$. *Further, let* $M(\delta)=1-\delta(1-\frac{1}{\Gamma(\delta+1)})$

$$
\begin{aligned}
\Pi &= \left(\mathbb{G}+\frac{1-\delta}{M(\delta)}[\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3]\right)\\
&\leq \left(\frac{1}{60}+\frac{0.5}{M(0.5)}[\frac{1}{60}+\frac{1}{60}+\frac{1}{60}]\right),\ \forall\,\delta\in(0,1]\\
&= 0.016267\\
&< 1,
\end{aligned}
$$

and

$$
\begin{aligned}
\Lambda &= \left[\mathbb{G}+\frac{(\mathbb{C}_1+\mathbb{C}_2+\mathbb{C}_3)}{M(\delta)}(1-\delta[1-\mathbf{T}])\right]\\
&\leq \left[\frac{1}{20}+\frac{\left(\frac{1}{25}+\frac{1}{10}+\frac{1}{10}\right)}{M(0.5)}(1-0)\right],\ \forall\,\delta\in(0,1]\\
&= 0.027535\\
&< 1.
\end{aligned}
$$

Hence, all the conditions of Theorems 3.2, 3.3, and 4.1 hold. So, Eq (6.1) *has at least unique solution and is UH type stable on* [0, 1].

Next, applying the scheme (5.4) *to approximate the solution of problem Eq* (6.1)*. In Figures 1 and 2, we present the approximate solution for the problem* (6.1) *using different fractional orders with step size* $h=0.05$.

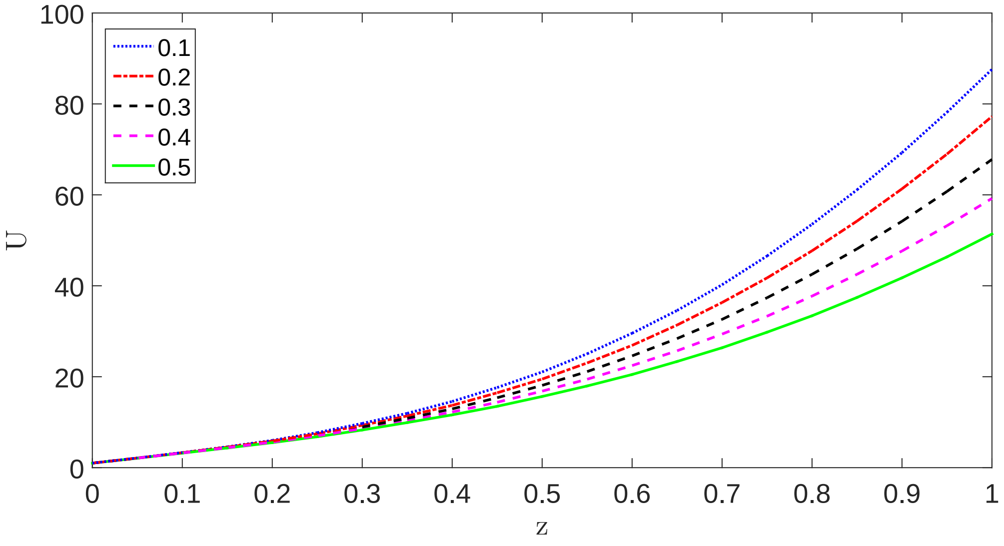

Figure 1. Numerical solutions at different fractional order for Eq (6.1).

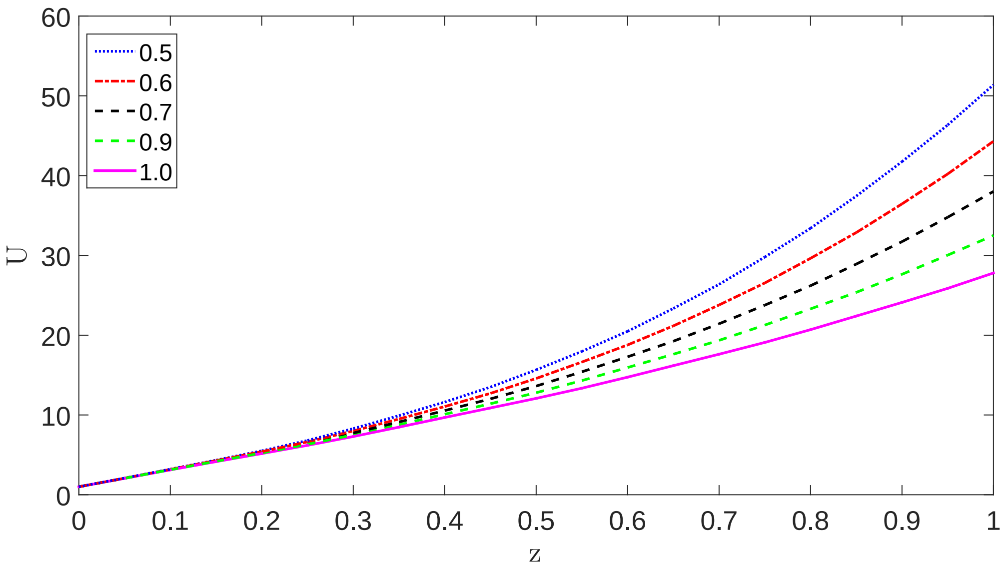

Figure 2. Numerical solutions at different fractional order for Eq (6.1).

Next, we take $h = 0.0003$ *and present the numerical solutions for the above example using different fractional orders in Figure 3.*

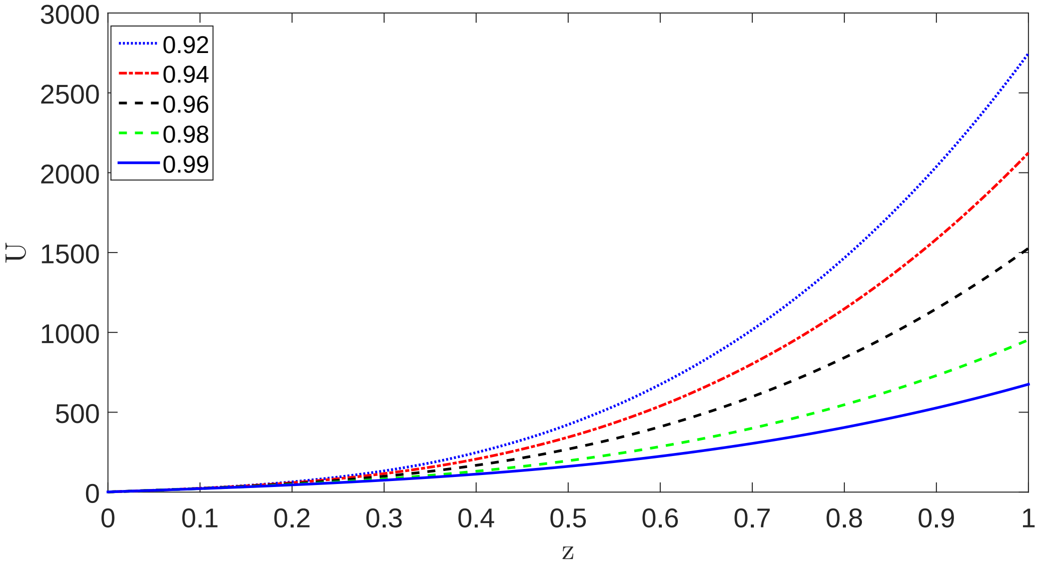

Figure 3. Numerical solutions at different fractional order for Eq (6.1).

We have graphically interpreted, the numerical solutions against various fractional orders using two step size.

Example 6.2. *Consider a houseflies model [26] with discrete and proportional delays*

$$\begin{cases} {}^{CF}\mathcal{D}_{0^+}^{\delta}\mathrm{U}(\mathrm{z}) = -d\mathrm{U}(\mathrm{z}) + b\mathrm{U}(\lambda \mathrm{z})\left[\kappa - b\mu\mathrm{U}(z-\tau)\right], \ z \in [0,1], \\ \mathrm{U}(0) = 1, \\ \mathrm{U}(\mathrm{z}) = 0, \mathrm{z} \in [-0.5, 0), \end{cases} \tag{6.2}$$

where $d = 0.147$ *represents death rate of adults,* $b = 1.81$ *denoted number of eggs laid par adult, and* $\kappa = 0.05107$ *shows the highest rate of egg-adult survival. Also,* $\tau = 5$ *stands for the duration of the adult developmental stage between oviposition and eclosion, and* $\lambda = 0.5$ *denotes probational delay. In addition,* $\mu = 0.000226$ *represents decrease rate in survival brought about by every extra egg. If we consider* $\lambda = 1, \tau = 0$, *then* (6.2) *becomes famous logistic equation which have widely been studied. Here,* $\mathrm{U}_0 = 1$ *and* $\Psi(\mathrm{U}) = 0$. *Here, we can deduce the assumptions for*

$$\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda \mathrm{z}), \mathrm{U}(\mathrm{z}-\tau)) = -0.147\mathrm{U}(\mathrm{z}) + 1.81\mathrm{U}(0.5\mathrm{z})\left[0.05107 - 1.81 \times 0.000226\mathrm{U}(z-0.5)\right]$$

as

$$|\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(\lambda \mathrm{z}), \mathrm{U}(\mathrm{z}-0.5)) - \Phi(\mathrm{z}, \widehat{\mathrm{U}}(\mathrm{z}), \widehat{\mathrm{U}}(0.5\mathrm{z}), \widehat{\mathrm{U}}(\mathrm{z}-0.5))| \leq 0.23945\left|\mathrm{U} - \widehat{\mathrm{U}}\right|.$$

$$\begin{aligned} \|\Phi(U)\| &= \sup_{\mathrm{z}\in\mathbf{J}}\left\{\left|\Phi(\mathrm{z}, \mathrm{U}(\mathrm{z}), \mathrm{U}(0.5\mathrm{z}), \mathrm{U}(\mathrm{z}-0.5))\right|\right\}, \\ &\leq \sup_{\mathrm{z}\in[0,1]}\left\{0.147|\mathrm{U}(\mathrm{z})| + 1.81 \times 0.05107|\mathrm{U}(0.5\mathrm{z})| + (1.81)^2 \times 0.000226|\mathrm{U}(0.5z)||\mathrm{U}(\mathrm{z}-0.5)|\right\}, \\ &\leq 0.2401770986\|\mathrm{U}\|. \end{aligned}$$

Hence Ψ *satisfies assumption* $(\mathbf{A_1})$ *with* $\mathbb{G} = 0$ *and also satisfies assumptions* $(\mathbf{A_2}, \mathbf{A_3})$ *with constants* $\mathcal{D}_1 = d$, $\mathcal{D}_2 = b\kappa = \mathbb{C}_2, \mathcal{D}_3 = b^2\mu = \mathbb{C}_3$ *and* $\mathcal{D}_4 = 0$. *Further, let* $M(\delta) = 1 - \delta(1 - \frac{1}{\Gamma(\delta+1)})$. *Then*

$$\Pi = \left(\mathbb{G} + \frac{1-\delta}{M(\delta)}[\mathbb{C}_1 + \mathbb{C}_2 + \mathbb{C}_3]\right) = 0.2401770986 < 1.$$

In the same way, we can compute $\Lambda = 0.13456 < 1$. *Hence, the said model has a unique solution. Also, existence of at least one solution is obvious. Moreover,* $\Pi < 1$, *conditions of UH, and generalized UH stability also hold. Next, in Figures 4–6, we present the results graphically using various fractional orders with step size* $h = 0.005$

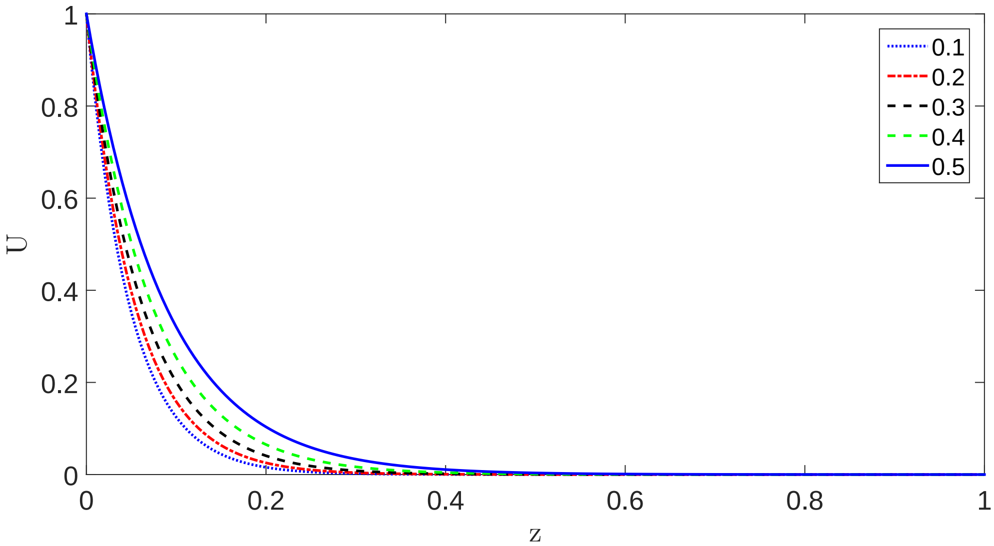

Figure 4. Numerical solutions at different fractional order for Eq (6.2).

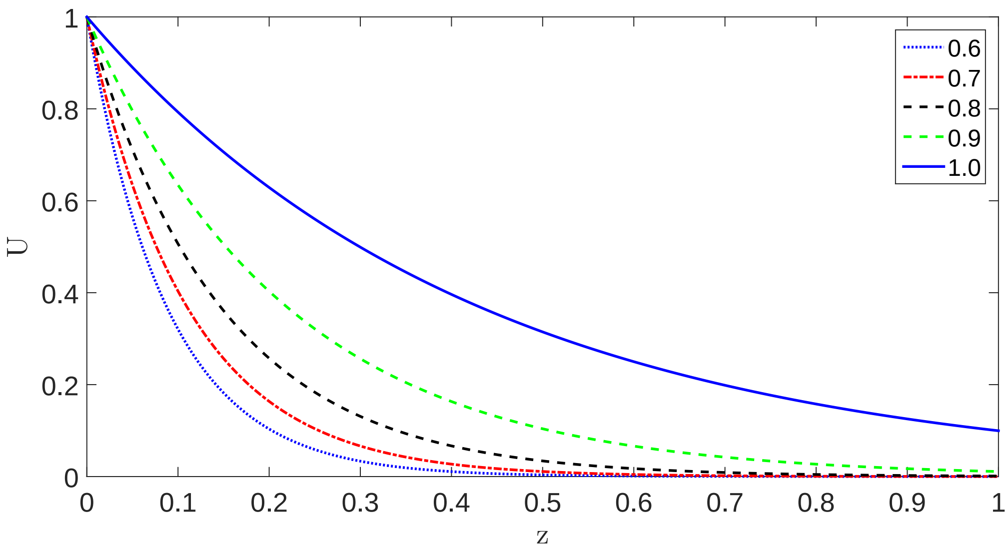

Figure 5. Numerical solutions at different fractional order for Eq (6.2).

Next, we take $h = 0.0003$ *and present the numerical solutions for the above example using different fractional orders in Figure 6.*

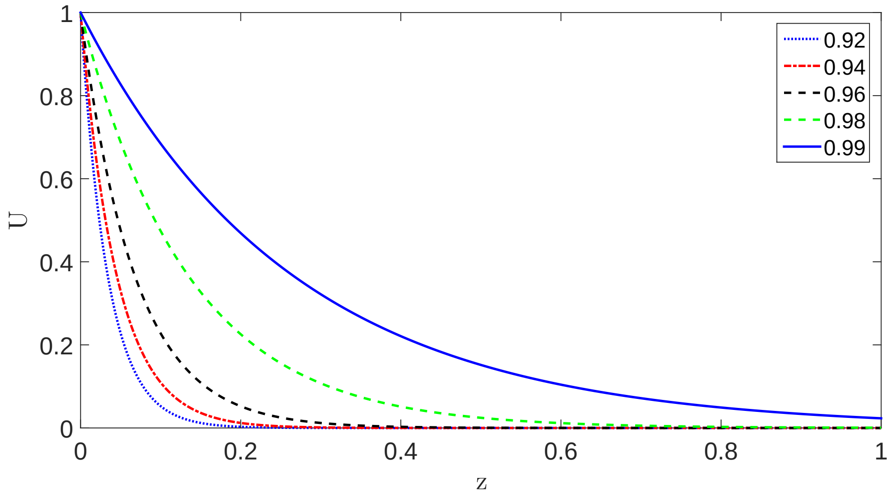

Figure 6. Numerical solutions at different fractional order for Eq (6.2).

From Figures 4–6, we observe that fractional orders and step size have great impact on the evolution of curves. Also, both have significant impact on the convergence behavior of the dynamical problems physically. In Figure 7, we compared the numerical solution by using the Euler method and Adam Bashforth method with the classical and fractional order solutions. We see that at fractional order solution the graphs are closely agreed of both methods. Here, we have considered step size $h = 0.01$ *for both methods. Here, the Euler method is used in standard Caputo sense for the Eq* (6.2)*, and the proposed method in sense of Caputo Fabrizio.*

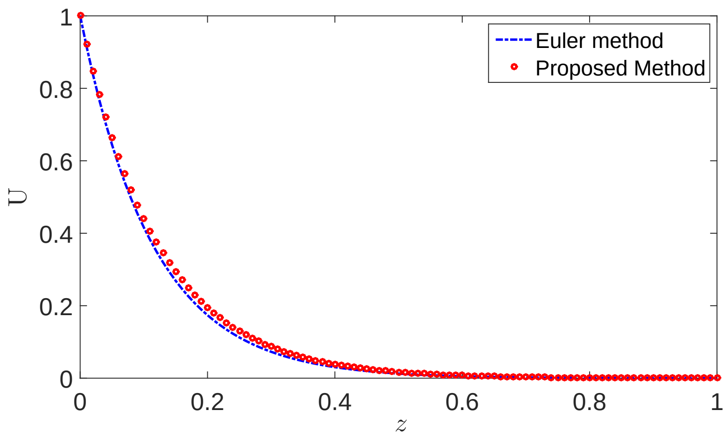

Figure 7. Comparison of approximate solutions at fractional order 0.29 using Euler and proposed method for Eq (6.2).

7. Conclusions

For existence theory and Ulam type stability, we thoroughly investigated a generalized fractional differential equation within the context of this study. Regarding the suggested model, both proportional and discrete delay terms have been investigated. The problem has also been analyzed for approximate solution using a numerical scheme based on the Lagrange interpolation method. In the end, two applicable concrete examples were testified by using the established analysis and computation to support the conclusions. The first example was devoted to the Cauchy problem and the second example was the famous houseflies model. We have compared the approximate solution using the Euler and proposed method. In the future, two delays will be used to investigate other infectious diseases model.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors have no conflict of interests regarding the publication of this paper.

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