AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 7152–7168.
DOI: 10.3934/math.2024349
Received: 14 December 2023
Revised: 28 January 2024
Accepted: 31 January 2024
Published: 19 February 2024

Research article

The bound of the correlation results of the roughness measure of the disturbation fuzzy set

Li Li[1,*], Hangyu Shi[1], Xiaona Liu[1] and Jingjun Shi[2,*]

[1] Mathematical Sciences College, Bohai University, No. 19 Keji Road, Songshan New District, Jinzhou 121013, China

[2] Lishu County Double River Township Middle School, Siping 136517, China

* **Correspondence:** Email: lll_dz@163.com.

Abstract: This paper mainly studies and proves the roughness bound of disturbation fuzzy sets. Firstly, based on the theory of determining self-increment and uncertain self-decrement operators, the problem that the execution subsets are not equal sets is effectively solved, which hinders the quantitative study of disturbed fuzzy sets and lays a foundation for the quantitative study of the related properties of disturbed fuzzy sets in the future. The boundary of roughness measure of disturbing fuzzy set is further studied and proved. The new territories proposed in this paper can effectively avoid the unnecessary calculation space outside the boundary in the calculation process, so as to improve the work efficiency.

Keywords: rough set; disturbance fuzzy set; roughness measure
Mathematics Subject Classification: 03E72

1. Introduction

Pawlak first proposed the rough set theory [1], which is the basis for testing the granularity of knowledge [2]. In recent years, many models related to rough sets have emerged, such as the rough set theory based on fuzzy covering [3–5]. The fuzzy set theory was first proposed by Zadeh [6]. Since then, theories and applications related to fuzzy sets [7] have also been widely studied, such as fuzzy soft sets [8], feature selection of fuzzy sets [9], outlier detection of fuzzy sets [10–12], decision application of fuzzy sets, etc. [13–16]. The relation and difference between fuzzy set theory and rough set theory is also a hot topic. An important component of this research is the roughness of the fuzzy set. Dubois et al first defined the concepts of the rough fuzzy set and fuzzy rough set [17]. The roughness measurement method of fuzzy sets proposed by Banerjee et al really makes the relationship between fuzzy sets and rough sets closer [18], and it has laid a solid foundation for subsequent researchers to explore the roughness measurement of fuzzy sets by applying fuzzy entropy [19], from the perspective of distance [20] and based on soft relation [21]. Li first mentioned the concept of disturbed fuzzy sets [22]. Liu and Chen formally described the concept of disturbed fuzzy sets [23]. Chen and Wu extended the tautology of fuzzy sets [24] to interval-valued fuzzy sets [25], intuitionistic fuzzy sets [26], and disturb fuzzy sets [27], respectively. It is found that the same result can be obtained only when the disturbation fuzzy set is generalized to the ordinary fuzzy set. Therefore, the disturbation fuzzy set shows excellent properties in the operation and has extensive research value, which is not found in any kind of fuzzy set, including the interval valued fuzzy set. Subsequently, Han et al. put forward the concept of disturbed fuzzy rough sets and the roughness measure of disturbed fuzzy sets [28]. It further enriches the theoretical basis of combining the fuzzy set and the rough set. However, there are few researches on the roughness measurement of the disturbance fuzzy set, and the application of disturbance fuzzy set roughness measurement is even less.

Upper and lower approximations of fuzzy sets are two important aspects in the study of fuzzy rough sets theory [29]. In this paper, there are two limitations: On the one hand, it is found in the exploration that the approximation of $\tilde{M} \cup \tilde{N}$ generally cannot be obtained by the approximation of $\tilde{M}$ and $\tilde{N}$, and these properties are the result of logical forms defined by assumptions in the domain of discussion expressed in an approximate manner [30]. Therefore they bring inconvenience and difficulty to the research in many fields, including the roughness measurement of disturbed fuzzy sets. On the other hand, when data analysis, data mining, decision support system, and machine learning are carried out, the datasets are usually huge, in order to solve the inconvenience caused by too large datasets. So in this paper, first, the related concepts of the rough set, disturbation fuzzy set, and roughness measurement are introduced. Second, by introducing two new operators designed by Zhang et al. and associating them with the disturbation fuzzy set, the limitation that the execution subset is not the equality of the set and cannot be studied quantitatively is effectively solved. Finally, the roughness measure of the disturbation fuzzy set is studied quantitatively, and its boundedness is obtained. Therefore, it is expected that using the boundary of roughness measure of the disturbing fuzzy set proved in this paper can effectively avoid the computing space outside the boundary and improve the computing efficiency.

2. Preparatory knowledge

In this section, some basic concepts related to approximate space, upper and lower approximations of fuzzy sets, the roughness measure of fuzzy sets, and disturbation fuzzy sets are given.

Definition 2.1. *(Approximate space) [1] The nonempty set U is called the discourse domain, S is the equivalence relation on U, and (U, S) is called an approximate space.*

Definition 2.2. *(Upper approximation, lower approximation, and boundary field) [1] (U, S) is the known approximation space, $M \subseteq U$, and in the approximation space $y_1, y_2, \cdots, y_k$ represents an equivalence class with respect to S. $\bar{S}\ (M)$ is the upper approximation of M and $\underline{S}(M)$ is the lower approximation of M. The boundary area BN_S is represented by*

$$\underline{S}\,(M) = \{y_i |\, [y_i]_S \subseteq M\}, \quad S\ (M) = \{y_i |\, [y_i]_S \cap M \neq \emptyset\}, \tag{2.1}$$

while

$$BN_S = \frac{\bar{S}\ (M)}{\underline{s}\,(M)}, \quad k = 1, 2, \cdots, m.$$

Definition 2.3. *(Upper and lower approximations of fuzzy sets) [1] In U, the upper and lower approximations of the fuzzy set M are defined as: $U/S \to [0,1]$ and*

$$\underline{S}(M)(y) = \inf_{y\in Y_k} M(x),$$

$$\bar{S}(M)(y) = \sup_{y\in Y_k} M(x), \quad k = 1,2,\cdots,m.$$

Definition 2.4. *(Roughness measure of fuzzy set) [1] (U,S) is the known approximation space, $M \subseteq U$, and the M roughness measure in (U,S) is*

$$\rho_M = 1 - \frac{|\underline{s}(M)|}{\left|\bar{S}(M)\right|},$$

where the set $|*|$ represents the cardinality of $*$.

Yao [2] once proposed that the roughness measure of a fuzzy set can be understood as the distance between the upper approximation and the lower approximation of the fuzzy set. If M: $U \to [0,1]$ is in U, $M(y), y \in U$ gives y membership in M.

Definition 2.5. *(Disturbed fuzzy sets) [23] If*

$$\tilde{P} : Z \mapsto \omega, z \mapsto \tilde{P}(z) \tag{2.2}$$

and

$$\omega = \left\{ \tilde{P}(z) = \left(\tilde{P}_\alpha(z), \tilde{P}_\beta(z)\right) \middle| \tilde{P}_\alpha(z), \tilde{P}_\beta(z) \in [0,1] \right\} \tag{2.3}$$

call $\tilde{P}$ a disturbed fuzzy set on Z, then all disturbed fuzzy sets on the discourse domain U are denoted as $\tilde{E}(U)$.

3. Roughness of the disturbance fuzzy set

Based on the concepts of upper approximation and lower approximation, this section introduces the roughness measure formula of the disturbed fuzzy set, the operation relations of upper approximation, and lower approximation, and the key properties of roughness measure of the disturbed fuzzy set.

Definition 3.1. *(Operation of disturbed fuzzy sets) [28] Let*

$$\omega = \left\{\mu = \left(\mu_\alpha, \mu_\beta\right)\right\},$$

the interval corresponding to $\left(\mu_\alpha, \mu_\beta\right)$ is

$$\left[\max\left(0, \mu_\alpha - \mu_\beta\right), \min\left(1, \mu_\alpha + \mu_\beta\right)\right] \tag{3.1}$$

for all

$$\mu = (\mu_\alpha, \mu_\beta), \quad \nu = (\nu_\alpha, \nu_\beta), \quad \mu, \nu \in \omega,$$

the operation on ω is defined as

$$\begin{aligned} \mu \wedge \nu &= \left(\min\{\mu_\alpha, \nu_\alpha\}, \max\left\{\mu_\beta, \nu_\beta\right\}\right), \\ \mu \vee \nu &= \left(\max\{\mu_\alpha, \nu_\alpha\}, \min\left\{\mu_\beta, \nu_\beta\right\}\right), \\ \mu^c &= \left(1 - \mu_\alpha, 1 - \mu_\beta\right). \end{aligned} \tag{3.2}$$

Definition 3.2. *(Relation of disturbed fuzzy sets) [28] The relationship between μ and ν is defined as*

$$\begin{aligned} &\mu = \nu \Leftrightarrow \mu_\alpha = \nu_\alpha, \ \ \mu_\beta = \nu_\beta, \\ &\mu \leq \nu \Leftrightarrow \mu_\alpha \leq \nu_\alpha, \ \ \mu_\beta \geq \nu_\beta, \\ &\mu < \nu \Leftrightarrow \mu_\alpha < \nu_\alpha, \ \ \mu_\beta \geq \nu_\beta \, or \, \mu_\alpha \leq \nu_\alpha, \ \ \mu_\beta < \nu_\beta, \end{aligned} \tag{3.3}$$

otherwise, we call it incomparable and denote by $U(\mu, \nu)$.

Obviously, when $(\omega, \leq)$, $\overline{0} = (0, 1)$ and $\overline{1} = (1, 0)$ are the minimum and maximum elements on ω, respectively.

Definition 3.3. *(Upper and lower approximations of disturbed fuzzy sets) [28] Let μ, ν be the two given parameters,*

$$\tilde{M} \in \tilde{E}(U), \quad \tilde{0} < \nu \leq \mu \leq \tilde{1},$$

and the (U, S) be the approximate space, defining the upper and lower approximations of a disturbed fuzzy set. The $\mu-$ cut sets and $\nu-$ cut sets of $\underline{S}\left(\tilde{M}\right)$ and $\bar{S}\left(\tilde{M}\right)$ are

$$\left(\underline{S}\left(\tilde{M}\right)\right)_\mu = \left\{y \in U | \left(\tilde{M}\right)(y) \geq \mu\right\}, \tag{3.4}$$

$$\left(\bar{S}\left(\tilde{M}\right)\right)_\nu = \left\{y \in U | \bar{S}\left(\tilde{M}\right)(y) \geq \nu\right\}, \tag{3.5}$$

where, $\left(\underline{S}\left(\tilde{M}\right)\right)_\mu$ and $\left(\bar{S}\left(\tilde{M}\right)\right)_\nu$ can be regarded as the sets of objects with μ and ν as the minimum membership degrees in the disturbance fuzzy set $\tilde{M}$.

Definition 3.4. *(Roughness of disturbed fuzzy set) [28] Let (U, S) be the approximate space,*

$$\tilde{M} \in \tilde{E}(U), \quad \bar{0} < \nu \leq \mu \leq \bar{1},$$

then the roughness of the disturbed fuzzy set $\tilde{M}$ on U in accordance with parameter μ, ν is

$$\tilde{\rho}_{\tilde{M}}^{\tilde{\mu}, \tilde{\nu}} = 1 - \frac{\left|\underline{S}\left(\tilde{M}\right)_{\tilde{\mu}}\right|}{\left|\bar{S}\left(\tilde{M}\right)_{\tilde{\nu}}\right|}. \tag{3.6}$$

Han et al. introduced several key properties of this roughness measure [28].

Proposition 3.1. *(Disturbation of upper and lower approximation of fuzzy sets) [28] Let μ, ν be two given parameters,*

$$\tilde{M} \in \tilde{E}(U), \quad 0 < \nu \leq \mu \leq 1,$$

and let $\left(\underline{S}\left(\tilde{M}\right)\right)_\mu$ and $\left(\bar{S}\left(\tilde{M}\right)\right)_\nu$ be the $\mu-$ cut sets and $\nu-$ cut sets of the upper and lower approximations of the disturbed fuzzy set $\underline{S}\left(\tilde{M}\right)$ and $\bar{S}\left(\tilde{M}\right)$, where

$$\left(\bar{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_\nu = \left(\bar{S}\left(\tilde{M}\right)\right)_\nu \cup \left(\bar{S}\left(\tilde{N}\right)\right)_\nu, \tag{3.7}$$

$$\left(\underline{S}\left(\tilde{M} \cap \tilde{N}\right)\right)_\mu = \left(\underline{S}\left(\tilde{M}\right)\right)_\mu \cap \left(\underline{S}\left(\tilde{N}\right)\right)_\mu, \tag{3.8}$$

$$\left(\underline{S}\left(\tilde{M}\right)\right)_\mu \cup \left(\underline{S}\left(\tilde{N}\right)\right)_\mu \subseteq \left(\underline{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_\mu, \tag{3.9}$$

$$\left(\bar{S}\left(\tilde{M} \cap \tilde{N}\right)\right)_\nu \subseteq \left(\bar{S}\left(\tilde{M}\right)\right)_\nu \cap \left(\bar{S}\left(\tilde{N}\right)\right)_\nu. \tag{3.10}$$

Property 3.1. *For disturbed fuzzy set $\tilde{M}, \tilde{N}$, there is [28]*

$$\tilde{\rho}^{\mu,\nu}_{\tilde{M}\cup\tilde{N}} = 1 - \frac{\left|\left(\underline{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_{\nu}\right|} = 1 - \frac{\left|\left(\underline{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu} \cup \left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|} \leq 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu} \cup \left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu} \cup \left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}, \tag{3.11}$$

$$\tilde{\rho}^{\mu,\nu}_{\tilde{M}\cap\tilde{N}} = 1 - \frac{\left|\left(\underline{S}\left(\tilde{M} \cap \tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M} \cap \tilde{N}\right)\right)_{\nu}\right|} = 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu} \cap \left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M} \cap \tilde{N}\right)\right)_{\nu}\right|} \leq 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu} \cap \left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu} \cap \left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}. \tag{3.12}$$

4. Determine the increment and indeterminate decrement operators

The pioneering study of fuzzy sets [1] derived as (3.8) and (3.9) in Proposition 3.1, which carry out the property that subsets are not equal sets, hindered the quantitative study of fuzzy sets. Because of this difficulty, Zhang et al. designed two new operators [31]. In this section, the new operator proposed by Zhang et al. is fully associated with the disturbed fuzzy set so as to effectively avoid the bad influence of this property in the roughness measurement process of the disturbed fuzzy set. The roughness measure of the disturbed fuzzy set can be studied quantitatively.

Definition 4.1. *(Determine the increment operator) [31] Let the discourse domain be U, S, the equivalence class on U, $P, Q \subseteq U$, when P is extended by Q (i.e., $P \cup Q$),*

$$\underline{X}_{(\cdot)}(\cdot) : U \times U \to U,$$

defining

$$\underline{X}_{(P)}(Q) = \cup \left\{ [p]_S \middle| p \in H(P), h_P(p) \not\subset Q \right\},$$

and $l_P(p) \subseteq Q$ is called the definite increment of P, where

$$H(P) = \cup \left\{ h_P(p) | p \in BN_S(P) \cap P \right\},$$

$$l_P(p) = [p]_S - P \text{ and } h_P(p) = [p]_S - l_P(p).$$

Definition 4.2. *(Uncertain decrement operator) [31] Let the discourse domain be U, S the equivalence class on U, $P, Q \subseteq U$, when P is cut by Q (i.e., $P \cap Q$),*

$$\underline{X}_{(\cdot)}(\cdot) : U \times U \to U,$$

defining

$$\bar{X}_{(P)}(Q) = \cup \left\{ [p]_S \middle| p \in H(P), h_P(p) \cap Q = \emptyset \right\}$$

and

$$l_P(p) \cap Q \neq \emptyset,$$

which is called the uncertainty decrement of P, where

$$H(P) = \cup \left\{ h_P(p) | p \in BN_S(P) \cap P \right\},$$

$$l_P(p) = [p]_S - P \text{ and } h_P(p) = [p]_S - l_P(p).$$

Property 4.1. *[31] $P, Q \subseteq U$, so*

$$\underline{X}_{(P)}(Q) = \underline{X}_{(Q)}(P), \tag{4.1}$$

$$\bar{X}_{(P)}(Q) = \bar{X}_{(Q)}(P). \tag{4.2}$$

Property 4.2. *[31]*

$$\underline{X}_{(P)}(\emptyset) = \emptyset, \tag{4.3}$$

$$\underline{X}_{(P)}(P) = \emptyset, \tag{4.4}$$

$$\underline{X}_{(P)}(\neg P) = BN_S(P) = \bar{S}P - \underline{S}P. \tag{4.5}$$

Property 4.3. *[31]*

$$\bar{X}_{(P)}(\emptyset) = \emptyset, \tag{4.6}$$

$$\bar{X}_{(P)}(P) = \emptyset, \tag{4.7}$$

$$\bar{X}_{(P)}(\neg P) = BN_S(P). \tag{4.8}$$

Theorem 4.1. *Let $\tilde{M}$ and $\tilde{N}$ be two disturbed fuzzy sets in the discourse domain U. Parameters μ, ν satisfy $0 < \nu \leq \mu \leq 1$, while $\underline{X}_{\tilde{M}_\mu}\left(\tilde{N}_\mu\right)$, $\bar{X}_{\tilde{M}_\nu}\left(\tilde{N}_\nu\right)$, $\underline{X}_{\tilde{N}_\mu}\left(\tilde{M}_\mu\right)$, and $\bar{X}_{\tilde{N}_\nu}\left(\tilde{M}_\nu\right)$ are, respectively, $\tilde{M}_\mu$, $\tilde{N}_\mu$ determines the increment and $\tilde{M}_\nu$, and the uncertainty of $\tilde{N}_\nu$ decreases so we can get*

$$\left(\underline{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_\mu = \left(\underline{S}\left(\tilde{M}\right)\right)_\mu \cup \left(\underline{S}\left(\tilde{N}\right)\right)_\mu \cup \underline{X}_{\tilde{M}_\mu}\left(\tilde{N}_\mu\right) = \left(\underline{S}\left(\tilde{M}\right)\right)_\mu \cup \left(\underline{S}\left(\tilde{N}\right)\right)_\mu \cup \underline{X}_{\tilde{N}_\mu}\left(\tilde{M}_\mu\right), \tag{4.9}$$

$$\left(\bar{S}\left(\tilde{M} \cap \tilde{N}\right)\right)_\nu = \left(\bar{S}\left(\tilde{M}\right)\right)_\nu \cap \left(\bar{S}\left(\tilde{N}\right)\right)_\nu - \bar{X}_{\tilde{M}_\nu}\left(\tilde{N}_\nu\right) = \left(\bar{S}\left(\tilde{M}\right)\right)_\nu \cap \left(\bar{S}\left(\tilde{N}\right)\right)_\nu - \bar{X}_{\tilde{N}_\nu}\left(\tilde{M}_\nu\right). \tag{4.10}$$

Property 4.4. *For disturbed fuzzy sets $\tilde{M}$ and $\tilde{N}$,*

$$\tilde{\rho}^{\mu,\nu}_{\tilde{M}\cup\tilde{N}} = 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_\mu \cup \left(\underline{S}\left(\tilde{N}\right)\right)_\mu \cup \underline{X}_{\tilde{M}_\mu}\left(\tilde{N}_\mu\right)\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_\nu \cup \left(\bar{S}\left(\tilde{N}\right)\right)_\nu\right|} = 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_\mu \cup \left(\underline{S}\left(\tilde{N}\right)\right)_\mu \cup \underline{X}_{\tilde{N}_\mu}\left(\tilde{M}_\mu\right)\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_\nu \cup \left(\bar{S}\left(\tilde{N}\right)\right)_{\nu'}\right|}, \tag{4.11}$$

$$\tilde{\rho}^{\mu,\nu}_{\tilde{M}\cap\tilde{N}} = 1 - \frac{\left|\left(\left(\tilde{M}\right)\right)_\mu \cap \left(\left(\tilde{N}\right)\right)_\mu\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_\nu \cap \left(\bar{S}\left(\tilde{N}\right)\right)_\nu - \bar{X}_{\tilde{M}_\nu}\left(\tilde{N}_\nu\right)\right|} = 1 - \frac{\left|\left(\left(\tilde{M}\right)\right)_\mu \cap \left(\left(\tilde{N}\right)\right)_\mu\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_\nu \cap \left(\bar{S}\left(\tilde{N}\right)\right)_\nu - \bar{X}_{\tilde{N}_\nu}\left(\tilde{M}_\nu\right)\right|}. \tag{4.12}$$

5. The boundary of the correlation results of the roughness measure of the perturbation fuzzy set

When calculating the roughness measurement of disturbed fuzzy sets, the datasets of many programs are huge and the measurement is very complicated and cumbersome work, which requires a lot of manpower and material resources. Therefore, this section presents the boundaries of some results necessary for the roughness measurement of disturbed fuzzy sets. Understanding the boundaries of these results before operation can greatly improve work efficiency. It has very important practical significance.

Theorem 5.1. *The upper bound of the roughness measure $\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}}$ of the disturbed fuzzy sets $\tilde{M}$ and $\tilde{N}$ in the discourse domain U is*

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \leq \frac{1-\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\,\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}}{2-\left(\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}+\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}\right)}$$

with respect to parameter μ,ν, satisfying $0<\nu\leq\mu\leq 1$.

Proof. From (3.7) in Proposition 3.1 and the fundamental properties of sets, we can get

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \leq 1-\frac{\max\left\{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|,\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|\right\}}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|+\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}, \tag{5.1}$$

if

$$\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right| \geq \left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|.$$

Thus

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \leq 1-\frac{1}{\left(\frac{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}+\frac{\left|\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}\right)}, \tag{5.2}$$

so, by Definition 3.4, we get

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \leq \frac{1-\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\,\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}}{2-\left(\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}+\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}\right)}. \tag{5.3}$$

□

Theorem 5.2. *The upper bound of the roughness measure $\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}}$ of the disturbed fuzzy sets $\tilde{M}$ and $\tilde{N}$ in the discourse domain U is*

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \leq \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}+\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}-1+U^{\circ}$$

with respect to parameter μ,ν, satisfying $0<\nu\leq\mu\leq 1$ and

$$U^{\circ}=\frac{\left|\left(\underline{s}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}.$$

Proof. From (3.8) in Proposition 3.1 and the fundamental properties of sets, we can get

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} = 1-\frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}-\frac{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}+\frac{\left|\left(\underline{S}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}. \tag{5.4}$$

Also, according to (3.10) in Proposition 3.1,

$$\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu} \subseteq \left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}, \tag{5.5}$$

$$\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu} \subseteq \left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}. \tag{5.6}$$

In other words, we have

$$\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right| \subseteq \left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|, \tag{5.7}$$

$$\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right| \subseteq \left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|, \tag{5.8}$$

so we can get

$$\frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} \leq \frac{\left|\left(\left(\underline{S}\tilde{M}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}, \tag{5.9}$$

$$\frac{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|} \leq \frac{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}. \tag{5.10}$$

Next, according to (3.9) in Proposition 3.1, it is obtained

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \leq 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} - \frac{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|} + \frac{\left|\left(\underline{S}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}. \tag{5.11}$$

According to Definition 3.4, we can get

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \leq \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} + \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - 1 + \frac{\left|\left(\underline{S}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}. \tag{5.12}$$

Therefore, to sum up,

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \leq \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} + \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - 1 + U^{\circ},$$

when

$$U^{\circ} = \frac{\left|\left(\underline{s}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}.$$

□

Remark 5.1. *The bounds of Theorem 5.1 depend on roughness measures of the disturbed fuzzy sets $\tilde{M}$ and $\tilde{N}$, and the bounds of Theorem 5.2 depend on roughness measures of the disturbed fuzzy sets $\tilde{M}$ and $\tilde{N}$ as well as $\left(\underline{S}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\mu}$ and $\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}$.*

Theorem 5.3. *The lower bound of the disturbed fuzzy sets $\tilde{M}$ and $\tilde{N}$ in the discourse domain U for the roughness measure $\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}}$ with respect to parameter μ, ν is*

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \geq \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} + \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - 1 + L_{\circ},$$

which satisfies $0 < \nu \leq \mu \leq 1$, and

$$L_{\circ} = \frac{\left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\max\left\{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|, \left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|\right\}}.$$

Proof. From (4.11) in Property 4.4 and the fundamental properties of sets, it is obtained that

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \geq 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right| + \left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right| + \left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\max\left\{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|, \left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|\right\}}, \tag{5.13}$$

We can obtain

$$\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right| \geq \left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right| \tag{5.14}$$

and

$$\begin{aligned}
\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} &\geq 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right| + \left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right| + \left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} \\
&= 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} - \frac{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} - \frac{\left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|},
\end{aligned} \tag{5.15}$$

According to Definition 3.4 and

$$\frac{\left|\left(\underline{s}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} \leq \frac{\left|\left(\underline{s}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|},$$

we can get

$$\begin{aligned}
\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} &\geq 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} - \frac{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} - \frac{\left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|} \\
&= 1 - \left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\right) - \left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}\right) - \frac{\left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|},
\end{aligned} \tag{5.16}$$

so

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \geq \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} + \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - 1 - \frac{\left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|}. \tag{5.17}$$

Likewise,

$$\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right| < \left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|, \tag{5.18}$$

we can get

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \geq 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right| + \left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right| + \left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}, \tag{5.19}$$

thus,

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \geq \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} + \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - 1 - \frac{\left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}. \tag{5.20}$$

To sum up,

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cup\tilde{N}} \geq \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} + \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - 1 + L_{\circ},$$

when

$$L_{\circ} = \frac{\left|\underline{X}_{\tilde{M}_{\mu}}\left(\tilde{N}_{\mu}\right)\right|}{\max\left\{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|, \left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|\right\}}.$$

□

Theorem 5.4. *The lower bound of the disturbed fuzzy sets $\tilde{M}$ and $\tilde{N}$ in the discourse domain U for the roughness measure $\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}}$ with respect to parameter μ, ν is*

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \geq 1 - \frac{1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} + \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}}{2 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - I_{\circ}\left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\right)\left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}\right)},$$

which satisfies $0 < \nu \leq \mu \leq 1$*, and*

$$I_{\circ} = \frac{\left|\left(\bar{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_{\nu}\right| + \left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|}{\min\left\{\left|\left(\underline{s}\left(\tilde{M}\right)\right)_{\mu}\right|, \left|\left(\underline{s}\left(\tilde{N}\right)\right)_{\mu}\right|\right\}}.$$

Proof. From (4.12) in Property 4.4 and the fundamental properties of sets, it is obtained that

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \geq 1 - \frac{\min\left\{\left|\underline{X}\left(\left(\tilde{M}\right)\right)_{\mu}\right|, \left|\underline{X}\left(\left(\tilde{N}\right)\right)_{\mu}\right|\right\}}{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right| + \left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right| - \left|\left(\bar{S}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\nu}\right| - \left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|}, \tag{5.21}$$

if

$$\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right| \leq \left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|, \tag{5.22}$$

we can get

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \geq 1 - \frac{1}{\frac{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|} + \frac{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|} - \frac{\left|\left(\bar{S}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\nu}\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|} - \frac{\left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}}. \tag{5.23}$$

According to Definition 3.4 and

$$\frac{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}{\left|\left(\underline{s}\left(\tilde{M}\right)\right)_{\mu}\right|} \geq \frac{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}{\left|\left(\left(\underline{s}\tilde{N}\right)\right)_{\mu}\right|},$$

we can get

$$\begin{aligned}
\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} &\geq 1 - \frac{1}{\frac{\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|} + \frac{\left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|}{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|} - \frac{\left|\left(\bar{S}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\nu}\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|} - \frac{\left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}} \\
&= 1 - \frac{1}{\frac{1}{1-\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}} + \frac{1}{1-\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}} - \frac{\left|\left(\bar{S}\left(\tilde{M}\cup\tilde{N}\right)\right)_{\nu}\right| + \left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|}{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}}.
\end{aligned} \tag{5.24}$$

Therefore, define

$$I_{\left|\left(\left(\tilde{M}\right)\right)_{\mu}\right|} = \frac{\left|\left(\bar{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_{\nu}\right| + \left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|}{\left|\left(\underline{s}\left(\tilde{M}\right)\right)_{\mu}\right|}, \tag{5.25}$$

so

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \geq 1 - \frac{1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}}{2 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - I_{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|}\left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\right)\left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}\right)}. \tag{5.26}$$

Likewise, for

$$\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right| > \left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|, \tag{5.27}$$

define

$$I_{\left|\left(\left(\tilde{N}\right)\right)_{\mu}\right|} = \frac{\left|\left(\bar{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_{\nu}\right| + \left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|}{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}, \tag{5.28}$$

so

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \geq 1 - \frac{1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}}{2 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - I_{\left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}\left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\right)\left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}\right)}. \tag{5.29}$$

Thus, to sum up

$$\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}} \geq 1 - \frac{1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} + \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}}{2 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}} - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}} - I_{\circ}\left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}}\right)\left(1 - \tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{N}}\right)},$$

when

$$I_{\circ} = \frac{\left|\left(\bar{S}\left(\tilde{M} \cup \tilde{N}\right)\right)_{\nu}\right| + \left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|}{\min\left\{\left|\left(\underline{s}\left(\tilde{M}\right)\right)_{\mu}\right|, \left|\left(\underline{s}\left(\tilde{N}\right)\right)_{\mu}\right|\right\}}.$$

□

Remark 5.2. *The lower bound of $\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}}$ is different from the upper bound of $\tilde{\rho}^{\tilde{\mu},\tilde{\nu}}_{\tilde{M}\cap\tilde{N}}$, and the roughness measure depends not only on the disturbation fuzzy sets $\tilde{M}$ and $\tilde{N}$, but also on $\left|\left(\bar{S}\left(\tilde{M}\right)\right)_{\nu}\right|, \left|\left(\bar{S}\left(\tilde{N}\right)\right)_{\nu}\right|, \left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\right|, \left|\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|$, and $\left|\bar{X}_{\tilde{M}_{\nu}}\left(\tilde{N}_{\nu}\right)\right|$.*

Remark 5.3. *In the study of the disturbed fuzzy set, it is fully understood that the roughness measure of the disturbed fuzzy set is bounded, and often roughness comparison can be made by roughly calculating the roughness measure limit of the disturbed fuzzy set, which can greatly reduce the calculation amount.*

6. Practical applications

In the previous section, it has been proved that the roughness measure of perturbed fuzzy sets is bounded, but the bound of the roughness measure of disturbed fuzzy sets can be fully applied in practical problems. Next, the superiority of the theory proposed in this paper is demonstrated more clearly through a practical application of grouping different students in a competition, as shown in Tables 1–5.

Example 6.1. *Due to receiving the notice that our province will soon hold a student learning competition to test the learning ability of two subjects of mathematics and Chinese, the school will send 6 students to participate in the competition. It is known that each student's ability level assessment of mathematics and Chinese constitutes a disturbance fuzzy set. The school will formulate two combinations, respectively,*

$$A : \{\{\text{student1}\}, \{\text{student2, student3, student5}\}, \{\text{student4, student6}\}\},$$

$$B : \{\{\text{student1, student4}\}, \{\text{student2, student5}\}, \{\text{student3, student6}\}\}.$$

If you want to know which combination is more likely to win, set parameter

$$(0.00, 0.00) < \mu = \nu \leq (0.60, 0.10),$$

(in real life, people usually think that 60 is a passing grade on a 100-point scale, and the parameter selection of different practical questions will be different). Table 1 is the assessment table of students' mathematical and language ability levels. The mathematics of disturbed fuzzy sets and the language of disturbed fuzzy sets are represented by $\tilde{M}$ and $\tilde{N}$, respectively.

Table 1. Assessment table of students' mathematical language ability level.

	Student 1	Student 2	Student 3	Student 4	Student 5	Student 6
$\tilde{M}$	(0.80, 0.09)	(0.30, 0.01)	(0.66, 0.06)	(0.61, 0.23)	(0.43, 0.05)	(0.60, 0.10)
$\tilde{N}$	(0.80, 0.09)	(0.75, 0.21)	(0.45, 0.08)	(0.70, 0.09)	(0.64, 0.03)	(0.50, 0.14)

Table 2. Mathematical and verbal approximations of disturbed fuzzy sets caused by A classification.

	Student 1	Student 2, Student 3, Student 5	Student 4, Student 6
$\underline{S}(\tilde{M})$	(0.80, 0.09)	(0.30, 0.06)	(0.60, 0.23)
$\bar{S}(\tilde{M})$	(0.80, 0.09)	(0.66, 0.01)	(0.61, 0.10)
$\underline{S}(\tilde{N})$	(0.66, 0.03)	(0.45, 0.21)	(0.50, 0.14)
$\bar{S}(\tilde{N})$	(0.66, 0.03)	(0.75, 0.03)	(0.70, 0.09)

Table 3. Approximate values of the intersection and union of mathematical and verbal disturbed fuzzy sets caused by A class classification.

	Student 1	Student 2, Student 3, Student 5	Student 4, Student 6
$\underline{S}(\tilde{M} \cup \tilde{N})$	(0.80, 0.03)	(0.64, 0.06)	(0.60, 0.09)
$\bar{S}(\tilde{M} \cup \tilde{N})$	(0.66, 0.09)	(0.75, 0.01)	(0.70, 0.01)
$\underline{S}(\tilde{M} \cap \tilde{N})$	(0.66, 0.09)	(0.30, 0.21)	(0.50, 0.23)
$\bar{S}(\tilde{M} \cap \tilde{N})$	(0.66, 0.09)	(0.45, 0.05)	(0.61, 0.14)

Table 4. Mathematical and verbal approximations of disturbed fuzzy sets caused by B classification.

	Student 1, Student 4	Student 2, Student 5	Student 3, Student 6
$\underline{S}(\tilde{M})$	(0.61, 0.23)	(0.30, 0.05)	(0.60, 0.10)
$\bar{S}(\tilde{M})$	(0.80, 0.09)	(0.43, 0.01)	(0.66, 0.06)
$\underline{S}(\tilde{N})$	(0.66, 0.09)	(0.64, 0.21)	(0.45, 0.14)
$\bar{S}(\tilde{N})$	(0.70, 0.03)	(0.75, 0.03)	(0.50, 0.08)

Table 5. Approximate values of the intersection and union of mathematical and verbal disturbed fuzzy sets caused by B class classification.

	Student 1, Student 4	Student 2, Student 5	Student 3, Student 6
$\underline{S}(\tilde{M}\cup\tilde{N})$	(0.70, 0.09)	(0.64, 0.03)	(0.60, 0.10)
$\bar{S}(\tilde{M}\cup\tilde{N})$	(0.80, 0.03)	(0.75, 0.01)	(0.66, 0.06)
$\underline{S}(\tilde{M}\cap\tilde{N})$	(0.61, 0.23)	(0.30, 0.21)	(0.45, 0.14)
$\bar{S}(\tilde{M}\cap\tilde{N})$	(0.66, 0.09)	(0.43, 0.05)	(0.50, 0.08)

So, according to Definition 3.4 and Tables 2 and 4,

$$\tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}}(A) = 1 - \frac{\left|\underline{S}\left(\tilde{M}\right)_{\tilde{\mu}}\right|}{\left|\bar{S}\left(\tilde{M}\right)_{\tilde{\nu}}\right|} = 1 - \frac{1}{6} = \frac{5}{6}, \qquad \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}}(A) = 1 - \frac{\left|\underline{S}\left(\tilde{N}\right)_{\tilde{\mu}}\right|}{\left|\bar{S}\left(\tilde{N}\right)_{\tilde{\nu}}\right|} = 1 - \frac{1}{6} = \frac{5}{6}, \tag{6.1}$$

$$\tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}}(B) = 1 - \frac{\left|\underline{S}\left(\tilde{M}\right)_{\tilde{\mu}}\right|}{\left|\bar{S}\left(\tilde{M}\right)_{\tilde{\nu}}\right|} = 1 - \frac{2}{4} = \frac{1}{2}, \qquad \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}}(B) = 1 - \frac{\left|\underline{S}\left(\tilde{N}\right)_{\tilde{\mu}}\right|}{\left|\bar{S}\left(\tilde{N}\right)_{\tilde{\nu}}\right|} = 1 - \frac{2}{4} = \frac{1}{2}. \tag{6.2}$$

From Theorems 5.1–5.4, it follows

$$\tilde{\rho}_{\tilde{M}\cap\tilde{N}}^{\tilde{\mu},\tilde{\nu}}(A) \leq \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}} + \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}} - 1 + U^{\circ} = \frac{20}{3}, \tag{6.3}$$

$$\tilde{\rho}_{\tilde{M}\cap\tilde{N}}^{\tilde{\mu},\tilde{\nu}}(A) \geq 1 - \frac{1 - \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}} - \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}} + \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}}\tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}}}{2 - \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}} - \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}} - I_{\circ}\left(1 - \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}}\right)\left(1 - \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}}\right)} = \frac{17}{18}, \tag{6.4}$$

$$\tilde{\rho}_{\tilde{M}\cap\tilde{N}}^{\tilde{\mu},\tilde{\nu}}(B) \leq \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}} + \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}} - 1 + U^{\circ} = 3, \tag{6.5}$$

$$\tilde{\rho}_{\tilde{M}\cap\tilde{N}}^{\tilde{\mu},\tilde{\nu}}(B) \geq 1 - \frac{1 - \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}} - \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}} + \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}}\tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}}}{2 - \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}} - \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}} - I_{\circ}\left(1 - \tilde{\rho}_{\tilde{M}}^{\tilde{\mu},\tilde{\nu}}\right)\left(1 - \tilde{\rho}_{\tilde{N}}^{\tilde{\mu},\tilde{\nu}}\right)} = \frac{6}{7}, \tag{6.6}$$

and to sum up,

$$\frac{17}{18} \leq \tilde{\rho}_{\tilde{M}\cap\tilde{N}}^{\tilde{\mu},\tilde{\nu}}(A) \leq \frac{20}{3}, \quad 3 \leq \tilde{\rho}_{\tilde{M}\cap\tilde{N}}^{\tilde{\mu},\tilde{\nu}}(B) \leq \frac{6}{7}.$$

Obviously, the roughness of B classification is smaller.

7. Comparative analysis

If the traditional disturbation fuzzy set roughness measure calculation method is

$$\tilde{\rho}^{\mu,\nu}_{\tilde{M}\cap\tilde{N}} = 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|} = 1 - \frac{\left|\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}\cap\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}\right|}{\left|\left(\bar{S}\left(\tilde{M}\cap\tilde{N}\right)\right)_{\nu}\right|}, \tag{7.1}$$

we need to calculate the number of equivalence classes after the intersection of $\left(\underline{S}\left(\tilde{M}\right)\right)_{\mu}$ and $\left(\underline{S}\left(\tilde{N}\right)\right)_{\mu}$. In Tables 3 and 5, the approximate values of the mathematical and verbal intersection of the disturbed fuzzy set caused by A classification and B classification are listed, respectively. It can be seen that the traditional method is more complicated to calculate. However, it can be seen from the example that using the method proposed in this paper to avoid complex calculation can effectively improve the work efficiency. This paper only lists 2 classification methods for 6 students. In practical problems, there may be tens of thousands of students' classification methods, etc. Therefore, when the sample size is large, the roughness measurement boundary of the disturbation fuzzy set proposed in this paper will greatly reduce the workload in operation. In practical problems with large datasets, such as when we need to do data mining, bioinformatics, cybersecurity, natural language processing, etc., the sample size is often huge. Therefore, it is usually better to determine the range of roughness first and then calculate in a small range.

8. Conclusions

First, this work effectively solves the problem that the execution subsets are not equal sets, which hindrances the quantitative study of disturbed fuzzy sets.

Second, through quantitative research, the new properties of the disturbation fuzzy set operation and the boundary of the roughness of the disturbation fuzzy set are established effectively, which can effectively reduce the workload in the operation when the actual data capacity is huge.

This paper proposes and proves that the roughness measure of the disturbed fuzzy set is bounded. In practical application, a full understanding of the roughness measure boundary of the disturbed fuzzy set can effectively avoid unnecessary computing space and greatly improve work efficiency. However, the roughness measurement of disturbed fuzzy sets depends on the choice of parameter μ, ν. The roughness measurement of disturbed fuzzy sets without parameters will be further explored in future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (No. 61603055).

Authors' contributions

Li Li: responsible for the planning, design and implementation of the research, providing financial and technical support. Hangyu Shi: designed research methods, processed and analyzed data, performed theorem proving, and wrote the first draft of the paper. Xiaona Liu: assisted in paper analysis and verification, and participated in paper revision. Jingjun Shi: provide partial data and coordinate the study as a whole.

Conflict of interest

The authors declare no conflicts of interest.

References

1. Z. Pawlak, Rough sets, *Int. J. Comput. Inf. Sci.*, **11** (1982), 341–356. http://doi.org/10.1007/BF01001956
2. Y. Yao, Rough sets, neighborhood systems and granular computing, *1999 IEEE Canadian Conference on Electrical and Computer Engineering*, 1999. http://doi.org/10.1109/CCECE.1999.804943
3. Z. Huang, J. Li, C. Wang, Robust feature selection using multigranulation variable-precision distinguishing indicators for fuzzy covering decision systems, *IEEE Trans. Syst. Man Cybern.*, **54** (2024), 903–914. http://doi.org/10.1109/TSMC.2023.3321315
4. Z. Huang, J. Li, Y. Qian, Noise-tolerant fuzzy-β-covering-based multigranulation rough sets and feature subset selection, *IEEE Trans. Fuzzy Syst.*, **30** (2022), 2721–2735. http://doi.org/10.1109/TFUZZ.2021.3093202
5. Z. Huang, J. Li, Discernibility measures for fuzzy β covering and their application, *IEEE Trans. Cybern.*, **52** (2022), 9722–9735. http://doi.org/10.1109/TCYB.2021.3054742
6. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 338–353. http://doi.org/10.1016/S0019-9958(65)90241-X
7. N. Jan, J. Gwak, D. Pamucar, A robust hybrid decision making model for human-computer interaction in the environment of bipolar complex picture fuzzy soft sets, *Inf. Sci.*, **645** (2023), 119163. http://doi.org/10.1016/j.ins.2023.119163
8. N. Jan, J. Gwak, D. Pamucar, L. Martínez, Hybrid integrated decision-making model for operating system based on complex intuitionistic fuzzy and soft information, *Inf. Sci.*, **651** (2023), 119592. http://doi.org/10.1016/j.ins.2023.119592
9. M. K. Ebrahimpour, M. Eftekhari, Ensemble of feature selection methods, *Soft Comput.*, **50** (2017), 300–312. http://doi.org/10.1016/j.asoc.2016.11.021
10. A. F. Jahromi, Z. E. Mimand, A new outlier detection method for high dimensional fuzzy databases based on LOF, *J. Math. Model.*, **2** (2018), 123–136. http://doi.org/10.22124/jmm.2018.8102.1108

11. Z. Yuan, H. Chen, T. Li, B. Sang, S. Wang, Outlier detection based on fuzzy rough granules in mixed attribute data, *IEEE Trans. Cybern.*, **52** (2022), 8399–8412. http://doi.org/10.1109/TCYB.2021.3058780

12. P. Zhang, T. Li, G. Wang, D. Wang, P. Lai, F. Zhang, A multi-source information fusion model for outlier detection, *Inf. Fusion*, **93** (2023), 192–208. http://doi.org/10.1016/j.inffus.2022.12.027

13. F. Xiao, EFMCDM: evidential fuzzy multicriteria decision making based on belief entrop, *IEEE Trans. Fuzzy Syst.*, **28** (2020), 1477–1491. http://doi.org/10.1109/TFUZZ.2019.2936368

14. F. Xiao, A distance measure for intuitionistic fuzzy sets and its application to pattern classification problems, *IEEE Trans. Syst. Man Cybern.*, **51** (2021), 3980–3992. http://doi.org/10.1109/TSMC.2019.2958635

15. X. Gou, Z. Xu, H. Liao, F. Herrera, Probabilistic double hierarchy linguistic term set and its use in designing an improved VIKOR method: the application in smart healthcare, *J. Oper. Res. Soc.*, **72** (2021), 2611–2630. http://doi.org/10.1080/01605682.2020.1806741

16. X. Gou, Z. Xu, P. Ren, The properties of continuous pythagorean fuzzy information, *Int. J. Intell. Syst.*, **31** (2016), 401–424. http://doi.org/10.1002/int.21788

17. D. Dubois, H. Prade, Rough fuzzy sets and fuzzy rough sets, *Int. J. Gen. Syst.*, **17** (1990), 191–209 http://doi.org/10.1080/03081079008935107

18. M. Banerjee, S. K. Pal, Roughness of a fuzzy set, *Inf. Sci.*, **93** (1996), 235–246. http://doi.org/10.1016/0020-0255(96)00081-3

19. W. Wei, J. Liang, Y. Qian, Can fuzzy entropies be effective measures for evaluating the roughness of a rough set? *Inf. Sci.*, **232** (2013), 143–166. http://doi.org/10.1016/J.INS.2012.12.036

20. J. Hu, W. Pedrycz, G. Wang, A roughness measure of fuzzy sets from the perspective of distance, *Int. J. Gen. Syst.*, **45** (2016), 352–367. http://doi.org/10.1080/03081079.2015.1086580

21. M. Z. Anwar, S. Bashir, M. Shabir, M. G. Alharbi, Multigranulation roughness of intuitionistic fuzzy sets by soft relations and their applications in decision making, *Mathematics*, **9** (2021), 2587. http://doi.org/10.3390/math9202587

22. Q. Li, Disturbation problems of membership functions and study of disturbation operators, *J. Dalian Univ. Technol.*, **41** (2001), 387–391.

23. X. Liu, T. Chen, Disturbance fuzzy logic and its non-operator, *Fuzzy Syst. Math.*, **16** (2002), 179–182.

24. G. J. Wang, *Non-classical mathematical logic and approximate reasoning*, Beijing Science Press, 2000.

25. T. Chen, Y. Meng, F. Wu, Generalized double blind expression in perturbed fuzzy propositional logic systems, *Fuzzy Syst. Math.*, **19** (2000), 86–89.

26. T. Chen, X. Li, Generalized quasi-tautologies of interval-valued fuzzy propositional logic and their classification, *J. Liaoning Normal Univ.*, **3** (2004), 22–25.

27. F. Wu, *Semantics of intuitionistic fuzzy propositional logic system*, Master thesis, Liaoning Normal University, 2005.

28. Y. Han, S. Chen, S. Chen, Roughness of disturbing fuzzy sets, *J. Appl. Math. Univ.*, **22** (2007), 498–504.

29. Y. Yang, R. John, Roughness bounds in rough set operations, *Inf. Sci.*, **176** (2006), 3256–3267. http://doi.org/10.1016/j.ins.2006.02.009

30. Z. Pawlak, *Rough sets: theoretical aspects of reasoning about data*, Springer Science & Business Media, 1992. http://doi.org/10.1007/978-94-011-3534-4

31. H. Zhang, H. Liang, D. Liu, Two new operators in rough set theory with applications to fuzzy sets, *Inf. Sci.*, **166** (2004), 147–165. http://doi.org/10.1016/j.ins.2003.11.003