Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 7253–7272.
DOI: 10.3934/math.2024352
Received: 26 November 2023
Revised: 06 February 2024
Accepted: 08 February 2024
Published: 20 February 2024

Research article

Complex symmetric difference of the weighted composition operators on weighted Bergman space of the half-plane

Zhi-jie Jiang[1,2,*]

[1] School of Mathematics and Statistics, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, China

[2] South Sichuan Center for Applied Mathematics, Sichuan University of Science and Engineering, Zigong, Sichuan, 643000, China

* **Correspondence:** Email: matjzj@126.com.

Abstract: The main goal of this paper was to completely characterize complex symmetric difference of the weighted composition operators induced by three type symbols on weighted Bergman space of the right half-plane with the conjugations $\mathcal{J}f(z) = \overline{f(\bar{z})}$, $\mathcal{J}_s f(z) = \overline{f(\bar{z} + is)}$, and $\mathcal{J}_* f(z) = \frac{1}{z^{\alpha+2}}\overline{f(\frac{1}{\bar{z}})}$. The special phenomenon that we focus on is that the difference is complex symmetric on weighted Bergman spaces of the half-plane with the related conjugation if and only if each weighted composition operator is complex symmetric.

Keywords: weighted Bergman space of the right half-plane; reproducing kernel; weighted composition operator; complex symmetric difference of operators
Mathematics Subject Classification: Primary 47B38; Secondary 47B33, 47B37, 30H05

1. Introduction

As usual, let H denote a complex Hilbert space and $\mathcal{B}(H)$ the set of all bounded linear operators on H. Let $\bar{z}$ be the usual conjugation of complex number z, that is, if $z = x + iy$, then $\bar{z} = x - iy$. Let T^* denote the adjoint operator of T for $T \in \mathcal{B}(H)$.

We continuously introduce some notations. For a complex number z, let $\Re z$ denote the real part of z and $\Im z$ the imaginary part of z. Let $\mathbb{C}$ denote the complex plane, Π the right half-plane $\{z \in \mathbb{C} : \Re z > 0\}$, and $A^2_\alpha(\Pi)$ the weighted Bergman space on Π.

We first need the following definition.

Definition 1.1. *An operator $C : H \to H$ is said to be a conjugation if it satisfies the following conditions:*

(i) conjugate-linear: $C(\alpha x+\beta y)=\bar{\alpha}C(x)+\bar{\beta}C(y)$, *for* $\alpha,\beta\in\mathbb{C}$ *and* $x,y\in H$;

(ii) isometric: $\|C(x)\|=\|x\|$, *for all* $x\in H$;

(iii) involutive: $C^2=I_d$, *where* I_d *is an identity operator.*

One can see [7] for more information about conjugations. Actually, there exists many conjugations on $A^2_\alpha(\Pi)$ such as $\mathcal{J}f(z)=\overline{f(\bar{z})}$.

Definition 1.2. *Let* $C:H\to H$ *be a conjugation and* $T\in\mathcal{B}(H)$. *If* $CTC=T^*$, *then* T *is said to be complex symmetric with* C.

Complex symmetric operators on abstract Hilbert space were studied by Garcia, Putinar, and Wogen in [7–10]. Afterwards, one started to consider such operators on function spaces. For example, Noor et al. in [24] characterized the complex symmetric composition operators on Hardy space of the right half-plane. See also [5,11–13,18,19,22,23,30,33] for the studies of such operators on function spaces. After a long time of research, people find that many operators are complex symmetric operators such as normal operators, Hankel operators, and Volterra integration operators. Complex symmetric operators have been extensively used in theoretic and application aspects (see [6]).

Ptak et al. in [26] introduced an interesting class of operators named complex normal operators and proved that the class of the complex normal operators properly contains complex symmetric operators.

Definition 1.3. *Let* C *be a conjugation on* H *and* $T\in\mathcal{B}(H)$. *If* $C(T^*T)=(TT^*)C$, *then* T *is said to be complex normal with* C.

In the recent paper [31], Wang et al. studied the structure of complex normal operators and provided a refined polar decomposition of complex normal operators. Recently, Bhuia in [1] studied complex normal weighted composition operators on Fock space and provided some properties of complex normal weighted composition operators. A direct proof shows that complex symmetric operators are always complex normal. Thus, complex normality can be viewed as a generalized complex symmetry. Also, in [26], basic properties of complex normal operators are developed. In particular, special attention is paid to complex normal operators on finite dimensional spaces, L^2 type spaces, and the Hardy space H^2.

Because of the above-mentioned studies, we can try to study the complex symmetric or complex normal weighted composition operators on other analytic function spaces. Coincidentally, when we are considering such problems, we find that Hai et al. in [14] studied the following conjugations on $A^2_\alpha(\Pi)$

$$\mathcal{J}g(z)=\overline{g(\bar{z})},\quad \mathcal{J}_s g(z)=\overline{g(\bar{z}+is)},\quad \mathcal{J}_* f(z)=\frac{1}{z^{\alpha+2}}\overline{g\Big(\frac{1}{\bar{z}}\Big)},\tag{1.1}$$

where $s\in\mathbb{R}$. Since it is difficult to give a proper description of the adjoint for the weighted composition operators with the general symbols on $A^2_\alpha(\Pi)$, in [14] they just characterized the complex symmetric weighted composition operators with the symbols in (I)–(III) on $A^2_\alpha(\Pi)$. These symbols are defined as follows:

(I)

$$\tau(z)=\frac{1}{(z-c)^{\alpha+2}},\quad \phi(z)=-a-\frac{b}{z-c},$$

where coefficients satisfy

$$\begin{cases} \text{either } \Re a = \Im b = 0,\ \Re b < 0,\ \Re c \le 0, \\ \text{or } \Re a < 0 \le -\Re c + \frac{\Re b + |b|}{2\Re a}. \end{cases} \tag{1.2}$$

(II)

$$\tau(z) = \frac{\delta}{(z + \mu + i\eta)^{\alpha+2}}, \quad \phi(z) = \mu,$$

where coefficients satisfy $\delta \in \mathbb{C}$, $\mu \in \Pi$, and $\eta \in \mathbb{R}$.

(III)

$$\tau(z) = \lambda, \quad \phi(z) = z + \gamma,$$

where coefficients satisfy $\lambda \in \mathbb{C}$ and $\gamma \in \Pi$.

Next, we will provide the research motivations of this paper. With the basic questions such as boundedness and compactness settled, more attention has been paid to the study of the topological structure of the composition operators or weighted composition operators in the operator norm topology. In this research background, Shapiro and Sundberg in [27] posed a question on whether two composition operators belong to the same connected component, when their difference is compact. In the study of difference of composition operators, some interesting phenomena were found. For example, there is no compact composition operators on weighted Bergman space on the half-plane (see [20]), but there is compact difference of composition operators on this space (see [3]); two noncompact composition operators can induce compact difference of composition operators on weighted Bergman space on the unit disk (see [21]). Perhaps due to these interesting phenomena, people initiated the study of difference of composition operators or weighted composition operators, which has become a very active topic (see [16, 21, 28]).

Motivated by the above-mentioned studies, a natural problem is to characterize complex symmetric difference of composition operators or weighted composition operators on analytic function spaces. To this end, we try to consider this problem on weighted Bergman space $A^2_\alpha(\Pi)$ by using the weighted composition operators with symbols in (I)–(III). As we expected, we find that the difference of such weighted composition operators is complex symmetric on weighted Bergman space $A^2_\alpha(\Pi)$ with the conjugations in (1.1) if and only if each weighted composition operator is complex symmetric. This is an interesting phenomenon, but it may be not right for the general case, that is, from the complex symmetry of the operator $T = T_1 + T_2$, where $T_1, T_2 \in \mathcal{B}(H)$, it cannot deduce the complex symmetries of the operators T_1 and T_2. On the other hand, it is well known that there is no compact composition operators on the weighted Bergman space $A^2_\alpha(\Pi)$. Maybe for that reason, there isarded as an useful supplement of the weighted composition operators on $A^2_\alpha(\Pi)$.

2. Preliminaries

Throughout the paper, we always assume that α is a nonnegative integer, since for any $w, z \in \mathbb{C}$ and $\alpha > 0$, $(wz)^\alpha \neq w^\alpha z^\alpha$ while the equality holds when α is a nonnegative integer.

Let $H(\Pi)$ be the set of all analytic functions on Π, dA be the area measure on Π, and $dA_\alpha(z) = \frac{2^\alpha(\alpha+1)}{\pi}(\Re z)^\alpha dA(z)$. The weighted Bergman space $A^2_\alpha(\Pi)$ consists of all $f \in H(\Pi)$ such that

$$\|f\|^2_{A^2_\alpha(\Pi)} = \int_\Pi |f(z)|^2 dA_\alpha(z) < \infty.$$

Moreover, this norm is induced by the inner product

$$\langle f, g\rangle_{A^2_\alpha(\Pi)} = \int_\Pi f(z)\overline{g(z)} dA_\alpha(z).$$

$A^2_\alpha(\Pi)$ is a Hilbert space with this inner product, and the reproducing kernel is

$$K^\alpha_w(z) = \frac{2^\alpha(\alpha+1)}{(z+\overline{w})^{\alpha+2}}, \quad z \in \Pi.$$

That is,

$$f(z) = \langle f, K^\alpha_z\rangle_{A^2_\alpha(\Pi)} = \int_\Pi f(w)\overline{K^\alpha_z(w)} dA_\alpha(w)$$

for any $f \in A^2_\alpha(\Pi)$ and $z \in \Pi$. One can see [4] for more information on $A^2_\alpha(\Pi)$.

Let φ be an analytic self-mapping of Π and $u \in H(\Pi)$. The weighted composition operator induced by the symbols u and φ on or between some subspaces of $H(\Pi)$ is defined by

$$W_{u,\varphi} f(z) = u(z) f(\varphi(z)).$$

From the definition, it follows that when $u \equiv 1$, $W_{u,\varphi}$ is the composition operator, denoted by C_φ; when $\varphi(z) = z$, $W_{u,\varphi}$ is the multiplication operator, denoted by M_u.

It is an interesting topic to provide the characterizations of the symbols u and φ which induce bounded or compact weighted composition operators. Recently, several authors have studied the composition operators and weighted composition operators on weighted Bergman space of the half-plane. For example, Elliott et al. in [4] characterized the bounded composition operators and proved that no composition operator on the weighted Bergman space of the upper half-plane is compact. Sharma et al. in [29] characterized the bounded weighted composition operators on vector-valued weighted Bergman spaces of the upper half-plane. Readers can also find some relevant studies about the operators on the weighted Bergman spaces of the upper half-plane and we will not repeat them anymore.

3. Complex symmetric difference of weighted composition operators

The following result can be directly obtained by utilizing the denseness of the linear span of the functions $\{K^\alpha_w : w \in \Pi\}$ in $A^2_\alpha(\Pi)$.

Lemma 3.1. *Let T be a bounded operator on $A^2_\alpha(\Pi)$. Then, T is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation C if and only if*

$$(CT - T^*C)K^\alpha_w(z) = 0 \tag{3.1}$$

for all $w, z \in \Pi$.

To study the difference of the operators $W_{\tau,\phi}$ with the symbols in (I)–(III) on $A^2_\alpha(\Pi)$, we need the following result, where (a) was proved in [32].

Lemma 3.2. *(a) Let* $\tau(z) = \frac{1}{(z-c)^{\alpha+2}}$ *and* $\phi(z) = -a - \frac{b}{z-c}$ *be the symbols defined in (I). Then, on* $A^2_\alpha(\Pi)$ *the following holds*

$$W^*_{\tau,\phi} = W_{\frac{1}{(z-\bar{a})^{\alpha+2}}, \frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}}.$$

(b) Let $\tau(z) = \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ *and* $\phi(z) = \mu$ *be the symbols defined in (II). Then, on* $A^2_\alpha(\Pi)$ *the following holds*

$$W^*_{\tau,\phi} = \bar{\delta} W_{\frac{1}{(z+\bar{\mu})^{\alpha+2}}, \bar{\mu}-i\eta}.$$

(c) Let $\tau(z) = \lambda$ *and* $\phi(z) = z + \gamma$ *be the symbols defined in (III). Then, on* $A^2_\alpha(\Pi)$ *the following holds*

$$W^*_{\tau,\phi} = \bar{\lambda} C_{z+\bar{\gamma}}.$$

Proof. (b). From Lemma 3.2 in [32], we have

$$W^*_{\tau,\phi} = W_{\frac{\bar{\delta}}{(z+\bar{\mu})^{\alpha+2}}, \bar{\mu}-i\eta} = \bar{\delta} W_{\frac{1}{(z+\bar{\mu})^{\alpha+2}}, \bar{\mu}-i\eta}.$$

(c). The proof can be similarly obtained, so we do not provide proof anymore. □

First, we characterize the complex symmetric difference of the operator $W_{\tau,\phi}$ with the symbols in (I) on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$.

Theorem 3.1. *Let* $\tau_j(z) = \frac{1}{(z-c_j)^{\alpha+2}}$ *and* $\phi_j(z) = -a_j - \frac{b_j}{z-c_j}$ *be the symbols in (I) for* $j = 1, 2$. *Then, the operator* $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ *is complex symmetric on* $A^2_\alpha(\Pi)$ *with the conjugation* $\mathcal{J}$ *if and only if* $a_1 = c_1$ *and* $a_2 = c_2$.

Proof. For all $w, z \in \Pi$, from Lemma 3.2 (a) the following equalities hold

$$\begin{aligned}
&\mathcal{J}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z) \\
&= \mathcal{J}\Big(\frac{1}{(z-c_1)^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{(-a_1 - \frac{b_1}{z-c_1} + \overline{w})^{\alpha+2}} - \frac{1}{(z-c_2)^{\alpha+2}} \frac{2^\alpha(\alpha+1)}{(-a_2 - \frac{b_2}{z-c_2} + \overline{w})^{\alpha+2}}\Big) \\
&= \mathcal{J}\Big(\frac{2^\alpha(\alpha+1)}{[(\overline{w}-a_1)z + a_1c_1 - b_1 - c_1\overline{w}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(\overline{w}-a_2)z + a_2c_2 - b_2 - c_2\overline{w}]^{\alpha+2}}\Big) \\
&= \frac{2^\alpha(\alpha+1)}{[(w-\bar{a}_1)z + \bar{a}_1\bar{c}_1 - \bar{b}_1 - \bar{c}_1 w]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(w-\bar{a}_2)z + \bar{a}_2\bar{c}_2 - \bar{b}_2 - \bar{c}_2 w]^{\alpha+2}}
\end{aligned} \tag{3.2}$$

and

$$(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^* \mathcal{J} K^\alpha_w(z)$$

$$= (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\Big(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}}\Big)$$
$$= \Big(W_{\frac{1}{(z-\bar{a}_1)^{\alpha+2}},\frac{\bar{a}_1\bar{c}_1-\bar{b}_1-\bar{c}_1 z}{z-\bar{a}_1}} - W_{\frac{1}{(z-\bar{a}_2)^{\alpha+2}},\frac{\bar{a}_2\bar{c}_2-\bar{b}_2-\bar{c}_2 z}{z-\bar{a}_2}}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}}\Big)$$
$$= \frac{2^\alpha(\alpha+1)}{[(w-\bar{c}_1)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(w-\bar{c}_2)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}}. \tag{3.3}$$

Hence, from (3.2), (3.3) and Lemma 3.1, it follows that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$ if and only if

$$\frac{1}{[(w-\bar{a}_1)z-\bar{c}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}} - \frac{1}{[(w-\bar{a}_2)z-\bar{c}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}}$$
$$= \frac{1}{[(w-\bar{c}_1)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}} - \frac{1}{[(w-\bar{c}_2)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}} \tag{3.4}$$

for all $w, z \in \Pi$.

Assume that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$. Then, from (3.4) we obtain

$$\frac{1}{[(w-\bar{a}_1)z-\bar{c}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}} - \frac{1}{[(w-\bar{c}_1)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}}$$
$$= \frac{1}{[(w-\bar{a}_2)z-\bar{c}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}} - \frac{1}{[(w-\bar{c}_2)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}} \tag{3.5}$$

for all $w, z \in \Pi$. From the formula

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}),$$

we have

$$\frac{(\bar{a}_1-\bar{c}_1)(z-w)\{[w-\bar{a}_1)z-\bar{c}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+1} + \cdots + [(w-\bar{c}_1)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+1}\}}{[(w-\bar{a}_1)z-\bar{c}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}[(w-\bar{c}_1)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}}$$
$$= \frac{(\bar{a}_2-\bar{c}_2)(z-w)\{[w-\bar{a}_2)z-\bar{c}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+1} + \cdots + [(w-\bar{c}_2)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+1}\}}{[(w-\bar{a}_2)z-\bar{c}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}[(w-\bar{c}_2)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}} \tag{3.6}$$

for all $w, z \in \Pi$. Clearly, if $a_2 \neq c_2$, then from (3.6) we have

$$\frac{\bar{a}_1-\bar{c}_1}{\bar{a}_2-\bar{c}_2} \equiv \frac{[(w-\bar{a}_2)z-\bar{c}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+1} + \cdots + [(w-\bar{c}_2)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+1}}{[w-\bar{a}_1)z-\bar{c}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+1} + \cdots + [(w-\bar{c}_1)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+1}}$$
$$\times \frac{[(w-\bar{a}_1)z-\bar{c}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}[(w-\bar{c}_1)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}}{[(w-\bar{a}_2)z-\bar{c}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}[(w-\bar{c}_2)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}} \tag{3.7}$$

for all $w, z \in \Pi$ and $w \neq z$. So, from the arbitrariness of w and z in (3.7), we deduce a contradiction. Then, we obtain $a_2 = c_2$. Similarly, we have $a_1 = c_1$.

Conversely, if $a_1 = c_1$ and $a_2 = c_2$, then it is easy to see that (3.4) holds. This shows that $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$. □

We have the following result, and we do not provide proof anymore.

Lemma 3.3. *Let $\tau(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\phi(z) = -a - \frac{b}{z-c}$ be the symbols in (I). Then, the operator $W_{\tau,\phi}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$ if and only if $a = c$.*

Remark 3.1. From Lemma 3.3 and Theorem 3.1, we see that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$ if and only if both W_{τ_1,ϕ_1} and W_{τ_2,ϕ_2} are complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$.

Example 3.1. Let $\tau_1(z) = \frac{1}{(z-i)^{\alpha+2}}$, $\phi_1(z) = -i + \frac{1}{z-i}$, $\tau_2(z) = \frac{1}{(z+1-i)^{\alpha+2}}$, and $\phi_2(z) = 1 - i - \frac{1+i}{2z+2-2i}$ be the symbols in (I). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$.

Proof. From the form of the symbols in (I), it follows that $a_1 = i$, $b_1 = -1$, $c_1 = i$, $a_2 = -1 + i$, $b_2 = \frac{1}{2} + \frac{i}{2}$ and $c_2 = -1 + i$. Then, from Theorem 3.1, the desired result follows. □

Now, we characterize the complex symmetry of the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s f(z) = \overline{f(\bar{z} + is)}$.

Theorem 3.2. *Let $\tau_j(z) = \frac{1}{(z-c_j)^{\alpha+2}}$ and $\phi_j(z) = -a_j - \frac{b_j}{z-c_j}$ be the symbols in (I) for $j = 1, 2$. Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$ if and only if $a_1 = c_1 - is$ and $a_2 = c_2 - is$.*

Proof. For all $w, z \in \Pi$, from Lemma 3.2 (a) the following equalities hold

$$
\begin{aligned}
&\mathcal{J}_s(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z)\\
&= \mathcal{J}_s\Big(\frac{1}{(z-c_1)^{\alpha+2}}\frac{2^\alpha(\alpha+1)}{(-a_1 - \frac{b_1}{z-c_1} + \overline{w})^{\alpha+2}} - \frac{1}{(z-c_2)^{\alpha+2}}\frac{2^\alpha(\alpha+1)}{(-a_2 - \frac{b_2}{z-c_2} + \overline{w})^{\alpha+2}}\Big)\\
&= \mathcal{J}_s\Big(\frac{2^\alpha(\alpha+1)}{[(\overline{w} - a_1)z + a_1c_1 - b_1 - c_1\overline{w}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(\overline{w} - a_2)z + a_2c_2 - b_2 - c_2\overline{w}]^{\alpha+2}}\Big)\\
&= \frac{2^\alpha(\alpha+1)}{[(w-\bar{a}_1)(z-is) + \bar{a}_1\bar{c}_1 - \bar{b}_1 - \bar{c}_1w]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(w-\bar{a}_2)(z-is) + \bar{a}_2\bar{c}_2 - \bar{b}_2 - \bar{c}_2w]^{\alpha+2}}\\
&= \frac{2^\alpha(\alpha+1)}{[(w-\bar{a}_1)z - (is+\bar{c}_1)w + i\bar{a}_1s + \bar{a}_1\bar{c}_1 - \bar{b}_1]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(w-\bar{a}_2)z - (is+\bar{c}_2)w + i\bar{a}_2s + \bar{a}_2\bar{c}_2 - \bar{b}_2]^{\alpha+2}}
\end{aligned}
\tag{3.8}
$$

and

$$
\begin{aligned}
&(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\mathcal{J}_sK^\alpha_w(z)\\
&= (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\Big(\frac{2^\alpha(\alpha+1)}{(z-is+w)^{\alpha+2}}\Big)\\
&= \Big(W_{\frac{1}{(z-\bar{a}_1)^{\alpha+2}},\frac{\bar{a}_1\bar{c}_1-\bar{b}_1-\bar{c}_1z}{z-\bar{a}_1}} - W_{\frac{1}{(z-\bar{a}_2)^{\alpha+2}},\frac{\bar{a}_2\bar{c}_2-\bar{b}_2-\bar{c}_2z}{z-\bar{a}_2}}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(z-is+w)^{\alpha+2}}\Big)\\
&= \frac{2^\alpha(\alpha+1)}{[(w-\bar{c}_1-is)z - \bar{a}_1w + \bar{a}_1\bar{c}_1 - \bar{b}_1 + is\bar{a}_1]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(w-\bar{c}_2-is)z - \bar{a}_2w + \bar{a}_2\bar{c}_2 - \bar{b}_2 + is\bar{a}_2]^{\alpha+2}}.
\end{aligned}
\tag{3.9}
$$

Hence, from (3.8), (3.9), and Lemma 3.1, we have that the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$ if and only if

$$\begin{aligned}
&\frac{1}{[(w-\bar{a}_1)z-(is+\bar{c}_1)w+i\bar{a}_1 s+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}}-\frac{1}{[(w-\bar{a}_2)z-(is+\bar{c}_2)w+i\bar{a}_2 s+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}}\\
&=\frac{1}{[(w-\bar{c}_1-is)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1+is\bar{a}_1]^{\alpha+2}}-\frac{1}{[(w-\bar{c}_2-is)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2+is\bar{a}_2]^{\alpha+2}} \qquad (3.10)
\end{aligned}$$

for all $w,z\in\Pi$.

Assume that the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$. Then, from (3.10) we obtain

$$\begin{aligned}
&\frac{1}{[(w-\bar{a}_1)z-(is+\bar{c}_1)w+i\bar{a}_1 s+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}}-\frac{1}{[(w-\bar{c}_1-is)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1+is\bar{a}_1]^{\alpha+2}}\\
&=\frac{1}{[(w-\bar{a}_2)z-(is+\bar{c}_2)w+i\bar{a}_2 s+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}}-\frac{1}{[(w-\bar{c}_2-is)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2+is\bar{a}_2]^{\alpha+2}} \qquad (3.11)
\end{aligned}$$

for all $w,z\in\Pi$. Also, applying the following formula in (3.11)

$$x^n-y^n=(x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}),$$

if $a_2\neq c_2-is$, then

$$\begin{aligned}
&\frac{[(w-\bar{a}_2)z-(is+\bar{c}_2)w+i\bar{a}_2 s+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+1}+\cdots+[(w-\bar{c}_2-is)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2+is\bar{a}_2]^{\alpha+1}}{[(w-\bar{a}_1)z-(is+\bar{c}_1)w+i\bar{a}_1 s+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+1}+\cdots+[(w-\bar{c}_1-is)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1+is\bar{a}_1]^{\alpha+1}}\\
&\times\frac{[(w-\bar{a}_1)z-(is+\bar{c}_1)w+i\bar{a}_1 s+\bar{a}_1\bar{c}_1-\bar{b}_1]^{\alpha+2}[(w-\bar{c}_1-is)z-\bar{a}_1 w+\bar{a}_1\bar{c}_1-\bar{b}_1+is\bar{a}_1]^{\alpha+2}}{[(w-\bar{a}_2)z-(is+\bar{c}_2)w+i\bar{a}_2 s+\bar{a}_2\bar{c}_2-\bar{b}_2]^{\alpha+2}[(w-\bar{c}_2-is)z-\bar{a}_2 w+\bar{a}_2\bar{c}_2-\bar{b}_2+is\bar{a}_2]^{\alpha+2}}\\
&\equiv\frac{\bar{c}_1-\bar{a}_1+is}{\bar{c}_2-\bar{a}_2+is} \qquad (3.12)
\end{aligned}$$

for all $w,z\in\Pi$ with $w\neq z$. So, from the arbitrariness of w and z in (3.12), we deduce a contradiction. Then, we obtain $a_2=c_2-is$. Similarly, we have $a_1=c_1-is$.

Conversely, if $a_1=c_1-is$ and $a_2=c_2-is$, then we see that (3.10) holds, which shows that $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$. □

Remark 3.2. If $\tau(z)=\frac{1}{(z-c)^{\alpha+2}}$ and $\phi(z)=-a-\frac{b}{z-c}$ are the symbols in (I), then the operator $W_{\tau,\phi}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$ if and only if $a=c-is$. Combining Theorem 3.2, we prove that if $\tau_j(z)=\frac{1}{(z-c_j)^{\alpha+2}}$ and $\phi_j(z)=-a_j-\frac{b_j}{z-c_j}$ are the symbols in (I) for $j=1,2$, then the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$ if and only if both W_{τ_1,ϕ_1} and W_{τ_2,ϕ_2} are complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$.

Example 3.2. Let $\tau_1(z)=\frac{1}{[z-(1+s)i]^{\alpha+2}}$, $\phi_1(z)=-i+\frac{1}{z-(1+s)i}$, $\tau_2(z)=\frac{1}{[z+1-(1+s)i]^{\alpha+2}}$, and $\phi_2(z)=1-i-\frac{1+i}{2z+2-2(1+s)i}$ be the symbols in (I). Then, the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$.

Proof. From the form of the symbols in (I), it follows that $a_1=i$, $b_1=-1$, $c_1=(1+s)i$, $a_2=-1+i$, $b_2=\frac{1}{2}+\frac{i}{2}$, and $c_2=-1+(1+s)i$. Then, from Theorem 3.2, the desired result follows. □

Now, we characterize the complex symmetric operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ induced by the symbols in (I) on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_* f(z) = \frac{1}{z^{\alpha+2}}\overline{f(\frac{1}{\bar{z}})}$.

Theorem 3.3. *Let $\tau_j(z) = \frac{1}{(z-c_j)^{\alpha+2}}$ and $\phi_j(z) = -a_j - \frac{b_j}{z-c_j}$ be the symbols in (I) for $j = 1, 2$. Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if $a_1c_1 - b_1 = 1$ and $a_2c_2 - b_2 = 1$.*

Proof. For all $w, z \in \Pi$, from Lemma 3.2 (a) it follows that

$$\begin{aligned}
&\mathcal{J}_*(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z)\\
&= \mathcal{J}_*\Big(\frac{1}{(z-c_1)^{\alpha+2}}\frac{2^\alpha(\alpha+1)}{(-a_1 - \frac{b_1}{z-c_1} + \overline{w})^{\alpha+2}} - \frac{1}{(z-c_2)^{\alpha+2}}\frac{2^\alpha(\alpha+1)}{(-a_2 - \frac{b_2}{z-c_2} + \overline{w})^{\alpha+2}}\Big)\\
&= \mathcal{J}_*\Big(\frac{2^\alpha(\alpha+1)}{[(\overline{w} - a_1)z + a_1c_1 - b_1 - c_1\overline{w}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(\overline{w} - a_2)z + a_2c_2 - b_2 - c_2\overline{w}]^{\alpha+2}}\Big)\\
&= \frac{2^\alpha(\alpha+1)}{[(\bar{a}_1\bar{c}_1 - \bar{b}_1)z - \bar{c}_1wz + w - \bar{a}_1]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[(\bar{a}_2\bar{c}_2 - \bar{b}_2)z - \bar{c}_2wz + w - \bar{a}_2]^{\alpha+2}}
\end{aligned}\tag{3.13}$$

and

$$\begin{aligned}
&(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\mathcal{J}_*K^\alpha_w(z) = (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big)\\
&= \Big(W_{\frac{1}{(z-\bar{a}_1)^{\alpha+2}},\frac{\bar{a}_1\bar{c}_1-\bar{b}_1-\bar{c}_1z}{z-\bar{a}_1}} - W_{\frac{1}{(z-\bar{a}_2)^{\alpha+2}},\frac{\bar{a}_2\bar{c}_2-\bar{b}_2-\bar{c}_2z}{z-\bar{a}_2}}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big)\\
&= \frac{2^\alpha(\alpha+1)}{[z - \bar{c}_1wz + (\bar{a}_1\bar{c}_1 - \bar{b}_1)w - \bar{a}_1]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)}{[z - \bar{c}_2wz + (\bar{a}_2\bar{c}_2 - \bar{b}_2)w - \bar{a}_2]^{\alpha+2}}.
\end{aligned}\tag{3.14}$$

Then, from (3.13), (3.14), and Lemma 3.1, it follows that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if

$$\begin{aligned}
&\frac{1}{[(\bar{a}_1\bar{c}_1 - \bar{b}_1)z - \bar{c}_1wz + w - \bar{a}_1]^{\alpha+2}} - \frac{1}{[(\bar{a}_2\bar{c}_2 - \bar{b}_2)z - \bar{c}_2wz + w - \bar{a}_2]^{\alpha+2}}\\
&= \frac{1}{[z - \bar{c}_1wz + (\bar{a}_1\bar{c}_1 - \bar{b}_1)w - \bar{a}_1]^{\alpha+2}} - \frac{1}{[z - \bar{c}_2wz + (\bar{a}_2\bar{c}_2 - \bar{b}_2)w - \bar{a}_2]^{\alpha+2}}
\end{aligned}\tag{3.15}$$

for all $w, z \in \Pi$. Clearly, (3.15) is equivalent to

$$\begin{aligned}
&\frac{1}{[(\bar{a}_1\bar{c}_1 - \bar{b}_1)z - \bar{c}_1wz + w - \bar{a}_1]^{\alpha+2}} - \frac{1}{[z - \bar{c}_1wz + (\bar{a}_1\bar{c}_1 - \bar{b}_1)w - \bar{a}_1]^{\alpha+2}}\\
&= \frac{1}{[(\bar{a}_2\bar{c}_2 - \bar{b}_2)z - \bar{c}_2wz + w - \bar{a}_2]^{\alpha+2}} - \frac{1}{[z - \bar{c}_2wz + (\bar{a}_2\bar{c}_2 - \bar{b}_2)w - \bar{a}_2]^{\alpha+2}}
\end{aligned}$$

for all $w, z \in \Pi$.

Now, assume that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$. Using the same method in the proof of Theorem 3.1, if $a_2c_2 - b_2 \neq 1$, then

$$\frac{\bar{a}_1\bar{c}_1 - \bar{b}_1 - 1}{\bar{a}_2\bar{c}_2 - \bar{b}_2 - 1} \equiv \frac{[(\bar{a}_2\bar{c}_2 - \bar{b}_2)z - \bar{c}_2wz + w - \bar{a}_2]^{\alpha+1} + \cdots + [z - \bar{c}_2wz + (\bar{a}_2\bar{c}_2 - \bar{b}_2)w - \bar{a}_2]^{\alpha+1}}{[(\bar{a}_1\bar{c}_1 - \bar{b}_1)z - \bar{c}_1wz + w - \bar{a}_1]^{\alpha+1} + \cdots + [z - \bar{c}_1wz + (\bar{a}_1\bar{c}_1 - \bar{b}_1)w - \bar{a}_1]^{\alpha+1}}$$

$$\times \frac{[(\bar{a}_1\bar{c}_1 - \bar{b}_1)z - \bar{c}_1 wz + w - \bar{a}_1]^{\alpha+2}[z - \bar{c}_1 wz + (\bar{a}_1\bar{c}_1 - \bar{b}_1)w - \bar{a}_1]^{\alpha+2}}{[(\bar{a}_2\bar{c}_2 - \bar{b}_2)z - \bar{c}_2 wz + w - \bar{a}_2]^{\alpha+2}[z - \bar{c}_2 wz + (\bar{a}_2\bar{c}_2 - \bar{b}_2)w - \bar{a}_2]^{\alpha+2}} \tag{3.16}$$

for all $w, z \in \Pi$ and $w \neq z$. Then, from the arbitrariness of w and z in (3.16), we deduce a contradiction. So, we obtain that $a_2c_2 - b_2 = 1$. Similarly, we also have that $a_1c_1 - b_1 = 1$.

Conversely, assume that $a_1c_1 - b_1 = 1$ and $a_2c_2 - b_2 = 1$. It is clear that (3.15) holds. This shows that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$. □

Remark 3.3. If $\tau(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\phi(z) = -a - \frac{b}{z-c}$ are the symbols in (I), then the operator $W_{\tau,\phi}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if $ac - b = 1$. Therefore, from Theorem 3.3, if $\tau_j(z) = \frac{1}{(z-c_j)^{\alpha+2}}$ and $\phi_j(z) = -a_j - \frac{b_j}{z-c_j}$ are the symbols in (I) for $j = 1, 2$, then the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if both W_{τ_1,ϕ_1} and W_{τ_2,ϕ_2} are complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$.

Example 3.3. Let $\tau_1(z) = \frac{1}{(z-i)^{\alpha+2}}$, $\phi_1(z) = -i + \frac{2}{z-i}$, $\tau_2(z) = \frac{1}{(z+1+i)^{\alpha+2}}$ and $\phi_2(z) = 1 - i - \frac{1}{z+1+i}$ be the symbols in (I). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$.

Proof. From the form of the symbols in (I), it follows that $a_1 = i$, $b_1 = -2$, $c_1 = i$, $a_2 = -1 + i$, $b_2 = 1$, and $c_2 = -1 - i$. From the calculations, we have $a_1c_1 - b_1 = 1$ and $a_2c_2 - b_2 = 1$. Then, from Theorem 3.3, the desired result follows. □

Next, we characterize the complex symmetric operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ induced by the symbols in (II) on $A^2_\alpha(\Pi)$. First, Lemma 3.2 (b) tells us that

$$W^*_{\tau,\phi} = W_{\frac{\bar{\delta}}{(z+\bar{\mu})^{\alpha+2}}, \bar{\mu} - i\eta} = \bar{\delta} W_{\frac{1}{(z+\bar{\mu})^{\alpha+2}}, \bar{\mu} - i\eta},$$

which shows that if $a = -\mu$, $b = 0$ and $c = -\mu - i\eta$, then

$$W^*_{\tau,\phi} = \bar{\delta} W_{\frac{1}{(z-\bar{a})^{\alpha+2}}, \frac{\bar{a}\bar{c} - \bar{b} - \bar{c}z}{z - \bar{a}}}.$$

Therefore, we can directly obtain the following several results.

Theorem 3.4. *Let $\tau_j(z) = \frac{\delta_j}{(z+\mu_j+i\eta_j)^{\alpha+2}}$ and $\phi_j(z) = \mu_j$ be the symbols in (II) for $j = 1, 2$. Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$ if and only if $\eta_1 = \eta_2 = 0$.*

Theorem 3.5. *Let $\tau_j(z) = \frac{\delta_j}{(z+\mu_j+i\eta_j)^{\alpha+2}}$ and $\phi_j(z) = \mu_j$ be the symbols in (II) for $j = 1, 2$. Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$ if and only if $\eta_1 = \eta_2 = -s$.*

For the conjugation $\mathcal{J}_*$, we assume that $\delta \neq 0$. Otherwise, it is trivial.

Theorem 3.6. *Let $\tau_j(z) = \frac{\delta_j}{(z+\mu_j+i\eta_j)^{\alpha+2}}$ and $\phi_j(z) = \mu_j$ be the symbols in (II) for $j = 1, 2$. Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if*

$$\mu_j = \sqrt{1 - \frac{\eta_j^2}{4}} - i\frac{\eta_j}{2}$$

with $\eta_j \in (-2, 2)$, $j = 1, 2$.

Remark 3.4. For the symbols in (II), although we do not give the proofs, we still see that the operators $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ are complex symmetric on $A^2_\alpha(\Pi)$ with the conjugations $\mathcal{J}$, $\mathcal{J}_s$, and $\mathcal{J}_*$ if and only if both W_{τ_1,ϕ_1} and W_{τ_2,ϕ_2} are complex symmetric on $A^2_\alpha(\Pi)$ with the conjugations $\mathcal{J}$, $\mathcal{J}_s$, and $\mathcal{J}_*$.

Example 3.4. Let $\tau_1(z) = \frac{1}{(z+\frac{\sqrt{3}}{2}+\frac{i}{2})^{\alpha+2}}$, $\phi_1(z) = \frac{\sqrt{3}}{2} - \frac{i}{2}$, $\tau_2(z) = \frac{1}{(z+\frac{\sqrt{15}}{4}+\frac{i}{4})^{\alpha+2}}$ and $\phi_2(z) = \frac{\sqrt{15}}{4} - \frac{i}{4}$ be the symbols in (II). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$.

Proof. From the form of the symbols in (II), it follows that $\mu_1 = \frac{\sqrt{3}}{2} - \frac{i}{2}$, $\eta_1 = 1$, $\mu_2 = \frac{\sqrt{15}}{4} - \frac{i}{4}$, and $\eta_2 = \frac{1}{2}$. Then, from Theorem 3.6, the desired result follows. □

Now, we discuss complex the symmetric operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ induced by the symbols in (III) on $A^2_\alpha(\Pi)$.

Theorem 3.7. *Let $\tau_j(z) = \lambda_j$ and $\phi_j(z) = z + \gamma_j$ be the symbols in (III) for $j = 1, 2$. Then, $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$.*

Proof. For all $w, z \in \Pi$, from Lemma 3.2 (c) we have

$$\begin{aligned}\mathcal{J}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z) &= \mathcal{J}\Big(\lambda_1 \frac{2^\alpha(\alpha+1)}{(z+\gamma_1+\overline{w})^{\alpha+2}} - \lambda_2 \frac{2^\alpha(\alpha+1)}{(z+\gamma_2+\overline{w})^{\alpha+2}}\Big)\\ &= \bar{\lambda}_1 \frac{2^\alpha(\alpha+1)}{(z+\bar{\gamma}_1+w)^{\alpha+2}} - \bar{\lambda}_2 \frac{2^\alpha(\alpha+1)}{(z+\bar{\gamma}_2+w)^{\alpha+2}}\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^* \mathcal{J} K^\alpha_w(z) &= (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^* \Big(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}}\Big)\\ &= \Big(\bar{\lambda}_1 C_{z+\bar{\gamma}_1} - \bar{\lambda}_2 C_{z+\bar{\gamma}_2}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}}\Big)\\ &= \bar{\lambda}_1 \frac{2^\alpha(\alpha+1)}{(z+\bar{\gamma}_1+w)^{\alpha+2}} - \bar{\lambda}_2 \frac{2^\alpha(\alpha+1)}{(z+\bar{\gamma}_2+w)^{\alpha+2}}.\end{aligned} \tag{3.18}$$

From (3.17) and (3.18), it follows that

$$\mathcal{J}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z) = (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^* \mathcal{J} K^\alpha_w(z)$$

for all $z \in \Pi$. The proof is completed. □

Of course, the following result is true. The proof is omitted.

Theorem 3.8. *Let $\tau_j(z) = \lambda_j$ and $\phi_j(z) = z + \gamma_j$ be the symbols in (III) for $j = 1, 2$. Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_s$.*

However, the result on the conjugation $\mathcal{J}_*$ is trivial since there exists the following theorem. Here, assume that $\lambda_j \neq 0$ for $j = 1, 2$. Otherwise, $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is a null operator.

Theorem 3.9. *Let $\tau_j(z) = \lambda_j$ and $\phi_j(z) = z + \gamma_j$ be the symbols in (III) for $j = 1, 2$. Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if $\gamma_1 = \gamma_2 = 0$.*

Proof. For all $w, z \in \Pi$, from Lemma 3.2 (c) it follows that

$$\begin{aligned}\mathcal{J}_*(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K_w^\alpha(z) &= \mathcal{J}_*\Big(\lambda_1 \frac{2^\alpha(\alpha+1)}{(z+\gamma_1+\overline{w})^{\alpha+2}} - \lambda_2 \frac{2^\alpha(\alpha+1)}{(z+\gamma_2+\overline{w})^{\alpha+2}}\Big)\\ &= \bar{\lambda}_1 \frac{2^\alpha(\alpha+1)}{(1+wz+\bar{\gamma}_1 z)^{\alpha+2}} - \bar{\lambda}_2 \frac{2^\alpha(\alpha+1)}{(1+wz+\bar{\gamma}_2 z)^{\alpha+2}} \end{aligned} \tag{3.19}$$

and

$$\begin{aligned}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^* \mathcal{J}_* K_w^\alpha(z) &= (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^* \Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big)\\ &= \Big(\bar{\lambda}_1 C_{z+\bar{\gamma}_1} - \bar{\lambda}_2 C_{z+\bar{\gamma}_2}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big)\\ &= \bar{\lambda}_1 \frac{2^\alpha(\alpha+1)}{(1+wz+\bar{\gamma}_1 w)^{\alpha+2}} - \bar{\lambda}_2 \frac{2^\alpha(\alpha+1)}{(1+wz+\bar{\gamma}_2 w)^{\alpha+2}}. \end{aligned} \tag{3.20}$$

Then, from (3.19), (3.20), and Lemma 3.1, it follows that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if

$$\bar{\lambda}_1\Big[\frac{1}{(1+wz+\bar{\gamma}_1 z)^{\alpha+2}} - \frac{1}{(1+wz+\bar{\gamma}_1 w)^{\alpha+2}}\Big] = \bar{\lambda}_2\Big[\frac{1}{(1+wz+\bar{\gamma}_2 z)^{\alpha+2}} - \frac{1}{(1+wz+\bar{\gamma}_2 w)^{\alpha+2}}\Big] \tag{3.21}$$

for all $w, z \in \Pi$.

Now, assume that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_*$. Using the above-mentioned method in the proof of Theorem 3.1, if $\gamma_2 \neq 0$, then

$$\frac{\bar{\lambda}_1\bar{\gamma}_1}{\bar{\lambda}_2\bar{\gamma}_2} \equiv \frac{(1+wz+\bar{\gamma}_2 w)^{\alpha+1} + \cdots + (1+wz+\bar{\gamma}_2 z)^{\alpha+1}}{(1+wz+\bar{\gamma}_1 w)^{\alpha+1} + \cdots + (1+wz+\bar{\gamma}_1 z)^{\alpha+1}} \frac{(1+wz+\bar{\gamma}_1 z)^{\alpha+2}(1+wz+\bar{\gamma}_1 w)^{\alpha+2}}{(1+wz+\bar{\gamma}_2 z)^{\alpha+2}(1+wz+\bar{\gamma}_2 w)^{\alpha+2}} \tag{3.22}$$

for all $w, z \in \Pi$ and $w \neq z$. Then, from the arbitrariness of w and z in (3.22), we deduce a contradiction. So, we obtain that $\gamma_2 = 0$. Similarly, we also have that $\gamma_1 = 0$.

Conversely, assume that $\gamma_1 = \gamma_2 = 0$. It is clear that (3.21) holds. This shows that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_*$. □

4. Complex symmetric difference of mixed types

In this section, we first consider complex symmetric difference induced by the symbols in (I) and (II). Assume that $\delta \neq 0$. Otherwise, $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2} = W_{\tau_1,\phi_1}$ in Theorem 4.1, whose complex symmetry has been studied in Lemma 3.3.

Theorem 4.1. *Let $\tau_1(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\phi_1(z) = -a - \frac{b}{z-c}$ be the symbols in (I), $\tau_2(z) = \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ and $\phi_2(z) = \mu$ the symbols in (II). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}$ if and only if $a = c$ and $\eta = 0$.*

Proof. For all $w, z \in \Pi$, it follows from Lemma 3.2 (a) and (b) that

$$\mathcal{J}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K_w^\alpha(z)$$

$$= \mathcal{J}\Big(\frac{1}{(z-c)^{\alpha+2}}\frac{2^{\alpha}(\alpha+1)}{(-a-\frac{b}{z-c}+\overline{w})^{\alpha+2}} - \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}\frac{2^{\alpha}(\alpha+1)}{(\overline{w}+\mu)^{\alpha+2}}\Big)$$
$$= \mathcal{J}\Big(\frac{2^{\alpha}(\alpha+1)}{[(\overline{w}-a)z+ac-b-c\overline{w}]^{\alpha+2}} - \frac{2^{\alpha}(\alpha+1)\delta}{[(\overline{w}+\mu)z+(\mu+i\eta)\overline{w}+\mu^2+(i\eta)\mu]^{\alpha+2}}\Big)$$
$$= \frac{2^{\alpha}(\alpha+1)}{[(w-\bar{a})z+\bar{a}\bar{c}-\bar{b}-\bar{c}w]^{\alpha+2}} - \frac{2^{\alpha}(\alpha+1)\bar{\delta}}{[(w+\bar{\mu})z+(\bar{\mu}-i\eta)w+\bar{\mu}^2-(i\eta)\bar{\mu}]^{\alpha+2}} \tag{4.1}$$

and

$$(W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2})^{*}\mathcal{J}K_w^{\alpha}(z) = (W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2})^{*}\Big(\frac{2^{\alpha}(\alpha+1)}{(z+w)^{\alpha+2}}\Big)$$
$$= \Big(W_{\frac{1}{(z-\bar{a})^{\alpha+2}},\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} - \bar{\delta}W_{\frac{1}{(z+\bar{\mu})^{\alpha+2}},\bar{\mu}-i\eta}\Big)\Big(\frac{2^{\alpha}(\alpha+1)}{(z+w)^{\alpha+2}}\Big)$$
$$= \frac{2^{\alpha}(\alpha+1)}{[(w-\bar{c})z-\bar{a}w+\bar{a}\bar{c}-\bar{b}]^{\alpha+2}} - \frac{2^{\alpha}(\alpha+1)\bar{\delta}}{[(\bar{\mu}-i\eta+w)z+\bar{\mu}w+\bar{\mu}^2-(i\eta)\bar{\mu}]^{\alpha+2}}. \tag{4.2}$$

Therefore, by Lemma 3.1, the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A_{\alpha}^2(\Pi)$ with the conjugation $\mathcal{J}$ if and only if

$$\frac{1}{[(w-\bar{a})z+\bar{a}\bar{c}-\bar{b}-\bar{c}w]^{\alpha+2}} - \frac{1}{[(w-\bar{c})z-\bar{a}w+\bar{a}\bar{c}-\bar{b}]^{\alpha+2}}$$
$$= \frac{\bar{\delta}}{[(w+\bar{\mu})z+(\bar{\mu}-i\eta)w+\bar{\mu}^2-i\eta\bar{\mu}]^{\alpha+2}} - \frac{\bar{\delta}}{[(\bar{\mu}-i\eta+w)z+\bar{\mu}w+\bar{\mu}^2-i\eta\bar{\mu}]^{\alpha+2}} \tag{4.3}$$

for all $w, z \in \Pi$.

Assume that the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A_{\alpha}^2(\Pi)$ with the conjugation $\mathcal{J}$. Then, by the formula

$$x^n - y^n = (x-y)(x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}),$$

(4.3) becomes

$$\frac{(\bar{a}-\bar{c})(z-w)\{[(w-\bar{a})z+\bar{a}\bar{c}-\bar{b}-\bar{c}w]^{\alpha+1}+\cdots+[(w-\bar{c})z-\bar{a}w+\bar{a}\bar{c}-\bar{b}]^{\alpha+1}\}}{[(w-\bar{a})z+\bar{a}\bar{c}-\bar{b}-\bar{c}w]^{\alpha+2}[(w-\bar{c})z-\bar{a}w+\bar{a}\bar{c}-\bar{b}]^{\alpha+2}}$$
$$= \frac{i\eta\bar{\delta}(w-z)\{[(w+\bar{\mu})z+(\bar{\mu}-i\eta)w+\bar{\mu}^2-i\eta\bar{\mu}]^{\alpha+1}+\cdots+[(\bar{\mu}-i\eta+w)z+\bar{\mu}w+\bar{\mu}^2-i\eta\bar{\mu}]^{\alpha+1}\}}{[(w+\bar{\mu})z+(\bar{\mu}-i\eta)w+\bar{\mu}^2-i\eta\bar{\mu}]^{\alpha+2}[(\bar{\mu}-i\eta+w)z+\bar{\mu}w+\bar{\mu}^2-i\eta\bar{\mu}]^{\alpha+2}} \tag{4.4}$$

for all $w, z \in \Pi$. Similar to the proof of Theorem 3.1, we see that if $a \neq c$, then we can deduce a contradiction. Similarly, we can obtain $\eta = 0$.

Conversely, if $a = c$ and $\eta = 0$, then it is easy to see that (4.3) holds. By Lemma 3.1, $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A_{\alpha}^2(\Pi)$ with the conjugation $\mathcal{J}$. The proof is completed. □

Example 4.1. Let $\tau_1(z) = \frac{1}{(z+2-i)^{\alpha+2}}$ and $\phi_1(z) = 2 - i - \frac{1}{z+2-i}$ be the symbols in (I), $\tau_2(z) = \frac{1}{(z+\frac{\sqrt{3}}{2}-\frac{i}{2})^{\alpha+2}}$ and $\phi_2(z) = \frac{\sqrt{3}}{2} - \frac{i}{2}$ be the symbols in (II). Then, the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A_{\alpha}^2(\Pi)$ with the conjugation $\mathcal{J}$.

Proof. It is clear that $a = c = -2 + i$, $\mu = \frac{\sqrt{3}}{2} - \frac{i}{2}$ and $\eta = 0$. From Theorem 4.1, the desired result follows. $\square$

Theorem 4.2. *Let $\tau_1(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\phi_1(z) = -a - \frac{b}{z-c}$ be the symbols in (I), $\tau_2(z) = \lambda$ and $\phi_2(z) = z + \gamma$ the symbols in (III). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$ if and only if $a = c$.*

Proof. For all $w, z \in \Pi$, it follows from Lemma 3.2 (a) and (c) that

$$\begin{aligned}\mathcal{J}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z) &= \mathcal{J}\Big(\frac{1}{(z-c)^{\alpha+2}}\frac{2^\alpha(\alpha+1)}{(-a-\frac{b}{z-c}+\overline{w})^{\alpha+2}} - \lambda\frac{2^\alpha(\alpha+1)}{(z+\gamma+\overline{w})^{\alpha+2}}\Big)\\ &= \mathcal{J}\Big(\frac{2^\alpha(\alpha+1)}{[(\overline{w}-a)z+ac-b-c\overline{w}]^{\alpha+2}} - \lambda\frac{2^\alpha(\alpha+1)}{(z+\gamma+\overline{w})^{\alpha+2}}\Big)\\ &= \frac{2^\alpha(\alpha+1)}{[(w-\bar{a})z+\bar{a}\bar{c}-\bar{b}-\bar{c}w]^{\alpha+2}} - \bar{\lambda}\frac{2^\alpha(\alpha+1)}{(z+\bar{\gamma}+w)^{\alpha+2}}\end{aligned}$$

and

$$\begin{aligned}&(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\mathcal{J}K^\alpha_w(z) = (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\Big(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}}\Big)\\ &= \Big(W_{\frac{1}{(z-\bar{a})^{\alpha+2}},\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} - \bar{\lambda}C_{z+\bar{\gamma}}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}}\Big)\\ &= \frac{2^\alpha(\alpha+1)}{[(w-\bar{c})z-\bar{a}w+\bar{a}\bar{c}-\bar{b}]^{\alpha+2}} - \bar{\lambda}\frac{2^\alpha(\alpha+1)}{(z+\bar{\gamma}+w)^{\alpha+2}}.\end{aligned}$$

Therefore, by Lemma 3.1, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$ if and only if

$$\frac{1}{[(w-\bar{a})z+\bar{a}\bar{c}-\bar{b}-\bar{c}w]^{\alpha+2}} = \frac{1}{[(w-\bar{c})z-\bar{a}w+\bar{a}\bar{c}-\bar{b}]^{\alpha+2}} \tag{4.5}$$

for all $w, z \in \Pi$.

Assume that $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$. Then, from (4.5) we obtain

$$(w-\bar{a})z+\bar{a}\bar{c}-\bar{b}-\bar{c}w = (w-\bar{c})z-\bar{a}w+\bar{a}\bar{c}-\bar{b}$$

for all $w, z \in \Pi$, that is, $a = c$.

Conversely, if $a = c$, then it is clear that (4.5) holds. By Lemma 3.1, $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$. The proof is completed. $\square$

Theorem 4.3. *Let $\tau_1(z) = \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ and $\phi_1(z) = \mu$ be the symbols in (II), $\tau_2(z) = \lambda$ and $\phi_2(z) = z+\gamma$ the symbols in (III). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$ if and only if $\eta = 0$.*

Proof. For $w, z \in \Pi$, by Lemma 3.2 (b) and (c),

$$\mathcal{J}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z) = \mathcal{J}\Big(\frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}\frac{2^\alpha(\alpha+1)}{(\overline{w}+\mu)^{\alpha+2}} - \lambda\frac{2^\alpha(\alpha+1)}{(z+\gamma+\overline{w})^{\alpha+2}}\Big)$$

$$= \frac{2^\alpha(\alpha+1)\bar{\delta}}{[(w+\bar{\mu})z+(\bar{\mu}-i\eta)w+\bar{\mu}^2-(i\eta)\bar{\mu}]^{\alpha+2}} - \bar{\lambda}\frac{2^\alpha(\alpha+1)}{(z+\bar{\gamma}+w)^{\alpha+2}}$$

and

$$\begin{aligned}
&(W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2})^*\mathcal{J}K_w^\alpha(z) = (W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2})^*\Big(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}}\Big)\\
&= \Big(\bar{\delta}W_{\frac{1}{(z+\bar{\mu})^{\alpha+2}},\bar{\mu}-i\eta} - \bar{\lambda}C_{z+\bar{\gamma}}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(z+w)^{\alpha+2}}\Big)\\
&= \frac{2^\alpha(\alpha+1)\bar{\delta}}{[(\bar{\mu}-i\eta+w)z+\bar{\mu}w+\bar{\mu}^2-(i\eta)\bar{\mu}]^{\alpha+2}} - \bar{\lambda}\frac{2^\alpha(\alpha+1)}{(z+\bar{\gamma}+w)^{\alpha+2}}.
\end{aligned}$$

Thus, from Lemma 3.1, it follows that $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}$ if and only if $\eta=0$. □

Next, we do not give the examples since one can easily give examples.

Theorem 4.4. *Let $\tau_1(z)=\frac{1}{(z-c)^{\alpha+2}}$ and $\phi_1(z)=-a-\frac{b}{z-c}$ be the symbols in (I), $\tau_2(z)=\frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ and $\phi_2(z)=\mu$ the symbols in (II). Then, the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if $ac-b=1$ and $\mu=\sqrt{1-\frac{\eta^2}{4}}-\frac{\eta}{2}i$, where $\eta\in(-2,2)$.*

Proof. For all $w,z\in\Pi$, from Lemma 3.2 (a) and (b), it follows that

$$\begin{aligned}
\mathcal{J}_*(W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2})K_w^\alpha(z) &= \mathcal{J}_*\Big(\frac{2^\alpha(\alpha+1)}{[(\overline{w}-a)z+ac-b-c\overline{w}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\delta}{[(\overline{w}+\mu)z+(\mu+i\eta)\overline{w}+\mu^2+i\eta\mu]^{\alpha+2}}\Big)\\
&= \frac{2^\alpha(\alpha+1)}{[(\bar{a}\bar{c}-\bar{b})z-\bar{c}wz+w-\bar{a}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\bar{\delta}}{[(\bar{\mu}-i\eta)wz+(\bar{\mu}^2-i\eta\bar{\mu})z+w+\bar{\mu}]^{\alpha+2}}
\end{aligned}$$

and

$$\begin{aligned}
&(W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2})^*\mathcal{J}_*K_w^\alpha(z) = (W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2})^*\Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big)\\
&= \Big(W_{\frac{1}{(z-\bar{a})^{\alpha+2}},\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} - \bar{\delta}W_{\frac{1}{(z+\bar{\mu})^{\alpha+2}},\bar{\mu}-i\eta}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big)\\
&= \frac{2^\alpha(\alpha+1)}{[z-\bar{c}wz+(\bar{a}\bar{c}-\bar{b})w-\bar{a}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\bar{\delta}}{[(\bar{\mu}-i\eta)wz+z+(\bar{\mu}^2-i\eta\bar{\mu})w+\bar{\mu}]^{\alpha+2}}.
\end{aligned}$$

By Lemma 3.1, the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if

$$\begin{aligned}
&\frac{1}{[(\bar{a}\bar{c}-\bar{b})z-\bar{c}wz+w-\bar{a}]^{\alpha+2}} - \frac{1}{[z-\bar{c}wz+(\bar{a}\bar{c}-\bar{b})w-\bar{a}]^{\alpha+2}}\\
&= \frac{\bar{\delta}}{[(\bar{\mu}-i\eta)wz+(\bar{\mu}^2-i\eta\bar{\mu})z+w+\bar{\mu}]^{\alpha+2}} - \frac{\bar{\delta}}{[(\bar{\mu}-i\eta)wz+z+(\bar{\mu}^2-i\eta\bar{\mu})w+\bar{\mu}]^{\alpha+2}}
\end{aligned} \tag{4.6}$$

for all $w,z\in\Pi$.

Assume that the operator $W_{\tau_1,\phi_1}-W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$. By using the same method, we obtain that $ac-b=1$ and $\mu=\sqrt{1-\frac{\eta^2}{4}}-\frac{\eta}{2}i$, where $\eta\in(-2,2)$.

Conversely, if $ac - b = 1$ and $\mu = \sqrt{1 - \frac{\eta^2}{4}} - \frac{\eta}{2}i$, where $\eta \in (-2, 2)$, then it is clear that (4.6) holds, which shows the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$. The proof is completed. □

Theorem 4.5. *Let $\tau_1(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\phi_1(z) = -a - \frac{b}{z-c}$ be the symbols in (I), $\tau_2(z) = \lambda$ and $\phi_2(z) = z + \gamma$ the symbols in (III). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if $ac - b = 1$ and $\gamma = 0$.*

Proof. For all $w, z \in \Pi$, from Lemma 3.2 (a) and (c), it follows that

$$\begin{aligned}\mathcal{J}_*(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z) &= \mathcal{J}_*\Big(\frac{2^\alpha(\alpha+1)}{[(\overline{w}-a)z + ac - b - c\overline{w}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\lambda}{(\overline{w}+z+\gamma)^{\alpha+2}}\Big)\\ &= \frac{2^\alpha(\alpha+1)}{[(\bar{a}\bar{c}-\bar{b})z - \bar{c}wz + w - \bar{a}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\bar{\lambda}}{(wz+\bar{\gamma}z+1)^{\alpha+2}}\end{aligned}$$

and

$$\begin{aligned}(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\mathcal{J}_*K^\alpha_w(z) &= (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^*\Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big) = \Big(W_{\frac{1}{(z-\bar{a})^{\alpha+2}},\frac{\bar{a}\bar{c}-\bar{b}-\bar{c}z}{z-\bar{a}}} - \bar{\lambda}C_{z+\bar{\gamma}}\Big)\Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big)\\ &= \frac{2^\alpha(\alpha+1)}{[z - \bar{c}wz + (\bar{a}\bar{c}-\bar{b})w - \bar{a}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\bar{\lambda}}{(wz+\bar{\gamma}w+1)^{\alpha+2}}.\end{aligned}$$

By Lemma 3.1, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if

$$\frac{1}{[(\bar{a}\bar{c}-\bar{b})z - \bar{c}wz + w - \bar{a}]^{\alpha+2}} - \frac{1}{[z - \bar{c}wz + (\bar{a}\bar{c}-\bar{b})w - \bar{a}]^{\alpha+2}} = \frac{\bar{\lambda}}{(wz+\bar{\gamma}z+1)^{\alpha+2}} - \frac{\bar{\lambda}}{(wz+\bar{\gamma}w+1)^{\alpha+2}} \tag{4.7}$$

for all $w, z \in \Pi$.

Assume that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$. By using the same method, we obtain that $ac - b = 1$ and $\gamma = 0$.

Conversely, if $ac - b = 1$ and $\gamma = 0$, then it is clear that (4.8) holds, which shows the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$. The proof is completed. □

Theorem 4.6. *Let $\tau_1(z) = \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ and $\phi_1(z) = \mu$ the symbols in (II), $\tau_2(z) = \lambda$ and $\phi_2(z) = z + \gamma$ the symbols in (III). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A^2_\alpha(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if $\gamma = 0$ and $\mu = \sqrt{1 - \frac{\eta^2}{4}} - \frac{\eta}{2}i$, where $\eta \in (-2, 2)$.*

Proof. For all $w, z \in \Pi$, from Lemma 3.2 (b) and (c), it follows that

$$\begin{aligned}\mathcal{J}_*(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})K^\alpha_w(z) &= \mathcal{J}_*\Big(\frac{2^\alpha(\alpha+1)\delta}{[(\overline{w}+\mu)z + (\mu+i\eta)\overline{w} + \mu^2 + i\eta\mu]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\lambda}{(\overline{w}+z+\gamma)^{\alpha+2}}\Big)\\ &= \frac{2^\alpha(\alpha+1)\bar{\delta}}{[(\bar{\mu}-i\eta)wz + (\bar{\mu}^2 - i\eta\bar{\mu})z + w + \bar{\mu}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\bar{\lambda}}{(wz+\bar{\gamma}z+1)^{\alpha+2}}\end{aligned}$$

and

$$(W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^* \mathcal{J}_* K_w^\alpha(z) = (W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2})^* \Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big) = \big(\bar{\delta} W_{\frac{1}{(z+\bar{\mu})^{\alpha+2}},\bar{\mu}-i\eta} - \bar{\lambda} C_{z+\bar{\gamma}}\big)\Big(\frac{2^\alpha(\alpha+1)}{(1+zw)^{\alpha+2}}\Big)$$
$$= \frac{2^\alpha(\alpha+1)\bar{\delta}}{[(\bar{\mu}-i\eta)wz + z + (\bar{\mu}^2 - i\eta\bar{\mu})w + \bar{\mu}]^{\alpha+2}} - \frac{2^\alpha(\alpha+1)\bar{\lambda}}{(wz+\bar{\gamma}w+1)^{\alpha+2}}.$$

By Lemma 3.1, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_*$ if and only if

$$\begin{aligned}&\frac{1}{[(\bar{a}\bar{c}-\bar{b})z - \bar{c}wz + w - \bar{a}]^{\alpha+2}} - \frac{1}{[z - \bar{c}wz + (\bar{a}\bar{c}-\bar{b})w - \bar{a}]^{\alpha+2}}\\ &= \frac{\bar{\lambda}}{(wz+\bar{\gamma}z+1)^{\alpha+2}} - \frac{\bar{\lambda}}{(wz+\bar{\gamma}w+1)^{\alpha+2}}\end{aligned} \tag{4.8}$$

for all $w, z \in \Pi$.

Assume that the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_*$. By using the same method, we obtain that $ac - b = 1$ and $\gamma = 0$.

Conversely, if $ac - b = 1$ and $\gamma = 0$, then it is clear that (4.8) holds, which shows the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_*$. The proof is completed. □

From the above proofs, the following results can be similarly proved.

Theorem 4.7. *Let $\tau_1(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\phi_1(z) = -a - \frac{b}{z-c}$ be the symbols in (I), $\tau_2(z) = \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ and $\phi_2(z) = \mu$ the symbols in (II). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_s$ if and only if $a = c - is$ and $\eta = -s$.*

Theorem 4.8. *Let $\tau_1(z) = \frac{1}{(z-c)^{\alpha+2}}$ and $\phi_1(z) = -a - \frac{b}{z-c}$ be the symbols in (I), $\tau_2(z) = \lambda$ and $\phi_2(z) = z + \gamma$ the symbols in (III). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_s$ if and only if $a = c - is$.*

Theorem 4.9. *Let $\tau_1(z) = \frac{\delta}{(z+\mu+i\eta)^{\alpha+2}}$ and $\phi_1(z) = \mu$ be the symbols in (II), $\tau_2(z) = \lambda$ and $\phi_2(z) = z+\gamma$ the symbols in (III). Then, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}_s$ if and only if $\eta = -s$.*

Remark 4.1. Considering the results in Section 3, the operator $W_{\tau_1,\phi_1} - W_{\tau_2,\phi_2}$ is complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}$, $\mathcal{J}_s$, and $\mathcal{J}_*$, respectively, if and only if both W_{τ_1,ϕ_1} and W_{τ_2,ϕ_2} are complex symmetric on $A_\alpha^2(\Pi)$ with the conjugation $\mathcal{J}$, $\mathcal{J}_s$, and $\mathcal{J}_*$, respectively.

5. Conclusions

Since it is impossible to give the proper description of the adjoint of the operator $W_{\tau,\phi}$ with the general symbols on $A_\alpha^2(\Pi)$, in this paper we just consider this problem for the operators $W_{\tau,\phi}$ with the symbols in (I)–(III) on $A_\alpha^2(\Pi)$. At the same time, by using these descriptions, we characterize complex symmetric difference of the operators $W_{\tau,\phi}$ with the symbols in (I)–(III) with the conjugations $\mathcal{J}$, $\mathcal{J}_s$, and $\mathcal{J}_*$ on $A_\alpha^2(\Pi)$. However, we still do not obtain any result for the general symbols. Therefore, we hope that the study can attract more attention for such a topic.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

The author thanks the anonymous referees for their time and comments.

This study was supported by Sichuan Science and Technology Program (2024NSFSC2314) and the Scientific Research and Innovation Team Program of Sichuan University of Science and Engineering (SUSE652B002).

Conflict of interest

The author declares that he has no competing interests.

References

1. S. R. Bhuia, A class of C-normal weighted composition operators on Fock space $\mathcal{F}^2(\mathbb{C})$, *J. Math. Anal. Appl.,* **508** (2022), 125896. http://dx.doi.org/10.1016/j.jmaa.2021.125896
2. M. Chō, E. Ko, J. Lee, On m-complex symmetric operators, *Mediterr. J. Math.,* **13** (2016), 2025–2038. http://dx.doi.org/10.1017/s0017089516000550
3. B. Choe, H. Koo, W. Smith, Difference of composition operators over the half-plane, *Trans. Amer. Math. Soc.,* **369** (2017), 3173–3205. http://dx.doi.org/10.1090/tran/6742
4. S. J. Elliott, A. Wynn, Composition operators on weighted Bergman spaces of a half-plane, *P. Edinburgh Math. Soc.,* **54** (2009), 373–379. http://dx.doi.org/10.1017/S0013091509001412
5. M. Fatehi, Complex symmetric weighted composition operators, *Complex Var. Elliptic,* **64** (2019), 710–720. http://dx.doi.org/10.1080/17476933.2018.1498087
6. S. R. Garcia, E. Prodan, M. Putinar, Mathematical and physical aspects of complex symmetric operators, *J. Phys. A-Math. Theor.,* **47** (2014), 353001. http://dx.doi.org/10.1088/1751-8113/47/35/353001
7. S. R. Garcia, M. Putinar, Complex symmetric operators and applications, *T. Am. Math. Soc.,* **358** (2006), 1285–1315. http://dx.doi.org/10.1090/s0002-9947-05-03742-6
8. S. R. Garcia, M. Putinar, Complex symmetric operators and applications II, *T. Am. Math. Soc.,* **359** (2007), 3913–3931. http://dx.doi.org/10.1090/s0002-9947-07-04213-4
9. S. R. Garcia, W. Wogen, Complex symmetric partial isometries, *J. Funct. Anal.,* **257** (2009), 1251–1260. http://dx.doi.org/10.1016/j.jfa.2009.04.005
10. S. R. Garcia, W. Wogen, Some new classes of complex symmetric operators, *T. Am. Math. Soc.,* **362** (2010), 6065–6077. http://dx.doi.org/10.1090/s0002-9947-2010-05068-8
11. Y. Gao, Z. Zhou, Complex symmetric composition operators induced by linear fractional maps, *Indiana U. Math. J.,* **69** (2020), 367–384. http://dx.doi.org/10.1512/iumj.2020.69.7622

12. A. Gupta, A. Malhotra, Complex symmetric weighted composition operators on the space $H_1^2(\mathbb{D})$, *Complex Var. Elliptic,* **65** (2020), 1488–1500. http://dx.doi.org/10.1080/17476933.2019.1664483

13. K. Han, M. Wang, Weighted composition-differentiation operators on the Hardy space, *Banach J. Math. Anal.,* **15** (2021), 44. http://dx.doi.org/10.1007/s43037-021-00131-z

14. P. V. Hai, O. R. Severiano, Complex symmetric weighted composition operators on Bergman spaces and Lebesgue spaces, *Anal. Math. Phys.,* **12** (2021), 43. http://dx.doi.org/10.1007/s13324-022-00651-3

15. J. W. Helton, *Operators with a representation as multiplication by x on a Sobolev space*, Colloquia Math. Soc., János Bolyai, 5, North-Holland, Amsterdam, 1972.

16. T. Hosokawa, Differences of weighted composition operators on the Bloch spaces, *Complex Anal. Oper. Th.,* **3** (2009), 847–866. http://dx.doi.org/10.1007/s11785-008-0062-1

17. L. Hu, S. Li, R. Yang, 2-complex symmetric composition operators on H^2, *Axioms,* **11** (2021), 358. http://dx.doi.org/10.3390/axioms11080358

18. S. Jung, Y. Kim, E. Ko, J. Lee, Complex symmetric weighted composition operators on $H^2(\mathbb{D})$, *J. Funct. Anal.,* **267** (2014), 323–351.

19. R. Lim, L. Khoi, Complex symmetric weighted composition operators on $H_\gamma(\mathbb{D})$, *J. Math. Anal. Appl.,* **464** (2018), 101–118. http://dx.doi.org/10.1016/j.jmaa.2018.03.071

20. V. Matache, Composition operators on Hardy spaces of a half-plane, *P. Am. Math. Soc.,* **127** (1999), 1483–1491. http://dx.doi.org/10.1090/s0002-9939-99-05060-1

21. J. Moorhouse, Compact differences of composition operators, *J. Funct. Anal.,* **219** (2005), 70–92. http://dx.doi.org/10.1016/j.jfa.2004.01.012

22. S. Narayan, D. Sievewright, D. Thompson, Complex symmetric composition operators on H^2, *J. Math. Anal. Appl.,* **443** (2016), 625–630. http://dx.doi.org/10.1016/j.jmaa.2016.05.046

23. S. Narayan, D. Sievewright, M. Tjani, Complex symmetric composition operators on weighted Hardy spaces, *P. Am. Math. Soc.,* **148** (2020), 2117–2127. http://dx.doi.org/10.1090/proc/14909

24. S. W. Noor, O. R. Severiano, Complex symmetry and cyclicity of composition operators on $H^2(\mathbb{C}_+)$, *P. Am. Math. Soc.,* **148** (2020), 2469–2476. http://dx.doi.org/10.1090/proc/14918

25. M. N. Oreshina, Spectral decomposition of normal operator in real Hilbert space, *Ufa Math. J.,* **9** (2017), 87–99. http://dx.doi.org/10.13108/2017-9-4-85

26. M. Ptak, K. Simik, A. Wicher, *C*-normal operators, *Electron J. Linear Al.,* **36** (2020), 67–79. http://dx.doi.org/10.13001/ela.2020.5045

27. J. H. Shapiro, C. Sundberg, Isolation amongst the composition operators, *Pac. J. Math.,* **145** (1990), 117–152. http://dx.doi.org/10.2140/pjm.1990.145.117

28. A. K. Sharma, R. Krishan, Difference of composition operators from the space of Cauchy integral transforms to the Dirichlet space, *Complex Anal. Oper. Th.,* **10** (2016), 141–152. http://dx.doi.org/10.1007/s11785-015-0487-2

29. S. D. Sharma, A. K. Sharma, Z. Abbas, Weighted composition operators on weighted vector-valued Bergman spaces, *Appl. Math. Sci.,* **4** (2010), 2049–2063.

30. D. Thompson, T. McClatchey, C. Holleman, Binormal, complex symmetric operators, *Linear Multilinear A.,* **69** (2021), 1705–1715. http://dx.doi.org/10.1080/03081087.2019.1635982

31. C. Wang, J. Y. Zhao, S. Zhu, Remarks on the structure of C-normal operators, *Linear Multilinear A.,* **70** (2020), 1682–1696. http://dx.doi.org/10.1080/03081087.2020.1771254

32. Y. F. Xu, Z. J. Jiang, C. S. Huang, 2-complex symmetric weighted composition operators on the weighted Bergman spaces of the half-plane, *Complex Anal. Oper. Th.,* **17** (2023), 119. http://dx.doi.org/10.1007/s11785-023-01418-9

33. X. Yao, Complex symmetric composition operators on a Hilbert space of Dirichlet series, *J. Math. Anal. Appl.,* **452** (2017), 1413–1419. http://dx.doi.org/10.1016/j.jmaa.2017.03.076

AIMS Press

© 2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)