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Research article

Certain geometric properties of the fractional integral of the Bessel function of the first kind

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Abstract: This paper revealed new fractional calculus applications of special functions in the geometric function theory. The aim of the study presented here was to introduce and begin the investigations on a new fractional calculus integral operator defined as the fractional integral of order λ for the Bessel function of the first kind. The focus of this research was on obtaining certain geometric properties that give necessary and sufficient univalence conditions for the new fractional calculus operator using the methods associated to differential subordination theory, also referred to as admissible functions theory, developed by Sanford S. Miller and Petru T. Mocanu. The paper discussed, in the proved theorems and corollaries, conditions that the fractional integral of the Bessel function of the first kind must comply in order to be a part of the sets of starlike functions, positive and negative order starlike functions, convex functions, positive and negative order convex functions, and close-to-convex functions, respectively. The geometric properties proved for the fractional integral of the Bessel function of the first kind recommend this function as a useful tool for future developments, both in geometric function theory in general, as well as in differential subordination and superordination theories in particular.

Keywords: Bessel function of the first kind; fractional integral; starlike function; convex function; close-to-convex function; integral operator; differential subordination; differential superordination; fractional calculus; special functions
Mathematics Subject Classification: 30C45, 30C80, 33C10

1. Introduction

Fractional calculus has recently become a prominent area of mathematical analysis with regard to theoretical studies and real-life applications. It has developed into an essential tool for modeling and analysis, having a significant impact on a wide range of disciplines. Recent detailed reviews of the subject [1, 2] examine its development and list numerous scientific and engineering fields in which it has been applied.

The recent review study by Srivastava [3] highlighted the advantages of adding fractional calculus to geometric function theory, and this has inspired and encouraged further research linking fractional calculus to univalent functions theory. Since Miller and Mocanu first proposed the theory of differential subordination in their publications from 1978 [4] and 1981 [5], it has received a lot of attention because it can be used to more easily obtain previously known results as well as generate remarkable results when combined with investigations involving analytic functions. A topic of study that evolved well in the differential subordination theory framework emerged when various kinds of operators were included into the study. As a recent survey report [6] demonstrates, integral operators are a crucial tool when such studies are taken into account.

The investigation addressed in this paper continues this research topic and geometric properties are established for an integral operator introduced using the fractional integral of order λ and the remarkable Bessel function of the first kind.

The classical fractional calculus operators, Riemann-Liouville fractional integral of order λ and Riemann-Liouville fractional derivative of order λ, were adapted for studies regarding analytic functions in [7]. The Riemann-Liouville fractional integral seen in [7] was combined with the generalized hypergeometric function in [8] and with the Gauss hypergeometric function in a general family of fractional integral operators that were introduced in [9]. A unified method on special functions and fractional calculus operators was introduced in [10] including a list of publications that can be read to follow the topic's evolution. Studies involving fractional operators in geometric function theory continued over the years. In recent developments on the topic, the Riemann-Liouville fractional integral was combined with the confluent hypergeometric function [11], Gauss hypergeometric function [12], Libera integral operator [13], Mittag-Leffler-Confluent hypergeometric functions [14], or with the q-hypergeometric function [15].

The present research continues the direction of study regarding the development of new fractional operators for conducting studies in geometric function theory. The methods are similar to those used in the previously cited works, but the new results embedding fractional calculus aspects are developed here using the prolific methods of the differential subordination or the admissible functions theory [16] and a well-known result due to Sakaguchi. Such combination of methods has not been used earlier for research on fractional integral operators.

Moreover, the novelty of the research is completed by the combination of the Riemann-Liouville fractional integral of order λ with the form introduced in [7] and the Bessel function of the first kind, resulting in a new fractional integral operator that has not been investigated before. As proved earlier, the Riemann-Liouville fractional integral is widely used in theoretical studies pertaining new fractional operators involving different hypergeometric functions with applications in the geometric function theory. The new application of this prolific investigation tool on the Bessel function of the first kind presented in this study is motivated by the initial studies on its geometric properties established by Baricz Á. [17–19] and more recent studies seen in papers [20–23].

The significant results published recently regarding an operator defined using the Atangana-Baleanu fractional integral applied to Bessel functions [24] constituted a confirmation of the hypothesis that an interesting fractional operator would be obtained by combining the fractional integral of order λ given in [7] and the Bessel function of the first kind. Further investigations conducted in this paper on the fractional integral operator introduced here involving the Bessel function of the first kind were inspired by the intriguing results recently published regarding geometric properties of integral operators defined involving the Bessel function [25–27].

With the topic of the study exposed and the motivation shown above, we now introduce the basic notions and denotations familiar to the geometric function theory.

The general class of holomorphic functions in the unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$ is identified by $H(U)$. The notations

$$\overline{U} = \{z \in \mathbb{C} : |z| \leq 1\} \text{ and } \partial U = \{z \in \mathbb{C} : |z| = 1\}$$

are associated to the unit disc U.

The special classes

$$H[a, n] = \{f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$$

and

$$A_n = \{f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + \ldots,\ z \in U\}$$

with $A_1 = A$ are also involved in this study. Furthermore, the study requires the following prominent classes:

$$S = \{f \in A : f \text{ univalent},\ f(0) = 0,\ f'(0) = 1\}, \text{ the class of univalent functions,}$$

$$S^*(\alpha) = \left\{ f \in A : \operatorname{Re} \frac{zf'(z)}{f(z)} > \alpha \right\},$$

with $\alpha < 1$, denoting the class of starlike functions of order α, obtaining the class of starlike functions S^* when $\alpha = 0$, and

$$K(\alpha) = \left\{ f \in A : \operatorname{Re} \left(\frac{zf''(z)}{f'(z)} + 1 \right) > \alpha \right\},$$

with $\alpha < 1$, denoting the class of convex functions of order α, obtaining the class of convex functions K when $\alpha = 0$.

Remark 1.1. *It should be mentioned that for the classes $S^*(\alpha)$, S^*, $K(\alpha)$, and K, a so-called Duality theorem was proved in* [28, Th. 4.4.4, p. 76], *which gives that for $\alpha \in [0, 1)$, $S^*(\alpha) \subset S^*$ and $K(\alpha) \subset K$, meaning that starlike and convex functions of order α are univalent functions. However, when $\alpha < 0$, the functions $f \in S^*(\alpha)$ and $f \in K(\alpha)$ are called starlike and convex of negative order, respectively, and these types of functions may not always be univalent.*

Finally, the study also includes reference to the class of Carathéodory functions,

$$\mathcal{P} = \{p \in H(U) : p(0) = 1,\ \operatorname{Re} p(z) > 0,\ z \in U\}$$

and the class of close-to-convex functions,

$$C = \left\{ f \in H(U) : \exists\, \varphi \in K,\ \operatorname{Re} \frac{f'(z)}{\varphi'(z)} > 0,\ z \in U \right\}.$$

If $\varphi(z) = z$, the condition becomes $\operatorname{Re} f'(z) > 0$, $z \in U$. Such a function is called close-to-convex with respect to the identical function and the condition gives that a close-to-convex function is also univalent in U.

A known result, which will be used in the following proofs, is given as:

Lemma 1.1. [28, p. 84] *A necessary and sufficient condition for a function* $f \in H(U)$, $f'(z) \neq 0$, *to be close-to-convex is:*

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[1 + \frac{zf''(z)}{f'(z)}\right] d\theta > -\pi,\ z = r^{i\theta},$$

whenever $0 \leq \theta_1 < \theta_2 < 2\pi$.

The basic concept of the differential subordination is known as the following:

Definition 1.1. [7, 29] *Let* f *and* F *be members of* $H(U)$. *The function* f *is said to be subordinate to* F, *written* $f \prec F$, $f(z) \prec F(z)$, *if there exists a function* w *analytic in* U, *with* $w(0) = 0$ *and* $|w(z)| < 1$, $z \in U$, *such that* $f(z) = F(w(z))$. *If* w *is univalent, then* $f \prec F$, *if and only if,* $f(0) = F(0)$ *and* $f(U) \subset F(U)$.

The remarkable result due to Sakaguchi is a notable tool for obtaining the new outcome of this study.

Lemma 1.2. [28, p. 49] *Let* $p \in H(U)$, *such that* $\mathrm{Re}\, p(0) > 0$, *and let* $\alpha \in \mathbb{R}$. *Then*

$$\mathrm{Re}\left[p(z) + \alpha\frac{zp'(z)}{p(z)}\right] > 0,\ z \in U,$$

implies

$$\mathrm{Re}\, p(z) > 0,\ z \in U.$$

The essential research tools are presented next.

Definition 1.2. [7, 8] *The fractional integral of order* λ ($\lambda > 0$) *is defined for a function* f *by the following expression:*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt, \tag{1.1}$$

where f *is an analytic function in a simply-connected region of the* z*-plane containing the origin and the multiplicity of* $(z-t)^{1-\lambda}$ *is removed by requiring* $\log(z-t)$ *to be real when* $z - t > 0$.

Definition 1.3. [17] *Consider the second-order differential equation*

$$z^2\, \frac{d^2y(z)}{dz^2} + z\, \frac{dy(z)}{dz} + (z^2 - \nu^2)y(z) = 0,$$

which is called Bessel's equation, where $\nu \in \mathbb{R}$ *or* $\nu \in \mathbb{C}$ *and* $z \in \mathbb{C}$. *The particular solutions of this equation are called Bessel functions.*

The Bessel function of the first kind and order ν *is given by:*

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{p=0}^{\infty} \frac{(-1)^p \left(\frac{z}{2}\right)^{2p}}{p!\Gamma(\nu + p + 1)},\ \nu \geq 0, \tag{1.2}$$

where Γ *is Euler's gamma function.*

The key findings of this investigation are revealed in the next section, starting with the introduction of the new fractional integral operator defined that considers Definitions 1.2 and 1.3. The first new results established in this paper give necessary and sufficient conditions for the newly defined operator to be starlike of positive and negative order, respectively. Next, similar conditions are obtained such that the newly defined operator is convex of positive and negative order, respectively. Furthermore, it is proved that the convexity of the fractional integral of the Bessel function of the first kind and order 0 implies its starlikeness. Finally, the conditions for the fractional integral of the Bessel function of the first kind and order 0 to be a close-to-convex function are obtained.

2. Results

The first new result introduced by this research shows the integral operator given as the fractional integral of the Bessel function of the first kind and order $\nu \geq 0$. The definition is obtained using Definitions 1.2 and 1.3 and the properties of the functions involved.

Definition 2.1. *Let $\nu \geq 0$, $z \in \mathbb{C}$, $\lambda > 0$, $p \in \mathbb{N}$, the fractional integral of order λ be given by (1.1), and the Bessel function of the first kind and order ν both be given by (1.2). The fractional integral of the Bessel function of the first kind and order ν are defined as:*

$$\begin{aligned} D_z^{-\lambda} I_\nu(z) &= \frac{1}{\Gamma(\lambda)} \int_0^z \frac{I_\nu(t)}{(z-t)^{1-\lambda}} dt \\ &= \frac{1}{\Gamma(\lambda)} \int_0^z \left(\sum_{p=0}^{\infty} \frac{(-1)^p t^{2p+\nu}}{2^{2p+\nu} p! \Gamma(\nu+p+1)} \cdot \frac{1}{(z-t)^{1-\lambda}} \right) dt \\ &= \frac{1}{\Gamma(\lambda)} \sum_{p=0}^{\infty} \left(\frac{(-1)^p}{2^{2p+\nu} p! \Gamma(\nu+p+1)} \int_0^z \frac{t^{2p+\nu}}{(z-t)^{1-\lambda}} \right) dt \\ &= \sum_{p=0}^{\infty} \frac{(-1)^p}{2^{2p+\nu} p! \Gamma(\nu+p+1)} \cdot \frac{\Gamma(2p+\nu+1)}{\Gamma(2p+\nu+\lambda+1)} \cdot z^{2p+\nu+\lambda} \\ &= \frac{z^{\nu+\lambda}}{2^\nu} \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma(2p+\nu+1)}{2^{2p} p! \Gamma(\nu+p+1) \Gamma(2p+\nu+\lambda+1)} \cdot z^{2p}. \end{aligned} \tag{2.1}$$

Remark 2.1. *For $\nu = 0$, we obtain:*

$$D_z^{-\lambda} I_0(z) = z^\lambda \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma(2p+1)}{2^{2p} (p!)^2 \Gamma(2p+\lambda+1)} \cdot z^{2p}. \tag{2.2}$$

Furthermore, for $\lambda = 1$, we get the following particular form, which is important for this investigation; the fractional integral of the Bessel function of the first kind and order 0*:*

$$D_z^{-1} I_0(z) = z \sum_{p=0}^{\infty} \frac{(-1)^p \Gamma(2p+1)}{2^{2p} (p!)^2 \Gamma(2p+2)} \cdot z^{2p}. \tag{2.3}$$

The following theorems highlight the necessary and sufficient conditions for the operator $D_z^{-\lambda} I_\nu(z)$ introduced in (2.1) to be starlike and starlike of positive and negative order, respectively.

Theorem 2.1. *Let $D_z^{-\lambda} I_\nu(z)$ be given by (2.1) and consider $q \in K$ as:*

$$q(z) = \frac{1+z}{1-z}, \; z \in U.$$

If

$$\frac{z(D_z^{-\lambda} I_\nu(z))'}{D_z^{-\lambda} I_\nu(z)} \prec \frac{1+z}{1-z}, \; z \in U, \tag{2.4}$$

then

$$\mathrm{Re}\,\frac{z(D_z^{-\lambda} I_\nu(z))'}{D_z^{-\lambda} I_\nu(z)} > 0, \; z \in U.$$

Proof. Since $q \in K$, $q(0) = 1$, and $q(U) = \{z \in \mathbb{C} : \mathrm{Re}\, z > 0\}$ is a convex domain, Relation (2.4) is equivalent to

$$\mathrm{Re}\,\frac{z(D_z^{-\lambda} I_\nu(z))'}{D_z^{-\lambda} I_\nu(z)} > \mathrm{Re}\, q(z) > 0, \; z \in U. \tag{2.5}$$

□

Remark 2.2. *Since $(D_z^{-\lambda} I_\nu)'(0) = 0$, Relation (2.5) doesn't imply that $D_z^{-\lambda} I_\nu(z) \in S^*$; hence, $D_z^{-\lambda} I_\nu(z) \notin S^*$. We can only conclude that $D_z^{-\lambda} I_\nu(z) \in \mathcal{P}$.*

For $\lambda = 1$, $\nu = 0$, and considering Relation (2.3), we have that $(D_z^{-1} I_0)'(0) = 1$, and the following corollary is obtained:

Corollary 2.1. *Let $D_z^{-1} I_0(z)$ as given by (2.3) and consider $q \in K$ as:*

$$q(z) = \frac{1+z}{1-z}, \; z \in U.$$

If

$$\frac{z(D_z^{-1} I_0(z))'}{D_z^{-1} I_0(z)} \prec \frac{1+z}{1-z}, \; z \in U,$$

then

$$\mathrm{Re}\,\frac{z(D_z^{-1} I_0(z))'}{D_z^{-1} I_0(z)} > 0, \; z \in U, \tag{2.6}$$

meaning that $D_z^{-1} I_0(z) \in S^*$.

Proof. Since $D_z^{-1} I_0(0) = 0$ and $(D_z^{-1} I_0)'(0) = 1 \neq 0$, Relation (2.6) implies that $D_z^{-1} I_0(z) \in S^*$ and $D_z^{-1} I_0(z) \in S$. □

Remark 2.3. *By letting*

$$g(z) = \frac{z(D_z^{-1} I_0(z))'}{D_z^{-1} I_0(z)},$$

we have that $g(0) = 1$, and by using (2.6), we can write $\mathrm{Re}\, g(z) > 0$, $z \in U$; hence, $g(z) \in \mathcal{P}$.

The following results demonstrate the necessary and sufficient conditions for the fractional integral of the Bessel function of the first kind and order ν given by (2.1) to be starlike of positive order.

Theorem 2.2. *Let $D_z^{-\lambda}I_\nu(z)$ as given by (2.1) and consider $q \in K$ as:*

$$q(z) = \frac{z}{1+z}, \ z \in U.$$

If

$$\frac{z(D_z^{-\lambda}I_\nu(z))'}{D_z^{-\lambda}I_\nu(z)} \prec \frac{z}{1+z}, \ z \in U, \tag{2.7}$$

then

$$\mathrm{Re}, \ \frac{z(D_z^{-\lambda}I_\nu(z))'}{D_z^{-\lambda}I_\nu(z)} > \frac{1}{2}.$$

Proof. First, it is proved that $q(z) = \dfrac{z}{1+z} \in K$. For that, we get

$$q'(z) = \frac{1}{(1+z)^2}, \quad q''(z) = \frac{-2}{(1+z)^3}$$

and

$$\mathrm{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right) = \mathrm{Re}\,\frac{1-z}{1+z} > 0. \tag{2.8}$$

Since $q(0) = 0$ and $q'(0) = 1 \neq 0$, Relation (2.8) gives that $q(z) \in K$. It follows that

$$q(U) = \left\{z \in \mathbb{C} : \mathrm{Re}\, z > \frac{1}{2}\right\}$$

is a convex domain and Relation (2.7) is equivalent to:

$$\mathrm{Re}\,\frac{z(D_z^{-\lambda}I_\nu(z))'}{D_z^{-\lambda}I_\nu(z)} > \frac{1}{2}, \ z \in U. \tag{2.9}$$

□

Remark 2.4. *Since $(D_z^{-\lambda}I_\nu)'(0) = 0$, Relation (2.9) doesn't imply that $D_z^{-\lambda}I_\nu(z) \in S^*\left(\frac{1}{2}\right)$; hence, $D_z^{-\lambda}I_\nu(z) \notin S^*\left(\frac{1}{2}\right)$.*

For $\lambda = 1$, $\nu = 0$, and considering Relation (2.3), we have that $(D_z^{-1}I_0)'(0) = 1$, and the following corollary is obtained:

Corollary 2.2. *Let $D_z^{-1}I_0(z)$ as given by (2.3) and consider $q \in K$ as:*

$$q(z) = \frac{z}{1+z}, \ z \in U.$$

If

$$\frac{z(D_z^{-1}I_0(z))'}{D_z^{-1}I_0(z)} \prec \frac{z}{1+z}, \ z \in U,$$

then

$$\mathrm{Re}\,\frac{z(D_z^{-1}I_0(z))'}{D_z^{-1}I_0(z)} > \frac{1}{2}, \ z \in U, \tag{2.10}$$

hence, $D_z^{-1}I_0(z) \in S^*\left(\frac{1}{2}\right)$, which implies that $D_z^{-1}I_0(z) \in S^*$.

Proof. Since $(D_z^{-1}I_0)'(0) = 1 \neq 0$, Inequality (2.10) shows that $D_z^{-1}I_0(z) \in S^*\left(\frac{1}{2}\right)$. We mentioned in Remark 1.1 from the Introduction that for $\alpha \in [0,1)$, we have that $S^*\left(\frac{1}{2}\right) \subset S^*$; hence, $D_z^{-1}I_0(z) \in S$. □

The next results demonstrate the necessary and sufficient conditions for the fractional integral of the Bessel function of the first kind and order ν given by (2.1) to be starlike of negative order.

Theorem 2.3. *Let $D_z^{-\lambda}I_\nu(z)$ as given by (2.1) and consider $q \in K$ as:*

$$q(z) = \frac{z}{1-z},\ z \in U.$$

If

$$\frac{z(D_z^{-\lambda}I_\nu(z))'}{D_z^{-\lambda}I_\nu(z)} \prec \frac{z}{1-z},\ z \in U, \tag{2.11}$$

then

$$\operatorname{Re}\frac{z(D_z^{-\lambda}I_\nu(z))'}{D_z^{-\lambda}I_\nu(z)} > -\frac{1}{2},\ z \in U.$$

Proof. The proof begins by proving that $q(z) = \dfrac{z}{1-z} \in K$. For that, we have

$$q'(z) = \frac{1}{(1-z)^2}, \quad q''(z) = \frac{2}{(1-z)^3}$$

and

$$\operatorname{Re}\left(\frac{zq''(z)}{q'(z)} + 1\right) = \operatorname{Re}\frac{1+z}{1-z} > 0,\ z \in U. \tag{2.12}$$

Since $q'(0) = 1 \neq 0$ Relation (2.12) implies that $q \in K$,

$$q(U) = \left\{z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{2}\right\},$$

is a convex domain and Relation (2.11) is equivalent to:

$$\operatorname{Re}\frac{z(D_z^{-\lambda}I_\nu(z))'}{D_z^{-\lambda}I_\nu(z)} > \operatorname{Re}\frac{z}{1-z} > -\frac{1}{2},\ z \in U. \tag{2.13}$$

□

Remark 2.5. *Since $(D_z^{-\lambda}I_\nu)'(0) = 0$, Relation (2.13) doesn't imply that $D_z^{-\lambda}I_\nu(z) \in S^*\left(-\frac{1}{2}\right)$.*

For $\lambda = 1$, $\nu = 0$, and considering Relation (2.3), we have that $(D_z^{-1}I_0)'(0) = 1 \neq 0$, and the following corollary is obtained:

Corollary 2.3. *Let $D_z^{-1}I_0(z)$ be given by (2.3) and consider $q \in K$ as:*

$$q(z) = \frac{z}{1-z},\ z \in U.$$

If

$$\frac{z(D_z^{-1}I_0(z))'}{D_z^{-1}I_0(z)} \prec \frac{z}{1-z},\ z \in U,$$

then

$$\operatorname{Re} \frac{z(D_z^{-1}I_0(z))'}{D_z^{-1}I_0(z)} > -\frac{1}{2},\ z \in U, \tag{2.14}$$

meaning that $D_z^{-1}I_0(z) \in S^*\left(-\frac{1}{2}\right)$.

Proof. Since $(D_z^{-1}I_0)'(0) = 1 \neq 0$, Inequality (2.14) gives that $D_z^{-1}I_0(z) \in S^*\left(-\frac{1}{2}\right)$. According to Remark 1.1, since $-\frac{1}{2} < 0$, $S^*\left(-\frac{1}{2}\right) \not\subset S^*$, and it is possible that $D_z^{-1}I_0(z)$ is not univalent in U. □

The next results provide necessary and sufficient conditions for the fractional integral of the Bessel function of the first kind and order ν to be a convex function of positive and negative order, respectively.

Theorem 2.4. *Let $D_z^{-\lambda}I_\nu(z)$ as given by (2.1) and consider $q \in K$ as:*

$$q(z) = \frac{1-z}{1+z},\ z \in U.$$

If

$$1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'} \prec \frac{1-z}{1+z},\ z \in U, \tag{2.15}$$

then

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'}\right] > 0,\ z \in U.$$

Proof. Since $q(z) = \frac{1-z}{1+z} \in K$, $q(U) = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ is a convex domain and Relation (2.15) is equivalent to

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'}\right] > \operatorname{Re} \frac{1-z}{1+z} > 0,\ z \in U. \tag{2.16}$$

□

Remark 2.6. *Since $(D_z^{-\lambda}I_\nu)'(0) = 0$, Relation (2.16) doesn't imply $D_z^{-\lambda}I_\nu(z) \in K$.*

For $\lambda = 1$, $\nu = 0$, and considering Relation (2.3), we get that $(D_z^{-1}I_0)'(0) = 1 \neq 0$, and the following corollary is obtained:

Corollary 2.4. *Let $D_z^{-1}I_0(z)$ be given by (2.3) and consider $q \in K$ as:*

$$q(z) = \frac{1-z}{1+z},\ z \in U.$$

If

$$1 + \frac{z(D_z^{-1}I_0(z))''}{(D_z^{-1}I_0(z))'} \prec \frac{1-z}{1+z},\ z \in U,$$

then

$$\mathrm{Re}\left[1 + \frac{z(D_z^{-1}I_0(z))''}{(D_z^{-1}I_0(z))'}\right] > 0,\ z \in U, \tag{2.17}$$

hence, $D_z^{-1}I_0(z) \in K$.

Proof. Since $(D_z^{-1}I_0)'(0) = 1 \neq 0$, (2.17) implies that $D_z^{-1}I_0(z) \in K$ and $D_z^{-1}I_0(z) \in S$. □

By applying the outcome of Corollary 2.4, we show that $D_z^{-1}I_0(z) \in K$ implies $D_z^{-1}I_0(z) \in S^*$.

Theorem 2.5. *Let $D_z^{-1}I_0(z)$ as given by (2.3). If $D_z^{-1}I_0(0) = 0$, $(D_z^{-1}I_0)'(0) = 1 \neq 0$, and*

$$\mathrm{Re}\left[1 + \frac{z(D_z^{-1}I_0(z))''}{(D_z^{-1}I_0(z))'}\right] > 0,\ z \in U,\ \textit{i.e.},\ D_z^{-1}I_0(z) \in K, \tag{2.18}$$

then

$$\mathrm{Re}\,\frac{z(D_z^{-1}I_0(z))'}{D_z^{-1}I_0(z)} > 0,\ z \in U,\ \textit{meaning that}\ D_z^{-1}I_0(z) \in S^*.$$

Proof. Let

$$g(z) = \frac{z(D_z^{-1}I_0(z))'}{D_z^{-1}I_0(z)},\ z \in U. \tag{2.19}$$

Applying (2.3) in (2.19), we obtain that $g(0) = 1 \neq 0$ and $\mathrm{Re}\, g(0) > 0$. By differentiating (2.19), after a few calculations, we get:

$$g(z) + \frac{zg'(z)}{g(z)} = 1 + \frac{z(D_z^{-1}I_0(z))''}{(D_z^{-1}I_0(z))'},\ z \in U. \tag{2.20}$$

Using Relation (2.18) in (2.20), we obtain:

$$\mathrm{Re}\left(g(z)\frac{zg'(z)}{g(z)}\right) > 0,\ z \in U. \tag{2.21}$$

Considering Lemma 1.2 with $\alpha = 1$, Inequality (2.21) implies

$$\mathrm{Re}\, g(z) > 0,\ z \in U. \tag{2.22}$$

Using Relation (2.19) in (2.22), we have:

$$\mathrm{Re}\,\frac{z(D_z^{-1}I_0(z))'}{D_z^{-1}I_0(z)} > 0,\ z \in U. \tag{2.23}$$

Since $D_z^{-1}I_0(0) = 0$ and $(D_z^{-1}I_0)' = 1 \neq 0$, Relation (2.23) implies that $D_z^{-1}I_0(z) \in S^*$; hence, $D_z^{-1}I_0(z) \in S$: a result already known from Corollary 2.1. □

Remark 2.7. *Since $D_z^{-1}I_0(0) = 0$, $(D_z^{-1}I_0)'(0) = 1$, and $D_z^{-1}I_0(z) \in A$, the Max-Strohhäcker theorem* [21, p. 194] *gives that*

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-1}I_0(z))''}{(D_z^{-1}I_0(z))'}\right] > 0 \textit{ implies } \operatorname{Re}\frac{z(D_z^{-1}I_0(z))'}{D_z^{-1}I_0(z)} > \frac{1}{2},$$

which is already known from Corollary 2.2.

The next results state the necessary and sufficient conditions for the fractional integral of the Bessel function of the first kind and order ν given by (2.1) to be a convex function of positive order $\alpha \in [0, 1)$.

Theorem 2.6. *Let $D_z^{-1}I_\nu(z)$ as given by (2.1) and consider $q \in K$ as:*

$$q(z)\frac{1}{1-z}, \ z \in U.$$

If

$$1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'} \prec \frac{1}{1-z}, \ z \in U, \tag{2.24}$$

then

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'}\right] > \frac{1}{2}, \ z \in U.$$

Proof. First, it is shown that $q(z) = \dfrac{1}{1-z} \in K$, then we get:

$$q'(z) = \frac{1}{(1-z)^2}, \quad q''(z) = \frac{2}{(1-z)^3}$$

and

$$\operatorname{Re}\left[\frac{zq''(z)}{q'(z)} + 1\right] > \operatorname{Re}\frac{1+z}{1-z} > 0. \tag{2.25}$$

Since $q'(0) = 1 \neq 0$, Relation (2.25) gives that $q(z) \in K$,

$$q(U) = \left\{z \in \mathbb{C} : \operatorname{Re} z > \frac{1}{2}\right\}$$

is a convex domain and differential subordination (2.24) is equivalent to:

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'}\right] > \operatorname{Re}\frac{1}{1+z} > \frac{1}{2}, \ z \in U. \tag{2.26}$$

□

Remark 2.8. *Since $(D_z^{-\lambda}I_\nu)'(0) = 0$, Relation (2.26) doesn't imply that $D_z^{-\lambda}I_\nu(z) \in K\left(\frac{1}{2}\right)$.*

For $\lambda = 1$, $\nu = 0$, and considering Relation (2.3), we get that $(D_z^{-1}I_0)'(0) = 1 \neq 0$, and the following corollary is obtained:

Corollary 2.5. *Let $D_z^{-1}I_0(z)$ as given by (2.3) and consider $q \in K$*

$$q(z) = \frac{1}{1-z}, \ z \in U.$$

If

$$1 + \frac{z(D_z^{-1}I_0(z))''}{(D_z^{-1}I_0(z))'} \prec \frac{1}{1-z}, \ z \in U,$$

then

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-1}I_0(z))''}{(D_z^{-1}I_0(z))'}\right] > \frac{1}{2}, \tag{2.27}$$

i.e., $D_z^{-1}I_0(z) \in K$.

Proof. Since $(D_z^{-1}I_0)'(0) = 1 \neq 0$, Inequality (2.27) gives that $D_z^{-1}I_0(z) \in K\left(\frac{1}{2}\right)$. According to Remark 1.1, $D_z^{-1}I_0(z) \in K$ since $\alpha = \frac{1}{2} \in [0,1)$. □

The next results establish the necessary and sufficient conditions for the fractional integral of the Bessel function of the first kind and order ν to be convex of negative order.

Theorem 2.7. *Let $D_z^{-\lambda}I_\nu(z)$ as given by (2.1) and consider $q \in K$ as:*

$$q(z) = \frac{1-2z}{1+z}, \ z \in U.$$

If

$$1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'} \prec \frac{1-2z}{1+z}, \ z \in U, \tag{2.28}$$

then

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'}\right] > -\frac{1}{2}, \ z \in U.$$

Proof. First, it is shown that the function $q(z) = \frac{1-2z}{1+z} \in K$. For that, we get:

$$q'(z) = \frac{-3}{(1+z)^2}, \quad q''(z) = \frac{6}{(1+z)^3}$$

and

$$\operatorname{Re}\left[\frac{zq''(z)}{q'(z)} + 1\right] = \operatorname{Re}\frac{1-z}{1+z} > 0, \ z \in U. \tag{2.29}$$

Since $q'(0) \neq 0$, Inequality (2.29) gives that $q(z) \in K$,

$$q(U) = \left\{z \in \mathbb{C} : \operatorname{Re} z > -\frac{1}{2}\right\}$$

is a convex domain, $\operatorname{Re} q(z) > -\frac{1}{2}$, and Relation (2.28) is equivalent to:

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-\lambda}I_\nu(z))''}{(D_z^{-\lambda}I_\nu(z))'}\right] > \operatorname{Re}\frac{1-2z}{1+z}, \ z \in U. \tag{2.30}$$

□

Remark 2.9. *Since $(D_z^{-\lambda} I_\nu)'(0) = 0$, Relation (2.30) doesn't imply that $D_z^{-\lambda} I_\nu(z) \in K\left(-\frac{1}{2}\right)$.*

For $\lambda = 1$, $\nu = 0$, and considering Relation (2.3), we obtain that $(D_z^{-1} I_0(z))'(0) = 1 \neq 0$, and the following corollary is obtained:

Corollary 2.6. *Let $D_z^{-\lambda} I_0(z)$ as given by (2.3) and consider $q \in K$ as:*

$$q(z) = \frac{1 - 2z}{1 + z}, \ z \in U.$$

If

$$1 + \frac{z(D_z^{-1} I_0(z))''}{(D_z^{-1} I_0(z))'} \prec \frac{1 - 2z}{1 + z}, \ z \in U,$$

then

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-1} I_0(z))''}{(D_z^{-1} I_0(z))'}\right] > -\frac{1}{2}, \tag{2.31}$$

meaning that $D_z^{-1} I_0(z) \in K\left(-\frac{1}{2}\right)$.

Proof. Since $(D_z^{-1} I_0)'(0) = 1 \neq 0$, Relation (2.31) implies that $D_z^{-1} I_0(z) \in K\left(-\frac{1}{2}\right)$. Since $\alpha = -\frac{1}{2} < 0$, according to Remark 1.1, the function $D_z^{-1} I_0(z)$ is not necessarily univalent. □

Using the outcome of Corollary 2.6, we next show that the function $D_z^{-1} I_0(z)$ given by (2.3) is a close-to-convex function and is univalent.

Theorem 2.8. *Let $D_z^{-1} I_0(z)$ as given by (2.3) satisfy:*

$$\operatorname{Re}\left[1 + \frac{z(D_z^{-1} I_0(z))''}{(D_z^{-1} I_0(z))'}\right] > -\frac{1}{2}, \ z \in U.$$

Thus, $D_z^{-1} I_0(z) \in C$ and $D_z^{-1} I_0(z) \in S$.

Proof. Lemma 1.1 is applied in order to prove this theorem. We evaluate

$$\int_{\theta_1}^{\theta_2} \operatorname{Re}\left[1 + \frac{z(D_z^{-1} I_0(z))''}{(D_z^{-1} I_0(z))'}\right] d\theta > \int_{\theta_1}^{\theta_2} -\frac{1}{2} d\theta = -\frac{1}{2}(\theta_2 - \theta_1) > -\pi,$$

since $0 \leq \theta_1 < \theta_2 < 2\pi$. Applying Lemma 1.1, we conclude that $D_z^{-1} I_0(z) \in C$; hence, $D_z^{-1} I_0(z) \in S$. □

3. Discussion

The main goal of the study is to present a novel fractional integral operator and to start investigations on its geometric properties by utilizing the differential subordination theory's methods. The fractional integral of order λ described in Definition 1.2 given by Relation (1.1) and the Bessel function of the first kind and order ν shown in Definition 1.3 given by Relation (1.2) are used to introduce, in Definition 2.1, the fractional integral of the Bessel function of the first kind denoted by $D_z^{-\lambda} I_\nu(z)$ given by Relation (2.1). In the Introduction, the relevant definitions for this study are displayed and the main tools for obtaining the new results are presented. In Section 2, 8 theorems and 6 corollaries contain new knowledge regarding the conditions for the univalence of the fractional integral of the Bessel function of the first kind. Necessary and sufficient conditions for $D_z^{-\lambda} I_\nu(z)$ to be starlike of positive and negative order, respectively, are found in Theorems 2.1, 2.2, and 2.3 in association with Corollaries 2.1, 2.2, and 2.3. The next theorems and associated corollaries provide necessary and sufficient conditions for $D_z^{-\lambda} I_\nu(z)$ to be convex of positive and negative order, respectively. By applying the outcome of Corollary 2.4, in Theorem 2.5 it is shown that the convexity of the particular form of the fractional integral of the Bessel function of the first kind $D_z^{-1} I_0(z)$ implies its starlikeness. Furthermore, by applying the knowledge given by Corollary 2.6, the last theorem stated in this study provides a condition for the operator $D_z^{-1} I_0(z)$ to be close-to-convex.

4. Conclusions

The results of this study are relevant to the research topic concerned with integrating fractional calculus and special functions in the geometric function theory. Future applications of the findings given here may establish new subclasses of analytical functions with particular geometric characteristics given by the properties of the operator $D_z^{-\lambda} I_\nu(z)$ given by (2.1) already illustrated in this article. Research related to the extensions of differential subordination theories, known as fuzzy differential subordination and strong differential subordination, that have been introduced in recent years can also be done on the operator $D_z^{-\lambda} I_\nu(z)$ defined in this paper. Recent works such as [30] and [31], which use specific fractional operators for studies involving the two different theories, respectively, could serve as an inspiration for these studies.

Studies to confirm that the Riemann-Liouville fractional derivative of order λ applied to the Bessel function of the first kind would provide notable outcome could be further conducted. Also, other fractional order differential operators might provide interesting outcome when applied to the Bessel function of the first kind.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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