AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 7084–7094.
DOI: 10.3934/math.2024345
Received: 15 December 2023
Revised: 26 January 2024
Accepted: 01 February 2024
Published: 19 February 2024

Research article

Approximate solution of a multivariable Cauchy-Jensen functional equation

Jae-Hyeong Bae[1] **and Won-Gil Park**[2,*]

[1] Humanitas College, Kyung Hee University, Yongin 17104, Republic of Korea

[2] Department of Mathematics Education, College of Education, Mokwon University, Daejeon 35349, Republic of Korea

* **Correspondence:** Email: wgpark@mokwon.ac.kr; Tel: +82428297547; Fax: +82428297038.

Abstract: Let n be an integer greater than 1. In this paper, we obtained the stability of the multivariable Cauchy-Jensen functional equation

$$nf\left(x_1 + \cdots + x_n, \frac{y_1 + \cdots + y_n}{n}\right) = \sum_{1 \le i,j \le n} f(x_i, y_j)$$

in Banach spaces, quasi-Banach spaces, and normed 2-Banach spaces.

Keywords: linear 2-normed space; quasi-normed space; Cauchy-Jensen mapping
Mathematics Subject Classification: 39B52, 39B82

1. Introduction

Hyers-Ulam stability in functional equations relates to the characteristic where a function may not perfectly satisfy a given equation but remains close to the actual solution if the error is within a limited range. This concept of stability can be applied to many types of functional equations. The concept of Hyers-Ulam stability is named after the mathematicians David Hyers and Stanislaw Ulam, who both discovered it in the 1940s. In 1978, Rassias [1] published the result that stability can be obtained even after generalizing the error from a constant to a variable. The stability problem where the error is a variable is called the Hyers-Ulam-Rassias stability.

This theory extends beyond the Cauchy equation to various other types of functional equations. For instance, it can be considered in more complex forms of functional equations involving power functions, trigonometric functions, exponential functions, and more. Additionally, this concept can be effectively applied in other mathematical structures like differential equations or integral equations.

Hyers-Ulam stability serves as an important tool in mathematical analysis and applied mathematics, playing a significant role in finding approximate solutions. It helps in finding exact solutions in real-world problems where errors are allowed and is useful in providing approximate interpretations in complex systems.

We would like to introduce the definition of the Cauchy-Jensen equation and past work.

Definition 1. *[2]* A mapping $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is called a *Cauchy-Jensen mapping* if f satisfies the system of equations

$$\begin{aligned} f(x+y,z) &= f(x,z) + f(y,z),\\ 2f\Big(x, \tfrac{y+z}{2}\Big) &= f(x,y) + f(x,z). \end{aligned}$$

In 2006, W. G. Park and J. H. Bae [2] obtained the general solution of the Cauchy-Jensen functional equation

$$2f\Big(x+y, \frac{z+w}{2}\Big) = f(x,z) + f(x,w) + f(y,z) + f(y,w) \tag{1.1}$$

and its stability. Subsequent papers have been published since 2007 by several authors [3–5].

In 2012, J. H. Bae and W. G. Park [6] introduced the following multivariable Cauchy-Jensen functional equation

$$n\,f\Big(x_1 + \cdots + x_n, \frac{y_1 + \cdots + y_n}{n}\Big) = \sum_{1 \le i,\, j \le n} f(x_i,\, y_j), \tag{1.2}$$

where n is an integer greater than 1. In Theorem 2.2 of [6], we proved that the functional Eqs (1.1) and (1.2) are equivalent. The functional Eq (1.1) is the case where $n = 2$ in the function Eq (1.2).

In 2011, W. G. Park [7] investigated the approximate additive and Jensen and quadratic mappings in 2-Banach spaces. In 2015, S. Yun [8] corrected the statements of results in [7] and proved the corrected theorems.

In this paper, we investigate the stability of the functional Eq (1.2) in Banach spaces, quasi-Banach spaces, and normed 2-Banach spaces.

2. Main results

Let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ a Banach space.

Theorem 1. *Let $r \in (0,1)$, $\varepsilon > 0$, and $\delta \ge 0$, and let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(x,0) = 0$ such that*

$$\left\| nf\Big(x_1 + \cdots + x_n, \frac{y_1 + \cdots + y_n}{n}\Big) - \sum_{1\le i,j\le n} f(x_i, y_j) \right\| \le \varepsilon + \delta\Big(\sum_{i=1}^{n} \|x_i\|^r + \sum_{j=1}^{n} \|y_j\|^r \Big) \tag{2.1}$$

for all $x, x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{X}$, then there exists a Cauchy-Jensen mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned} &\|f(x,y) - f(x,0) - F(x,y)\| \\ &\qquad \le (n^2 + 3n + 1)\left[\frac{2\varepsilon}{n(n^2 + n - 1)} + \frac{\delta\|x\|^r}{n(n+1) - n^r} \right] + \frac{2\delta[3n + 1 + 4(n+1)^r]\|y\|^r}{n(n+1) - (n+1)^r} \end{aligned} \tag{2.2}$$

for all $x, y \in \mathcal{X}$.

Proof. Let $g(x,y) := f(x,y) - f(0,y)$ for all $x, y \in \mathcal{X}$, then $g(x,0) = g(0,y) = 0$ for all $x, y \in \mathcal{X}$. By (2.1), g satisfies

$$\left\| ng\left(x_1 + \cdots + x_n, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{1 \le i,j \le n} g(x_i, y_j) \right\| \le 2\varepsilon + \delta\left(\sum_{i=1}^{n} \|x_i\|^r + 2 \sum_{j=1}^{n} \|y_j\|^r \right) \tag{2.3}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{X}$. Letting $x_1 = \cdots = x_n = x$ in (2.3), we gain

$$\left\| g\left(nx, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{j=1}^{n} g(x, y_j) \right\| \le \frac{1}{n}\left[2\varepsilon + \delta\left(n\|x\|^r + 2 \sum_{j=1}^{n} \|y_j\|^r \right) \right] \tag{2.4}$$

for all $x, y_1, \cdots, y_n \in \mathcal{X}$. Putting $y_1 = y, y_2 = -y$, and $y_3 = \cdots = y_n = 0$ in (2.4), we get

$$\|g(x,y) + g(x,-y)\| \le \frac{1}{n}[2\varepsilon + \delta(n\|x\|^r + 4\|y\|^r)]$$

for all $x, y \in \mathcal{X}$. Setting $y_1 = \cdots = y_n = y$ in (2.4), we have

$$\|g(nx,y) - ng(x,y)\| \le \frac{2}{n}\varepsilon + \delta\,(\|x\|^r + 2\|y\|^r)$$

for all $x, y \in \mathcal{X}$. By the above two inequalities, we obtain

$$\|g(nx,y) + ng(x,-y)\| \le (n+1)\left(\frac{2}{n}\varepsilon + \delta\|x\|^r\right) + 6\delta\|y\|^r \tag{2.5}$$

for all $x, y \in \mathcal{X}$. Taking $y_1 = y, y_2 = -(n+1)y$ and $y_j = 0$ $(3 \le j \le n)$ in (2.4), we gain

$$\|g(nx,-y) - g(x,y) - g(x,-(n+1)y)\| \le \frac{1}{n}\left[2\varepsilon + \delta\left(n\|x\|^r + 2[1 + (n+1)^r]\|y\|^r\right)\right]$$

for all $x, y \in \mathcal{X}$. By (2.5) and the above inequality, we get

$$\|(n+1)g(x,y) + g(x,-(n+1)y)\| \le (n+2)\left(\frac{2}{n}\varepsilon + \delta\|x\|^r\right) + 2\delta\left(3 + \frac{1}{n}[1 + (n+1)^r]\right)\|y\|^r \tag{2.6}$$

for all $x, y \in \mathcal{X}$. Replacing y by $(n+1)y$ in (2.5), we have

$$\|g\,(nx,(n+1)y) + ng\,(x,-(n+1)y)\| \le (n+1)\left(\frac{2}{n}\varepsilon + \delta\|x\|^r\right) + 6\delta(n+1)^r\|y\|^r$$

for all $x, y \in \mathcal{X}$. By (2.6) and the above inequality, we obtain

$$\|n(n+1)g(x,y) - g(nx,(n+1)y)\| \le \left(n + 3 + \frac{1}{n}\right)(2\varepsilon + n\delta\|x\|^r) + 2\delta[3n + 1 + 4(n+1)^r]\|y\|^r$$

for all $x, y \in \mathcal{X}$. Replacing x by $n^k x$ and y by $(n+1)^k y$ in the above inequality and dividing $n^{k+1}(n+1)^{k+1}$, we see that

$$\left\| \frac{1}{n^k(n+1)^k} g(n^k x, (n+1)^k y) - \frac{1}{n^{k+1}(n+1)^{k+1}} g(n^{k+1}x, (n+1)^{k+1}y) \right\|$$

$$\leq \frac{n^2+3n+1}{n^{k+2}(n+1)^{k+1}}(2\varepsilon+\delta n^{1+kr}\|x\|^r)+\frac{2\delta[3n+1+4(n+1)^r]}{n^{k+1}(n+1)^{k(1-r)+1}}\|y\|^r$$

for all $x, y \in \mathcal{X}$ and all nonnegative integers k. Thus, we have

$$\begin{aligned}&\left\|\frac{1}{n^l(n+1)^l}g(n^l x,(n+1)^l y)-\frac{1}{n^m(n+1)^m}g\left(n^m x,(n+1)^m y\right)\right\|\\ &\qquad\leq \sum_{k=l}^{m-1}\left(\frac{n^2+3n+1}{n^{k+2}(n+1)^{k+1}}(2\varepsilon+\delta n^{1+kr}\|x\|^r)+\frac{2\delta[3n+1+4(n+1)^r]}{n^{k+1}(n+1)^{k(1-r)+1}}\|y\|^r\right)\end{aligned}\tag{2.7}$$

for all integers l, $m\,(0 \leq l < m)$, and all $x, y \in \mathcal{X}$. Thus, the sequence $\{\frac{1}{n^k(n+1)^k}g(n^k x,(n+1)^k y)\}$ is a Cauchy sequence for all $x,\ y \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{n^k(n+1)^k}g(n^k x,(n+1)^k y)\}$ converges for all $x,\ y \in \mathcal{X}$. Define $F : \mathcal{X}\times\mathcal{X} \to \mathcal{Y}$ by

$$F(x,y) := \lim_{k\to\infty}\frac{1}{n^k(n+1)^k}g(n^k x,(n+1)^k y)$$

for all $x, y \in \mathcal{X}$. By (2.3), we have

$$\begin{aligned}&\frac{1}{n^k(n+1)^k}\left\|ng\Big(n^k(x_1+\cdots+x_n),(n+1)^k\frac{y_1+\cdots+y_n}{n}\Big)-\sum_{1\leq i,j\leq n}g\left(n^k x_i,(n+1)^k y_j\right)\right\|\\ &\qquad\leq \frac{1}{n^k(n+1)^k}\left[2\varepsilon+\delta\Big(n^{kr}\sum_{i=1}^{n}\|x_i\|^r+2(n+1)^{kr}\sum_{j=1}^{n}\|y_j\|^r\Big)\right]\end{aligned}$$

for all $x_1,\cdots,x_n,y_1,\cdots,y_n \in \mathcal{X}$, and all nonnegative integers k. Letting $k\to\infty$ in the above inequality, we obtain that F satisfies (1.2). By [6], F is a Cauchy-Jensen mapping. Setting $l = 0$ and taking $m\to\infty$ in (2.7), one can obtain the inequality (2.2).

In [9–11], one can find the concept of quasi-Banach spaces.

Definition 2. Let $\mathcal{X}$ be a real vector space. A *quasi-norm* is a real-valued function on $\mathcal{X}$ satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in \mathcal{X}$, and $\|x\| = 0$ if, and only if, $x = 0$.

(ii) $\|\lambda x\| = |\lambda|\|x\|$ for all $\lambda \in \mathbb{R}$ and all $x \in \mathcal{X}$.

(iii) There is a constant $K \geq 1$ such that $\|x+y\| \leq K(\|x\|+\|y\|)$ for all $x,\ y \in \mathcal{X}$.

Definition 3. The pair $(\mathcal{X}, \|\cdot\|)$ is called a *quasi-normed space* if $\|\cdot\|$ is a quasi-norm on $\mathcal{X}$. The smallest possible K is called the *modulus of concavity* of $\|\cdot\|$. A *quasi-Banach space* is a complete quasi-normed space. A quasi-norm $\|\cdot\|$ is called a *p-norm* $(0 < p \leq 1)$ if

$$\|x+y\|^p \leq \|x\|^p + \|y\|^p$$

for all $x,\ y \in \mathcal{X}$. In this case, a quasi-Banach space is called a *p-Banach space*.

We will use the following lemma in the proof of the next theorem.

Lemma 1. [12] Let $0 \leq p \leq 1$ and let $x_1, x_2, \cdots, x_n$ be nonnegative real numbers, then

$$(x_1+x_2+\cdots+x_n)^p \leq x_1^p + x_2^p + \cdots + x_n^p.$$

From now on, assume that $\mathcal{X}$ is a quasi-normed space with quasi-norm $\|\cdot\|$ and that $\mathcal{Y}$ is a p-Banach space with p-norm $\|\cdot\|_{\mathcal{Y}}$.

The following theorem proves the stability of the Eq (1.2) in quasi-Banach spaces.

Theorem 2. *Let $r \in (0,1)$, $\varepsilon > 0$, and $\delta \geq 0$, and let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(x,0) = 0$ such that*

$$\left\| nf\left(x_1 + \cdots + x_n, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{1 \leq i,j \leq n} f(x_i, y_j) \right\|_{\mathcal{Y}} \leq \varepsilon + \delta\left(\sum_{i=1}^{n} \|x_i\|^r + \sum_{j=1}^{n} \|y_j\|^r\right) \tag{2.8}$$

for all $x, x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{X}$, then there exists a Cauchy-Jensen mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned} \|f(x,y) - f(0,y) - F(x,y)\|_{\mathcal{Y}} \leq \Bigg(&\left(n + 3 + \frac{1}{n}\right)\left[\frac{2\varepsilon^p}{n^p(n+1)^p - 1} + \frac{\delta^p n^p \|x\|^{rp}}{n^p(n+1)^p - n^{rp}}\right] \\ &+ \frac{2\delta^p \|y\|^{rp}}{n^p(n+1)^p - (n+1)^{rp}}\left((2^p n + n^p)\left[\frac{1}{n}(n+1)^{rp} + 1\right] + [1 + (n+1)^r]^p\right)\Bigg)^{\frac{1}{p}} \end{aligned} \tag{2.9}$$

for all $x, y \in \mathcal{X}$.

Proof. Let $g(x,y) := f(x,y) - f(0,y)$ for all $x, y \in \mathcal{X}$, then $g(x,0) = g(0,y) = 0$ for all $x, y \in \mathcal{X}$. By (2.8) and by using Lemma 1, g satisfies

$$\left\| ng\left(x_1 + \cdots + x_n, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{1 \leq i,j \leq n} g(x_i, y_j) \right\|_{\mathcal{Y}}^p \leq 2\varepsilon^p + \delta^p\left[\left(\sum_{i=1}^{n} \|x_i\|^r\right)^p + 2\left(\sum_{j=1}^{n} \|y_j\|^r\right)^p\right] \tag{2.10}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{X}$. Letting $x_1 = \cdots = x_n = x$ in (2.10), we gain

$$\left\| g\left(nx, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{j=1}^{n} g(x, y_j) \right\|_{\mathcal{Y}}^p \leq \frac{1}{n}\left(2\varepsilon^p + \delta^p\left[n^p \|x\|^{rp} + 2\left(\sum_{j=1}^{n} \|y_j\|^r\right)^p\right]\right) \tag{2.11}$$

for all $x, y_1, \cdots, y_n \in \mathcal{X}$. Putting $y_1 = y, y_2 = -y$, and $y_3 = \cdots = y_n = 0$ in (2.11), we get

$$\|g(x,y) + g(x,-y)\|_{\mathcal{Y}}^p \leq \frac{1}{n}[2\varepsilon^p + \delta^p(n^p\|x\|^{rp} + 2^{p+1}\|y\|^{rp})]$$

for all $x, y \in \mathcal{X}$. Setting $y_1 = \cdots = y_n = y$ in (2.11), we have

$$\|g(nx,y) - ng(x,y)\|_{\mathcal{Y}}^p \leq \frac{1}{n}[2\varepsilon^p + \delta^p(n^p\|x\|^{rp} + 2n^p\|y\|^{rp})]$$

for all $x, y \in \mathcal{X}$. By the above two inequalities, we obtain

$$\|g(nx,y) + ng(x,-y)\|_{\mathcal{Y}}^p \leq \frac{1}{n}[(n+1)(2\varepsilon^p + \delta^p n^p \|x\|^{rp}) + 2(2^p n + n^p)\delta^p \|y\|^{rp}] \tag{2.12}$$

for all $x, y \in \mathcal{X}$. Taking $y_1 = y, y_2 = -(n+1)y$, and $y_3 = \cdots = y_n = 0$ in (2.11), we gain

$$\|g(nx,-y) - g(x,y) - g(x,-(n+1)y)\|_{\mathcal{Y}}^p \leq \frac{1}{n}\left[2\varepsilon^p + \delta^p\left(n^p\|x\|^{rp} + 2[1 + (n+1)^r]^p\|y\|^{rp}\right)\right]$$

for all $x, y \in \mathcal{X}$. By (2.12) and the above inequality, we get

$$\left\|(n+1)g(x,y) + g(x,-(n+1)y)\right\|_{\mathcal{Y}}^p \leq \left(1 + \frac{2}{n}\right)(2\varepsilon^p + \delta^p n^p \|x\|^{rp}) + \frac{2\delta^p}{n}\left(2^p n + n^p + [1 + (n+1)^r]^p\right)\|y\|^{rp} \tag{2.13}$$

for all $x, y \in \mathcal{X}$. Replacing y by $(n+1)y$ in (2.12), we have

$$\|g(nx,(n+1)y) + ng(x,-(n+1)y)\|_{\mathcal{Y}}^p \leq \frac{1}{n}[(n+1)(2\varepsilon^p + \delta^p n^p\|x\|^{rp}) + 2(2^p n + n^p)(n+1)^{rp}\delta^p\|y\|^{rp}]$$

for all $x, y \in \mathcal{X}$. By (2.13) and the above inequality, we obtain

$$\begin{aligned}
&\left\|n(n+1)g(x,y) - g(nx,(n+1)y)\right\|_{\mathcal{Y}}^p \\
&\leq \left(n+3+\frac{1}{n}\right)(2\varepsilon^p + \delta^p n^p\|x\|^{rp}) + 2\delta^p\left(\frac{1}{n}(2^p n + n^p)(n+1)^{rp} + 2^p n + n^p + [1+(n+1)^r]^p\right)\|y\|^{rp}
\end{aligned}$$

for all $x, y \in \mathcal{X}$. Replacing x by $n^k x$ and y by $(n+1)^k y$ in the above inequality and dividing $n^{(k+1)p}(n+1)^{(k+1)p}$, we see that

$$\begin{aligned}
&\left\|\frac{1}{n^k(n+1)^k}g\left(n^k x,(n+1)^k y\right) - \frac{1}{n^{k+1}(n+1)^{k+1}}g(n^{k+1}x,(n+1)^{k+1}y)\right\|_{\mathcal{Y}}^p \\
&\leq \frac{1}{n^{(k+1)p}(n+1)^{(k+1)p}}\left(n+3+\frac{1}{n}\right)(2\varepsilon^p + \delta^p n^{p(1+kr)}\|x\|^{rp}) \\
&\quad + \frac{2\delta^p(n+1)^{krp}}{n^{(k+1)p}(n+1)^{(k+1)p}}\left(\frac{1}{n}(2^p n + n^p)(n+1)^{rp} + 2^p n + n^p + [1+(n+1)^r]^p\right)\|y\|^{rp}
\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all nonnegative integers k. Thus, we have

$$\begin{aligned}
&\left\|\frac{1}{n^l(n+1)^l}g\left(n^l x,(n+1)^l y\right) - \frac{1}{n^m(n+1)^m}g\left(n^m x,(n+1)^m y\right)\right\|_{\mathcal{Y}}^p \\
&\leq \sum_{k=l}^{m-1}\left[\frac{1}{n^{(k+1)p}(n+1)^{(k+1)p}}\left(n+3+\frac{1}{n}\right)(2\varepsilon^p + \delta^p n^{p(1+kr)}\|x\|^{rp})\right. \\
&\quad \left. + \frac{2\delta^p(n+1)^{krp}}{n^{(k+1)p}(n+1)^{(k+1)p}}\left(\frac{1}{n}(2^p n + n^p)(n+1)^{rp} + 2^p n + n^p + [1+(n+1)^r]^p\right)\|y\|^{rp}\right]
\end{aligned} \tag{2.14}$$

for all integers $l, m\,(0 \leq l < m)$, and all $x, y \in \mathcal{X}$. Thus, the sequence $\{\frac{1}{n^k(n+1)^k}g(n^k x,(n+1)^k y)\}$ is a Cauchy sequence for all $x,\ y \in \mathcal{X}$. Since $\mathcal{Y}$ is complete, the sequence $\{\frac{1}{n^k(n+1)^k}g(n^k x,(n+1)^k y)\}$ converges for all $x,\ y \in \mathcal{X}$. Define $F : \mathcal{X}\times\mathcal{X} \to \mathcal{Y}$ by

$$F(x,y) := \lim_{k\to\infty}\frac{1}{n^k(n+1)^k}g(n^k x,(n+1)^k y)$$

for all $x, y \in \mathcal{X}$. By (2.10), we have

$$\begin{aligned}
&\frac{1}{n^{kp}(n+1)^{kp}}\left\|ng\left(n^k(x_1+\cdots+x_n),(n+1)^k\frac{y_1+\cdots+y_n}{n}\right) - \sum_{1\leq i,j\leq n}g\left(n^k x_i,(n+1)^k y_j\right)\right\|_{\mathcal{Y}}^p \\
&\leq \frac{1}{n^{kp}(n+1)^{kp}}\left(2\varepsilon^p + \delta^p\left[n^{krp}\left(\sum_{i=1}^n\|x_i\|^r\right)^p + 2(n+1)^{krp}\left(\sum_{j=1}^n\|y_j\|^r\right)^p\right]\right)
\end{aligned}$$

for all $x_1,\cdots,x_n,y_1,\cdots,y_n \in \mathcal{X}$, and all nonnegative integers k. Letting $k\to\infty$ in the above inequality, we obtain that F satisfies (1.2). By [6], F is a Cauchy-Jensen mapping. Setting $l = 0$ and taking $m\to\infty$ in (2.14), one can obtain the inequality (2.9).

Taking $n = 2$ and $\delta = 0$ in Theorem 2, we obtain the following corollary.

Corollary 1. *Let $\varepsilon > 0$ be fixed. Suppose that a mapping $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ is a mapping satisfying $f(x, 0) = 0$ such that*

$$\left\| 2f\left(x + y, \frac{z + w}{2}\right) - f(x, z) - f(x, w) - f(y, z) - f(y, w) \right\|_{\mathcal{Y}} \leq \varepsilon$$

for all $x, y, z, w \in \mathcal{X}$, then there exists a unique bi-additive mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ satisfying

$$\|f(x, y) - f(0, y) - F(x, y)\|_{\mathcal{Y}} \leq \varepsilon \left(\frac{11}{6^p - 1}\right)^{\frac{1}{p}}$$

for all $x, y \in \mathcal{X}$.

We introduce some definitions on 2-Banach spaces [13–15].

Definition 4. Let $\mathcal{X}$ be a real vector space with $\dim \mathcal{X} \geq 2$ and $\|\cdot, \cdot\| : \mathcal{X}^2 \to \mathbb{R}$ be a function, then $(\mathcal{X}, \|\cdot, \cdot\|)$ is called a *linear 2-normed space* if the following conditions hold:

(a) $\|x, y\| = 0$ if, and only if, x and y are linearly dependent,

(b) $\|x, y\| = \|y, x\|$,

(c) $\|\alpha x, y\| = |\alpha| \|x, y\|$,

(d) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$

for all $\alpha \in \mathbb{R}$ and $x, y, z \in \mathcal{X}$. In this case, the function $\|\cdot, \cdot\|$ is called a *2-norm* on $\mathcal{X}$.

Definition 5. A sequence $\{x_n\}$ in a linear 2-normed space $\mathcal{X}$ is called a *Cauchy sequence* if there are two linearly independent points $y, z \in \mathcal{X}$ such that for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$, $\|x_m - x_n, y\| < \varepsilon$, and $\|x_m - x_n, z\| < \varepsilon$. For convenience, we will write $\lim_{m,n\to\infty} \|x_n - x_m, y\| = 0$ for a Cauchy sequence $\{x_n\}$.

Definition 6. Let $\{x_n\}$ be a sequence in a linear 2-normed space $\mathcal{X}$. The sequence $\{x_n\}$ is said to *convergent* in $\mathcal{X}$ if there exits an element $x \in \mathcal{X}$ such that

$$\lim_{n\to\infty} \|x_n - x, y\| = 0$$

for all $y \in \mathcal{X}$. In this case, we say that a sequence $\{x_n\}$ converges to the limit x, simply denoted by $\lim_{n\to\infty} x_n = x$.

Definition 7. A *2-Banach space* is defined to be a linear 2-normed space in which every Cauchy sequence is convergent. $(\mathcal{X}, \|\cdot\|, \|\cdot, \cdot\|)$ is called a *normed 2-Banach space* if $(\mathcal{X}, \|\cdot\|)$ is a normed space such that $(\mathcal{X}, \|\cdot, \cdot\|)$ is a 2-Banach space.

In the following lemma, we obtain some basic properties in a linear 2-normed space, which will be used to prove the stability results.

Lemma 2. *[7]* Let $(\mathcal{X}, \|\cdot, \cdot\|)$ be a linear 2-normed space and $x \in \mathcal{X}$.

(a) If $\|x, y\| = 0$ for all $y \in \mathcal{X}$, then $x = 0$.

(b) $\big| \|x, z\| - \|y, z\| \big| \leq \|x - y, z\|$ for all $x, y, z \in \mathcal{X}$.

(c) If a sequence $\{x_n\}$ is convergent in $\mathcal{X}$, then $\lim_{n\to\infty} \|x_n, y\| = \| \lim_{n\to\infty} x_n, y\|$ for all $y \in \mathcal{X}$.

From now on, let $\mathcal{X}$ be a normed space and $\mathcal{Y}$ a normed 2-Banach space. The following theorem proves the stability of the Eq (1.2) in normed 2-Banach spaces.

Theorem 3. *Let $r \in (0,1)$, $\varepsilon > 0$, δ, and $\eta \geq 0$, and let $f : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ be a mapping satisfying $f(x,0) = 0$ such that*

$$\left\| nf\left(x_1 + \cdots + x_n, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{1\le i,j\le n} f(x_i, y_j), z \right\| \le \varepsilon + \delta\left(\sum_{i=1}^{n} \|x_i\|^r + \sum_{j=1}^{n} \|y_j\|^r\right) + \eta\|z\| \tag{2.15}$$

for all $x, x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{X}$, and all $z \in \mathcal{Y}$, then there exists a Cauchy-Jensen mapping $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ such that

$$\begin{aligned} &\|f(x,y) - f(0,y) - F(x,y), z\| \\ &\quad \le (n^2 + 3n + 1)\left[\frac{2(\varepsilon + \eta\|z\|)}{n(n^2 + n - 1)} + \frac{\delta\|x\|^r}{n(n+1) - n^r}\right] + \frac{2\delta[3n + 1 + 4(n+1)^r]\|y\|^r}{n(n+1) - (n+1)^r} \end{aligned} \tag{2.16}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$.

Proof. Let $g(x,y) := f(x,y) - f(0,y)$ for all $x, y \in \mathcal{X}$, then $g(x,0) = g(0,y) = 0$ for all $x, y \in \mathcal{X}$. By (2.15), g satisfies

$$\left\| ng\left(x_1 + \cdots + x_n, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{1\le i,j\le n} g(x_i, y_j), z \right\| \le 2\varepsilon + \delta\left(\sum_{i=1}^{n} \|x_i\|^r + 2\sum_{j=1}^{n} \|y_j\|^r\right) + 2\eta\|z\| \tag{2.17}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{X}$, and all $z \in \mathcal{Y}$. Letting $x_1 = \cdots = x_n = x$ in (2.17), we gain

$$\left\| g\left(nx, \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{j=1}^{n} g(x, y_j), z \right\| \le \frac{1}{n}\left[2\varepsilon + \delta\left(n\|x\|^r + 2\sum_{j=1}^{n} \|y_j\|^r\right) + 2\eta\|z\|\right] \tag{2.18}$$

for all $x, y_1, \cdots, y_n \in \mathcal{X}$, and all $z \in \mathcal{Y}$. Putting $y_1 = y, y_2 = -y$, and $y_3 = \cdots = y_n = 0$ in (2.18), we get

$$\|g(x,y) + g(x,-y), z\| \le \frac{1}{n}[\, 2\varepsilon + \delta(n\|x\|^r + 4\|y\|^r) + 2\eta\|z\| \,]$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Setting $y_1 = \cdots = y_n = y$ in (2.18), we have

$$\|g(nx,y) - ng(x,y), z\| \le \frac{1}{n}[2\varepsilon + n\delta(\|x\|^r + 2\|y\|^r) + 2\eta\|z\|]$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. By the above two inequalities, we obtain

$$\|g(nx,y) + ng(x,-y), z\| \le \left(1 + \frac{1}{n}\right)(2\varepsilon + n\delta\|x\|^r + 2\eta\,\|z\|) + 6\delta\|y\|^r \tag{2.19}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Taking $y_1 = y, y_2 = -(n+1)y$ and $y_3 = \cdots = y_n = 0$ in (2.18), we gain

$$\|g(nx,-y) - g(x,y) - g(x,-(n+1)y), z\| \le \frac{1}{n}\left[2\varepsilon + \delta\left(n\|x\|^r + 2[1 + (n+1)^r]\|y\|^r\right) + 2\eta\|z\|\right]$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. By (2.19) and the above inequality, we get

$$\|(n+1)g(x,y) + g(x,-(n+1)y), z\|$$

$$\leq \left(1 + \frac{2}{n}\right)(2\varepsilon + \delta n\|x\|^r + 2\eta\|z\|) + \left[3 + \frac{1 + (n+1)^r}{n}\right]2\delta\|y\|^r \tag{2.20}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Replacing y by $(n+1)y$ in (2.19), we have

$$\|g\,(nx, (n+1)y) + ng\,(x, -(n+1)y)\,, z\| \leq \left(1 + \frac{1}{n}\right)(2\varepsilon + n\delta\|x\|^r + 2\eta\,\|z\|) + 6\delta(n+1)^r\|y\|^r$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. By (2.20) and the above inequality, we obtain

$$\begin{aligned}&\|n(n+1)g(x, y) - g(nx, (n+1)y), z\| \\ &\qquad\leq \left(n + 3 + \frac{1}{n}\right)(2\varepsilon + \delta n\|x\|^r + 2\eta\|z\|) + [3n + 1 + 4(n+1)^r]2\delta\|y\|^r\end{aligned}$$

for all $x, y \in \mathcal{X}$ and all $z \in \mathcal{Y}$. Replacing x by $n^k x$ and y by $(n+1)^k y$ in the above inequality and dividing $n^{k+1}(n+1)^{k+1}$, we see that

$$\begin{aligned}&\left\|\frac{1}{n^k(n+1)^k} g\left(n^k x, (n+1)^k y\right) - \frac{1}{n^{k+1}(n+1)^{k+1}} g(n^{k+1}x, (n+1)^{k+1}y), z\right\| \\ &\qquad\leq \frac{n^2 + 3n + 1}{n^{k+2}(n+1)^{k+1}}(2\varepsilon + \delta n^{1+kr}\|x\|^r + 2\eta\|z\|) + \frac{3n + 1 + 4(n+1)^r}{n^{k+1}(n+1)^{k(1-r)+1}} 2\delta\|y\|^r\end{aligned}$$

for all $x, y \in \mathcal{X}$, all $z \in \mathcal{Y}$, and all nonnegative integers k. Thus, we have

$$\begin{aligned}&\left\|\frac{1}{n^l(n+1)^l} g\left(n^l x, (n+1)^l y\right) - \frac{1}{n^m(n+1)^m} g\,(n^m x, (n+1)^m y)\,, z\right\| \\ &\leq \sum_{k=l}^{m-1} \left[\frac{n^2 + 3n + 1}{n^{k+2}(n+1)^{k+1}}\left(2\varepsilon + \delta n^{1+kr}\|x\|^r + 2\eta\|z\|\right) + \frac{3n + 1 + 4(n+1)^r}{n^{k+1}(n+1)^{k(1-r)+1}} 2\delta\|y\|^r\right]\end{aligned} \tag{2.21}$$

for all integers l, $m\,(0 \leq l < m)$, all $x, y \in \mathcal{X}$, and all $z \in \mathcal{Y}$. Thus, the sequence $\{\frac{1}{n^k(n+1)^k} g(n^k x, (n+1)^k y)\}$ is a Cauchy sequence in $(\mathcal{Y}, \|\cdot, \cdot\|)$ for all x, $y \in \mathcal{X}$. Since $(\mathcal{Y}, \|\cdot, \cdot\|)$ is complete, the sequence $\{\frac{1}{n^k(n+1)^k} g(n^k x, (n+1)^k y)\}$ converges for all x, $y \in \mathcal{X}$.

Define $F : \mathcal{X} \times \mathcal{X} \to \mathcal{Y}$ by

$$F(x, y) := \lim_{k\to\infty} \frac{1}{n^k(n+1)^k} g(n^k x, (n+1)^k y)$$

for all $x, y \in \mathcal{X}$. By (2.17), we have

$$\begin{aligned}&\frac{1}{n^k(n+1)^k}\left\|ng\left(n^k(x_1 + \cdots + x_n), (n+1)^k \frac{y_1 + \cdots + y_n}{n}\right) - \sum_{1\leq i,j\leq n} g\left(n^k x_i, (n+1)^k y_j\right), z\right\| \\ &\qquad\leq \frac{1}{n^k(n+1)^k}\left(2\varepsilon + \delta\left[n^{kr}\sum_{i=1}^n \|x_i\|^r + 2(n+1)^{kr}\sum_{j=1}^n \|y_j\|^r\right] + 2\eta\|z\|\right)\end{aligned}$$

for all $x_1, \cdots, x_n, y_1, \cdots, y_n \in \mathcal{X}$, all $z \in \mathcal{Y}$, and all nonnegative integers k. Letting $k \to \infty$ in the above inequality and using Lemma 2, we obtain that F satisfies (1.2). By [6], F is a Cauchy-Jensen mapping. Setting $l = 0$ and taking $m \to \infty$ in (2.21), using Lemma 2, one can obtain the inequality (2.16).

3. Conclusions

This paper deals with the stability of the Cauchy-Jensen functional equation, presenting connections to three different types of spaces: normed spaces, quasi-normed spaces, and normed 2-normed spaces. These spaces differ in the criteria used to measure the distance and magnitude of the function. A norm provide functions that measure the length or size of vectors, which play an important role in many mathematical analyses and optimization problems. Quasi-normed spaces have less stringent conditions and do not have to satisfy all the properties of a norm, allowing for analysis in more general situations. A 2-normed space is a space whose norms are defined specifically by Euclidean norms (or 2-norms), which are a generalization of distances and angles in vector spaces.

By proving the stability of the Cauchy-Jensen functional equation on these three spaces, the paper shows how the equation can be applied in these different contexts. In other words, the paper shows that the Cauchy-Jensen equation maintains consistent properties in these different types of spaces, and gives a set of conditions that guarantee the existence of the solution of the equation in each space. This suggests that the equation is a powerful tool with a wide range of applications across a variety of mathematical structures.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors state no conflict of interest.

References

1. T. M. Rassias, On the stability of the linear mapping in Banach spaces, *Proc. Amer. Math. Soc.*, **72** (1978), 297–300. https://doi.org/10.2307/2042795
2. W. G. Park, J. H. Bae, On a Cauchy-Jensen functional equation and its stability, *J. Math. Anal. Appl.*, **323** (2006), 634–643. https://doi.org/10.1016/j.jmaa.2005.09.028
3. A. Bahyrycz, J. Olko, On stability and hyperstability of an equation characterizing multi-Cauchy-Jensen mappings, *Results Math.*, **73** (2018), 55. https://doi.org/10.1007/s00025-018-0815-8
4. K. W. Jun, J. R. Lee, Y. H. Lee, On the generalized Hyers-Ulam stability of the Cauchy-Jensen functional equation II, *Pure Appl. Math.*, **16** (2009), 167–178.
5. Y. H. Lee, On the Hyers-Ulam-Rassias stability of a Cauchy-Jensen functional equation, *J. Chungcheong. Math. Soc.*, **20** (2007), 163–172.
6. J. H. Bae, W. G. Park, A fixed point approach to the stability of a Cauchy-Jensen functional equation, *Abstract Appl. Anal.*, **2012** (2012), 205160. https://doi.org/10.1155/2012/205160
7. W. G. Park, Approximate additive mappings in 2-Banach spaces and related topics, *J. Math. Anal. Appl.*, **376** (2011), 193–202. https://doi.org/10.1016/j.jmaa.2010.10.004

8. S. Yun, Approximate additive mappings in 2-Banach spaces and related topics: revisited, *Korean J. Math.*, **23** (2015), 393–399. https://doi.org/10.11568/kjm.2015.23.3.393
9. Y. Benyamini, J. Lindenstrauss, *Geometric nonlinear functional analysis*, American Mathematical Socity, 2000.
10. N. Kalton, Quasi-Banach spaces In: *Handbook of the geometry of Banach spaces*, 2003. https://doi.org/10.1016/S1874-5849(03)80032-3
11. S. Rolewicz, *Metric linear spaces*, 1985.
12. A. Najati, M. B. Moghimi, Stability of a functional equation deriving from quadratic and additive functions in quasi-Banach spaces, *J. Math. Anal. Appl.*, **337** (2008), 399–415. https://doi.org/10.1016/j.jmaa.2007.03.104
13. S. Gähler, 2-metrische Räume und ihre topologische Struktur, *Math. Nachr.*, **26** (1963), 115–148. https://doi.org/10.1002/mana.19630260109
14. S. Gähler, Lineare 2-normierte Räumen, *Math. Nachr.*, **28** (1964), 1–43. https://doi.org/10.1002/mana.19640280102
15. A. White, 2-Banach spaces, *Math. Nachr.*, **42** (1969), 43–60. https://doi.org/10.1002/mana.19690420104

AIMS Press