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Research article

Oncolysis by SARS-CoV-2: modeling and analysis

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Abstract: The relationship between cancer and the severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) infection is controversial. While SARS-CoV-2 can worsen the status of a cancer patient, many remission cases after SARS-CoV-2 infection have been recorded. It has been suggested that SARS-CoV-2 could have oncolytic properties, which needs further investigations. Mathematical modeling is a powerful tool that can significantly enhance experimental and medical studies. Our objective was to propose and analyze a mathematical model for oncolytic SARS-CoV-2 with immunity. The basic properties of this model, including existence, uniqueness, nonnegativity, and boundedness of the solutions, were confirmed. The equilibrium points were computed, and their existence conditions were determined. The global stability of the equilibria was proven using the Lyapunov theory. Numerical simulations were implemented to validate the theoretical results. It was found that the model has thirteen equilibrium points that reflect different infection states. Based on the model's results, the infection of cancer cells by SARS-CoV-2 can lead to a reduction in the concentration of cancer cells. Additionally, the induction of cytotoxic T lymphocytes (CTLs) decreases the number of cancer cells, potentially resulting in cancer remission or an improvement in the overall health of cancer patients. This theoretical result aligns with numerous studies highlighting the oncolytic role of SARS-CoV-2. In addition, given the limited availability of real data, further studies are essential to better comprehend the role of immune responses and their impact on the oncolytic role of SARS-CoV-2.

Keywords: SARS-CoV-2; cancer; virotherapy; stability; CTL
Mathematics Subject Classification: 34D20, 34D23, 37N25, 92B05

Abbreviation

N: nutrient, it is produced from a source at a fixed rate, cells grow as a result of consuming nutrient; M: epithelial cells, the type of cells in the lungs that are infected by SARS-CoV-2; C cancer cells: the cells characterized by uncontrolled growth that become infected by SARS-CoV-2; V: the free SARS-

CoV-2 particles, the virus responsible for COVID-19, these particles infect epithelial and cancer cells; T: anti-cancer CTLs, the immune cells that specifically target and eliminate cancer cells; A: antibodies, they are used by the immune system to eliminate virus particles from the body

1. Introduction

COVID-19 is a disease caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2). While there has been a decline in reported cases and deaths globally, SARS-CoV-2 is still spreading in many countries [1]. SARS-CoV-2 passes into host cells by way of a transmembrane protein known as the angiotensin-converting enzyme 2 receptor [2]. It principally causes infection in alveolar epithelial type-II cells of the lungs [3, 4]. However, other organs can be infected by SARS-CoV-2. The impact of COVID-19 on cancer patients has opened up a wide field of research. This group of patients is vulnerable to COVID-19 due to weakened immune system or ongoing anti-cancer treatments [5]. The question of whether SARS-CoV-2 induces remission in cancer patients or exacerbates the severity of the disease remains controversial [5, 6].

The interconnection between cancer and viruses has become one of the most important topics in oncology and virology [7]. Some viruses, called oncolytic viruses (OVs), have the ability to infect cancer cells. These viruses can be found in nature or genetically modified to replicate in cancer cells without infecting normal cells [5, 8]. OVs kill cancer cells after massive replication inside them and by inducing a specific antitumor immune response [5, 8]. Examples of OVs include adenovirus, vaccinia virus, Coxsackievirus, and herpes simplex virus [8, 9]. Talimogene laherparepvec is the only approved oncolytic virotherapy [6, 10]. Talimogene laherparepvec is an engineered herpes virus used to treat advanced melanoma through immediate injection into the tumor [10].

The impact of COVID-19 on cancer patients is bidirectional. It has been indicated that SARS-CoV-2 infection can enhance cancer progression [5, 7, 8, 11]. On the other hand, cancer remission after SARS-CoV-2 infection has been reported in many patients [6, 12–14]. For example, Pasin et al. [15] reported the case of a patient with refractory natural killer (NK)/T-cell lymphoma who experienced a transient remission during SARS-CoV-2 infection. As angiotensin-converting enzyme 2 is expressed in NK cells, the authors in [15] proposed that SARS-CoV-2 could own some oncolytic properties. Challenor and Tucker [16] presented the case of a remission in a patient with classical Hodgkin lymphoma after SARS-CoV-2 infection. Another case was reported by Sollini et al. [17] involving a patient with follicular lymphoma who achieved full remission after SARS-CoV-2 infection. The authors in [16, 17] supposed that the infection stimulated an antitumor immune response. Kandeel et al. [18] reported the remission of two cases with acute leukemia. Antwi-Amoabeng et al. [19] presented a case in which a patient with multiple myeloma had remission following SARS-CoV-2 infection. The patient received a single dose of chemotherapy. However, the authors mentioned that the remission in this case was parallel to the remission in patients who got four doses of chemotherapy. Ohadi et al. [20] reported a case of mycosis fungoides that went into remission after the coronavirus infection. Other cases of remission were reported in [6].

The above remission cases suggest that SARS-CoV-2 could have an oncolytic role in many types of cancer. It may infect and destroy cancer cells to expose the tumor-associated antigens. These antigens stimulate an immune response against cancer cells, leading to cancer remission [18]. Hence, there is an urgent need to understand the relationship between SARS-CoV-2 infection and cancer.

This understanding may enable the engineering of SARS-CoV-2 for use as an efficient therapy against certain types of cancer. Mathematical modeling is a strong tool that is often employed to assist experiments and medical research [21, 22]. Mathematical models have been used to understand the dynamics of many infectious diseases and test hypotheses that may be challenging to assess experimentally. The analysis of these models can offer predictions of outcomes and aid in identifying optimal treatment strategies.

Many mathematical models about SARS-CoV-2 [23–25], cancer [26], SARS-CoV-2/cancer [27], and oncolytic virotherapy [28–32] have been constructed and studied. However and to the best of our knowledge, no oncolytic SARS-CoV-2 models have been established yet. Such models are important to understand the effect of the infection of cancer cells by SARS-CoV-2 and the role of different immune responses during this coinfection. In this work, we propose an oncolytic SARS-CoV-2 virotherpy model. The construction of this model follows similar principles of those used in [33]. We conduct a comprehensive mathematical analysis of this model including assessments of boundedness, nonnegativity, and global stability of equilibrium points. In addition, we implement some numerical simulations.

This paper is structured as follows. Section 2 introduces the model under consideration. Section 3 demonstrates that all solutions are bounded and have zero or positive values. Furthermore, it computes the equilibrium solutions of the proposed model. Section 4 verifies the global properties of these solutions. Section 5 is dedicated to numerical simulations. The last section discusses the results and provides a glimpse of the future vision.

2. Oncolytic SARS-CoV-2 model with immune responses

In formulating the model, we consider the following assumptions:

(i) The nutrient is produced from a source at a fixed rate, while it is depleted due to its consumption by epithelial cells and cancer cells. Additionally, depletion occurs as a result of natural death;

(ii) Epithelial cells proliferate as a result of nutrient consumption, and their numbers decrease due to either viral infection or natural death;

(iii) Cancer cells replicate as a result of nutrient utilization, and their numbers decline due to viral infection, attacks by cytotoxic T lymphocytes (CTLs), or natural cell death;

(iv) Free virus particles increase as a consequence of infecting epithelial cells and cancer cells, but they diminish due to the removal by antibodies or natural death;

(v) CTLs are stimulated by infected cancer cells, while antibodies are stimulated by free virus particles;

(vi) CTLs and antibodies undergo decay through natural processes;

(vii) The induction of CTLs by SARS-CoV-2 infection is implied in the stimulation rate of CTLs;

(viii) The model does not contain infected components for epithelial cells and cancer cells.

The proposed ordinary differential equation (ODE) model consists of six nonlinear equations and takes the form

$$
\begin{cases}
\dfrac{dN(t)}{dt} = \alpha - \lambda_1 N(t)M(t) - \lambda_2 N(t)C(t) - \omega N(t), \\
\dfrac{dM(t)}{dt} = \theta\lambda_1 N(t)M(t) - \epsilon_1 M(t)V(t) - (\omega + \omega_1)M(t), \\
\dfrac{dC(t)}{dt} = \theta\lambda_2 N(t)C(t) - \epsilon_2 C(t)V(t) - \xi_1 C(t)T(t) - (\omega + \omega_2)C(t), \\
\dfrac{dV(t)}{dt} = p\epsilon_1 M(t)V(t) + p\epsilon_2 C(t)V(t) - \xi_2 V(t)A(t) - (\omega + \omega_3)V(t), \\
\dfrac{dT(t)}{dt} = s_1\xi_1 C(t)T(t) - (\omega + \omega_4)T(t), \\
\dfrac{dA(t)}{dt} = s_2\xi_2 V(t)A(t) - (\omega + \omega_5)A(t),
\end{cases}
\tag{2.1}
$$

where $N(t)$, $M(t)$, $C(t)$, $V(t)$, $T(t)$, and $A(t)$ typify the concentrations of nutrient, epithelial cells, cancer cells, SARS-CoV-2 particles, anti-cancer CTLs, and antibodies. The nutrient is released from its source at rate α and declines at rate ωN. Epithelial cells expend nutrient at rate $\lambda_1 NM$, reproduce at rate $\theta\lambda_1 NM$, and get infected by SARS-CoV-2 at rate $\epsilon_1 MV$. Cancer cells expend nutrient at rate $\lambda_2 NC$, grow at rate $\theta\lambda_2 NC$, and become infected at rate $\epsilon_2 CV$. SARS-CoV-2 replicates as a result of infecting epithelial cells and cancer cells at rates $p\epsilon_1 MV$ and $p\epsilon_2 CV$, respectively. CTLs kill cancer cells at rate $\xi_1 CT$ and reproduce at rate $s_1\xi_1 CT$. Antibodies eliminate SARS-CoV-2 at rate $\xi_2 VA$ and get stimulated at rate $s_2\xi_2 VA$. Epithelial cells, cancer cells, SARS-CoV-2 particles, CTLs, and antibodies die at natural rates $\omega_1 M$, $\omega_2 C$, $\omega_3 V$, $\omega_4 T$, and $\omega_5 A$, respectively. The parameters in model (2.1) are postulated to take positive values. Figure 1 provides a schematic diagram of the model. For simplicity, we consider the following

$$a_1 \equiv \omega + \omega_1,\ a_2 \equiv \omega + \omega_2,\ a_3 \equiv \omega + \omega_3,\ a_4 \equiv \omega + \omega_4,\ a_5 \equiv \omega + \omega_5.$$

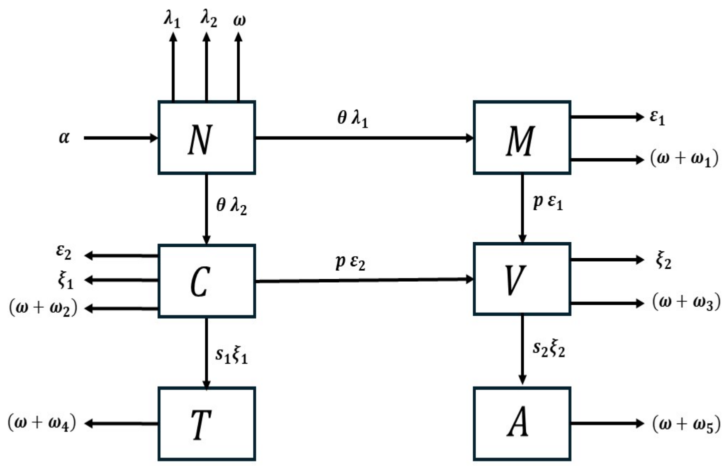


Figure 1. Schematic diagram of model (2.1).

3. Basic properties

The following theorem demonstrates the existence and uniqueness of the solutions of model (2.1).

Theorem 1. *Assume that the initial values* $(N_0, M_0, C_0, V_0, T_0, A_0) \in \mathbb{R}^6$ *are given. There exists* $t_0 > 0$ *and continuously differentiable functions* N, M, C, V, T, A*:* $[0, t_0) \to \mathbb{R}$ *such that* (N, M, C, V, T, A) *satisfies model* (2.1) *and*

$$(N(0), M(0), C(0), V(0), T(0), A(0)) = (N_0, M_0, C_0, V_0, T_0, A_0)\,.$$

Proof. As the system of ODEs given in (2.1) is autonomous, it is enough to prove that the function f: $\mathbb{R}^6 \to \mathbb{R}^6$ defined by

$$f(z) = \begin{bmatrix} \alpha - \lambda_1 z_1 z_2 - \lambda_2 z_1 z_3 - \omega z_1 \\ \theta\lambda_1 z_1 z_2 - \epsilon_1 z_2 z_4 - a_1 z_2 \\ \theta\lambda_2 z_1 z_3 - \epsilon_2 z_3 z_4 - \xi_1 z_3 z_5 - a_2 z_3 \\ p\epsilon_1 z_2 z_4 + p\epsilon_2 z_3 z_4 - \xi_2 z_4 z_6 - a_3 z_4 \\ s_1\xi_1 z_3 z_5 - a_4 z_5 \\ s_2\xi_2 z_4 z_6 - a_5 z_6 \end{bmatrix}$$

is locally Lipschitz in its z argument. We observe that the Jacobian matrix

$$\nabla f(z) = \begin{bmatrix} -\lambda_1 z_2 - \lambda_2 z_3 - \omega & -\lambda_1 z_1 & -\lambda_2 z_1 & 0 & 0 & 0 \\ \theta\lambda_1 z_2 & \theta\lambda_1 z_1 - \epsilon_1 z_4 - a_1 & 0 & -\epsilon_1 z_2 & 0 & 0 \\ \theta\lambda_2 z_3 & 0 & \theta\lambda_2 z_1 - \epsilon_2 z_4 - \xi_1 z_5 - a_2 & -\epsilon_2 z_3 & -\xi_1 z_3 & 0 \\ 0 & p\epsilon_1 z_4 & p\epsilon_2 z_4 & p\epsilon_1 z_2 + p\epsilon_2 z_3 - \xi_2 z_6 - a_3 & 0 & -\xi_2 z_4 \\ 0 & 0 & s_1\xi_1 z_5 & 0 & s_1\xi_1 z_3 - a_4 & 0 \\ 0 & 0 & 0 & s_2\xi_2 z_6 & 0 & s_2\xi_2 z_4 - a_5 \end{bmatrix}$$

is linear in z and consequently locally bounded for all $z \in \mathbb{R}^6$. Therefore, f has a continuous and bounded derivative on any compact subset of $\mathbb{R}^6$, and so f is locally Lipschitz in z. According to the classical Picard-Lindelöf theorem [34], there exists a unique solution $z(t)$ to the ODE

$$\frac{dz(t)}{dt} = f(z(t))$$

on the time interval $[0, t_0]$ for some $t_0 > 0$ [35]. □

Next, we prove the nonnegativity and boundedness of the solutions of model (2.1).

Theorem 2. *Let* $\tau_i > 0$ *(*$i = 1, 2, 3, 4, 5$*), then the set*

$$\Omega = \left\{(N, M, C, V, T, A) \in \mathbb{R}_+^6 : 0 \le N(t) \le \tau_1, 0 \le M(t), C(t) \le \tau_2, 0 \le V(t) \le \tau_3, 0 \le T(t) \le \tau_4, 0 \le A(t) \le \tau_5\right\}$$

is positively invariant set for system (2.1).

Proof. For system (2.1), we obtain

$$\frac{dN}{dt}|_{N=0} = \alpha > 0, \quad \frac{dM}{dt}|_{M=0} = 0, \quad \frac{dC}{dt}|_{C=0} = 0, \quad \frac{dV}{dt}|_{V=0} = 0, \quad \frac{dT}{dt}|_{T=0} = 0, \quad \frac{dA}{dt}|_{A=0} = 0.$$

This shows that

$$(N(t), M(t), C(t), V(t), T(t), A(t)) \in \mathbb{R}_+^6$$

for $t \geq 0$ whenever

$$(N(0), M(0), C(0), V(0), T(0), A(0)) \in \mathbb{R}_+^6.$$

To prove the boundedness, we consider the function

$$\chi(t) = N(t) + \frac{1}{\theta}M(t) + \frac{1}{\theta}C(t) + \frac{1}{\theta p}V(t) + \frac{1}{\theta s_1}T(t) + \frac{1}{\theta p s_2}A(t).$$

By computing $\dfrac{d\chi(t)}{dt}$, we get

$$\begin{aligned}
\frac{d\chi(t)}{dt} &= \alpha - \omega N(t) - \frac{a_1}{\theta}M(t) - \frac{a_2}{\theta}C(t) - \frac{a_3}{\theta p}V(t) - \frac{a_4}{\theta s_1}T(t) - \frac{a_5}{\theta p s_2}A(t) \\
&\leq \alpha - \kappa\left[N(t) + \frac{1}{\theta}M(t) + \frac{1}{\theta}C(t) + \frac{1}{\theta p}V(t) + \frac{1}{\theta s_1}T(t) + \frac{1}{\theta p s_2}A(t)\right] \\
&= \alpha - \kappa\chi(t),
\end{aligned}$$

where

$$\kappa = \min\{\omega, a_1, a_2, a_3, a_4, a_5\}.$$

This implicates that

$$0 \leq \chi(t) \leq \tau_1 \quad \text{if} \quad \chi(0) \leq \tau_1, \text{ for } t \geq 0,$$

where

$$\tau_1 = \frac{\alpha}{\kappa}.$$

Consequently, we have $N(t) \leq \tau_1$, $M(t) \leq \tau_2$, $C(t) \leq \tau_2$, $V(t) \leq \tau_3$, $T(t) \leq \tau_4$, and $A(t) \leq \tau_5$, where $\tau_2 = \theta\tau_1$, $\tau_3 = \theta p\tau_1$, $\tau_4 = \theta s_1\tau_1$, and $\tau_5 = \theta p s_2\tau_1$. Thus, the set Ω is positively invariant [36]. □

Theorem 3. *Model* (2.1) *has thirteen equilibrium points as follows:*

(1) The trivial equilibrium E_0 always exists;

(2) The uninfected-epithelial equilibrium E_1 exists if $R_0 > 1$;

(3) The uninfected-cancer equilibrium E_2 exists if $R_1 > 1$;

(4) The infected-epithelial equilibrium E_3 exists if

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1};$$

(5) The infected-cancer equilibrium E_4 exists if

$$R_1 > 1 + \frac{\lambda_2 a_3}{\omega p \epsilon_2};$$

(6) The uninfected-cancer equilibrium with CTLs E_5 exists if

$$R_1 > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1};$$

(7) The infected epithelial-cancer equilibrium without immunity E_6 exists if

$$R_1 + \frac{a_1\epsilon_2}{a_2\epsilon_1} + \frac{a_1\lambda_2 a_3}{\omega p\epsilon_1 a_2} < 1 + \frac{\lambda_2 a_3}{\omega p\epsilon_2} + \frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2},$$

$$R_0 + \frac{a_2\epsilon_1}{a_1\epsilon_2} + \frac{\lambda_1 a_2 a_3}{\omega p a_1\epsilon_2} < 1 + \frac{\lambda_1 a_3}{\omega p\epsilon_1} + \frac{\alpha\theta\epsilon_1\lambda_2}{\omega a_1\epsilon_2},$$

$$\frac{a_1\epsilon_2}{a_2\epsilon_1} > 1, \quad \frac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1} > 1, \quad \textit{and} \quad \frac{a_1\lambda_2}{a_2\lambda_1} > 1;$$

(8) The uninfected epithelial-cancer equilibrium with CTLs E_7 exists if

$$R_0 > 1 + \frac{\lambda_2 a_4}{\omega s_1\xi_1} \quad \textit{and} \quad \frac{a_1\lambda_2}{a_2\lambda_1} > 1;$$

(9) The infected-epithelial equilibrium with antibodies E_8 exists if

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p\epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2\xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2\xi_2};$$

(10) The infected-cancer equilibrium with antibodies E_9 exists if

$$R_1 > 1 + \frac{\lambda_2 a_3}{\omega p\epsilon_2} + \frac{\epsilon_2 a_5}{a_2 s_2\xi_2} + \frac{\lambda_2 a_3 a_5}{\omega p a_2 s_2\xi_2};$$

(11) The infected-cancer equilibrium with CTLs and antibodies E_{10} exists if

$$R_1 > 1 + \frac{\lambda_2 a_4}{\omega s_1\xi_1} + \frac{\epsilon_2 a_5}{a_2 s_2\xi_2} + \frac{\epsilon_2\lambda_2 a_4 a_5}{\omega s_1\xi_1 a_2 s_2\xi_2} \quad \textit{and} \quad \frac{p\epsilon_2 a_4}{s_1\xi_1 a_3} > 1;$$

(12) The infected epithelial-cancer equilibrium with CTLs E_{11} exists if

$$\frac{\lambda_2\epsilon_1}{\lambda_1\epsilon_2} + \frac{s_1\xi_1 a_3}{p\epsilon_2 a_4} + \frac{\omega\epsilon_1 s_1\xi_1}{\lambda_1\epsilon_2 a_4} > 1,$$

$$R_0 + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} > 1 + \frac{\lambda_2 a_4}{\omega s_1\xi_1} + \frac{\lambda_1 a_3}{\omega p\epsilon_1}$$

and

$$R_1 + \frac{a_1\epsilon_2}{a_2\epsilon_1} + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} + \frac{a_1\lambda_1\epsilon_2 a_3}{\omega p\epsilon_1^2 a_2} + \frac{a_1\epsilon_2\lambda_2 a_4}{\omega\epsilon_1 s_1\xi_1 a_2} > 1 + \frac{\lambda_1 a_3}{\omega p\epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1\xi_1} + \frac{a_1\lambda_1\epsilon_2^2 a_4}{\omega\epsilon_1^2 s_1\xi_1 a_2} + \frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2};$$

(13) The infected epithelial-cancer equilibrium with CTLs and antibodies E_{12} exists if

$$R_0 > 1 + \frac{\epsilon_1 a_5}{a_1 s_2\xi_2} + \frac{\lambda_2 a_4}{\omega s_1\xi_1} + \frac{\epsilon_1\lambda_2 a_4 a_5}{\omega a_1 s_1\xi_1 s_2\xi_2},$$

$$\frac{a_1\lambda_2}{a_2\lambda_1} + \frac{\epsilon_1\lambda_2 a_5}{\lambda_1 a_2 s_2\xi_2} > 1 + \frac{\epsilon_2 a_5}{a_2 s_2\xi_2}$$

and

$$R_0 + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} + \frac{\lambda_1\epsilon_2 a_4 a_5}{\omega a_1 s_1\xi_1 s_2\xi_2} > 1 + \frac{\epsilon_1 a_5}{a_1 s_2\xi_2} + \frac{\lambda_1 a_3}{\omega p\epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1\xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1\lambda_2 a_4 a_5}{\omega a_1 s_1\xi_1 s_2\xi_2}.$$

Proof. To get the equilibria of system (2.1), we solve the following system:

$$\begin{cases} 0 = \alpha - \lambda_1 NM - \lambda_2 NC - \omega N, \\ 0 = \theta\lambda_1 NM - \epsilon_1 MV - a_1 M, \\ 0 = \theta\lambda_2 NC - \epsilon_2 CV - \xi_1 CT - a_2 C, \\ 0 = p\epsilon_1 MV + p\epsilon_2 CV - \xi_2 VA - a_3 V, \\ 0 = s_1\xi_1 CT - a_4 T, \\ 0 = s_2\xi_2 VA - a_5 A. \end{cases}$$

Then, we obtain

(1) The trivial equilibrium

$$E_0 = (N_0, 0, 0, 0, 0, 0) = (\frac{\alpha}{\omega}, 0, 0, 0, 0, 0).$$

This point always exists. This point has no biological meaning as all components vanish except for the nutrient.

(2) The uninfected-epithelial equilibrium $E_1 = (N_1, M_1, 0, 0, 0, 0)$. The components N_1 and M_1 are defined as:

$$N_1 = \frac{a_1}{\theta\lambda_1}, \quad M_1 = \frac{\omega}{\lambda_1}\left(R_0 - 1\right),$$

where

$$R_0 = \frac{\alpha\theta\lambda_1}{\omega a_1}.$$

As $N_1 > 0$, the equilibrium E_1 exists if $R_0 > 1$. This equilibrium represents a healthy individual without cancer or SARS-CoV-2 infection.

(3) The uninfected-cancer equilibrium $E_2 = (N_2, 0, C_2, 0, 0, 0)$, where

$$N_2 = \frac{a_2}{\theta\lambda_2}, \quad C_2 = \frac{\omega}{\lambda_2}\left(R_1 - 1\right),$$

where

$$R_1 = \frac{\alpha\theta\lambda_2}{\omega a_2}.$$

As $N_2 > 0$, the point E_2 exists if $R_1 > 1$. This equilibrium represents the case of a person who has cancer, but without SARS-CoV-2 infection.

(4) The infected-epithelial equilibrium $E_3 = (N_3, M_3, 0, V_3, 0, 0)$, where

$$N_3 = \frac{\alpha p\epsilon_1}{\omega p\epsilon_1 + \lambda_1 a_3}, \quad M_3 = \frac{a_3}{p\epsilon_1},$$

and

$$V_3 = \frac{\omega p a_1}{\omega p\epsilon_1 + \lambda_1 a_3}\left(R_0 - 1 - \frac{\lambda_1 a_3}{\omega p\epsilon_1}\right).$$

Clearly, $N_3 > 0$, $M_3 > 0$, and $V_3 > 0$ if

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p\epsilon_1}.$$

Hence, E_3 exists if

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1}.$$

The person here suffers from SARS-CoV-2 infection, but he is cancer-free.

(5) The infected-cancer equilibrium $E_4 = (N_4, 0, C_4, V_4, 0, 0)$, where

$$N_4 = \frac{\alpha p \epsilon_2}{\omega p \epsilon_2 + \lambda_2 a_3}, \quad C_4 = \frac{a_3}{p \epsilon_2},$$

and

$$V_4 = \frac{\omega p a_2}{\omega p \epsilon_2 + \lambda_2 a_3}\left(R_1 - 1 - \frac{\lambda_2 a_3}{\omega p \epsilon_2}\right).$$

Notably, $N_4 > 0$, $C_4 > 0$, and $V_4 > 0$ if

$$R_1 > 1 + \frac{\lambda_2 a_3}{\omega p \epsilon_2}.$$

Thus, E_4 exists if

$$R_1 > 1 + \frac{\lambda_2 a_3}{\omega p \epsilon_2}.$$

In this scenario, the cancer patient is experiencing a SARS-CoV-2 infection with the disappearance of healthy epithelial cells.

(6) The uninfected-cancer equilibrium with CTLs $E_5 = (N_5, 0, C_5, 0, T_5, 0)$, where

$$N_5 = \frac{\alpha s_1 \xi_1}{\lambda_2 a_4 + \omega s_1 \xi_1}, \quad C_5 = \frac{a_4}{s_1 \xi_1},$$

and

$$T_5 = \frac{\omega s_1 a_2}{\lambda_2 a_4 + \omega s_1 \xi_1}\left(R_1 - 1 - \frac{\lambda_2 a_4}{\omega s_1 \xi_1}\right).$$

Clearly, $N_5 > 0$, $C_5 > 0$, and $T_5 > 0$ if

$$R_1 > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1}.$$

Hence, E_5 exists if

$$R_1 > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1}.$$

CTLs are activated to eliminate cancer cells in a patient who has experienced the disappearance of healthy epithelial cells.

(7) The infected epithelial-cancer equilibrium without immunity $E_6 = (N_6, M_6, C_6, V_6, 0, 0)$, where

$$N_6 = \frac{a_2\left(\dfrac{a_1\epsilon_2}{a_2\epsilon_1} - 1\right)}{\theta\lambda_2\left(\dfrac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1} - 1\right)},$$

$$M_6 = \frac{\omega\epsilon_2\left(1 + \dfrac{\lambda_2 a_3}{\omega p\epsilon_2} + \dfrac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2} - R_1 - \dfrac{a_1\epsilon_2}{a_2\epsilon_1} - \dfrac{a_1\lambda_2 a_3}{\omega p\epsilon_1 a_2}\right)}{\epsilon_1\lambda_2\left(\dfrac{a_1\epsilon_2}{a_2\epsilon_1} - 1\right)\left(\dfrac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1} - 1\right)},$$

$$C_6 = \frac{\omega a_1\epsilon_2\left(1 + \dfrac{\lambda_1 a_3}{\omega p\epsilon_1} + \dfrac{\alpha\theta\epsilon_1\lambda_2}{\omega a_1\epsilon_2} - R_0 - \dfrac{a_2\epsilon_1}{a_1\epsilon_2} - \dfrac{\lambda_1 a_2 a_3}{\omega p a_1\epsilon_2}\right)}{\epsilon_1 a_2\lambda_2\left(\dfrac{a_1\epsilon_2}{a_2\epsilon_1} - 1\right)\left(\dfrac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1} - 1\right)},$$

$$V_6 = \frac{\lambda_1 a_2\left(\dfrac{a_1\lambda_2}{a_2\lambda_1} - 1\right)}{\epsilon_1\lambda_2\left(\dfrac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1} - 1\right)}.$$

We note that the components are positive if

$$R_1 + \frac{a_1\epsilon_2}{a_2\epsilon_1} + \frac{a_1\lambda_2 a_3}{\omega p\epsilon_1 a_2} < 1 + \frac{\lambda_2 a_3}{\omega p\epsilon_2} + \frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2},$$

$$R_0 + \frac{a_2\epsilon_1}{a_1\epsilon_2} + \frac{\lambda_1 a_2 a_3}{\omega p a_1\epsilon_2} < 1 + \frac{\lambda_1 a_3}{\omega p\epsilon_1} + \frac{\alpha\theta\epsilon_1\lambda_2}{\omega a_1\epsilon_2},$$

$$\frac{a_1\epsilon_2}{a_2\epsilon_1} > 1, \quad \frac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1} > 1, \quad \text{and} \quad \frac{a_1\lambda_2}{a_2\lambda_1} > 1.$$

Thus, E_6 is defined when the above conditions are met. In this scenario, the cancer patient has SARS-CoV-2 infection with inactive immune responses.

(8) The uninfected epithelial-cancer equilibrium with CTLs $E_7 = (N_7, M_7, C_7, 0, T_7, 0)$, where

$$N_7 = \frac{a_1}{\theta\lambda_1}, \quad M_7 = \frac{\omega}{\lambda_1}\left(R_0 - 1 - \frac{\lambda_2 a_4}{\omega s_1\xi_1}\right), \quad C_7 = \frac{a_4}{s_1\xi_1}, \quad T_7 = \frac{a_2}{\xi_1}\left(\frac{a_1\lambda_2}{a_2\lambda_1} - 1\right).$$

Accordingly, $N_7 > 0$, $C_7 > 0$, $M_7 > 0$ if

$$R_0 > 1 + \frac{\lambda_2 a_4}{\omega s_1\xi_1},$$

and $T_7 > 0$ if

$$\frac{a_1\lambda_2}{a_2\lambda_1} > 1.$$

Therefore, E_7 exists when

$$R_0 > 1 + \frac{\lambda_2 a_4}{\omega s_1\xi_1} \quad \text{and} \quad \frac{a_1\lambda_2}{a_2\lambda_1} > 1.$$

Here, the cancer patient has active CTL immunity that works on killing cancer cells.

(9) The infected-epithelial equilibrium with antibodies $E_8 = (N_8, M_8, 0, V_8, 0, A_8)$, where

$$N_8 = \frac{\epsilon_1 a_5 + a_1 s_2 \xi_2}{\theta \lambda_1 s_2 \xi_2}, \quad M_8 = \frac{\omega a_1 s_2 \xi_2}{\lambda_1 (\epsilon_1 a_5 + a_1 s_2 \xi_2)} \left(R_0 - 1 - \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} \right), \quad V_8 = \frac{a_5}{s_2 \xi_2},$$
$$A_8 = \frac{\omega p a_1 \epsilon_1 s_2}{\lambda_1 (\epsilon_1 a_5 + a_1 s_2 \xi_2)} \left(R_0 - 1 - \frac{\lambda_1 a_3}{\omega p \epsilon_1} - \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} - \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2} \right).$$

Thus, the components are positive and E_8 exists if

$$R_0 > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} \quad \text{and} \quad R_0 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2}.$$

Notably, the first condition is naturally satisfied when the second condition is met.

The SARS-CoV-2 patient has active antibody immunity against the virus.

(10) The infected-cancer equilibrium with antibodies $E_9 = (N_9, 0, C_9, V_9, 0, A_9)$, where

$$N_9 = \frac{\epsilon_2 a_5 + a_2 s_2 \xi_2}{\theta \lambda_2 s_2 \xi_2}, \quad C_9 = \frac{\omega a_2 s_2 \xi_2}{\lambda_2 (\epsilon_2 a_5 + a_2 s_2 \xi_2)} \left(R_1 - 1 - \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} \right), \quad V_9 = \frac{a_5}{s_2 \xi_2},$$
$$A_9 = \frac{\omega p a_2 \epsilon_2 s_2}{\lambda_2 (\epsilon_2 a_5 + a_2 s_2 \xi_2)} \left(R_1 - 1 - \frac{\lambda_2 a_3}{\omega p \epsilon_2} - \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} - \frac{\lambda_2 a_3 a_5}{\omega p a_2 s_2 \xi_2} \right).$$

We note that E_9 is defined when

$$R_1 > 1 + \frac{\lambda_2 a_3}{\omega p \epsilon_2} + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} + \frac{\lambda_2 a_3 a_5}{\omega p a_2 s_2 \xi_2},$$

where the other condition

$$R_1 > 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}$$

is naturally met when the previous condition is satisfied.

The infected cancer patient who suffers from the disappearance of epithelial cells has active antibody immunity against the virus.

(11) The infected-cancer equilibrium with CTLs and antibodies $E_{10} = (N_{10}, 0, C_{10}, V_{10}, T_{10}, A_{10})$, where

$$N_{10} = \frac{\alpha s_1 \xi_1}{\lambda_2 a_4 + \omega s_1 \xi_1}, \quad C_{10} = \frac{a_4}{s_1 \xi_1}, \quad V_{10} = \frac{a_5}{s_2 \xi_2},$$
$$T_{10} = \frac{\omega s_1 a_2}{\lambda_2 a_4 + \omega s_1 \xi_1} \left(R_1 - 1 - \frac{\lambda_2 a_4}{\omega s_1 \xi_1} - \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} - \frac{\epsilon_2 \lambda_2 a_4 a_5}{\omega s_1 \xi_1 a_2 s_2 \xi_2} \right),$$
$$A_{10} = \frac{a_3}{\xi_2} \left(\frac{p \epsilon_2 a_4}{s_1 \xi_1 a_3} - 1 \right).$$

Thus, E_{10} is biologically accepted when

$$R_1 > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} + \frac{\epsilon_2 \lambda_2 a_4 a_5}{\omega s_1 \xi_1 a_2 s_2 \xi_2} \quad \text{and} \quad \frac{p \epsilon_2 a_4}{s_1 \xi_1 a_3} > 1.$$

This point represents the case of a cancer patient who is infected, with active immune responses, but experiencing the disappearance of epithelial cells.

(12) The infected epithelial-cancer equilibrium with CTLs $E_{11} = (N_{11}, M_{11}, C_{11}, V_{11}, T_{11}, 0)$, where

$$N_{11} = \frac{\alpha\epsilon_1 s_1\xi_1}{\lambda_1\epsilon_2 a_4\left(\frac{\lambda_2\epsilon_1}{\lambda_1\epsilon_2} + \frac{s_1\xi_1 a_3}{p\epsilon_2 a_4} + \frac{\omega\epsilon_1 s_1\xi_1}{\lambda_1\epsilon_2 a_4} - 1\right)},$$

$$M_{11} = \frac{\epsilon_2 a_4}{\epsilon_1 s_1\xi_1}\left(\frac{s_1\xi_1 a_3}{p\epsilon_2 a_4} - 1\right), \quad C_{11} = \frac{a_4}{s_1\xi_1},$$

$$V_{11} = \frac{\omega a_1 s_1\xi_1\left(R_0 + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} - 1 - \frac{\lambda_2 a_4}{\omega s_1\xi_1} - \frac{\lambda_1 a_3}{\omega p\epsilon_1}\right)}{\lambda_1\epsilon_2 a_4\left(\frac{\lambda_2\epsilon_1}{\lambda_1\epsilon_2} + \frac{s_1\xi_1 a_3}{p\epsilon_2 a_4} + \frac{\omega\epsilon_1 s_1\xi_1}{\lambda_1\epsilon_2 a_4} - 1\right)},$$

$$T_{11} = \frac{\omega\epsilon_1 s_1 a_2}{\lambda_1\epsilon_2 a_4\left(\frac{\lambda_2\epsilon_1}{\lambda_1\epsilon_2} + \frac{s_1\xi_1 a_3}{p\epsilon_2 a_4} + \frac{\omega\epsilon_1 s_1\xi_1}{\lambda_1\epsilon_2 a_4} - 1\right)}\left(R_1 + \frac{a_1\epsilon_2}{a_2\epsilon_1} + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} + \frac{a_1\lambda_1\epsilon_2 a_3}{\omega p\epsilon_1^2 a_2} + \frac{a_1\epsilon_2\lambda_2 a_4}{\omega\epsilon_1 s_1\xi_1 a_2}\right.$$
$$\left. - 1 - \frac{\lambda_1 a_3}{\omega p\epsilon_1} - \frac{\lambda_2 a_4}{\omega s_1\xi_1} - \frac{a_1\lambda_1\epsilon_2^2 a_4}{\omega\epsilon_1^2 s_1\xi_1 a_2} - \frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2}\right).$$

It is easy to observe that N_{11}, V_{11}, and T_{11} are positive and E_{11} exists when

$$\frac{\lambda_2\epsilon_1}{\lambda_1\epsilon_2} + \frac{s_1\xi_1 a_3}{p\epsilon_2 a_4} + \frac{\omega\epsilon_1 s_1\xi_1}{\lambda_1\epsilon_2 a_4} > 1,$$

$$R_0 + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} > 1 + \frac{\lambda_2 a_4}{\omega s_1\xi_1} + \frac{\lambda_1 a_3}{\omega p\epsilon_1}$$

and

$$R_1 + \frac{a_1\epsilon_2}{a_2\epsilon_1} + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} + \frac{a_1\lambda_1\epsilon_2 a_3}{\omega p\epsilon_1^2 a_2} + \frac{a_1\epsilon_2\lambda_2 a_4}{\omega\epsilon_1 s_1\xi_1 a_2} > 1 + \frac{\lambda_1 a_3}{\omega p\epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1\xi_1} + \frac{a_1\lambda_1\epsilon_2^2 a_4}{\omega\epsilon_1^2 s_1\xi_1 a_2} + \frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2}.$$

At this point, the infected cancer patient has active CTLs and inactive antibody immune response.

(13) The infected epithelial-cancer equilibrium with CTLs and antibodies $E_{12} = (N_{12}, M_{12}, C_{12}, V_{12}, T_{12}, A_{12})$, where

$$N_{12} = \frac{\epsilon_1 a_5 + a_1 s_2\xi_2}{\theta\lambda_1 s_2\xi_2},$$

$$M_{12} = \frac{\omega a_1 s_2\xi_2}{\lambda_1(\epsilon_1 a_5 + a_1 s_2\xi_2)}\left(R_0 - 1 - \frac{\epsilon_1 a_5}{a_1 s_2\xi_2} - \frac{\lambda_2 a_4}{\omega s_1\xi_1} - \frac{\epsilon_1\lambda_2 a_4 a_5}{\omega a_1 s_1\xi_1 s_2\xi_2}\right),$$

$$C_{12} = \frac{a_4}{s_1\xi_1}, \quad V_{12} = \frac{a_5}{s_2\xi_2},$$

$$T_{12} = \frac{a_2 s_2}{s_1\xi_1}\left(\frac{a_1\lambda_2}{a_2\lambda_1} + \frac{\epsilon_1\lambda_2 a_5}{\lambda_1 a_2 s_2\xi_2} - 1 - \frac{\epsilon_2 a_5}{a_2 s_2\xi_2}\right),$$

$$A_{12} = \frac{\omega p a_1\epsilon_1 s_2}{\lambda_1(\epsilon_1 a_5 + a_1 s_2\xi_2)}\left(R_0 + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} + \frac{\lambda_1\epsilon_2 a_4 a_5}{\omega a_1 s_1\xi_1 s_2\xi_2} - 1 - \frac{\epsilon_1 a_5}{a_1 s_2\xi_2} - \frac{\lambda_1 a_3}{\omega p\epsilon_1}\right.$$
$$\left. - \frac{\lambda_2 a_4}{\omega s_1\xi_1} - \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} - \frac{\epsilon_1\lambda_2 a_4 a_5}{\omega a_1 s_1\xi_1 s_2\xi_2}\right).$$

We observe that the components are positive and E_{12} is defined if

$$R_0 > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2},$$

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} > 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{\lambda_1 \epsilon_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2}.$$

This point imitates the case of cancer patient with SARS-CoV-2 infection and active immune responses. □

In the next sections, we will focus our analysis on the equilibria E_0, E_1, E_3, E_6, E_7, E_8, E_{11}, and E_{12} as we are interested in the points where the epithelial cells component (M) does not vanish.

4. Global properties

The following theorems are aimed to establish the global stability of equilibria through nominating Lyapunov functions. Let $Y_i^{'}$ be the largest invariant subset of

$$Y_i = \left\{ (N, M, C, V, T, A) \mid \frac{d\Sigma_i}{dt} = 0 \right\},$$

where $i = 0, 1, 3, 6, 7, 8, 11, 12$.

Theorem 4. *The equilibrium E_0 is globally asymptotically stable (GS) when $R_0 \leq 1$ and $R_1 \leq 1$.*

Proof. We consider

$$\Sigma_0(t) = N_0 \left(\frac{N}{N_0} - 1 - \ln \frac{N}{N_0} \right) + \frac{1}{\theta} M + \frac{1}{\theta} C + \frac{1}{\theta p} V + \frac{1}{\theta s_1} T + \frac{1}{\theta p s_2} A.$$

Then, we get

$$\begin{aligned}
\frac{d\Sigma_0}{dt} =& \left(1 - \frac{N_0}{N}\right)(\alpha - \lambda_1 NM - \lambda_2 NC - \omega N) + \frac{1}{\theta}(\theta \lambda_1 NM - \epsilon_1 MV - a_1 M) \\
&+ \frac{1}{\theta}(\theta \lambda_2 NC - \epsilon_2 CV - \xi_1 CT - a_2 C) + \frac{1}{\theta p}(p\epsilon_1 MV + p\epsilon_2 CV - \xi_2 VA - a_3 V) \\
&+ \frac{1}{\theta s_1}(s_1 \xi_1 CT - a_4 T) + \frac{1}{\theta p s_2}(s_2 \xi_2 VA - a_5 A) \\
=& \left(1 - \frac{N_0}{N}\right)(\alpha - \omega N) + \lambda_1 N_0 M + \lambda_2 N_0 C - \frac{a_1}{\theta} M - \frac{a_2}{\theta} C - \frac{a_3}{\theta p} V - \frac{a_5}{\theta p s_2} A \\
=& -\frac{\omega (N - N_0)^2}{N} + \frac{a_1}{\theta}(R_0 - 1) M + \frac{a_2}{\theta}(R_1 - 1) C - \frac{a_3}{\theta p} V - \frac{a_5}{\theta p s_2} A.
\end{aligned}$$

We see that

$$\frac{d\Sigma_0}{dt} \leq 0$$

if $R_0 \leq 1$ and $R_1 \leq 1$. Furthermore,

$$\frac{d\Sigma_0}{dt} = 0$$

when $N = N_0$ and $M = C = V = A = 0$. This gives

$$\frac{dC}{dt} = 0.$$

From the third equation of model (2.1), we obtain $T = 0$. Hence, $Y_0^{'} = \{E_0\}$ and by LaSalle's invariance principle (LP) [37], E_0 is GS if $R_0 \leq 1$ and $R_1 \leq 1$. □

Theorem 5. *Let $R_0 > 1$. Then, the equilibrium E_1 is GS if*

$$R_0 \leq 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1}$$

and

$$\frac{a_1\lambda_2}{a_2\lambda_1} \leq 1.$$

Proof. We opt

$$\Sigma_1(t) = N_1\left(\frac{N}{N_1} - 1 - \ln\frac{N}{N_1}\right) + \frac{1}{\theta}M_1\left(\frac{M}{M_1} - 1 - \ln\frac{M}{M_1}\right) + \frac{1}{\theta}C + \frac{1}{\theta p}V + \frac{1}{\theta s_1}T + \frac{1}{\theta p s_2}A.$$

Then, we get

$$\begin{aligned}\frac{d\Sigma_1}{dt} =& \left(1 - \frac{N_1}{N}\right)(\alpha - \lambda_1 NM - \lambda_2 NC - \omega N) + \frac{1}{\theta}\left(1 - \frac{M_1}{M}\right)(\theta\lambda_1 NM - \epsilon_1 MV - a_1 M)\\ &+ \frac{1}{\theta}\left(\theta\lambda_2 NC - \epsilon_2 CV - \xi_1 CT - a_2 C\right) + \frac{1}{\theta p}\left(p\epsilon_1 MV + p\epsilon_2 CV - \xi_2 VA - a_3 V\right)\\ &+ \frac{1}{\theta s_1}\left(s_1\xi_1 CT - a_4 T\right) + \frac{1}{\theta p s_2}\left(s_2\xi_2 VA - a_5 A\right).\end{aligned} \tag{4.1}$$

At equilibrium, E_1 solves the system

$$\begin{cases}\alpha = \lambda_1 N_1 M_1 + \omega N_1,\\ \lambda_1 N_1 M_1 = \dfrac{a_1}{\theta}M_1.\end{cases}$$

Applying the above equations to collect (4.1) gives

$$\begin{aligned}\frac{d\Sigma_1}{dt} =& \left(1 - \frac{N_1}{N}\right)(\omega N_1 - \omega N) + \lambda_1 N_1 M_1\left(2 - \frac{N_1}{N} - \frac{N}{N_1}\right)\\ &+ \left(\lambda_2 N_1 - \frac{a_2}{\theta}\right)C + \left(\frac{\epsilon_1}{\theta}M_1 - \frac{a_3}{\theta p}\right)V - \frac{a_4}{\theta s_1}T - \frac{a_5}{\theta p s_2}A\end{aligned}$$

$$= -\frac{\omega (N-N_1)^2}{N} + \lambda_1 N_1 M_1 \left(2 - \frac{N_1}{N} - \frac{N}{N_1}\right) + \frac{a_2}{\theta}\left(\frac{a_1\lambda_2}{a_2\lambda_1} - 1\right) C$$
$$+ \frac{\omega\epsilon_1}{\theta\lambda_1}\left(R_0 - 1 - \frac{\lambda_1 a_3}{\omega p \epsilon_1}\right) V - \frac{a_4}{\theta s_1} T - \frac{a_5}{\theta p s_2} A.$$

Hence,

$$\frac{d\Sigma_1}{dt} \leq 0$$

if

$$R_0 \leq 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} \quad \text{and} \quad \frac{a_1\lambda_2}{a_2\lambda_1} \leq 1.$$

In addition,

$$\frac{d\Sigma_1}{dt} = 0$$

if $N = N_1$ and $C = V = T = A = 0$. Consequently,

$$\frac{dN}{dt} = 0$$

and the first equation of (2.1) gives $M = M_1$. Therefore, $Y_1^{'} = \{E_1\}$ and E_1 is GS when

$$R_0 > 1, \quad R_0 \leq 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} \quad \text{and} \quad \frac{a_1\lambda_2}{a_2\lambda_1} \leq 1$$

according to LP [37]. □

Theorem 6. *Let*

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1}.$$

Then, the equilibrium E_3 is GS if

$$R_0 + \frac{a_2\epsilon_1}{a_1\epsilon_2} + \frac{\lambda_1 a_2 a_3}{\omega p a_1 \epsilon_2} \geq 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\alpha\theta\epsilon_1\lambda_2}{\omega a_1 \epsilon_2}$$

and

$$R_0 \leq 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2}.$$

Proof. We construct

$$\Sigma_3(t) = N_3\left(\frac{N}{N_3} - 1 - \ln\frac{N}{N_3}\right) + \frac{1}{\theta} M_3 \left(\frac{M}{M_3} - 1 - \ln\frac{M}{M_3}\right) + \frac{1}{\theta} C + \frac{1}{\theta p} V_3 \left(\frac{V}{V_3} - 1 - \ln\frac{V}{V_3}\right) + \frac{1}{\theta s_1} T + \frac{1}{\theta p s_2} A.$$

Then, we get

$$\begin{aligned}\frac{d\Sigma_3}{dt} =& \left(1 - \frac{N_3}{N}\right)(\alpha - \lambda_1 NM - \lambda_2 NC - \omega N) + \frac{1}{\theta}\left(1 - \frac{M_3}{M}\right)(\theta\lambda_1 NM - \epsilon_1 MV - a_1 M) \\ &+ \frac{1}{\theta}(\theta\lambda_2 NC - \epsilon_2 CV - \xi_1 CT - a_2 C) + \frac{1}{\theta p}\left(1 - \frac{V_3}{V}\right)(p\epsilon_1 MV + p\epsilon_2 CV - \xi_2 VA - a_3 V) \\ &+ \frac{1}{\theta s_1}(s_1\xi_1 CT - a_4 T) + \frac{1}{\theta p s_2}(s_2\xi_2 VA - a_5 A).\end{aligned} \tag{4.2}$$

The equilibrium conditions at E_3 are

$$\begin{cases} \alpha = \lambda_1 N_3 M_3 + \omega N_3, \\ \lambda_1 N_3 M_3 = \dfrac{\epsilon_1}{\theta} M_3 V_3 + \dfrac{a_1}{\theta} M_3, \\ \dfrac{\epsilon_1}{\theta} M_3 V_3 = \dfrac{a_3}{\theta p} V_3. \end{cases} \tag{4.3}$$

By using (6), Eq (4.2) becomes

$$\begin{aligned} \frac{d\Sigma_3}{dt} =& \left(1 - \frac{N_3}{N}\right)(\omega N_3 - \omega N) + \lambda_1 N_3 M_3 \left(2 - \frac{N_3}{N} - \frac{N}{N_3}\right) + \left(\lambda_2 N_3 - \frac{a_2}{\theta} - \frac{\epsilon_2}{\theta} V_3\right) C \\ &+ \left(\frac{\xi_2}{\theta p} V_3 - \frac{a_5}{\theta p s_2}\right) A - \frac{a_4}{\theta s_1} T \\ =& - \frac{\omega (N - N_3)^2}{N} + \lambda_1 N_3 M_3 \left(2 - \frac{N_3}{N} - \frac{N}{N_3}\right) \\ &+ \frac{\omega p a_1 \epsilon_2}{\theta (\lambda_1 a_3 + \omega p \epsilon_1)} \left(1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\alpha \theta \epsilon_1 \lambda_2}{\omega a_1 \epsilon_2} - R_0 - \frac{a_2 \epsilon_1}{a_1 \epsilon_2} - \frac{\lambda_1 a_2 a_3}{\omega p a_1 \epsilon_2}\right) C \\ &+ \frac{\omega a_1 \xi_2}{\theta (\lambda_1 a_3 + \omega p \epsilon_1)} \left(R_0 - 1 - \frac{\lambda_1 a_3}{\omega p \epsilon_1} - \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} - \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2}\right) A - \frac{a_4}{\theta s_1} T. \end{aligned}$$

We observe that

$$\frac{d\Sigma_3}{dt} \leq 0$$

if

$$R_0 + \frac{a_2 \epsilon_1}{a_1 \epsilon_2} + \frac{\lambda_1 a_2 a_3}{\omega p a_1 \epsilon_2} \geq 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\alpha \theta \epsilon_1 \lambda_2}{\omega a_1 \epsilon_2}$$

and

$$R_0 \leq 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2}.$$

In addition, $\dfrac{d\Sigma_3}{dt} = 0$ when $N = N_3$, $M = M_3$, $V = V_3$ and $C = T = A = 0$. Thus, $Y_3' = \{E_3\}$ and LP [37] implies that E_3 is GS when

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1}$$

with the above conditions. □

Theorem 7. *Let the existence conditions of E_6 be satisfied. Then, the equilibrium E_6 is GS if*

$$R_1 + \frac{a_1 \epsilon_2}{a_2 \epsilon_1} + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2 a_3}{\omega p \epsilon_1^2 a_2} + \frac{a_1 \epsilon_2 \lambda_2 a_4}{\omega \epsilon_1 s_1 \xi_1 a_2} \leq 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2^2 a_4}{\omega \epsilon_1^2 s_1 \xi_1 a_2} + \frac{\alpha \theta \lambda_1 \epsilon_2}{\omega \epsilon_1 a_2}$$

and

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} \leq 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}.$$

Proof. See Appendix A. □

Theorem 8. *Let*

$$R_0 > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} \quad \textit{and} \quad \frac{a_1 \lambda_2}{a_2 \lambda_1} > 1.$$

Then, the equilibrium E_7 is GS if

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} \leq 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3}{\omega p \epsilon_1}.$$

Proof. See Appendix B. □

Theorem 9. *Let*

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2}.$$

Then, E_8 is GS if

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} \leq 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}.$$

Proof. See Appendix C. □

Theorem 10. *Let*

$$\frac{\lambda_2 \epsilon_1}{\lambda_1 \epsilon_2} + \frac{s_1 \xi_1 a_3}{p \epsilon_2 a_4} + \frac{\omega \epsilon_1 s_1 \xi_1}{\lambda_1 \epsilon_2 a_4} > 1,$$

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3}{\omega p \epsilon_1}$$

and

$$R_1 + \frac{a_1 \epsilon_2}{a_2 \epsilon_1} + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2 a_3}{\omega p \epsilon_1^2 a_2} + \frac{a_1 \epsilon_2 \lambda_2 a_4}{\omega \epsilon_1 s_1 \xi_1 a_2} > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2^2 a_4}{\omega \epsilon_1^2 s_1 \xi_1 a_2} + \frac{\alpha \theta \lambda_1 \epsilon_2}{\omega \epsilon_1 a_2}.$$

Then, E_{11} is GS if

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{\lambda_1 \epsilon_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} \leq 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2}.$$

Proof. See Appendix D. □

Theorem 11. *Let*

$$R_0 > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2},$$

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} > 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{\lambda_1 \epsilon_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2}.$$

Then, the equilibrium E_{12} is GS.

Proof. We nominate

$$\begin{aligned}\Sigma_{12}(t) =& N_{12}\left(\frac{N}{N_{12}}-1-\ln\frac{N}{N_{12}}\right)+\frac{1}{\theta}M_{12}\left(\frac{M}{M_{12}}-1-\ln\frac{M}{M_{12}}\right)+\frac{1}{\theta}C_{12}\left(\frac{C}{C_{12}}-1-\ln\frac{C}{C_{12}}\right)\\ &+\frac{1}{\theta p}V_{12}\left(\frac{V}{V_{12}}-1-\ln\frac{V}{V_{12}}\right)+\frac{1}{\theta s_1}T_{12}\left(\frac{T}{T_{12}}-1-\ln\frac{T}{T_{12}}\right)+\frac{1}{\theta p s_2}A_{12}\left(\frac{A}{A_{12}}-1-\ln\frac{A}{A_{12}}\right).\end{aligned}$$

Then, we get

$$\begin{aligned}\frac{d\Sigma_{12}}{dt} =& \left(1-\frac{N_{12}}{N}\right)(\alpha-\lambda_1 NM-\lambda_2 NC-\omega N)+\frac{1}{\theta}\left(1-\frac{M_{12}}{M}\right)(\theta\lambda_1 NM-\epsilon_1 MV-a_1 M)\\ &+\frac{1}{\theta}\left(1-\frac{C_{12}}{C}\right)(\theta\lambda_2 NC-\epsilon_2 CV-\xi_1 CT-a_2 C)+\frac{1}{\theta p}\left(1-\frac{V_{12}}{V}\right)(p\epsilon_1 MV+p\epsilon_2 CV-\xi_2 VA-a_3 V)\\ &+\frac{1}{\theta s_1}\left(1-\frac{T_{12}}{T}\right)(s_1\xi_1 CT-a_4 T)+\frac{1}{\theta p s_2}\left(1-\frac{A_{12}}{A}\right)(s_2\xi_2 VA-a_5 A).\end{aligned} \tag{4.4}$$

By using the conditions of equilibrium state at E_{12}

$$\begin{cases}\alpha=\lambda_1 N_{12}M_{12}+\lambda_2 N_{12}C_{12}+\omega N_{12},\\ \lambda_1 N_{12}M_{12}=\frac{\epsilon_1}{\theta}M_{12}V_{12}+\frac{a_1}{\theta}M_{12},\\ \lambda_2 N_{12}C_{12}=\frac{\epsilon_2}{\theta}C_{12}V_{12}+\frac{\xi_1}{\theta}C_{12}T_{12}+\frac{a_2}{\theta}C_{12},\\ \frac{\epsilon_1}{\theta}M_{12}V_{12}+\frac{\epsilon_2}{\theta}C_{12}V_{12}=\frac{\xi_2}{\theta p}V_{12}A_{12}+\frac{a_3}{\theta p}V_{12},\\ \frac{\xi_1}{\theta}C_{12}T_{12}=\frac{a_4}{\theta s_1}T_{12},\\ \frac{\xi_2}{\theta p}V_{12}A_{12}=\frac{a_5}{\theta p s_2}A_{12}.\end{cases}$$

Equation (4.4) is transformed into

$$\frac{d\Sigma_{12}}{dt}=-\frac{\omega(N-N_{12})^2}{N}+\lambda_1 N_{12}M_{12}\left(2-\frac{N_{12}}{N}-\frac{N}{N_{12}}\right)+\lambda_2 N_{12}C_{12}\left(2-\frac{N_{12}}{N}-\frac{N}{N_{12}}\right).$$

We note that $\frac{d\Sigma_{12}}{dt}\le 0$ and $\frac{d\Sigma_{12}}{dt}=0$ at E_{12}. Based on LP [37], E_{12} is GS when the existence conditions are met. □

5. Numerical simulations

The ode45 solver of Matlab is utilized to effectuate the numerical simulations. ode45 is the default solver for ODEs in Matlab. It utilizes an explicit Runge-Kutta formula and generally performs well with a wide range of ODE problems. Nevertheless, when dealing with stiff problems or situations demanding high accuracy, alternative solvers like ode15s, ode23s, and ode23t may prove more efficient. We consider three different groups of initial conditions:

(1) $(N(0),M(0),C(0),V(0),T(0),A(0))=(0.0001,0.01,0.03,0.01,0.001,0.001)$;

(2) $(N(0), M(0), C(0), V(0), T(0), A(0)) = (0.1, 0.03, 0.06, 0.05, 0.002, 0.003)$;

(3) $(N(0), M(0), C(0), V(0), T(0), A(0)) = (0.6, 0.06, 0.1, 0.06, 0.03, 0.01)$.

The chosen sets of initial conditions are arbitrary, as the global stability is guaranteed for any initial values. To affirm the global stability of E_0, E_1, E_3, E_6, E_7, E_8, E_{11}, and E_{12}, we partition the numerical simulations into eight classes. We change the values of λ_1, λ_2, s_1, s_2, ϵ_1, ϵ_2, ξ_2, ω_2, ω_3, and ω_5 to obtain the global stability of the equilibrium in each case. The other values are fixed and given in Table 1.

Table 1. Parameters' values of system (2.1).

Parameter	Value	Source
α	0.02	[33]
λ_1	Varied	–
λ_2	Varied	–
ω	0.02	[33]
ω_1	0.01	[38]
ω_2	Varied	–
ω_3	Varied	–
ω_4	0.1	[38]
ω_5	Varied	–
θ	0.8	[33]
ϵ_1	Varied	–
ϵ_2	Varied	–
p	0.24	[39]
ξ_1	0.5	[38]
ξ_2	Varied	–
s_1	Varied	–
s_2	Varied	–

Thus, we have

(i) We opt $\lambda_1 = 0.03$, $\lambda_2 = 0.03$, $s_1 = 0.1$, $s_2 = 0.2$, $\epsilon_1 = 0.55$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.17$, $\omega_3 = 0.6$, and $\omega_5 = 0.05$. This yields $R_0 = 0.8 < 1$ and $R_1 = 0.1263 < 1$. Thus, $E_0 = (1, 0, 0, 0, 0, 0)$ is GS as indicated in Theorem 4 (see Figure 2). This point has no significant biological interpretation as all populations, except the nutrient's component, tend to zero.

(ii) We choose $\lambda_1 = 0.05$, $\lambda_2 = 0.03$, $s_1 = 0.1$, $s_2 = 0.2$, $\epsilon_1 = 1 \times 10^{-5}$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.17$, $\omega_3 = 0.9$, and $\omega_5 = 0.05$. The corresponding thresholds are

$$R_0 = 1.33 > 1, \quad R_0 < 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} = 9.58 \times 10^5 \quad \text{and} \quad \frac{a_1 \lambda_2}{a_2 \lambda_1} = 0.0947 < 1.$$

Thus, $E_1 = (0.75, 0.1333, 0, 0, 0, 0)$ is GS which matches with Theorem 5 (see Figure 3). This simulates the case of an individual who neither has SARS-CoV-2 infection nor cancer.

(iii) We nominate $\lambda_1 = 0.07$, $\lambda_2 = 0.03$, $s_1 = 0.1$, $s_2 = 0.2$, $\epsilon_1 = 0.5$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$,

$\omega_2 = 0.17$, $\omega_3 = 1 \times 10^{-4}$, and $\omega_5 = 0.2$. This gives

$$R_0 = 1.8667 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} = 1.5863,$$

$$R_0 + \frac{a_2 \epsilon_1}{a_1 \epsilon_2} + \frac{\lambda_1 a_2 a_3}{\omega p a_1 \epsilon_2} = 10.9996 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\alpha \theta \epsilon_1 \lambda_2}{\omega a_1 \epsilon_2} = 2.3135$$

and

$$R_0 < 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2} = 5.9593 \times 10^8.$$

This causes $E_3 = (0.6304, 0.1675, 0, 0.01061, 0, 0)$ to be GS as verified in Theorem 6 (see Figure 4). This imitates the case of a patient with SARS-CoV-2 infection but without cancer.

(iv) We pick out $\lambda_1 = 0.06$, $\lambda_2 = 0.05$, $s_1 = 0.1$, $s_2 = 0.2$, $\epsilon_1 = 1 \times 10^{-5}$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.001$, $\omega_3 = 0.001$, and $\omega_5 = 0.05$. This corresponds to

$$R_1 + \frac{a_1 \epsilon_2}{a_2 \epsilon_1} + \frac{a_1 \lambda_2 a_3}{\omega p \epsilon_1 a_2} = 1.0982 \times 10^5 < 1 + \frac{\lambda_2 a_3}{\omega p \epsilon_2} + \frac{\alpha \theta \lambda_1 \epsilon_2}{\omega \epsilon_1 a_2} = 1.2572 \times 10^5,$$

$$R_0 + \frac{a_2 \epsilon_1}{a_1 \epsilon_2} + \frac{\lambda_1 a_2 a_3}{\omega p a_1 \epsilon_2} = 1.9341 < 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\alpha \theta \epsilon_1 \lambda_2}{\omega a_1 \epsilon_2} = 2.6251 \times 10^4,$$

$$\frac{a_1 \epsilon_2}{a_2 \epsilon_1} = 7.8571 \times 10^4 > 1, \quad \frac{\lambda_1 \epsilon_2}{\lambda_2 \epsilon_1} = 66000 > 1, \quad \frac{a_1 \lambda_2}{a_2 \lambda_1} = 1.1905 > 1,$$

$$\begin{aligned} &R_1 + \frac{a_1 \epsilon_2}{a_2 \epsilon_1} + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2 a_3}{\omega p \epsilon_1^2 a_2} + \frac{a_1 \epsilon_2 \lambda_2 a_4}{\omega \epsilon_1 s_1 \xi_1 a_2} \\ &= 2.0634 \times 10^9 < 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2^2 a_4}{\omega \epsilon_1^2 s_1 \xi_1 a_2} + \frac{\alpha \theta \lambda_1 \epsilon_2}{\omega \epsilon_1 a_2} \\ &= 3.1114 \times 10^{10} \end{aligned}$$

and

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} = 2.8473 \times 10^3 < 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} = 1.8784 \times 10^8.$$

In parallel with Theorem 7, $E_6 = (0.625, 0.06742, 0.1591, 0.0073, 0, 0)$ is GS (see Figure 5). This simulates the case of a cancer patient who has SARS-CoV-2 infection with inactive immune responses.

(v) We opt $\lambda_1 = 0.06$, $\lambda_2 = 0.05$, $s_1 = 1.2$, $s_2 = 0.2$, $\epsilon_1 = 1 \times 10^{-5}$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 1 \times 10^{-4}$, $\omega_3 = 0.9$, and $\omega_5 = 0.05$. The resultant thresholds are

$$R_0 = 1.6 > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} = 1.5, \quad \frac{a_1 \lambda_2}{a_2 \lambda_1} = 1.2438 > 1$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} = 3.3 \times 10^4 < 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} = 1.15 \times 10^6.$$

In agreement with Theorem 8, $E_7 = (0.625, 0.0333, 0.2, 0, 0.0098, 0)$ is GS (see Figure 6). In this scenario, the cancer patient does not have SARS-CoV-2 infection, and the CTL immunity against cancer cells is active.

(vi) We select $\lambda_1 = 0.07$, $\lambda_2 = 0.03$, $s_1 = 0.4$, $s_2 = 1.6$, $\epsilon_1 = 0.5$, $\epsilon_2 = 0.55$, $\xi_2 = 1.6$, $\omega_2 = 0.17$, $\omega_3 = 1 \times 10^{-6}$, and $\omega_5 = 1 \times 10^{-8}$. This gives

$$R_0 = 1.8667 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2} = 1.7895$$

and

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} = 0.0765 < 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} = 1.0226.$$

As indicated in Theorem 9, $E_8 = (0.6055, 0.1862, 0, 0.0078, 0, 0.0015)$ is GS (see Figure 7). Here, the patient is solely affected by SARS-CoV-2 infection with an active immune response against the virus.

(vii) We consider $\lambda_1 = 0.06$, $\lambda_2 = 0.06$, $s_1 = 1.6$, $s_2 = 0.2$, $\epsilon_1 = 1.4 \times 10^{-1}$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.001$, $\omega_3 = 0.001$, and $\omega_5 = 0.05$. This gives

$$\frac{\lambda_2 \epsilon_1}{\lambda_1 \epsilon_2} + \frac{s_1 \xi_1 a_3}{p \epsilon_2 a_4} + \frac{\omega \epsilon_1 s_1 \xi_1}{\lambda_1 \epsilon_2 a_4} = 1.8808 > 1,$$

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} = 3.3679 > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} = 3.325,$$

$$R_1 + \frac{a_1 \epsilon_2}{a_2 \epsilon_1} + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2 a_3}{\omega p \epsilon_1^2 a_2} + \frac{a_1 \epsilon_2 \lambda_2 a_4}{\omega \epsilon_1 s_1 \xi_1 a_2} = 22.7143 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2^2 a_4}{\omega \epsilon_1^2 s_1 \xi_1 a_2} + \frac{\alpha \theta \lambda_1 \epsilon_2}{\omega \epsilon_1 a_2} = 22.2262$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{\lambda_1 \epsilon_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} = 5.917 \times 10^7 < 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} = 1.1129 \times 10^8.$$

As proved in Theorem 10, $E_{11} = (0.6422, 0.0357, 0.15, 0.0059, 0.01316, 0)$ is GS (Figure 8). The CTL immunity against cancer cells is activated in the cancer patient infected with SARS-CoV-2.

(viii) We pick up $\lambda_1 = 0.06$, $\lambda_2 = 0.06$, $s_1 = 1.6$, $s_2 = 1.9$, $\epsilon_1 = 1.4 \times 10^{-1}$, $\epsilon_2 = 1.2$, $\xi_2 = 1.9$, $\omega_2 = 0.001$, $\omega_3 = 0.001$, and $\omega_5 = 0.000001$. The corresponding thresholds are

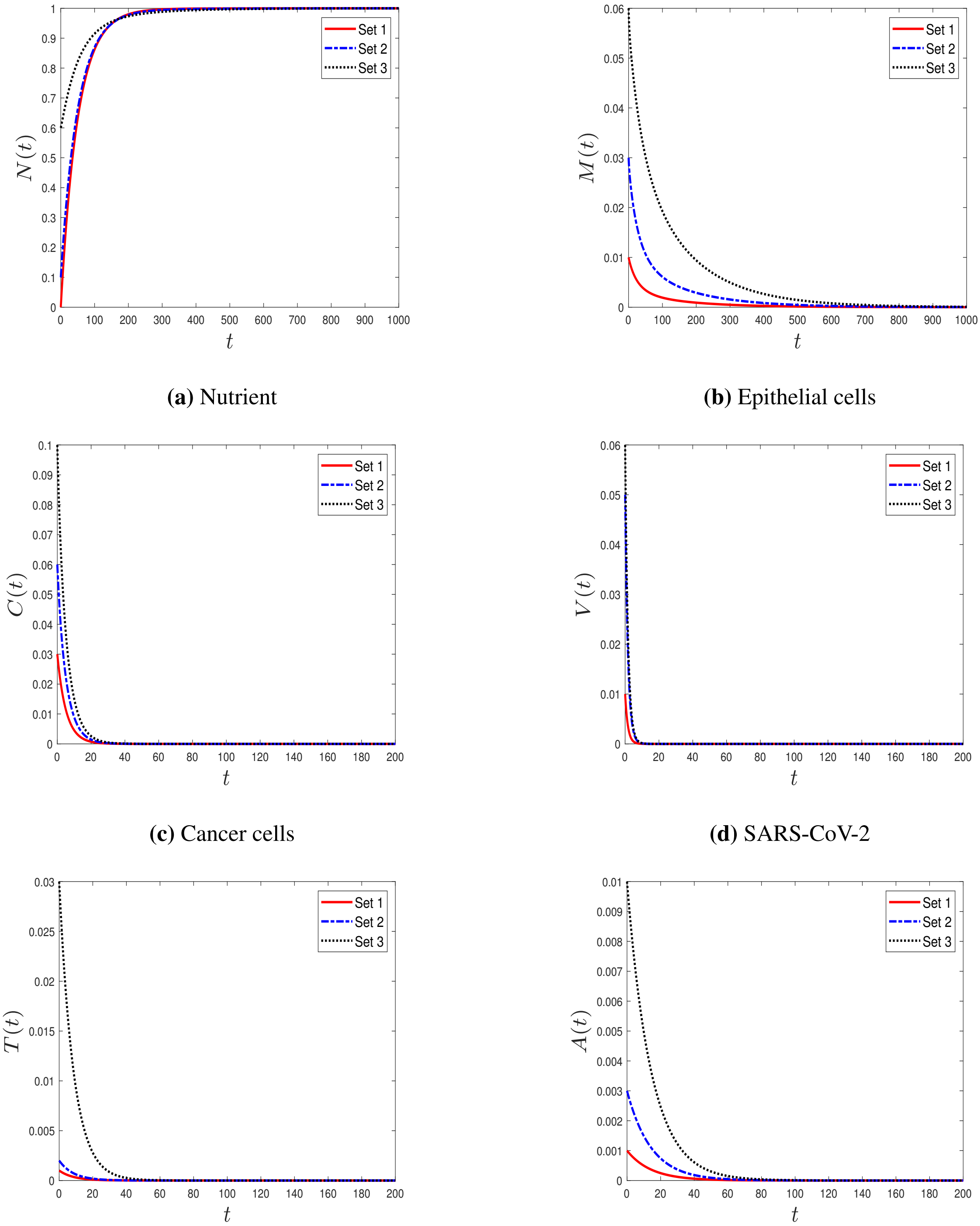
$$R_0 = 1.6 > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} = 1.4875,$$

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} = 1.4655 > 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} = 1.3166$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{\lambda_1 \epsilon_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} = 5.5569 > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} = 3.4110.$$

As indicated in Theorem 11, $E_{12} = (0.641, 0.03657, 0.1499, 0.006, 0.006, 0.0123)$ is GS (see Figure 9). The CTL and antibody immune responses are activated in the cancer patient infected with SARS-CoV-2.



(a) Nutrient

(b) Epithelial cells

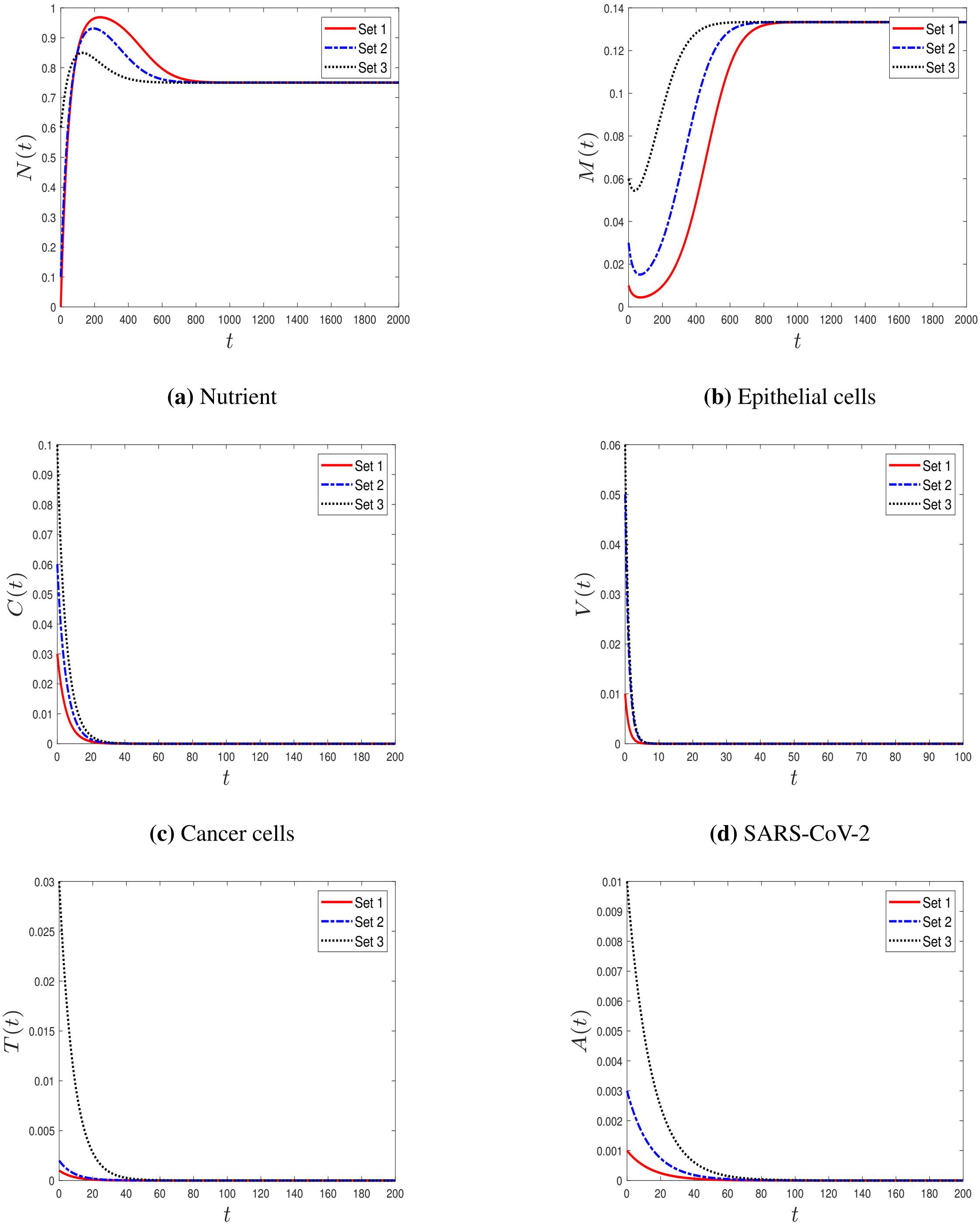
(c) Cancer cells

(d) SARS-CoV-2

(e) Anti-cancer CTLs

(f) Antibodies

Figure 2. The numerical results of system (2.1) for $\lambda_1 = 0.03$, $\lambda_2 = 0.03$, $s_1 = 0.1$, $s_2 = 0.2$, $\epsilon_1 = 0.55$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.17$, $\omega_3 = 0.6$, and $\omega_5 = 0.05$. The point $E_0 = (1, 0, 0, 0, 0, 0)$ is GS.



(a) Nutrient

(b) Epithelial cells

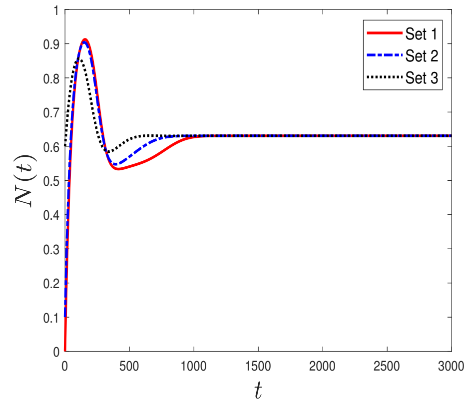
(c) Cancer cells

(d) SARS-CoV-2

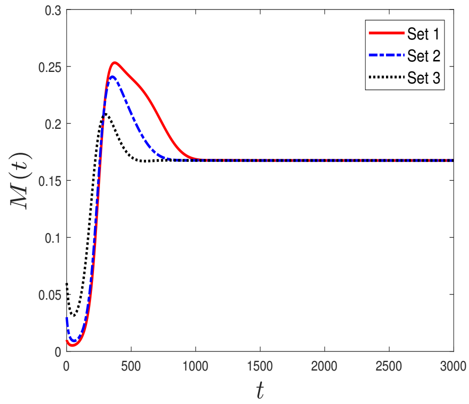
(e) Anti-cancer CTLs

(f) Antibodies

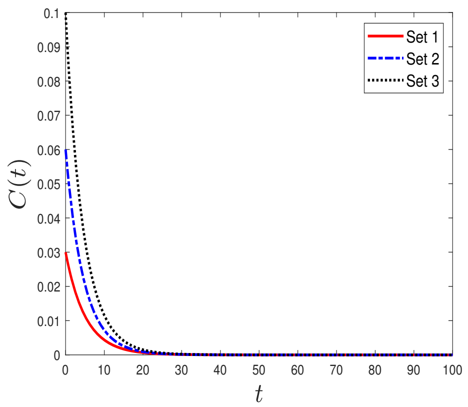
Figure 3. The numerical results of system (2.1) for $\lambda_1 = 0.05$, $\lambda_2 = 0.03$, $s_1 = 0.1$, $s_2 = 0.2$, $\epsilon_1 = 1 \times 10^{-5}$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.17$, $\omega_3 = 0.9$, and $\omega_5 = 0.05$. The point $E_1 = (0.75, 0.1333, 0, 0, 0, 0)$ is GS.



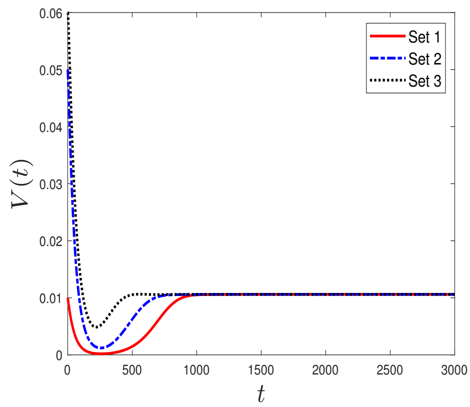
(a) Nutrient



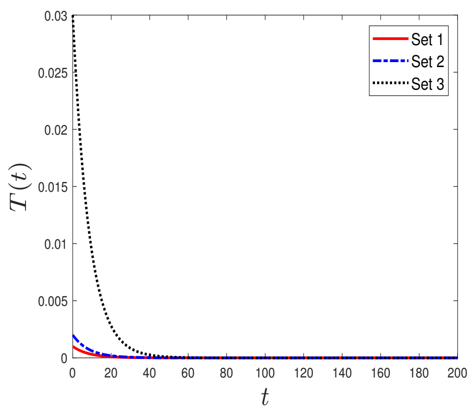
(b) Epithelial cells



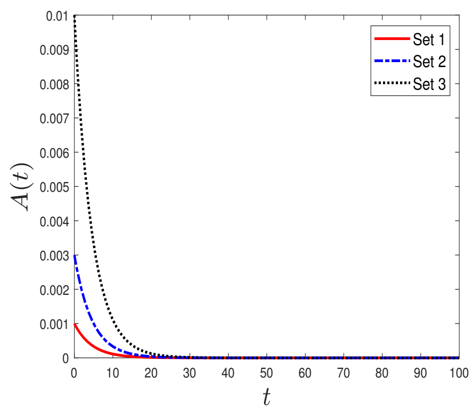
(c) Cancer cells



(d) SARS-CoV-2

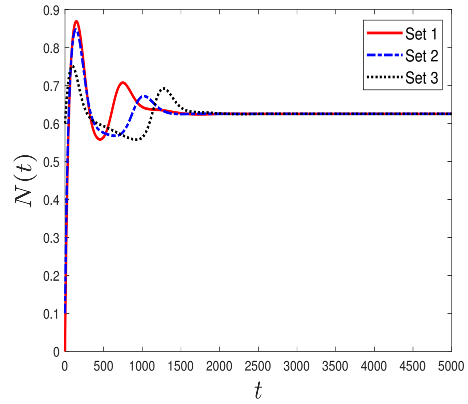


(e) Anti-cancer CTLs

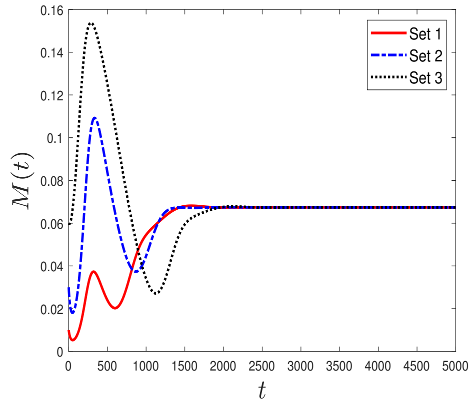


(f) Antibodies

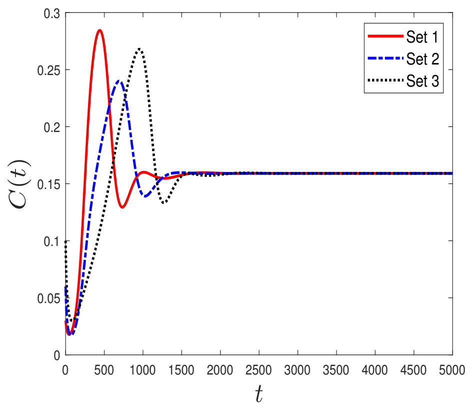
Figure 4. The numerical results of system (2.1) for $\lambda_1 = 0.07$, $\lambda_2 = 0.03$, $s_1 = 0.1$, $s_2 = 0.2$, $\epsilon_1 = 0.5$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.17$, $\omega_3 = 1 \times 10^{-4}$, and $\omega_5 = 0.2$. The point $E_3 = (0.6304, 0.1675, 0, 0.01061, 0, 0)$ is GS.



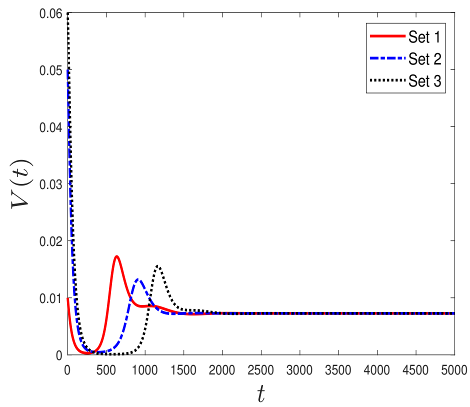
(a) Nutrient



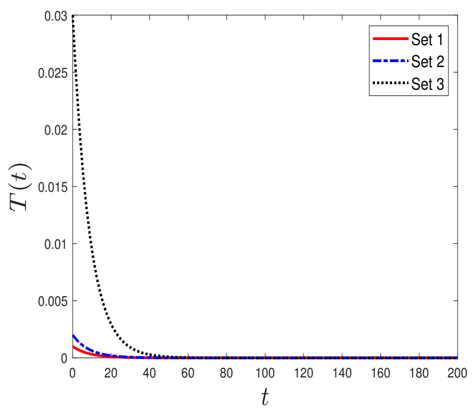
(b) Epithelial cells



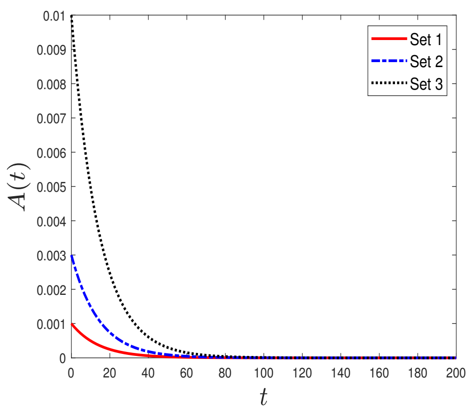
(c) Cancer cells



(d) SARS-CoV-2

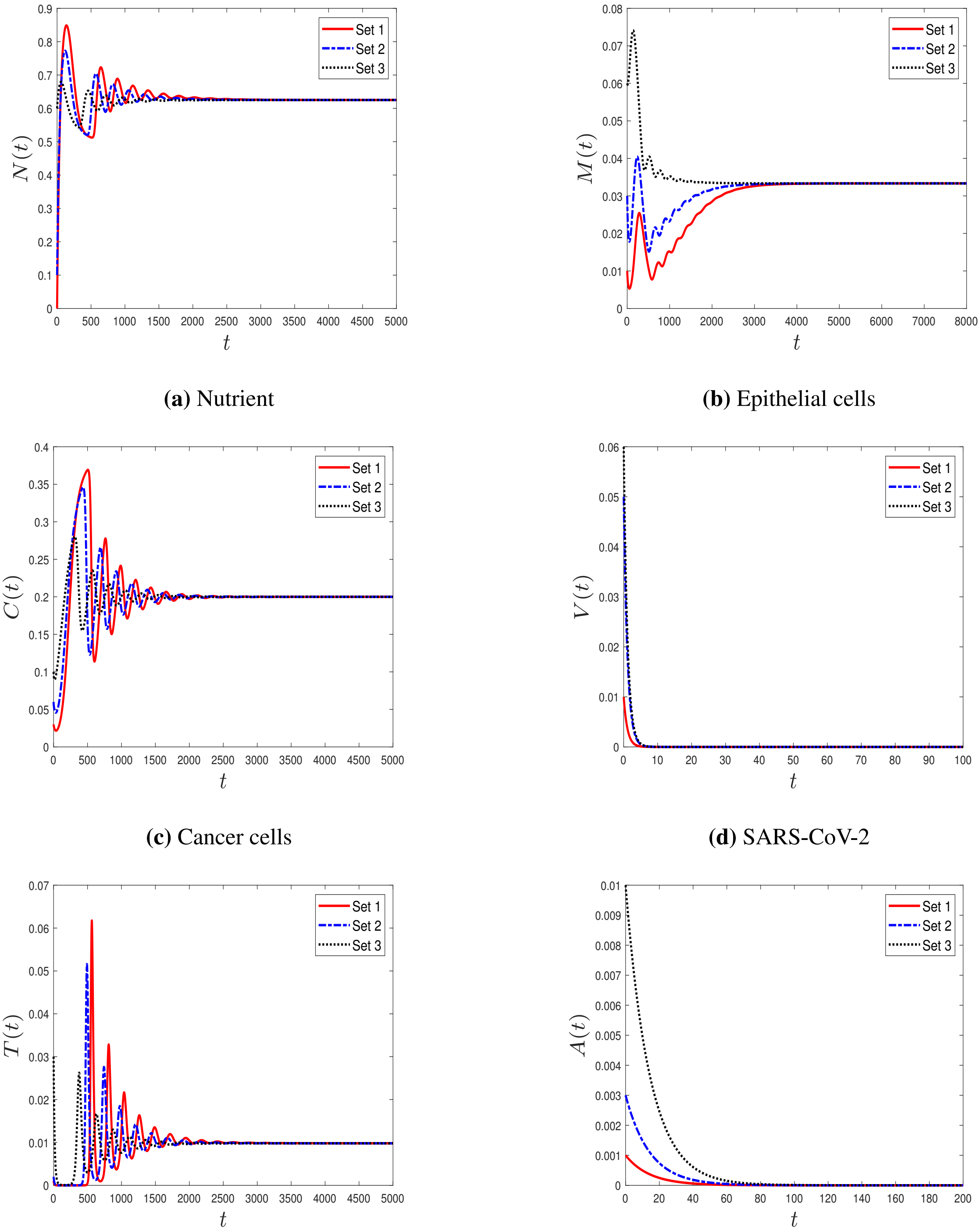


(e) Anti-cancer CTLs



(f) Antibodies

Figure 5. The numerical results of system (2.1) for $\lambda_1 = 0.06$, $\lambda_2 = 0.05$, $s_1 = 0.1$, $s_2 = 0.2$, $\epsilon_1 = 1 \times 10^{-5}$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.001$, $\omega_3 = 0.001$, and $\omega_5 = 0.05$. The point $E_6 = (0.625, 0.06742, 0.1591, 0.0073, 0, 0)$ is GS.



(a) Nutrient

(b) Epithelial cells

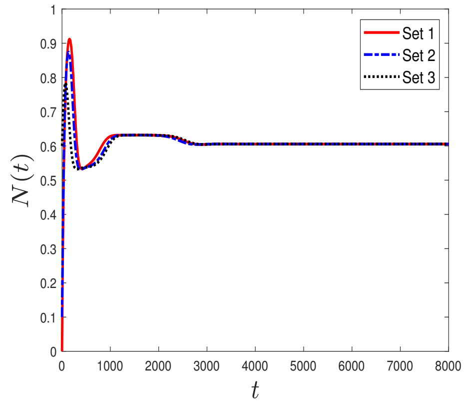
(c) Cancer cells

(d) SARS-CoV-2

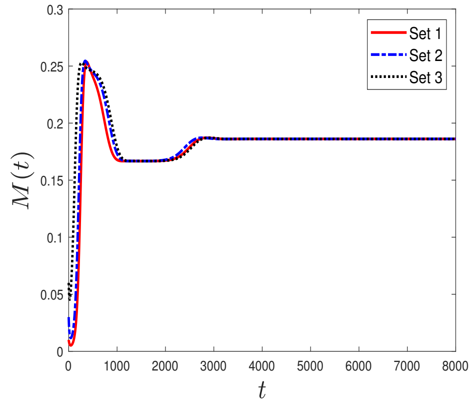
(e) Anti-cancer CTLs

(f) Antibodies

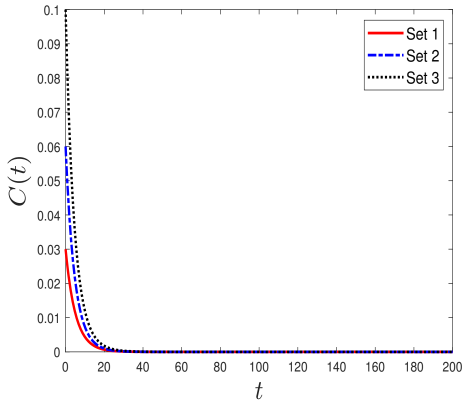
Figure 6. The numerical results of system (2.1) for $\lambda_1 = 0.06$, $\lambda_2 = 0.05$, $s_1 = 1.2$, $s_2 = 0.2$, $\epsilon_1 = 1 \times 10^{-5}$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 1 \times 10^{-4}$, $\omega_3 = 0.9$, and $\omega_5 = 0.05$. The point $E_7 = (0.625, 0.0333, 0.2, 0, 0.0098, 0)$ is GS.



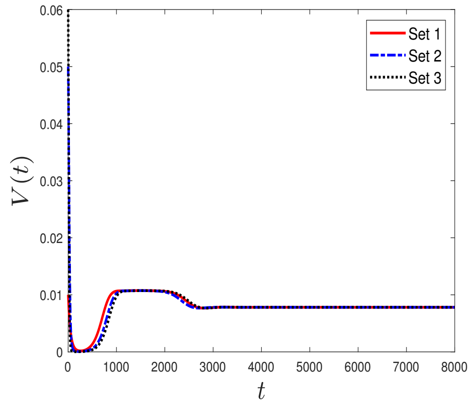
(a) Nutrient



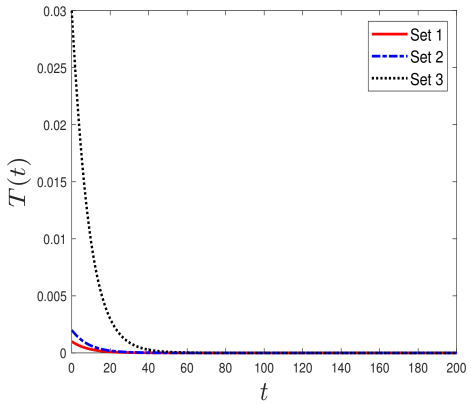
(b) Epithelial cells



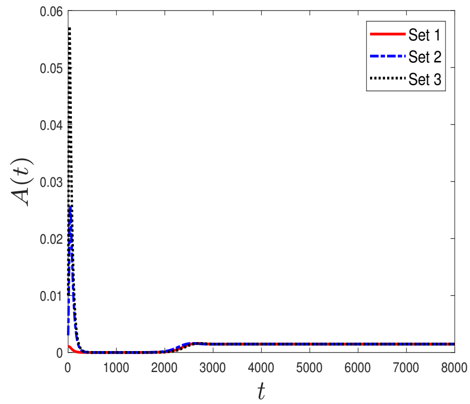
(c) Cancer cells



(d) SARS-CoV-2

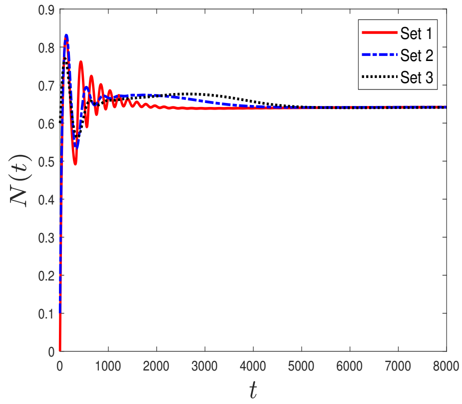


(e) Anti-cancer CTLs

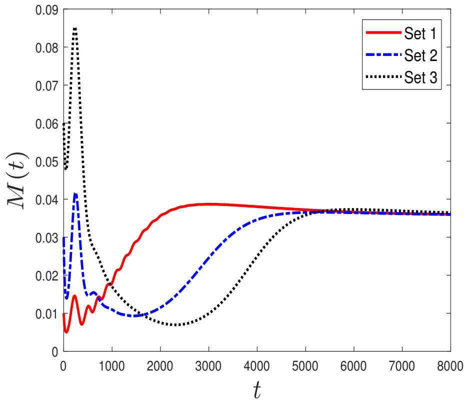


(f) Antibodies

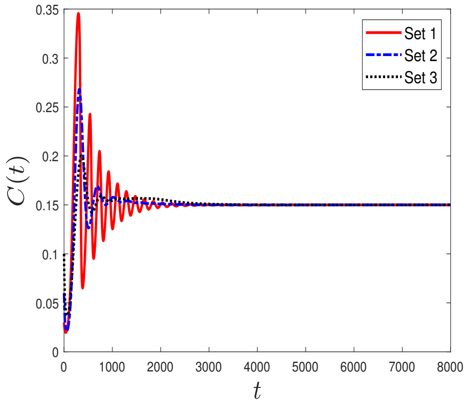
Figure 7. The numerical results of system (2.1) for $\lambda_1 = 0.07$, $\lambda_2 = 0.03$, $s_1 = 0.4$, $s_2 = 1.6$, $\epsilon_1 = 0.5$, $\epsilon_2 = 0.55$, $\xi_2 = 1.6$, $\omega_2 = 0.17$, $\omega_3 = 1 \times 10^{-6}$, and $\omega_5 = 1 \times 10^{-8}$. The point $E_8 = (0.6055, 0.1862, 0, 0.0078, 0, 0.0015)$ is GS.



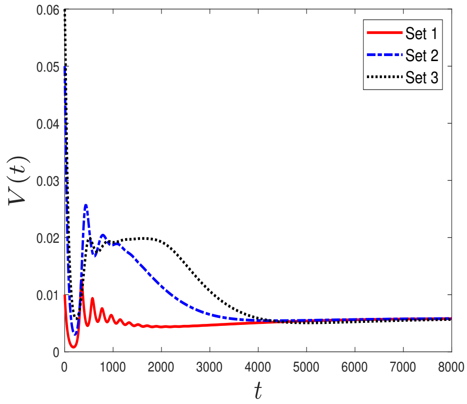
(a) Nutrient



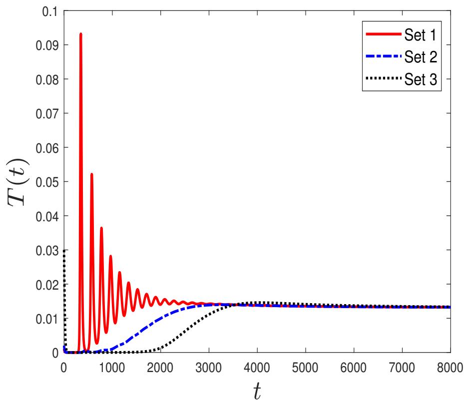
(b) Epithelial cells



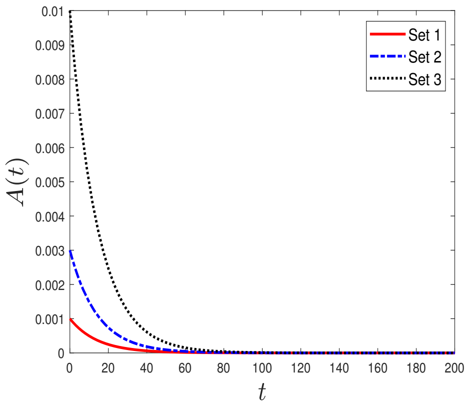
(c) Cancer cells



(d) SARS-CoV-2

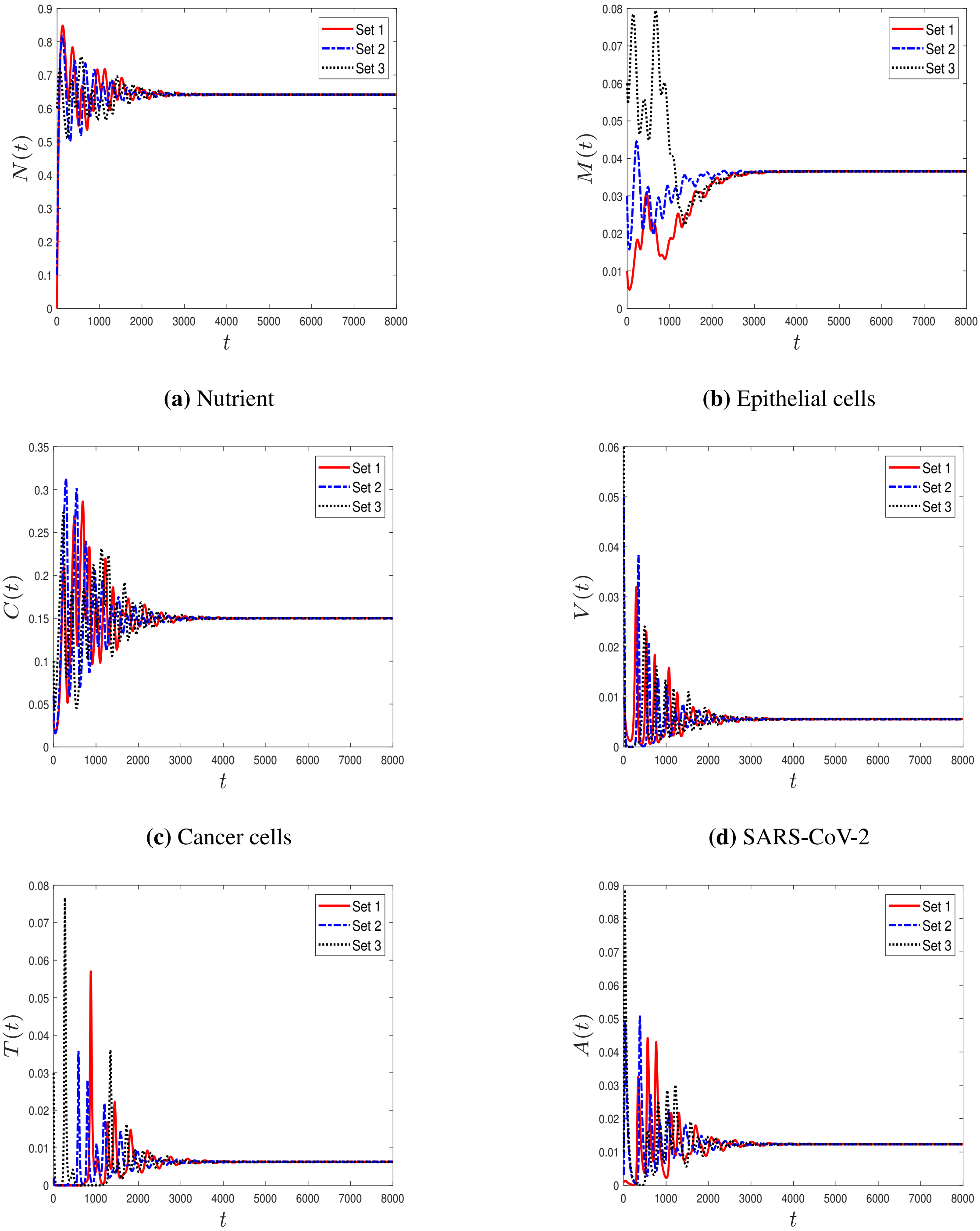


(e) Anti-cancer CTLs



(f) Antibodies

Figure 8. The numerical results of system (2.1) for $\lambda_1 = 0.06$, $\lambda_2 = 0.06$, $s_1 = 1.6$, $s_2 = 0.2$, $\epsilon_1 = 1.4 \times 10^{-1}$, $\epsilon_2 = 0.55$, $\xi_2 = 4.88 \times 10^{-8}$, $\omega_2 = 0.001$, $\omega_3 = 0.001$, and $\omega_5 = 0.05$. The point $E_{11} = (0.6422, 0.0357, 0.15, 0.0059, 0.01316, 0)$ is GS.



(a) Nutrient

(b) Epithelial cells

(c) Cancer cells

(d) SARS-CoV-2

(e) Anti-cancer CTLs

(f) Antibodies

Figure 9. The numerical results of system (2.1) for $\lambda_1 = 0.06$, $\lambda_2 = 0.06$, $s_1 = 1.6$, $s_2 = 1.9$, $\epsilon_1 = 1.4 \times 10^{-1}$, $\epsilon_2 = 1.2$, $\xi_2 = 1.9$, $\omega_2 = 0.001$, $\omega_3 = 0.001$, and $\omega_5 = 0.000001$. The point $E_{12} = (0.641, 0.03657, 0.1499, 0.006, 0.006, 0.0123)$ is GS.

To observe the impact of ϵ_2 (the infection rate of cancer cells by SARS-CoV-2) on the concentration of cancer cells before stimulating any immune responses, we increase the value of ϵ_2 in case (iv). When we set $\epsilon_2 = 0.7$, we get $C_6 = 0.125$. Additionally, if we raise ϵ_2 to 0.9, we find $C_6 = 0.097$. Figure 10 shows the impact of increasing ϵ_2 on the decrease in the concentration of cancer cells for other values of ϵ_2. Thus, the infection of cancer cells by SARS-CoV-2 can lead to a reduction in cancer cells concentration, consequently resulting in a remission or an improvement in the patient's situation. Similarly, when we increase the value of ξ_1 (the killing rate of cancer cells by CTLs) in case (viii), the concentration of cancer cells decreases to lower values (See Figure 11). In fact, these results align with many studies that suggest the ability of SARS-CoV-2 to infect cancer cells and induce immune responses, leading to cancer remission [15–19].

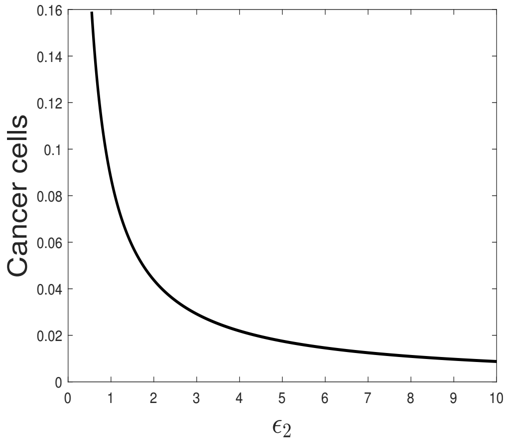


Figure 10. The effect of varying the infection rate of cancer cells by SARS-CoV-2 (ϵ_2) on the concentration of cancer cells in case (iv).

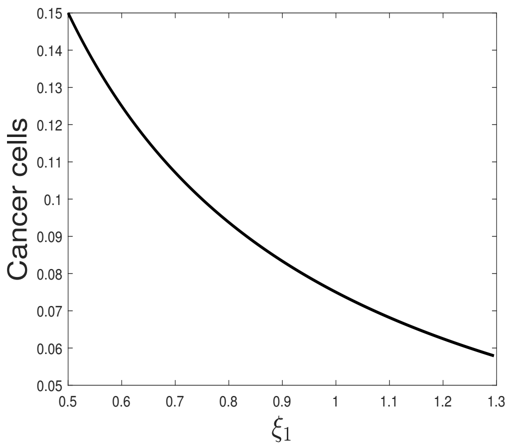


Figure 11. The effect of varying the killing rate of cancer cells by CTLs (ξ_1) on the concentration of cancer cells in case (viii).

6. Conclusions and discussion

Cancer remission after SARS-CoV-2 infection has been observed in many patients. This remission has been transient or complete, and it has been recorded with various types of cancer such as NK/T-cell lymphoma [15], Hodgkin lymphoma [16], follicular lymphoma [17], acute leukemia [18], and other types of cancer [6]. This has raised an urgent need to understand the relationship between cancer and SARS-CoV-2. This paper proposes and analyzes an oncolytic SARS-CoV-2 model. The model has 13 equilibrium points, and we focused our analysis on the points with the most important biological significance as follows:

(1) The trivial equilibrium E_0 which is GS if $R_0 \le 1$ and $R_1 \le 1$. At this point, all populations disappear except for the nutrient.

(2) The uninfected-epithelial equilibrium E_1 is GS if $R_0 > 1$, $R_0 \le 1 + \dfrac{\lambda_1 a_3}{\omega p \epsilon_1}$, and $\dfrac{a_1\lambda_2}{a_2\lambda_1} \le 1$. Here, the patient is free from both SARS-CoV-2 infection and cancer.

(3) The infected-epithelial equilibrium E_3 is GS if

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1}, \quad R_0 + \frac{a_2\epsilon_1}{a_1\epsilon_2} + \frac{\lambda_1 a_2 a_3}{\omega p a_1 \epsilon_2} \ge 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\alpha\theta\epsilon_1\lambda_2}{\omega a_1 \epsilon_2}$$

and

$$R_0 \le 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2}.$$

The patient here has only SARS-CoV-2 infection.

(4) The infected epithelial-cancer equilibrium without immunity E_6 is GS when

$$R_1 + \frac{a_1\epsilon_2}{a_2\epsilon_1} + \frac{a_1\lambda_2 a_3}{\omega p \epsilon_1 a_2} < 1 + \frac{\lambda_2 a_3}{\omega p \epsilon_2} + \frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2},$$

$$R_0 + \frac{a_2\epsilon_1}{a_1\epsilon_2} + \frac{\lambda_1 a_2 a_3}{\omega p a_1 \epsilon_2} < 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\alpha\theta\epsilon_1\lambda_2}{\omega a_1 \epsilon_2},$$

$$\frac{a_1\epsilon_2}{a_2\epsilon_1} > 1, \quad \frac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1} > 1, \quad \frac{a_1\lambda_2}{a_2\lambda_1} > 1,$$

$$R_1 + \frac{a_1\epsilon_2}{a_2\epsilon_1} + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} + \frac{a_1\lambda_1\epsilon_2 a_3}{\omega p \epsilon_1^2 a_2} + \frac{a_1\epsilon_2\lambda_2 a_4}{\omega\epsilon_1 s_1 \xi_1 a_2} \le 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{a_1\lambda_1\epsilon_2^2 a_4}{\omega\epsilon_1^2 s_1\xi_1 a_2} + \frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2}$$

and

$$\frac{a_1\lambda_2}{a_2\lambda_1} + \frac{\epsilon_1\lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} \le 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}.$$

Here, the cancer patient has SARS-CoV-2 infection with inactive immunity.

(5) The uninfected epithelial-cancer equilibrium with CTLs E_7 is GS when

$$R_0 > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1}, \quad \frac{a_1\lambda_2}{a_2\lambda_1} > 1$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} \leq 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3}{\omega p \epsilon_1}.$$

Here, the cancer patient with active CTL immunity does not have SARS-CoV-2 infection.

(6) The infected-epithelial equilibrium with antibodies E_8 is GS if

$$R_0 > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2 \xi_2}$$

and

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} \leq 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}.$$

The patient is cancer-free and suffers only from SARS-CoV-2 infection with active immunity against the virus.

(7) The infected epithelial-cancer equilibrium with CTLs E_{11} is GS when

$$\frac{\lambda_2 \epsilon_1}{\lambda_1 \epsilon_2} + \frac{s_1 \xi_1 a_3}{p \epsilon_2 a_4} + \frac{\omega \epsilon_1 s_1 \xi_1}{\lambda_1 \epsilon_2 a_4} > 1,$$

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} > 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3}{\omega p \epsilon_1},$$

$$R_1 + \frac{a_1 \epsilon_2}{a_2 \epsilon_1} + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2 a_3}{\omega p \epsilon_1^2 a_2} + \frac{a_1 \epsilon_2 \lambda_2 a_4}{\omega \epsilon_1 s_1 \xi_1 a_2} > 1 + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{a_1 \lambda_1 \epsilon_2^2 a_4}{\omega \epsilon_1^2 s_1 \xi_1 a_2} + \frac{\alpha \theta \lambda_1 \epsilon_2}{\omega \epsilon_1 a_2}$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{\lambda_1 \epsilon_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} \leq 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2}.$$

In this case, the cancer patient has SARS-CoV-2 with active CTLs against the cancer cells.

(8) The infected epithelial-cancer equilibrium with CTLs and antibodies E_{12} is GS if

$$R_0 > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2},$$

$$\frac{a_1 \lambda_2}{a_2 \lambda_1} + \frac{\epsilon_1 \lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} > 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{\lambda_1 \epsilon_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} > 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2}.$$

The cancer patient has SARS-CoV-2 infection with active immune responses against the cancer cells and virus particles.

We found complete agreement between the theoretical contributions and numerical simulations. The global stability conditions of equilibrium points determine various infection scenarios, such as patients having only SARS-CoV-2, cancer, both SARS-CoV-2 and cancer, or no infections. These conditions are dependent on the parameters of model (2.1), emphasizing the importance of carefully selecting their values. Furthermore, our findings indicate that the infection rate of cancer cells by SARS-CoV-2 (ϵ_2) and the killing rate of these cells by CTLs (ξ_1) contribute to the reduction in the concentration of cancer cells. Based on these results, SARS-CoV-2 has the potential to lead to cancer remission or improve health conditions by either infecting cancer cells or inducing an anti-cancer immune response. This outcome aligns with recent studies suggesting an oncolytic role of SARS-CoV-2 [15–19]. In comparison to existing works, our model is the first to propose and analyze the oncolytic effect of SARS-CoV-2 in cancer patients. As such, these results warrant further investigation and comparison with the outcomes of experimental studies. Then, the model can be utilized in studies aiming to employ SARS-CoV-2 as oncolytic virotherapy to target cancer cells. However, a main limitation of this work is the absence of real data to estimate the values of the parameters in model (2.1), given the limited availability of such data in this direction. We utilized values from the literature and made assumptions for some parameters. Consequently, model (1) can be developed by:

(i) Estimating parameter values through fitting with real data once sufficient information becomes available;

(ii) Testing the model results against real data;

(iii) Studying the effect of immune responses on the oncolytic role of SARS-CoV-2 and when they can be supportive;

(iv) Including the direct induction of CTLs by SARS-CoV-2;

(v) Adding more components to the model, such as infected cancer cells and infected epithelial cells, for a deeper understanding of the model's dynamics;

(vi) Considering time delays that occur during different biological processes;

(vii) Accounting for parameters and model uncertainties by performing sensitivity analysis and other methods once experimental or real data becomes available.

These enhancements would contribute to a better understanding of the model and facilitate improved predictions.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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Appendix

Appendix A

Proof of Theorem 7. We nominate

$$\begin{aligned}\Sigma_6(t) =&N_6\left(\frac{N}{N_6}-1-\ln\frac{N}{N_6}\right)+\frac{1}{\theta}M_6\left(\frac{M}{M_6}-1-\ln\frac{M}{M_6}\right)+\frac{1}{\theta}C_6\left(\frac{C}{C_6}-1-\ln\frac{C}{C_6}\right)\\&+\frac{1}{\theta p}V_6\left(\frac{V}{V_6}-1-\ln\frac{V}{V_6}\right)+\frac{1}{\theta s_1}T+\frac{1}{\theta p s_2}A.\end{aligned}$$

By evaluating $\frac{d\Sigma_6}{dt}$, we get

$$\begin{aligned}\frac{d\Sigma_6}{dt} =&\left(1-\frac{N_6}{N}\right)(\alpha-\lambda_1 NM-\lambda_2 NC-\omega N)+\frac{1}{\theta}\left(1-\frac{M_6}{M}\right)(\theta\lambda_1 NM-\epsilon_1 MV-a_1 M)\\&+\frac{1}{\theta}\left(1-\frac{C_6}{C}\right)(\theta\lambda_2 NC-\epsilon_2 CV-\xi_1 CT-a_2 C)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\theta p}\left(1-\frac{V_6}{V}\right)(p\epsilon_1 MV+p\epsilon_2 CV-\xi_2 VA-a_3V)\\
&+\frac{1}{\theta s_1}\left(s_1\xi_1 CT-a_4T\right)+\frac{1}{\theta p s_2}\left(s_2\xi_2 VA-a_5A\right).
\end{aligned}
\tag{A.1}
$$

At equilibrium, E_6 fulfills the equations:

$$
\begin{cases}
\alpha=\lambda_1 N_6 M_6+\lambda_2 N_6 C_6+\omega N_6,\\
\lambda_1 N_6 M_6=\dfrac{\epsilon_1}{\theta}M_6V_6+\dfrac{a_1}{\theta}M_6,\\
\lambda_2 N_6 C_6=\dfrac{\epsilon_2}{\theta}C_6V_6+\dfrac{a_2}{\theta}C_6,\\
\dfrac{\epsilon_1}{\theta}M_6V_6+\dfrac{\epsilon_2}{\theta}C_6V_6=\dfrac{a_3}{\theta p}V_6.
\end{cases}
$$

Thus, Eq (A.1) can be collected as

$$
\begin{aligned}
\frac{d\Sigma_6}{dt}=&\left(1-\frac{N_6}{N}\right)(\omega N_6-\omega N)+\lambda_1 N_6 M_6\left(2-\frac{N_6}{N}-\frac{N}{N_6}\right)+\lambda_2 N_6 C_6\left(2-\frac{N_6}{N}-\frac{N}{N_6}\right)\\
&+\left(\frac{\xi_1}{\theta}C_6-\frac{a_4}{\theta s_1}\right)T+\left(\frac{\xi_2}{\theta p}V_6-\frac{a_5}{\theta p s_2}\right)A\\
=&-\frac{\omega\left(N-N_6\right)^2}{N}+\lambda_1 N_6 M_6\left(2-\frac{N_6}{N}-\frac{N}{N_6}\right)+\lambda_2 N_6 C_6\left(2-\frac{N_6}{N}-\frac{N}{N_6}\right)\\
&+\frac{\omega\xi_1}{\theta\lambda_2\left(\frac{a_1\epsilon_2}{a_2\epsilon_1}-1\right)\left(\frac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1}-1\right)}\Bigg(R_1+\frac{a_1\epsilon_2}{a_2\epsilon_1}+\frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1}+\frac{a_1\lambda_1\epsilon_2 a_3}{\omega p\epsilon_1^2 a_2}+\frac{a_1\epsilon_2\lambda_2 a_4}{\omega\epsilon_1 s_1\xi_1 a_2}\\
&-1-\frac{\lambda_1 a_3}{\omega p\epsilon_1}-\frac{\lambda_2 a_4}{\omega s_1\xi_1}-\frac{a_1\lambda_1\epsilon_2^2 a_4}{\omega\epsilon_1^2 s_1\xi_1 a_2}-\frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2}\Bigg)T\\
&+\frac{\lambda_1 a_2\xi_2}{\theta p\epsilon_1\lambda_2\left(\frac{\lambda_1\epsilon_2}{\lambda_2\epsilon_1}-1\right)}\left(\frac{a_1\lambda_2}{a_2\lambda_1}+\frac{\epsilon_1\lambda_2 a_5}{\lambda_1 a_2 s_2\xi_2}-1-\frac{\epsilon_2 a_5}{a_2 s_2\xi_2}\right)A.
\end{aligned}
$$

Thus,

$$
\frac{d\Sigma_6}{dt}\le 0
$$

if

$$
R_1+\frac{a_1\epsilon_2}{a_2\epsilon_1}+\frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1}+\frac{a_1\lambda_1\epsilon_2 a_3}{\omega p\epsilon_1^2 a_2}+\frac{a_1\epsilon_2\lambda_2 a_4}{\omega\epsilon_1 s_1\xi_1 a_2}\le 1+\frac{\lambda_1 a_3}{\omega p\epsilon_1}+\frac{\lambda_2 a_4}{\omega s_1\xi_1}+\frac{a_1\lambda_1\epsilon_2^2 a_4}{\omega\epsilon_1^2 s_1\xi_1 a_2}+\frac{\alpha\theta\lambda_1\epsilon_2}{\omega\epsilon_1 a_2}
$$

and

$$
\frac{a_1\lambda_2}{a_2\lambda_1}+\frac{\epsilon_1\lambda_2 a_5}{\lambda_1 a_2 s_2\xi_2}\le 1+\frac{\epsilon_2 a_5}{a_2 s_2\xi_2}.
$$

Also, it is easy to observe that $\frac{d\Sigma_6}{dt}=0$ when $(N,M,C,V,T,A)=(N_6,M_6,C_6,V_6,0,0)$. Hence, $Y_6^{'}=\{E_6\}$ and E_6 is GS when the existence conditions and the global stability conditions are met based on LP [37]. □

Appendix B

Proof of Theorem 8. We select

$$\begin{aligned}\Sigma_7(t) =& N_7\left(\frac{N}{N_7} - 1 - \ln\frac{N}{N_7}\right) + \frac{1}{\theta}M_7\left(\frac{M}{M_7} - 1 - \ln\frac{M}{M_7}\right) + \frac{1}{\theta}C_7\left(\frac{C}{C_7} - 1 - \ln\frac{C}{C_7}\right)\\ &+ \frac{1}{\theta p}V + \frac{1}{\theta s_1}T_7\left(\frac{T}{T_7} - 1 - \ln\frac{T}{T_7}\right) + \frac{1}{\theta p s_2}A.\end{aligned}$$

Then, we obtain

$$\begin{aligned}\frac{d\Sigma_7}{dt} =& \left(1 - \frac{N_7}{N}\right)(\alpha - \lambda_1 NM - \lambda_2 NC - \omega N) + \frac{1}{\theta}\left(1 - \frac{M_7}{M}\right)(\theta\lambda_1 NM - \epsilon_1 MV - a_1 M)\\ &+ \frac{1}{\theta}\left(1 - \frac{C_7}{C}\right)(\theta\lambda_2 NC - \epsilon_2 CV - \xi_1 CT - a_2 C) + \frac{1}{\theta p}\left(p\epsilon_1 MV + p\epsilon_2 CV - \xi_2 VA - a_3 V\right)\\ &+ \frac{1}{\theta s_1}\left(1 - \frac{T_7}{T}\right)(s_1\xi_1 CT - a_4 T) + \frac{1}{\theta p s_2}\left(s_2\xi_2 VA - a_5 A\right).\end{aligned} \tag{B.1}$$

By applying the equilibrium conditions at E_7

$$\begin{cases}\alpha = \lambda_1 N_7 M_7 + \lambda_2 N_7 C_7 + \omega N_7,\\ \lambda_1 N_7 M_7 = \dfrac{a_1}{\theta}M_7,\\ \lambda_2 N_7 C_7 = \dfrac{\xi_1}{\theta}C_7 T_7 + \dfrac{a_2}{\theta}C_7,\\ \dfrac{\xi_1}{\theta}C_7 T_7 = \dfrac{a_4}{\theta s_1}T_7.\end{cases}$$

Equation (B.1) can be collected as

$$\begin{aligned}\frac{d\Sigma_7}{dt} =& \left(1 - \frac{N_7}{N}\right)(\omega N_7 - \omega N) + \lambda_1 N_7 M_7\left(2 - \frac{N_7}{N} - \frac{N}{N_7}\right) + \lambda_2 N_7 C_7\left(2 - \frac{N_7}{N} - \frac{N}{N_7}\right)\\ &+ \left(\frac{\epsilon_1}{\theta}M_7 + \frac{\epsilon_2}{\theta}C_7 - \frac{a_3}{\theta p}\right)V - \frac{a_5}{\theta p s_2}A\\ =& -\frac{\omega\left(N - N_7\right)^2}{N} + \lambda_1 N_7 M_7\left(2 - \frac{N_7}{N} - \frac{N}{N_7}\right) + \lambda_2 N_7 C_7\left(2 - \frac{N_7}{N} - \frac{N}{N_7}\right)\\ &+ \frac{\omega\epsilon_1}{\theta\lambda_1}\left(R_0 + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} - 1 - \frac{\lambda_2 a_4}{\omega s_1\xi_1} - \frac{\lambda_1 a_3}{\omega p\epsilon_1}\right)V - \frac{a_5}{\theta p s_2}A.\end{aligned}$$

Thus,

$$\frac{d\Sigma_7}{dt} \le 0$$

if

$$R_0 + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} \le 1 + \frac{\lambda_2 a_4}{\omega s_1\xi_1} + \frac{\lambda_1 a_3}{\omega p\epsilon_1}.$$

Also, $\dfrac{d\Sigma_7}{dt} = 0$ when $(N, M, C, V, T, A) = (N_7, M_7, C_7, 0, T_7, 0)$. Hence, $Y_7^{'} = \{E_7\}$ and by LP [37], E_7 is GS if

$$R_0 > 1 + \frac{\lambda_2 a_4}{\omega s_1\xi_1}, \quad \frac{a_1\lambda_2}{a_2\lambda_1} > 1$$

and

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} \leq 1 + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3}{\omega p \epsilon_1}.$$

□

Appendix C

Proof of Theorem 9. We nominate

$$\begin{aligned}\Sigma_8(t) =& N_8\left(\frac{N}{N_8} - 1 - \ln\frac{N}{N_8}\right) + \frac{1}{\theta}M_8\left(\frac{M}{M_8} - 1 - \ln\frac{M}{M_8}\right) + \frac{1}{\theta}C \\ &+ \frac{1}{\theta p}V_8\left(\frac{V}{V_8} - 1 - \ln\frac{V}{V_8}\right) + \frac{1}{\theta s_1}T + \frac{1}{\theta p s_2}A_8\left(\frac{A}{A_8} - 1 - \ln\frac{A}{A_8}\right).\end{aligned}$$

Then, we have

$$\begin{aligned}\frac{d\Sigma_8}{dt} =& \left(1 - \frac{N_8}{N}\right)(\alpha - \lambda_1 NM - \lambda_2 NC - \omega N) + \frac{1}{\theta}\left(1 - \frac{M_8}{M}\right)(\theta\lambda_1 NM - \epsilon_1 MV - a_1 M) \\ &+ \frac{1}{\theta}(\theta\lambda_2 NC - \epsilon_2 CV - \xi_1 CT - a_2 C) + \frac{1}{\theta p}\left(1 - \frac{V_8}{V}\right)(p\epsilon_1 MV + p\epsilon_2 CV - \xi_2 VA - a_3 V) \\ &+ \frac{1}{\theta s_1}(s_1\xi_1 CT - a_4 T) + \frac{1}{\theta p s_2}\left(1 - \frac{A_8}{A}\right)(s_2\xi_2 VA - a_5 A). \end{aligned} \tag{C.1}$$

At equilibrium, E_8 fulfills the following:

$$\begin{cases}\alpha = \lambda_1 N_8 M_8 + \omega N_8, \\ \lambda_1 N_8 M_8 = \dfrac{\epsilon_1}{\theta}M_8 V_8 + \dfrac{a_1}{\theta}M_8, \\ \dfrac{\epsilon_1}{\theta}M_8 V_8 = \dfrac{\xi_2}{\theta p}V_8 A_8 + \dfrac{a_3}{\theta p}V_8, \\ \dfrac{\xi_2}{\theta p}V_8 A_8 = \dfrac{a_5}{\theta p s_2}A_8.\end{cases}$$

Thus, Eq (C.1) can be collected as

$$\begin{aligned}\frac{d\Sigma_8}{dt} =& \left(1 - \frac{N_8}{N}\right)(\omega N_8 - \omega N) + \lambda_1 N_8 M_8\left(2 - \frac{N_8}{N} - \frac{N}{N_8}\right) + \left(\lambda_2 N_8 - \frac{a_2}{\theta} - \frac{\epsilon_2}{\theta}V_8\right)C - \frac{a_4}{\theta s_1}T \\ =& -\frac{\omega(N - N_8)^2}{N} + \lambda_1 N_8 M_8\left(2 - \frac{N_8}{N} - \frac{N}{N_8}\right) + \frac{a_2}{\theta}\left(\frac{a_1\lambda_2}{a_2\lambda_1} + \frac{\epsilon_1\lambda_2 a_5}{\lambda_1 a_2 s_2 \xi_2} - 1 - \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}\right)C - \frac{a_4}{\theta s_1}T.\end{aligned}$$

Hence,

$$\frac{d\Sigma_8}{dt} \leq 0$$

if

$$\frac{a_1\lambda_2}{a_2\lambda_1} + \frac{\epsilon_1\lambda_2 a_5}{\lambda_1 a_2 s_2\xi_2} \leq 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2} \quad \text{and} \quad \frac{d\Sigma_8}{dt} = 0$$

when $(N, M, C, V, T, A) = (N_8, M_8, 0, V_8, 0, A_8)$. Therefore, $Y_8' = \{E_8\}$ and based on LP [37], E_8 is GS when it exists and

$$\frac{a_1\lambda_2}{a_2\lambda_1} + \frac{\epsilon_1\lambda_2 a_5}{\lambda_1 a_2 s_2\xi_2} \leq 1 + \frac{\epsilon_2 a_5}{a_2 s_2 \xi_2}.$$

□

Appendix D

Proof of Theorem 10. We select

$$\Sigma_{11}(t) = N_{11}\left(\frac{N}{N_{11}} - 1 - \ln\frac{N}{N_{11}}\right) + \frac{1}{\theta}M_{11}\left(\frac{M}{M_{11}} - 1 - \ln\frac{M}{M_{11}}\right) + \frac{1}{\theta}C_{11}\left(\frac{C}{C_{11}} - 1 - \ln\frac{C}{C_{11}}\right)$$
$$+ \frac{1}{\theta p}V_{11}\left(\frac{V}{V_{11}} - 1 - \ln\frac{V}{V_{11}}\right) + \frac{1}{\theta s_1}T_{11}\left(\frac{T}{T_{11}} - 1 - \ln\frac{T}{T_{11}}\right) + \frac{1}{\theta p s_2}A.$$

Then, we have

$$\begin{aligned}\frac{d\Sigma_{11}}{dt} =& \left(1 - \frac{N_{11}}{N}\right)(\alpha - \lambda_1 NM - \lambda_2 NC - \omega N) + \frac{1}{\theta}\left(1 - \frac{M_{11}}{M}\right)(\theta\lambda_1 NM - \epsilon_1 MV - a_1 M)\\ &+ \frac{1}{\theta}\left(1 - \frac{C_{11}}{C}\right)(\theta\lambda_2 NC - \epsilon_2 CV - \xi_1 CT - a_2 C)\\ &+ \frac{1}{\theta p}\left(1 - \frac{V_{11}}{V}\right)(p\epsilon_1 MV + p\epsilon_2 CV - \xi_2 VA - a_3 V)\\ &+ \frac{1}{\theta s_1}\left(1 - \frac{T_{11}}{T}\right)(s_1\xi_1 CT - a_4 T) + \frac{1}{\theta p s_2}\left(s_2\xi_2 VA - a_5 A\right).\end{aligned} \tag{D.1}$$

At equilibrium, E_{11} fulfills the system

$$\begin{cases}\alpha = \lambda_1 N_{11}M_{11} + \lambda_2 N_{11}C_{11} + \omega N_{11},\\ \lambda_1 N_{11}M_{11} = \dfrac{\epsilon_1}{\theta}M_{11}V_{11} + \dfrac{a_1}{\theta}M_{11},\\ \lambda_2 N_{11}C_{11} = \dfrac{\epsilon_2}{\theta}C_{11}V_{11} + \dfrac{\xi_1}{\theta}C_{11}T_{11} + \dfrac{a_2}{\theta}C_{11},\\ \dfrac{\epsilon_1}{\theta}M_{11}V_{11} + \dfrac{\epsilon_2}{\theta}C_{11}V_{11} = \dfrac{a_3}{\theta p}V_{11},\\ \dfrac{\xi_1}{\theta}C_{11}T_{11} = \dfrac{a_4}{\theta s_1}T_{11}.\end{cases}$$

Thus, Eq (D.1) can be collected as

$$\begin{aligned}\frac{d\Sigma_{11}}{dt} =& \left(1 - \frac{N_{11}}{N}\right)(\omega N_{11} - \omega N) + \lambda_1 N_{11}M_{11}\left(2 - \frac{N_{11}}{N} - \frac{N}{N_{11}}\right) + \lambda_2 N_{11}C_{11}\left(2 - \frac{N_{11}}{N} - \frac{N}{N_{11}}\right)\\ &+ \left(\frac{\xi_2}{\theta p}V_{11} - \frac{a_5}{\theta p s_2}\right)A\\ =& -\frac{\omega\left(N - N_{11}\right)^2}{N} + \lambda_1 N_{11}M_{11}\left(2 - \frac{N_{11}}{N} - \frac{N}{N_{11}}\right) + \lambda_2 N_{11}C_{11}\left(2 - \frac{N_{11}}{N} - \frac{N}{N_{11}}\right)\\ &+ \frac{\omega a_1 s_1\xi_1\xi_2}{\theta p\lambda_1\epsilon_2 a_4\left(\dfrac{\lambda_2\epsilon_1}{\lambda_1\epsilon_2} + \dfrac{s_1\xi_1 a_3}{p\epsilon_2 a_4} + \dfrac{\omega\epsilon_1 s_1\xi_1}{\lambda_1\epsilon_2 a_4} - 1\right)}\Bigg(R_0 + \frac{\lambda_1\epsilon_2 a_4}{\omega\epsilon_1 s_1\xi_1} + \frac{\lambda_1\epsilon_2 a_4 a_5}{\omega a_1 s_1\xi_1 s_2\xi_2} - 1 - \frac{\epsilon_1 a_5}{a_1 s_2\xi_2}\\ &- \frac{\lambda_1 a_3}{\omega p\epsilon_1} - \frac{\lambda_2 a_4}{\omega s_1\xi_1} - \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} - \frac{\epsilon_1\lambda_2 a_4 a_5}{\omega a_1 s_1\xi_1 s_2\xi_2}\Bigg)A.\end{aligned}$$

Thus,

$$\frac{d\Sigma_{11}}{dt} \leq 0$$

if

$$R_0 + \frac{\lambda_1 \epsilon_2 a_4}{\omega \epsilon_1 s_1 \xi_1} + \frac{\lambda_1 \epsilon_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2} \leq 1 + \frac{\epsilon_1 a_5}{a_1 s_2 \xi_2} + \frac{\lambda_1 a_3}{\omega p \epsilon_1} + \frac{\lambda_2 a_4}{\omega s_1 \xi_1} + \frac{\lambda_1 a_3 a_5}{\omega p a_1 s_2} + \frac{\epsilon_1 \lambda_2 a_4 a_5}{\omega a_1 s_1 \xi_1 s_2 \xi_2}.$$

Also,

$$\frac{d\Sigma_{11}}{dt} = 0$$

when $(N, M, C, V, T, A) = (N_{11}, M_{11}, C_{11}, V_{11}, T_{11}, 0)$. Therefore, $Y'_{11} = \{E_{11}\}$ and by LP [37], E_{11} is GS when the existence and stability conditions are met. □

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