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Research article

Global structure of positive solutions for third-order semipositone integral boundary value problems

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Abstract: In this paper, we were concerned with the global behavior of positive solutions for third-order semipositone problems with an integral boundary condition

$$\begin{aligned} &y''' + \beta y'' + \alpha y' + \lambda f(t,y) = 0, \quad t \in (0,1),\\ &y(0) = y'(0) = 0, \quad y(1) = \chi \int_0^1 y(s)ds, \end{aligned}$$

where $\alpha \in (0,\infty)$ and $\beta \in (-\infty,\infty)$ are two constants, λ, χ are two positive parameters, and $f \in C([0,1]\times[0,\infty),\mathbb{R})$ with $f(t,0) < 0$. Our analysis mainly relied on the bifurcation theory.

Keywords: positive solutions; semipositone; third-order integral boundary value problems; bifurcation
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1. Introduction

Third-order boundary value problems emerge in applied mathematics and physics, which have been investigated via various methods; see [1–7]. In [5], by using the disconjugacy theory [8], Ma and Lu obtained the optimal intervals to guarantee that the Green's functions corresponding to the third order linear problem are of one sign. Such results have been extended in [1].

Lemma 1.1. ([1, Theorem 2.1]) Suppose that there exist two constants $\alpha \in (0,\infty)$ and $\beta \in (-\infty,\infty)$, which satisfy

$$0 < 4\alpha - \beta^2 < \pi^2 \ \text{ and } \ e^{-\frac{\beta}{2}}\sqrt{4\alpha-\beta^2} + \beta \sin\left(\frac{\sqrt{4\alpha-\beta^2}}{2}\right) \geq \sqrt{4\alpha-\beta^2}\cos\left(\frac{\sqrt{4\alpha-\beta^2}}{2}\right). \tag{1.1}$$

The property of disconjugacy of $y''' + \beta y'' + \alpha y' = 0$ is valid.

To our knowledge, the behavior of positive solutions for second-order boundary value problems under the semipositone condition have been discussed extensively; see [9–15]. It is worth noting that based upon the bifurcation theory, excluding Ambrosetti et al. [9] who explored semipositone elliptic problems under the linear boundary condition, recently, Ma and Wang [14, 15] proved the solvability for second-order problems with a nonlinear boundary condition. In comparison to second-order cases, the challenge in studying third-order cases lies in that the third-order differential operator is non-self-adjoint.

Inspired by the aforementioned literature, we investigate the behavior of positive solutions for the following problems

$$\begin{aligned} &y''' + \beta y'' + \alpha y' + \lambda f(t, y) = 0, \quad t \in (0, 1), \\ &y(0) = y'(0) = 0, \quad y(1) = \chi \int_0^1 y(s)ds, \end{aligned} \tag{1.2}$$

where $\alpha \in (0, \infty)$ and $\beta \in (-\infty, \infty)$ are two constants, λ, χ are two positive parameters, and

$$f : [0, 1] \times [0, \infty) \to \mathbb{R}$$

is continuous and satisfies that the following assumption holds.

(F_1) (semipositone) $f(t, 0) < 0, \ \forall\, t \in [0, 1]$.

The integral boundary conditions can be decomposed into various different situations, such as multipoint and nonlocal boundary conditions. As far as we are concerned, few literatures on the solvability of third-order semipositone problems with an interval boundary condition since the maximum principle may fail. In order to overcome this difficulty, as a first step, Henderson [3] translated the third-order semipositone problems into the positone problems under the three-point boundary condition. Concurrently, we found that they only showed that the discussed problems have at least one positive solution. No detailed information about the global behavior of solutions was investigated since the spectral structure of third-order linear eigenvalue problems has not yet been found. Meanwhile, in order to use the fixed point index theory and bifurcation theory to show our results, we need to obtain a suitable cone utilizing the property of Green's function in a Banach space. Recently, Cabada used a new technique to construct a smaller positive cone (see [16, 17]).

To sum up, we first obtain the relationship between the Green's function of (1.2) and the Green's function corresponding to $(1, 2)$, $(2, 1)$-conjugate boundary conditions. As we will see, the expression of $G(t, s)$ of (1.2) is so complex with two parameters that it is hard to obtain the suitable cone. Fortunately, we look for a condition showed in [18] to avoid the complex computations by the form of $G(t, s)$ of (1.2). We will be concerned with the limits of $G(t, s)/G(1, s)$ as $s = 0$ and $s = 1$.

(P_G) There is a continuous function ϕ: $[0, 1] \to (0, \infty)$ and $k_1, k_2 \in C[0, 1]$ such that $0 < k_1(s) < k_2(s)$ satisfies

$$\phi(t)k_1(s) \leq G(t, s) \leq \phi(t)k_2(s), \quad \forall (t, s) \in [0, 1] \times [0, 1].$$

Finally, relying on the linear behavior of $f(t, y)$ as $y \to \infty$, we are employed to consider three cases. All conclusions are derived that a global branch of solutions for (1.2) exists and bifurcates from infinity. As λ nears to the bifurcation point, we demonstrate that the solutions of large norms are still positive, enabling the application of bifurcation theory or topological methods.

2. Study of the sign of the Green's function

From Lemma 1.1, by means of the method introduced in [8, Page 105–106], we enunciate the following expression of the Green's function.

Theorem 2.1. Assume that Lemma 1.1 is satisfied, then,

$$\begin{aligned} &y'''(t) + \beta y''(t) + \alpha y'(t) = h_1(t), \quad t \in (0,1), \\ &y(0) = y'(0) = y(1) = 0, \end{aligned} \tag{2.1}$$

has a unique solution given by

$$y(t) = \int_0^1 G_1(t,s)h_1(s)ds,$$

where $h_1 \in X$ and $G_1(t,s)$ is the Green's function of (2.1), that is,

$$G_1(t,s) = \begin{cases} -\frac{e^{-\beta(s-t)}g(t)g(1-s)}{\alpha\sqrt{4\alpha-\beta^2}g(1)}, & 0 \le t \le s \le 1, \\ -\frac{e^{-\beta(s-t)}(g(t)g(1-s)-g(1)g(t-s))}{\alpha\sqrt{4\alpha-\beta^2}g(1)}, & 0 \le s \le t \le 1, \end{cases} \tag{2.2}$$

where

$$g(\xi) = \sqrt{4\alpha-\beta^2} - \sqrt{4\alpha-\beta^2}e^{-\frac{\beta\xi}{2}}\cos\left(\frac{\sqrt{4\alpha-\beta^2}}{2}\xi\right) - \beta e^{-\frac{\beta\xi}{2}}\sin\left(\frac{\sqrt{4\alpha-\beta^2}}{2}\xi\right), \quad \xi \in [0,1].$$

Corollary 2.1. The following properties of the function $G_1(t,s)$ are satisfied.

(1) $G_1(t,s)$ is negative on $(t,s) \in (0,1) \times (0,1)$.

(2) $G_1(0,s) = \frac{\partial G_1}{\partial t}(0,s) = G_1(1,s) = 0$, $\forall s \in (0,1)$.

(3) $G_1(t,1) = \frac{\partial G_1}{\partial s}(t,1) = G_1(t,0) = 0$, $\forall t \in (0,1)$.

(4) $\frac{\partial^2 G_1}{\partial t^2}(0,s) < 0 < \frac{\partial G_1}{\partial t}(1,s)$, $\forall s \in (0,1)$.

(5) $\frac{\partial G_1}{\partial s}(t,0) < 0$ and $\frac{\partial^2 G_1}{\partial s^2}(t,1) < 0$, $\forall t \in (0,1)$.

First of all, we point out that the following problem

$$\begin{aligned} &y'''(t) + \beta y''(t) + \alpha y'(t) = 0, \quad t \in (0,1), \\ &y(0) = y'(0) = 0, \quad y(1) = 1, \end{aligned} \tag{2.3}$$

has no solution if, and only if, Lemma 1.1 is fulfilled.

In another case, (2.3) has a unique solution ω, that is,

$$\omega(t) = \frac{g(t)}{g(1)}. \tag{2.4}$$

Obviously, $\omega(t)$ is positive on $t \in (0,1]$. Moreover, by denoting

$$D := \int_0^1 \omega(\tau)d\tau > 0,$$

we have that it is given by the following expression

$$D = \frac{(2\alpha-\beta)\sqrt{4\alpha-\beta^2} + 2\beta\sqrt{4\alpha-\beta^2}e^{-\frac{\beta}{2}}\cos\left(\frac{\sqrt{4\alpha-\beta^2}}{2}\right) - 2(2\alpha-\beta^2)e^{-\frac{\beta}{2}}\sin\left(\frac{\sqrt{4\alpha-\beta^2}}{2}\right)}{2\alpha g(1)}. \tag{2.5}$$

From the disconjugacy theory, if we investigate the following problem ($h_2 \in X$),

$$\begin{aligned} &v''' + \beta v'' + \alpha v' = h_2(t), \quad t \in (0,1), \\ &v(0) = v(1) = v'(1) = 0, \end{aligned} \tag{2.6}$$

one can know that (2.6) is the adjoint of (2.1) and the Green's function of (2.6) satisfies

$$G_2(t,s) = -G_1(s,t).$$

Furthermore, we have

$$z(s) = \int_0^1 G_2(s,\eta)d\eta = -\int_0^1 G_1(\eta,s)d\eta \tag{2.7}$$

as the unique solution satisfying the following problem

$$\begin{aligned} &z'''(s) + \beta z''(s) + \alpha z'(s) = 1, \quad t \in (0,1), \\ &z(0) = z(1) = z'(1) = 0. \end{aligned} \tag{2.8}$$

Moreover, from Lemma 1.1 and [8, Theorem 11], $z(s) > 0$ on $s \in (0,1)$, $z'(s) > 0$, and $z''(1) > 0$.

Lemma 2.1. Suppose that Lemma 1.1 is satisfied. The problem

$$\begin{aligned} &y'''(t) + \beta y''(t) + \alpha y'(t) + \hbar(t) = 0, \ t \in (0,1), \\ &y(0) = y'(0) = 0, \ y(1) = \chi\int_0^1 y(s)ds, \end{aligned} \tag{2.9}$$

is equivalent to the following integral equation if, and only if, $\chi D \neq 1$, where $\hbar \in X$, i.e.,

$$y(t) = \int_0^1 G(t,s)\hbar(s)ds,$$

where $G(t,s)$ represents the Green's function of (2.9), i.e.,

$$G(t,s) = -G_1(t,s) - \frac{\chi\omega(t)}{1-\lambda D}\int_0^1 G_1(\eta,s)d\eta, \tag{2.10}$$

and $\omega(t)$ and D are given in (2.4) and (2.5), respectively.

Proof. Since (2.1) and (2.4) have the solution v and ω, respectively, then the unique solution of (2.9) is

$$y = v + \chi\omega\int_0^1 y(s)ds.$$

Furthermore,

$$y = -\int_0^1 G_1(t,s)\hbar(s)ds + \chi\omega(t)\int_0^1 y(s)ds. \tag{2.11}$$

Put

$$A := \int_0^1 y(s)ds,$$

then integrating (2.11) on the interval $(0, 1)$, we get

$$\begin{aligned} A &= -\int_0^1\int_0^1 G_1(\eta,s)\hbar(s)dsd\eta + \chi\int_0^1 \omega(\eta)\int_0^1 y(s)dsd\eta \\ &= -\int_0^1\int_0^1 G_1(\eta,s)\hbar(s)dsd\eta + \chi A\int_0^1 \omega(\eta)d\eta, \end{aligned}$$

and, furthermore,

$$A = -\frac{\int_0^1 \hbar(s)\int_0^1 G_1(\eta,s)d\eta ds}{1-\chi\int_0^1\omega(\eta)d\eta}.$$

Replacing A in (2.11), we arrive at

$$y = -\int_0^1 G_1(t,s)\hbar(s)ds - \chi\omega(t)\frac{\int_0^1 \hbar(s)\int_0^1 G_1(\eta,s)d\eta ds}{1-\chi\int_0^1\omega(\eta)d\eta}.$$

□

Next, we give a careful analysis of $G(t, s)$, and the following theorem is fulfilled.

Theorem 2.2. From Lemma 1.1 and $\chi \in (0, \frac{1}{D})$, we derive that $G(t, s) > 0$ on $(t, s) \in [0, 1] \times [0, 1]$.

Proof. From Corollary 2.1, $G_1(t, s)$ is negative. Next, by the property of (P_G), we show that there is a positive constant $\mathcal{R}$ and a function $l \in X$ satisfying $l(t) > 0$ on $t \in (0, 1]$ and $l(0) = 0$, for which the following result is satisfied, that is,

$$l(t)\frac{\chi}{1-\chi D}z(s) \le G(t,s) \le \mathcal{R}\frac{\chi}{1-\chi D}z(s), \quad (t,s)\in[0,1]\times[0,1]. \tag{2.12}$$

Now, set

$$\psi(t,s) := \frac{G(t,s)}{G(1,s)} = \frac{1-\chi D}{\chi}\frac{G_1(t,s)}{\int_0^1 G_1(\eta,s)d\eta} + \omega(t). \tag{2.13}$$

Obviously, $\psi(t, s)$ is continuous and

$$G(1,s) = -\frac{\chi}{1-\chi D}\int_0^1 G_1(\eta,s)d\eta.$$

By Corollary 2.1, $z(s)$ is the solution of (2.8). From the L'Hôptial rule, we derive

$$\lim_{s\to0^+}\frac{G_1(t,s)}{\int_0^1 G_1(\eta,s)d\eta} = \lim_{s\to0^+}\frac{-G_1(t,s)}{z(s)} = \lim_{s\to0^+}\frac{\frac{-\partial G_1}{\partial s}(t,s)}{z'(s)} = \frac{\frac{-\partial G_1}{\partial s}(t,0)}{z'(0)} > 0.$$

Thus,

$$\lim_{s\to 0^+}\psi(t,s)=\frac{1-\chi D}{\chi}\left(\frac{-\frac{\partial G_1}{\partial s}(t,0)}{z'(0)}\right)+\omega(t):=\psi_1(t)>0.$$

Analogously,

$$\lim_{s\to 1^-}\frac{G_1(t,s)}{\int_0^1 G_1(\eta,s)d\eta}=\lim_{s\to 1^-}\frac{-G_1(t,s)}{z(s)}=\lim_{s\to 1^-}\frac{-\frac{\partial^2 G_1}{\partial s^2}(t,s)}{z''(s)}=\frac{\frac{-\partial^2 G_1}{\partial s^2}(t,1)}{z''(1)}>0$$

and

$$\lim_{s\to 1^-}\psi(t,s)=\frac{1-\chi D}{\chi}\left(\frac{-\frac{\partial^2 G_1}{\partial s^2}(t,1)}{z''(1)}\right)+\omega(t):=\psi_2(t)>0.$$

Since the limits functions $\psi_1(t)$ and $\psi_2(t)$ exist and are finite, then at $s=0,1$, $\psi(t,s)$ has removable discontinuities, so it can be extended to a function $\bar{\psi}\in C((0,1)\times(0,1))$. Therefore,

$$l(t)=\min_{s\in[0,1]}\bar{\psi}(t,s)$$

is continuous such that

$$l(0)=0,\quad 0<l(t)\le\bar{\psi}(t,s)\le\mathcal{R}:=\max_{(t,s)\in[0,1]\times[0,1]}\bar{\psi}(t,s).$$

□

Corollary 2.2. From Lemma 1.1 and $\chi\in(0,\frac{1}{D})$, for all constant $\delta\in(0,1)$, there is a constant $\gamma\in(0,1)$ depending on δ, such that the following result is satisfied, that is,

$$\gamma\frac{\chi}{1-\chi D}z(s)\le G(t,s),\quad (t,s)\in[\delta,1]\times[0,1]. \tag{2.14}$$

Proof. The result is satisfied from the definition of the function l. □

Define a cone

$$P:=\left\{y\in X|y(t)\ge 0, t\in[0,1] \text{ and } \min_{t\in[\delta,1]}y(t)\ge\gamma\|y\|\right\}, \tag{2.15}$$

where δ satisfies Corollary 2.2.

Remark 2.1. In this paper, in view of the boundary conditions, the integral boundary condition is more widely used than the obtained third-order conjugate boundary condition in [1,5].

3. Nonlinear problems

The work space is $X=C[0,1]$ with the norm

$$\|y\|:=\max_{t\in[0,1]}|y(t)|.$$

We also set

$$B_r:=\{y\in X:\|y\|<r\}$$

with $r > 0$.

Next, by means of the Krein-Rutman theorem [19, Theorem 19.3 (a)], we discuss the existence of the principal eigenvalue for the linear eigenvalue problem as follows

$$\begin{aligned} & -y''' - \beta y'' - \alpha y' = \lambda b(t) y, \quad t \in (0,1), \\ & y(0) = y'(0) = 0, \quad y(1) = \chi \int_0^1 y(s) ds. \end{aligned} \tag{3.1}$$

Denote $\mathcal{A}$: $\mathcal{P} \to X$ as the map,

$$\mathcal{A}y(t) := \lambda \int_0^1 G(t,s) b(s) y(s) ds, \quad t \in [0,1].$$

Lemma 3.1. Assume that $\chi \in (0, \frac{1}{D})$ and Lemma 1.1 are satisfied, (3.1) has a principal eigenvalue λ_1, and the corresponding eigenfunction $\phi_1(t)$ is positive.

Proof. It follows from (2.15), then $\mathcal{P}$ is normal and has nonempty interior, so $X = \overline{\mathcal{P} - \mathcal{P}}$. From Theorem 2.2, $\mathcal{A}$ is a strong positive operator and $\mathcal{A} \in \operatorname{int} \mathcal{P}$. By the Krein-Rutman theorem, the spectral radius $r(\mathcal{A})$ is positive, and $\phi_1 \in X$ exists, satisfying $\phi_1 > 0$ and $\mathcal{A}\phi_1 = r(\mathcal{A})\phi_1$. Thus,

$$\lambda_1 = (r(\mathcal{A}))^{-1} > 0.$$

Let $\mathcal{A}^*$ be the conjugate operator of $\mathcal{A}$, then $\mathcal{A}^*\phi_2 = r(\mathcal{A}^*)\phi_2$, where $\phi_2 \in X$ such that $\phi_2 > 0$ on $(0,1)$, corresponding to λ_1. Since

$$\int_0^1 (\mathcal{A}\phi_1)\phi_2 dt = \lambda_1 \int_0^1 \phi_1 \phi_2 dt = \int_0^1 \phi_1 (\mathcal{A}^* \phi_2) dt,$$

then the algebraic multiplicity of λ_1 is 1. Thus, λ_1 is the principal eigenvalue of (3.1). □

Denote a nonlinear operator $\mathcal{K}$: $X \to X$ by

$$y := \mathcal{K}\hbar.$$

From the above notation, it follows that (1.2) is equivalent to

$$y - \lambda \mathcal{K} f(\cdot, y) = 0, \quad y \in X. \tag{3.2}$$

Throughout the paper, we will use the same symbol to represent both the function and the corresponding Nemytskii operator.

We denote if there is a sequence (μ_n, y_n) with $\mu_n \to \lambda_\infty$ and $y_n \in X$, such that $y_n - \mu_n \mathcal{K} f(y_n) = 0$ and $\|y_n\| \to \infty$, then λ_∞ is a bifurcation from infinity for (3.2).

In some cases, such as what we will later discuss in detail, by application of an appropriate rescaling to look for bifurcation from infinity based on the Leray-Schauder topological degree, which represents $\deg(\cdot,\cdot,\cdot)$, it is worth noting that $\mathcal{K}$ is continuous and compact. Thus it is reasonable to investigate the topological degree of $I - \lambda\mathcal{K}f$, where I is the identity map.

3.1. Asymptotically linear problems

Theorem 3.1. Assume that $\chi \in (0, \frac{1}{D})$ and Lemma 1.1 are satisfied. We suppose that

$$f \in C([0,1] \times [0, \infty), \mathbb{R})$$

satisfies (F_1) and (F_2).

(F_2) There exists a positive function $c \in X$ such that

$$\lim_{y \to \infty} \frac{f(\cdot, y)}{y} = c.$$

There is a positive constant ε such that (1.2) exist positive solutions if either

(I) $\upsilon_1 > 0$ (possibly ∞) in $[0,1]$ and $\lambda \in [\lambda_\infty - \varepsilon, \lambda_\infty)$,

or

(II) $\upsilon_2 < 0$ (possibly $-\infty$) in $[0,1]$ and $\lambda \in (\lambda_\infty, \lambda_\infty + \varepsilon]$,

where

$$\lambda_\infty := \frac{\lambda_1}{c}$$

and

$$\upsilon_1(t) := \liminf_{y \to \infty}(f(t,y) - cy), \quad \upsilon_2(t) := \limsup_{y \to \infty}(f(t,y) - cy).$$

In order to show Theorem 3.1 is valid, we first extend $f(t, \cdot)$ to $\mathbb{R}$ and set

$$F(t,y) := f(t, |y|) \tag{3.3}$$

and

$$\Phi(\lambda, y) := y - \lambda \mathcal{K} F(t,y), \quad y \in X. \tag{3.4}$$

Obviously, the solution $y > 0$ of $\Phi(\lambda, y) = 0$ is equivalent to a positive solution of (1.2).

Lemma 3.2. Assume that $\chi \in (0, \frac{1}{D})$ and Lemma 1.1 are satisfied. For every compact interval $\aleph \subset [0, +\infty) \backslash \{\lambda_\infty\}$, there is a positive constant r such that, for all $\lambda \in \aleph$, $\|y\| \geq r$, $\Phi(\lambda, y) \neq 0$. Moreover,

(1) if $\upsilon_1 > 0$, then $\aleph = [\lambda_\infty, \lambda]$, for $\lambda > \lambda_\infty$;

(2) if $\upsilon_2 < 0$, then $\aleph = [0, \lambda_\infty]$.

Proof. Let $\mu_n \to \mu \geq 0 \in \aleph$, that is, $\mu \neq \lambda_\infty$ and $\|y_n\| \to \infty$ be such that

$$y_n = \mu_n \mathcal{K} F(t, y_n).$$

Setting

$$w_n := \frac{y_n}{\|y_n\|},$$

it follows that

$$w_n = \mu_n \|y_n\|^{-1} \mathcal{K} F(t, y_n).$$

From (F_2) and (3.3), up to a subsequence in X, $w_n \to w$ is fulfilled for which w satisfies the problem

$$\begin{aligned} &-w''' - \beta w'' - \alpha w' = \mu c|w|, \quad t \in (0,1), \\ &w(0) = w'(0) = 0, \quad w(1) = \chi \int_0^1 w(s)ds, \end{aligned}$$

and $\|w\| = 1$. By Theorem 2.2, $w \geq 0$, then $\mu c = \lambda_1$, i.e., $\mu = \lambda_\infty$, which contradicts with the assumption of $\mu \neq \lambda_\infty$.

Next, we give a short illustration of Lemma 3.2 (1), and (2) follows similarly. Now, assume that there is a sequence $(\mu_n, y_n) \in (0,\infty) \times X$, where $\mu_n \to \lambda_\infty$, $\|y_n\| \to \infty$, and $\mu_n > \lambda_\infty$, such that

$$\Phi(\mu_n, y_n) = 0. \tag{3.5}$$

Note that $y_n \in X$ has a unique decomposition

$$y_n = v_n + s_n\phi_1, \tag{3.6}$$

where $s_n \in \mathbb{R}$, since $y_n > 0$, $\phi_2 > 0$, and

$$\int_0^1 v_n(t)\phi_2(t)dt = 0,$$

and by (3.6), we obtain

$$s_n = \left(\int_0^1 u_n(t)\phi_2(t)dt\right)\left(\int_0^1 \phi_1(t)\phi_2(t)dt\right)^{-1} > 0, \quad n \in \mathbb{N}. \tag{3.7}$$

From (3.5), it follows that

$$\int_0^1 \phi_2(t)y_n(t)dt = \mu_n \int_0^1 \phi_2(t)\mathcal{K}F(t, y_n(t))dt.$$

Since

$$\int_0^1 \phi_2(t)\mathcal{K}F(t, y_n(t))dt = -\lambda_1 \int_0^1 \phi_2(t)y_n(t)dt,$$

we obtain

$$\begin{aligned} -\lambda_1 \int_0^1 \phi_2(t)y_n(t)dt &= \int_0^1 (y_n''' + \beta y_n'' + \alpha y_n')\phi_2(t)dt \\ &= -\int_0^1 \mu_n f(t, y_n(t))\phi_2(t)dt \\ &= -\int_0^1 \mu_n(f(t, y_n(t)) - cy_n(t))\phi_2(t)dt - \int_0^1 \mu_n y_n(t)c\phi_2(t)dt, \end{aligned}$$

then

$$(\mu_n c - \lambda_1)\int_0^1 \phi_2(t)y_n(t)dt = -\int_0^1 \mu_n(f(t, y_n(t)) - cy_n(t))\phi_2(t)dt.$$

For n large enough, $\mu_n > \lambda_\infty$ and

$$\int_0^1 y_n(t)\phi_2(t)dt > 0$$

hold, then we infer that

$$\int_0^1 (f(t, y_n(t)) - cy_n(t))\phi_2(t)dt < 0$$

and from the Fatou lemma, we yield

$$0 \geq \liminf_{n\to\infty} \int_0^1 (f(t, y_n) - cy_n)\phi_2 dt \geq \int_0^1 \upsilon_1\phi_2 dt,$$

which contradicts with $\upsilon_1 > 0$. □

Lemma 3.3. Assume that $\chi \in (0, \frac{1}{D})$ and Lemma 1.1 are satisfied. For $\lambda \in (\lambda_\infty, \infty)$, there is a positive constant r such that

$$\Phi(\lambda, y) \neq \tau\phi_1, \text{ for all } \tau \geq 0, \ \|y\| \geq r.$$

Proof. If there is a sequence $\{y_n\} \in X$ with $\|y_n\| \to \infty$ and numbers $\tau_n \geq 0$ satisfy $\Phi(\lambda, y_n) = \tau_n\phi_1$, then

$$-y_n''' - \beta y_n'' - \alpha y_n' = \lambda F(t, y_n) + \tau_n\lambda_1\phi_1.$$

Since $F(t, y) \approx c|y| \to \infty$ and $\tau_n\lambda_1\phi_1 \geq 0$, to the maximum principle, $y_n > 0$ for all $t \in (0, 1)$.

Choose $\epsilon > 0$ such that

$$\lambda_\infty < \lambda(1 - \epsilon).$$

From condition (F_2), there is a constant $R_0 > 0$ such that

$$f(t, y) \geq (1 - \epsilon)cy, \ \ \forall\, y > R_0, \ \ t \in (0, 1).$$

By $\|y_n\| \to \infty$, we know there is a positive constant N^* such that

$$y_n > R_0, \quad \forall\, n \geq N^*$$

and

$$f(t, y_n) \geq (1 - \epsilon)cy_n. \tag{3.8}$$

By (3.7) and (3.8), we derive

$$\begin{aligned}
s_n\lambda_1 \int_0^1 \phi_1(t)\phi_2(t)dt &= \int_0^1 -(y_n''' + \beta y_n'' + \alpha y_n')\phi_2(t)dt \\
&= \lambda \int_0^1 F(t, y_n)\phi_2(t)dt + \tau\lambda_1 \int_0^1 \phi_1(t)\phi_2(t)dt \\
&\geq \lambda \int_0^1 F(t, y_n)\phi_2(t)dt \\
&\geq \lambda \int_0^1 (1 - \epsilon)cy_n(t)\phi_2(t)dt \\
&= \lambda(1 - \epsilon)cs_n \int_0^1 \phi_1(t)\phi_2(t)dt.
\end{aligned}$$

Thus,

$$\lambda_\infty \geq \lambda(1-\epsilon),$$

which is a contradiction. □

For $y \neq 0$, we set

$$\mathcal{Z} := \frac{y}{||y||^2}.$$

Set

$$\begin{aligned}\Psi(\lambda, \mathcal{Z}) := \frac{\Phi(\lambda, y)}{||y||^2} &= \frac{y - \lambda \mathcal{K} F(t, y)}{||y||^2}\\ &= \mathcal{Z} - \lambda ||\mathcal{Z}||^2 \mathcal{K} F\left(t, \frac{\mathcal{Z}}{||\mathcal{Z}||^2}\right).\end{aligned}$$

λ_∞ is a bifurcation from infinity for (3.4) if, and only if, λ_∞ is a bifurcation from $\mathcal{Z} = 0$ for $\Psi = 0$. From Lemma 3.2, for $\lambda \in (0, \lambda_\infty)$, by homotopy, we have

$$\begin{aligned}\deg(\Psi(\lambda, \cdot), B_{\frac{1}{r}}, 0) &= \deg(\Psi(0, \cdot), B_{\frac{1}{r}}, 0)\\ &= \deg(I, B_{\frac{1}{r}}, 0)\\ &= 1.\end{aligned} \tag{3.9}$$

Similarly, from Lemma 3.2, for $\tau \in [0, 1]$ and $\lambda \in (\lambda_\infty, \infty)$,

$$\begin{aligned}\deg(\Psi(\lambda, \cdot), B_{\frac{1}{r}}, 0) &= \deg(\Psi(0, \cdot) - \tau\phi_2, B_{\frac{1}{r}}, 0)\\ &= \deg(\Psi(0, \cdot) - \phi_1, B_{\frac{1}{r}}, 0)\\ &= 0.\end{aligned} \tag{3.10}$$

Set

$$\Sigma := \{(\lambda, y) \in [0, \infty) \times X : y \neq 0, \Phi(\lambda, y) = 0\}.$$

It follows from (3.9) and (3.10) that the following lemma is fulfilled.

Lemma 3.4. λ_∞ is a bifurcation from infinity for (3.4). To be precise, there is an unbounded, closed, connected set $\Sigma_\infty \subset \Sigma$ which bifurcates from infinity. Furthermore, if $\upsilon_1 > 0$, Σ_∞ bifurcates to the left (respectively, if $\upsilon_2 < 0$, Σ_∞ bifurcates to the right).

Proof of Theorem 3.1. From Lemmas 3.2–3.4, it is sufficient to demonstrate that if $\mu_n \to \lambda_\infty$ and $||y_n|| \to \infty$, for all $t \in (0, 1)$ and n large enough, y_n is positive. Set

$$w_n = \frac{y_n}{||y_n||}.$$

By utilizing the preceding results up to subsequence in X, $w_n \to w$ and $w = \vartheta\phi_1$ with $\vartheta > 0$ are satisfied, then, $y_n > 0$ for n large enough. □

3.2. Superlinear problems

Theorem 3.2. Assume that $\chi \in (0, \frac{1}{D})$ and Lemma 1.1 are satisfied. We suppose that

$$f \in C([0,1] \times [0,\infty), \mathbb{R})$$

satisfies (F_1) and (F_3).

(F_3) There is a positive function $c \in X$ such that

$$\lim_{y\to\infty} \frac{f(\cdot, y)}{y^p} = c, \quad p \in (1,\infty).$$

There is a constant $\lambda^* > 0$, such that for $\lambda \in (0, \lambda^*]$, (1.2) exist positive solutions. More specifically, there exist a connected set of positive solutions for (1.2), which bifurcates from infinity at $\lambda_\infty = 0$.

Set

$$\mathcal{F}(t,y) := F(t,y) - c|y|^p,$$

where $F(t,y)$ is denoted as (3.3).

Next, using the rescaling $w = dy$ and $\lambda = d^{p-1}$ with $d > 0$, shows that $\lambda_\infty = 0$ is a bifurcation from infinity for

$$y - \lambda \mathcal{K} F(t,y) = 0, \tag{3.11}$$

which is equivalent to (λ, y), which is a solution of (3.11) if, and only if,

$$w - \mathcal{K}\widehat{\mathcal{F}}(d,w) = 0, \tag{3.12}$$

where

$$\widehat{\mathcal{F}}(d,w) := c|w|^p + d^p \mathcal{F}(d^{-1}w).$$

As $d = 0$, we set

$$\widehat{\mathcal{F}}(0,w) := c|w|^p.$$

By (F_3), we know that $\widehat{\mathcal{F}}(d,w)$ is continuous for $(d,w) \in [0,\infty) \times \mathbb{R}$. Let

$$\tilde{\mathcal{S}}(d,w) := w - \mathcal{K}\widehat{\mathcal{F}}(d,w), \quad d \in (0,\infty).$$

Thus, $\tilde{\mathcal{S}}(d,\cdot)$ is compact. For $d = 0$, solution of $\tilde{\mathcal{S}}(0,w) = 0$ are nothing but solutions of

$$\begin{aligned} &-w''' - \beta w'' - \alpha w' = c|w|^p, \quad t \in (0,1),\\ &w(0) = w'(0) = 0, \quad w(1) = \chi \int_0^1 w(s)ds. \end{aligned} \tag{3.13}$$

Now, we show that the following results are fulfilled:

$$\tilde{\mathcal{S}}(0,w) \neq 0, \quad \text{for all} \quad \|w\| \geq R_2, \tag{3.14}$$

$$\tilde{\mathcal{S}}(0,w) \neq 0, \quad \text{for all} \quad 0 < \|w\| \leq R_1, \tag{3.15}$$

$$\deg(\tilde{\mathcal{S}}(0,w), \mathcal{P}_R \backslash \overline{\mathcal{P}}_r, 0) = -1, \quad \text{for } r \in (0,R_1], \ R \in [R_2,\infty), \tag{3.16}$$

where R_1, R_2 are two constants with $0 < R_1 < R_2$.

First, we claim that there exists $R > 0$ such that for $\|w\| \geq R$, $\tilde{\mathcal{S}}(0, w) \neq 0$.

Assume that (3.13) has a sequence $\{w_n\}$ satisfying

$$\lim_{n\to\infty} \|w_n\| = \infty,$$

i.e.,

$$\begin{aligned} & -w_n''' - \beta w_n'' - \alpha w_n' = (c|w_n|^{p-1})w_n, \quad t \in (0,1), \\ & w_n(0) = w_n'(0) = 0, \quad w_n(1) = \chi \int_0^1 w_n(s)ds. \end{aligned}$$

Note that

$$\lim_{n\to\infty} c|w_n|^{p-1} = \infty, \quad t \in (0,1).$$

By the remarks in the final paragraph on [20, Page 56], w_n must change its sign in $(0, 1)$, which is a contradiction.

Second, we prove that for $0 < \|w\| \leq R_1$, $\tilde{\mathcal{S}}(0, w) \neq 0$, where $R_1 > 0$ is a constant.

On the contrary, if (3.15) does not hold, then (3.13) exists a sequence of solutions w_n, which satisfies

$$\|w_n\| \to 0, \quad n \to \infty. \tag{3.17}$$

Let

$$v_n = \frac{w_n}{\|w_n\|}.$$

From (3.13), we have

$$\begin{aligned} & -v_n''' - \beta v_n'' - \alpha v_n' = c\frac{|w_n|^p}{\|w_n\|}, \quad t \in (0,1), \\ & v_n(0) = v_n'(0) = 0, \ v_n(1) = \chi \int_0^1 v_n(s)ds. \end{aligned}$$

From (3.17), we have

$$\lim_{n\to\infty} v_n = 0$$

uniformly in $t \in (0, 1)$.

According to the standard argument, after taking a subsequence and relabeling if necessary, $v_* \in X$ exists and $\|v_*\| = 1$, satisfying

$$v_n \to v_*, \ \ n \to \infty$$

and

$$\begin{aligned} & -v_*''' - \beta v_*'' - \alpha v_*' = 0, \quad t \in (0,1), \\ & v_*(0) = v_*'(0) = 0, \quad v_*(1) = \chi \int_0^1 v_*(s)ds, \end{aligned}$$

which shows that $v_* = 0$ holds. However, this is a contradiction. Hence, (3.15) holds.

In the end, we show (3.16) is valid. Denote

$$\mathcal{P}_r := \{y \in \mathcal{P} : \|y\| < r\}.$$

Now, by (3.14) and (3.15), we derive that for all $w \in \partial\mathcal{P}_R$ and $w \in \partial\mathcal{P}_r$, $\tilde{\mathcal{S}}(0, w) \neq 0$ is valid. So for all

$$w \in \partial(\mathcal{P}_R \backslash \overline{\mathcal{P}}_r),\ \tilde{\mathcal{S}}(0, w) \neq 0$$

is still fulfilled. Hence, $\deg(\tilde{\mathcal{S}}(0, w), \mathcal{P}_R \backslash \overline{\mathcal{P}}_r, 0)$ is well defined.

Note that $\tilde{f}(w) = |w|^p$, so we claim that

$$\deg(\tilde{\mathcal{S}}(0, w), \mathcal{P}_R \backslash \overline{\mathcal{P}}_r, 0) = -1.$$

It is easy to verify the following conditions:

(H_1) $f_0 := \lim\limits_{w \to 0^+} \frac{\tilde{f}(w)}{w} = 0$;

(H_2) $f_\infty := \lim\limits_{w \to +\infty} \frac{\tilde{f}(w)}{w} = \infty$.

Choose $M_1 > 0$, which satisfies

$$M_1 \gamma \frac{\chi}{1 - \chi D} \int_0^1 z(s)c(s)ds > 1.$$

From (H_2), there is a constant $R_2 > 0$, such that $\forall w \geq R_2$, $f(w) > M_1 w$ is satisfied. Choosing $R > \max\{R_1, R_2\}$, we claim that

$$\|\mathcal{K}\widehat{\mathcal{F}}(0, w)\| > \|w\|$$

for $w \in \partial\mathcal{P}_R$. In fact, for $w \in \partial\mathcal{P}_R$,

$$\begin{aligned}(\mathcal{K}\widehat{\mathcal{F}}(0, w))(t) &= \int_0^1 G(t, s)c(s)|w|^p ds \\ &\geq M_1 \gamma \frac{\chi}{1 - \chi D} \|w\| \int_0^1 z(s)c(s)ds \\ &> \|w\|.\end{aligned}$$

Hence, it follows from the fixed point index theorem of [19] that

$$i(\mathcal{K}\widehat{\mathcal{F}}(0, \cdot), \mathcal{P}_R, \mathcal{P}) = 0. \tag{3.18}$$

From (H_1), there exists a constant $\iota > 0$ such that $w \in [0, \iota]$, and

$$\tilde{f}(w) \leq M_2 w,$$

for which $M_2 > 0$ satisfies

$$M_2 \mathcal{R} \frac{\chi}{1 - \chi D} \int_0^1 z(s)c(s)ds \leq 1.$$

Choose $0 < r < \min\{\iota, \frac{R}{2}\}$, for $w \in \partial\mathcal{P}_r$,

$$\begin{aligned}\|\mathcal{K}\widehat{\mathcal{F}}(0,w)\| &= \max_{t\in[\delta,1]}\int_0^1 G(t,s)c(s)|w|^p ds\\ &\leq M_2\mathcal{R}\frac{\chi}{1-\chi D}\|w\|\int_\delta^1 z(s)c(s)ds\\ &\leq \|w\|.\end{aligned}$$

Obviously, $\mathcal{K}\widehat{\mathcal{F}}(0,w) \neq w$ for $w \in \partial\mathcal{P}_r$. By the fixed point index theorem of [19],

$$i(\mathcal{K}\widehat{\mathcal{F}}(0,\cdot),\mathcal{P}_r,\mathcal{P}) = 1. \tag{3.19}$$

From the additivity of the fixed point index, (3.18), and (3.19), we get

$$i(\mathcal{K}\widehat{\mathcal{F}}(0,\cdot),\mathcal{P}_R\backslash\overline{\mathcal{P}}_r,\mathcal{P}) = -1. \tag{3.20}$$

From (3.20) and

$$\tilde{\mathcal{S}}(0,w) : X \to \mathcal{P}_R\backslash\overline{\mathcal{P}}_r,$$

we obtain

$$\deg(\tilde{\mathcal{S}}(0,w),\mathcal{P}_R\backslash\overline{\mathcal{P}}_r,0) = -1.$$

Lemma 3.5. Assume that $\chi \in (0,\frac{1}{D})$ and Lemma 1.1 are fulfilled, then there is a constant $d_0 > 0$ such that

(i) $\deg(\tilde{\mathcal{S}}(d,\cdot),\mathcal{P}_R\backslash\overline{\mathcal{P}}_r,0) = -1,\ \forall d \in [0,d_0]$;

(ii) if $\tilde{\mathcal{S}}(d,w) = 0, d \in [0,d_0], \|w\| \in [r,R]$, then $w > 0$.

Proof. On the contrary, there is a sequence (d_n,w_n), where $d_n \to 0$, $\|w_n\| \in \{r,R\}$, and

$$w_n = \mathcal{K}\widehat{\mathcal{F}}(d_n,w_n).$$

Since the operator $\mathcal{K}$ is compact, up to a subsequence, $w_n \to w$ and

$$\tilde{\mathcal{S}}(0,w) = 0, \quad \|w_n\| \in \{r,R\},$$

which contradicts with (3.14) and (3.15). Therefore, (i) holds.

In order to show (ii) is valid, we once argue by means of contradiction. From the preceding results, we can present a sequence $\{w_n\} \in X$ with $\{t \in [0,1] : w_n \leq 0\} \neq \emptyset$, which satisfies $w_n \to w$, $\|w\| \in \{r,R\}$ and $\tilde{\mathcal{S}}(0,w) = 0$; that is, w is a solution of (3.13). From Theorem 2.2, $w > 0$ holds. Therefore, we get a contradiction that for n large enough, w_n is positive. □

Proof of Theorem 3.2. From Lemma 2.1, $\forall d \in [0,d_0]$, and (3.12) exists a positive solution w_d. Recalling for $d > 0$ and $\lambda = d^{p-1}$, $y = \frac{w}{d}$ is a solution (λ, y_λ) of (3.11) for

$$0 < \lambda < \lambda^* := d_0^{p-1}.$$

For $w_d > 0$, (λ, y_λ) is the positive solution of (1.2). So, $\forall d \in [0,d_0]$, $\|w_d\| \geq d$ implies that

$$\|y_d\| = \|w_d\|/d \to \infty$$

as $d \to 0$. □

3.3. Sublinear problems

Theorem 3.3. Assume that $\chi \in (0, \frac{1}{D})$ and Lemma 1.1 are satisfied. We suppose that

$$f \in C([0,1] \times [0,\infty), \mathbb{R})$$

satisfies (F_1) and (F_4).

(F_4) There is a positive function $c \in X$, such that

$$\lim_{y\to\infty} \frac{f(\cdot, y)}{y^q} = c, \quad q \in [0,1).$$

There is a constant $\lambda_* > 0$, such that for $\lambda \in [\lambda_*, \infty)$, positive solutions of (1.2) exist. To be precise, there is a connected set of positive solutions of (1.2), which bifurcates from infinity for $\lambda_\infty = \infty$.

In this case, we will show that (1.2) exist positive solutions, which branch off from ∞ as $\lambda_\infty = \infty$. As the same treatment as the superlinear problems, we again use $w = dy$, $\lambda = d^{q-1}$ with q replacing p. In the case of superlinear problems, (λ, y) is the solution of (3.11) if (d, w) satisfies (3.12). Now, by $q \in [0,1)$,

$$\lambda \to \infty \Leftrightarrow d \to 0. \tag{3.21}$$

Lemma 3.6. Assume that $\chi \in (0, \frac{1}{D})$ and Lemma 1.1 are satisfied. For $q \in (0,1)$, the problem

$$\begin{aligned} &-w'''(t) - \beta w''(t) - \alpha w'(t) = c(t)w^q(t), \quad t \in (0,1), \\ &w(0) = w'(0) = 0, \quad w(1) = \chi \int_0^1 w(s)ds \end{aligned} \tag{3.22}$$

has a unique positive solution w_0.

Proof. Assume that w_1, w_2 are two positive solutions of (3.22), i.e.,

$$-w_1'''(t) - \beta w_1''(t) - \alpha w_1'(t) = c(t)w_1^q, \;\; w_1(0) = w_1'(0) = 0, \;\; w_1(1) = \chi \int_0^1 w_1(s)ds,$$

$$-w_2'''(t) - \beta w_2''(t) - \alpha w_2'(t) = c(t)w_2^q, \;\; w_2(0) = w_2'(0) = 0, \;\; w_2(1) = \chi \int_0^1 w_2(s)ds.$$

We will show that $w_1 \geq w_2$ and $w_2 \geq w_1$. If $w_1 \ngeq w_2$, we discuss that the element $\bar{w}$ satisfies the following form, that is,

$$\bar{w}(t) = w_1(t) - \epsilon w_2(t), \quad t \in [0,1],$$

where $\epsilon \in (0,1)$. Let there exist a point $\zeta_0 \in (0,1)$ such that

$$\bar{w}(\zeta_0) = w_1(\zeta_0) - \epsilon_0 w_2(\zeta_0) = 0. \tag{3.23}$$

On the other hand,

$$\begin{aligned} -\bar{w}'''(t) - \beta\bar{w}''(t) - \alpha\bar{w}'(t) &= -(w_1(t) - \epsilon_0 w_2(t))''' - \beta(w_1(t) - \epsilon_0 w_2(t))'' - \alpha(w_1(t) - \epsilon_0 w_2(t))' \\ &= c(t)[w_1^q(t) - \epsilon_0 w_2^q(t)] \\ &\geq c(t)[\epsilon_0^q w_2^q(t) - \epsilon_0 w_2^q(t)] \\ &> 0, \\ \bar{w}(0) = \bar{w}'(0) = 0,& \\ \bar{w}(1) = \chi \int_0^1 \bar{w}(s)ds.& \end{aligned}$$

So, $\bar{w}(t) > 0$, which contradicts with (3.23). Therefore, $w_1 \geq w_2$. By the same method, we may prove that $w_1 \leq w_2$. □

As the same treatment as the superlinear case, we also can obtain that the following results are fulfilled:

$$\tilde{\mathcal{S}}(0, w) \neq 0 \quad \text{for all} \quad \|w\| \geq R_3, \tag{3.24}$$

$$\tilde{\mathcal{S}}(0, w) \neq 0 \quad \text{for all} \quad 0 < \|w\| \leq R_4, \tag{3.25}$$

and

$$\deg(\tilde{\mathcal{S}}(0, w), \mathcal{P}_R \backslash \overline{\mathcal{P}}_r, 0) = 1 \quad \text{for } r \in (0, R_3], \quad R \in [R_4, \infty), \tag{3.26}$$

where R_3 and R_4 are two constants satisfying $0 < R_3 < R_4$. Therefore, the following results can be obtained.

Lemma 3.7. Assume that $\chi \in (0, \frac{1}{D})$ and Lemma 1.1 are satisfied. There exists $d_0 > 0$ such that

(i) $\deg(\tilde{\mathcal{S}}(d, \cdot), \mathcal{P}_R \backslash \overline{\mathcal{P}}_r, 0) = 1, \ \forall d \in [0, d_0]$;

(ii) If $\tilde{\mathcal{S}}(d, w) = 0, d \in [0, d_0], \|w\| \in [r, R]$, then $w > 0$.

Proof of Theorem 3.3. By the continuation, a connected subset Γ of solutions of $\tilde{\mathcal{S}}(d, w) = 0$ satisfies $(0, w_0) \in \Gamma$. From Lemma 3.7, we derive that there is a positive constant d_0, such that for $d < d_0$, these solutions are positive. Since $\lambda = d^{q-1}$, $u = \frac{w}{d}$, then a connected subset of solutions of (1.2) exists, which it can be transformed by Γ, so for

$$\lambda > \lambda^* := d_0^{q-1},$$

these solutions are positive. Furthermore, from (3.21), Σ_∞ bifurcates from infinity for $\lambda_\infty = +\infty$. □

Remark 3.1. Comparing with [3], we no longer consider the nonlinear term as a translation and we obtain the optimal interval of the parameter λ under the three cases.

3.4. Example

In order to illustrate that Theorems 3.1–3.3 are valid, we take $\alpha = 1$ and $\beta = 0$, then Lemma 1.1 holds and

$$\chi \in \left(0, \frac{1 - \cos(1)}{1 - \sin(1)}\right).$$

If

$$f(t, y) = 2y^{\tilde{p}} + t\ln(1 + y) - t - 1, \quad \tilde{p} \in [0, \infty),$$

then $f(t, 0) = -t - 1 < 0$ is satisfied for $t \in [0, 1]$. Let λ_1 be the first positive eigenvalue corresponding to the linear problem (3.1) and ϕ_1 be the positive eigenfunction corresponding to λ_1. Next, we will check that all conditions in Theorems 3.1–3.3 are fulfilled. In fact,

$$c = \lim_{y \to \infty} \frac{f(t, y)}{y^{\tilde{p}}} = 2.$$

In view of Theorem 3.1, $\lambda_\infty = \frac{\lambda_1}{2}$, where $\tilde{p} = 1$,

$$v_1(t) = \liminf_{y \to \infty}(f(t, y) - cy) = \liminf_{y \to \infty}(t\ln(1 + y) - t - 1) > 0, \quad t \in [0, 1].$$

Thus, Theorem 3.1 is valid. In the case of the superlinear problem,

$$\lambda_\infty := \frac{\lambda_1}{2} y^{1-\tilde{p}},$$

where $\tilde{p} > 1$, so we derive that

$$\lim_{y\to\infty} \lambda_\infty = \frac{\lambda_1}{2} y^{1-\tilde{p}} = 0.$$

Therefore, Theorem 3.2 is valid. Similarly, we also show that Theorem 3.3 is fulfilled.

4. Conclusions and discussion

By virtue of bifurcation theory or topological methods, we show the global behavior of positive solutions of (1.2) in the cases of asymptotically linear, superlinear, and sublinear as $y \to \infty$, and for λ near the bifurcation value, where the solutions norms are indeed positive. The domain of (1.2) offers potential for further studies. For example, in this kind of semipositone case, $f(t, 0) < 0$ is bounded. A natural question is whether or not there exist positive solutions for (1.2) if $f(t, 0) \to -\infty$. Furthermore, the methods from this paper can be used for studying neutral-type and impulse differential equations as forms in [21–25]. We can also further discuss the stability analysis of the obtained positive solutions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests in this paper.

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