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Research article

Synchronization of generalized fractional complex networks with partial subchannel losses

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Abstract: This article focuses on the synchronization problem for two classes of complex networks with subchannel losses and generalized fractional derivatives. Initially, a new stability theorem for generalized fractional nonlinear system is formulated using the properties of generalized fractional calculus and the generalized Laplace transform. This result is also true for classical fractional cases. Subsequently, synchronization criteria for the generalized fractional complex networks are attained by the proposed stability theorem and the state layered method. Lastly, two numerical examples with some new kernel functions are given to validate the synchronization results.

Keywords: generalized fractional calculus; stability; complex network; synchronization; subchannel losses
Mathematics Subject Classification: 26A33, 34K24, 34K37, 65R10, 68R10

1. Introduction

Complex networks, such as metabolic networks [1], the World Wide Web [2], social networks [3], and the genetic regulatory networks [4], have significant impacts on our lives. In theory, these complex networks can be described in terms of nodes and edges, where each node denotes a fundamental unit, and the edges represent connections between the nodes. In recent decades, various interesting dynamic behaviors of complex networks have attracted increasing interest [5, 6], such as stability, control, synchronization, etc. In such scenarios, synchronization is a paradigmatic dynamic behavior of complex networks. It not only can explain many emerging collective behaviors well but also has a wide range of applications in the real world, such as secure communication [7], nuclear magnetic resonance [8], information science [9], and multi-robot coordination [10].

The correspondence between L'Hôpital and Leibniz in 1695, discussing the implications of a derivative order of $\frac{1}{2}$, led to the birth of fractional calculus. Compared to classic integer-order calculus, fractional calculus exhibits a long memory property and offers more degrees of freedom [11]. Since then, fractional calculus has been extensively studied in the field of mathematics. In recent decades, it has garnered increasing interest from scholars due to its wide-ranging applications in chemistry, biology, electrical engineering, control [12–15], etc. Various definitions of fractional calculus have been proposed to describe different non-local characteristics in practical problems, such as Riemann-Liouville [11], Caputo [16], and exponential fractional derivatives [17], the Hadamard derivative [18], and the Caputo Hadamard derivative [19]. To unify these definitions, generalized fractional operators (ψ-fractional calculus) with a general kernel function $\psi(t)$ have been introduced [20–22], including the generalized fractional integral $\mathscr{I}_{t_0,t}^{\alpha,\psi}$, generalized Riemann-Liouville derivative ${}_{RL}\mathscr{D}_{t_0,t}^{\alpha,\psi}$, and generalized Caputo derivative ${}_{C}\mathscr{D}_{t_0,t}^{\alpha,\psi}$. It is worth noting that fractional order theory is introduced into complex networks to describe memory properties in real networks more accurately [23, 24]. In this paper, generalized fractional calculus is introduced into complex networks to characterize more extensive memory features.

In complex networks, due to numerous physical limitations and unpredictable environmental fluctuations, many communication constraints exist in the synchronization process, such as random perturbations [25], information transmission with time delay [26], discontinuous subsystems, pulse perturbations [27], and uncertain parameters [28]. However, the synchronization problem of a complex network with partial communication channel losses is still not fully investigated. The information of each node encompasses multiple levels of information, which requires multiple communication channels to transmit the corresponding levels of information. For example, in sensor networks, the inner coupling divides into multiple information channels to transmit the multiple information for each agent [29, 30]. Most existing research [25–28, 31] assumes that all the channels of the connection can transmit information, which is inconsistent with the real world. Unfortunately, the phenomenon of partial communication channel losses is ubiquitous. For instance, a study found that only 5% of synapse excitations can be transmitted perfectly between two connected cortical regions of brain networks [32]. Therefore, the research of partial communication channel losses can provide a deep understanding of the underlying physical mechanisms. Notably, [33] studied synchronization and consensus behaviors with partial information transmission in complex networks. In [34], the finite-time synchronization of complex networks with partial communication channels failure is studied. However, the above studies mainly focus on integer order complex networks. The synchronization problem of fractional complex networks with partial information losses has not been discussed until now.

Based on the preceding discussions, this paper presents a study of generalized complex networks with partial information transmission. The primary contributions of this work can be listed as follows:

- Generalized fractional calculus can more accurately portray complex long memory and genetic traits in networks. This work introduces generalized fractional derivatives into complex networks for the first time. The generalized fractional complex network models are more general than the existing models and further fill the gap in the field of complex network models.
- A generalized stability theorem for nonlinear fractional systems is proved, which broadens the existing results in the study of kinetics of fractional systems. Using this method, asymptotic stability of such fractional systems can be easily obtained.
- By employing the new stability theorem and a state layered method, synchronization criteria for

two generalized complex networks with partial information losses are obtained.

- The generalized fractional complex networks greatly enrich the dynamic behavior of the networks. Moreover, two numerical examples with different kernel functions are given to verify the validity and universality of the proposed results.

The article is structured as follows. In Section 2, necessary preparations are presented. Section 3 proves the stability theorem for nonlinear generalized fractional systems. Synchronization of generalized fractional complex networks is studied in the following section. Two numerical examples are shown in Section 5. The last section concludes this paper.

Notations: Let diag$\{\cdots\}$ represent a diagonal matrix. The superscript T represents the transpose. We use $\lambda_{\max}(\cdot)$ and $\lambda_2(\cdot)$ to denote the maximum and second largest eigenvalues of a real symmetric matrix, respectively. $sign(\cdot)$ denotes the $sign$ function. $\mathbf{1}_N = \underbrace{(1, 1, \ldots, 1)^T}_{N}$.

2. Preliminaries

The required concepts of generalized fractional calculus, lemmas, and graph theory knowledge will be recalled. The function space $X_c^p(t_0, T)$ is defined in [35].

Definition 2.1. *[20,21] For a given function $x(t) \in X_c^p(t_0, T)$, the definitions of a generalized fractional integral and the generalized Caputo fractional derivative of order β can be expressed as:*

$$\mathscr{I}_{t_0,t}^{\beta,\psi} x(t) = \frac{1}{\Gamma(\beta)} \int_{t_0}^{t} (\psi(t) - \psi(s))^{\beta-1} x(s)\psi'(s)ds \tag{2.1}$$

and

$${}_C\mathscr{D}_{t_0,t}^{\beta,\psi} x(t) = \frac{1}{\Gamma(1-\beta)} \int_{t_0}^{t} (\psi(t) - \psi(s))^{-\beta} x'(s)ds, \tag{2.2}$$

respectively, where $0 < \beta < 1, t \in [t_0, T], \psi(t) \in C^1[t_0, T]$ is an increasing function, and $\psi'(t) \neq 0$, for all $t \in [t_0, T]$.

Remark 2.1. *Generalized fractional calculus depends on the kernel function ψ, and for specific functions ψ, we can obtain some classic fractional calculus formulations like Riemann-Liouville, Caputo, Hadamard, Caputo-Hadamard, and exponential fractional calculus. In addition, the generalized fractional calculus still retains the non-local behavior and semi-group properties of classic fractional calculus. It has been realized that these types of fractional operators have been successfully used to describe and simulate many societal and natural phenomena [36].*

Lemma 2.1. *[20,21] If $x(t) \in C^1[t_0, T]$, then*

$$\mathscr{I}_{t_0,t}^{\beta,\psi}\, {}_C\mathscr{D}_{t_0,t}^{\beta,\psi} x(t) = x(t) - x(t_0), \quad \beta \in (0, 1). \tag{2.3}$$

Lemma 2.2. *[37] Let $x(t) \in X_c^p(t_0, T)$ and $\mathscr{L}_\psi$ be the generalized Laplace transform. Then,*

$$\text{(i) } \mathscr{L}_\psi\{\mathscr{I}_{t_0,t}^{\beta,\psi} x(t)\} = \frac{\mathscr{L}_\psi\{x(t)\}}{s^\beta}; \tag{2.4}$$

$$\text{(ii) } \mathscr{L}_\psi\{{}_C\mathscr{D}_{t_0,t}^{\beta,\psi} x(t)\} = s^\beta \mathscr{L}_\psi\{x(t)\} - s^{\beta-1} x(t_0), \tag{2.5}$$

where $0 < \beta < 1$.

Lemma 2.3. *[37] Let* $E_{\beta,\gamma}(z) = \sum_{i=0}^{\infty} \frac{z^i}{\Gamma(\beta i+\gamma)}$ *be the Mittag-Leffler function. Then,*

$$\mathscr{L}_{\psi}\left\{(\psi(t)-\psi(t_0))^{\gamma-1} E_{\beta,\gamma}\left(\pm\lambda\,(\psi(t)-\psi(t_0))^{\beta}\right)\right\} = \frac{s^{\beta-\gamma}}{s^{\beta}\mp\lambda}, \tag{2.6}$$

where $\Re(\beta) > 0, |\frac{\lambda}{s^\beta}| < 1, z \in \mathbb{C}$.

Lemma 2.4. *[38] If* $n \in \mathbb{N}, \rho > 1, x_k \in \mathbb{R}^+, k = 1, 2, \ldots, n$*, then*

$$\left\|\sum_{k=1}^{n} x_k\right\|^{\rho} \le n^{\rho-1}\sum_{k=1}^{n}\|x_k\|^{\rho}. \tag{2.7}$$

In particular, when $\rho = 2$*, one has*

$$\left\|\sum_{k=1}^{n} x_k\right\|^{2} \le n\sum_{k=1}^{n}\|x_k\|^{2}. \tag{2.8}$$

Lemma 2.5. *[39] If* $x(t) \in \mathbb{R}^n$ *is a differentiable vector value function, then the inequality*

$${}_{C}\mathscr{D}^{\beta,\psi}_{t_0,t}\left(x^T(t)Mx(t)\right) \le 2x(t)^T M\, {}_{C}\mathscr{D}^{\beta,\psi}_{t_0,t}x(t) \tag{2.9}$$

holds, where $\beta \in (0,1), t > t_0$*, and* $M \in \mathbb{R}^{n\times n}$ *is a symmetric and positive definite matrix.*

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ be an undirected graph, in which $\mathcal{V} = \{1, 2, \ldots, N\}, \mathcal{E} \subset \mathcal{V}\times\mathcal{V} = \{(i, j) \mid i, j \in \mathcal{V}\}$, and $\mathcal{A} = \left\{a_{ij}\right\} \in \mathbb{R}^{N\times N}$ represent the graph nodes collection, the set of edges, and weighted adjacency matrix of $\mathcal{G}$, respectively. The edge $e = (i, j) \in \mathcal{E}$ means that information can be exchanged between nodes i and j. If $(j, i) \in \mathcal{E}$, then $a_{ij} > 0$, that is, the element a_{ij} of matrix $\mathcal{A}$ is determined by the connection between nodes. Assume that there is no self-loop ($a_{ii} = 0, i \in \mathcal{V}$), and $\mathcal{G}$ is connected.

The Laplace matrix $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ of graph $\mathcal{G}$ is expressed as

$$l_{ij} = \begin{cases} \sum\limits_{k=1}^{N} a_{ik}, & j = i, \\ -a_{ij}, & j \ne i. \end{cases}$$

3. Stability of nonlinear generalized fractional system

In this part, a new theorem of Mittag-Leffler stability for a generalized fractional system is discussed.

Consider the following initial value problem with generalized Caputo derivative:

$$\begin{cases} {}_{C}\mathscr{D}^{\beta,\psi}_{t_0,t}x(t) = h(t, x(t)), \\ x(t_0) = x_a, \end{cases} \tag{3.1}$$

in which $\beta \in (0, 1), \Omega \subseteq \mathbb{R}^n, h(t, x) : [t_0, \infty)\times\Omega \to \mathbb{R}^n$ is piecewise continuous with t and locally Lipschitz with x, and the origin $x = 0 \in \Omega$.

For convenience, let the equilibrium of system (3.1) be $x_e = 0$.

Theorem 3.1. *Consider the generalized Caputo fractional system* (3.1). *Suppose that* $G(x(t))$ *is a continuous, positive definite function and satisfies*

$$a_1\|x(t)\|^{a_2} \leq G(x(t)), \tag{3.2}$$

$$_C\mathscr{D}_{t_0,t}^{\beta,\psi}G(x(t)) \leq -a_3G^{\mu}(x(t)), \tag{3.3}$$

where $\beta \in (0,1), \mu \in (0,1], a_1, a_2, a_3 \in \mathbb{R}^+$. *Then,* $x_e = 0$ *of* (3.1) *is Mittag-Leffler stable.* $G(x(t))$ *can be estimated as*

$$G(x(t)) \leq G\left(x\left(t_0\right)\right) E_\beta\left(-a_3 G^{\mu-1}\left(x\left(t_0\right)\right)\left(\psi(t)-\psi\left(t_0\right)\right)^\beta\right).$$

Proof. Let $G(t) = G(x(t))$. According to (3.3), one obtains

$$_C\mathscr{D}_{t_0,t}^{\beta,\psi}G(t) = -a_3G^{\mu}(t) - m_1(t), \tag{3.4}$$

where $m_1(t) \geq 0$. Taking the β-order generalized fractional integral on (3.4), one can derive that

$$\begin{aligned} G(t) &= G\left(t_0\right) - a_3\mathscr{I}_{t_0,t}^{\beta,\psi}G^{\mu}(t) - \mathscr{I}_{t_0,t}^{\beta,\psi}m_1(t) \\ &= G\left(t_0\right) - \frac{1}{\Gamma(\beta)}\int_{t_0}^{t}\frac{a_3G^{\mu}(s)+m_1(s)}{(\psi(t)-\psi(s))^{1-\beta}}\psi'(s)ds \\ &\leq G\left(t_0\right). \end{aligned} \tag{3.5}$$

If $G\left(t_0\right) = 0$, then it follows from (3.5) that $G(t) = 0$, which implies the solution of (3.1) is $x(t) = 0$. If $G\left(t_0\right) \neq 0$, that is, $G\left(t_0\right) > 0$, then it follows from (3.5) that

$$G^{\mu-1}(t) \geq G^{\mu-1}\left(t_0\right), \quad 0 < \mu \leq 1. \tag{3.6}$$

Multiplying both sides of (3.6) by $G(t)$, one has

$$G^{\mu}(t) \geq G^{\mu-1}\left(t_0\right)G(t). \tag{3.7}$$

Substituting (3.7) into (3.3), the following can be derived:

$$_C\mathscr{D}_{t_0,t}^{\beta,\psi}G(t) \leq -a_3G^{\mu-1}\left(t_0\right)G(t).$$

Furthermore,

$$_C\mathscr{D}_{t_0,t}^{\beta,\psi}G(t) = -a_3G^{\mu-1}\left(t_0\right)G(t) - m_2(t), \tag{3.8}$$

where $m_2(t) \geq 0$. From the generalized Laplace transform, one has

$$s^{\beta}G(s) = s^{\beta-1}G(t_0) - a_3G^{\mu-1}\left(t_0\right)G(s) - m_2(s)G(s) = \frac{s^{\beta-1}G(t_0) - m_2(s)}{s^{\beta} + a_3G^{\mu-1}\left(t_0\right)},$$

in which $\mathscr{L}_{\psi}\{G(t)\} = G(s)$. According to Lemma 2.3 and the generalized convolution theorem [37], the solution of (3.8) is

$$G(t) = G(t_0) E_\beta \left(-a_3 G^{\mu-1}(t_0) (\psi(t) - \psi(t_0))^\beta\right) - \int_{t_0}^{t} (\psi(t) - \psi(s))^{\beta-1} E_{\beta,\beta} \left(-a_3 G^{\mu-1}(t_0) (\psi(t) - \psi(s))^\beta\right) m_2(s) ds. \tag{3.9}$$

Since $m_2(t) \geq 0$ and

$$E_{\beta,\beta} \left(-a_3 G^{\mu-1}(t_0) (\psi(t) - \psi(t_0))^\beta\right) \geq 0,$$

one gets

$$G(t) \leq G(t_0) E_\beta \left(-a_3 G^{\mu-1}(t_0) (\psi(t) - \psi(t_0))^\beta\right). \tag{3.10}$$

Combining (3.2) and (3.10) yields

$$\|x(t)\| \leq \left[\frac{G(t_0)}{a_1} E_\beta \left(-a_3 G^{\mu-1}(t_0) (\psi(t) - \psi(t_0))^\beta\right)\right]^{\frac{1}{a_2}}.$$

Let $m_3 = \frac{G(t_0)}{a_1} \geq 0$, and then we have $\|x(t)\| \leq \left[m_3 E_\beta \left(-a_3 G^{\mu-1}(t_0) (\psi(t) - \psi(t_0))^\beta\right)\right]^{\frac{1}{a_2}}$, where $m_3 = 0$ holds if and only if $G(t_0) = 0$, which implies the Mittag-Leffler stability of system (3.1). □

Remark 3.1. *Theorem 3.1 presented in this paper is an extension or improvement of the existing results. Specifically, when* $\mu = 1$*, inequality* (3.3) *becomes* ${}_C\mathscr{D}_{t_0,t}^{\beta,\psi} G(x(t)) \leq -a_3 G(x(t))$*, which has been studied in [39]. When* $\psi(t) = t, 0 < \mu \leq 1$*, Theorem 3.1 also holds for the classic Caputo fractional derivative, which has not been discussed until now. When* $\psi(t) = t, \mu = 1$*, the inequality* (3.3) *coincides with inequality* ${}_C\mathscr{D}_{t_0,t}^{\beta} G(x(t)) \leq -a_3 G(x(t))$*, which has been presented in [40]. Compared with existing stability results, Theorem 3.1 has wider applications in stability analysis of fractional systems.*

4. Synchronization of complex networks with generalized Caputo derivative

In this section, two complex network models under a new communication constraint are considered, which are composed of N coupled nodes. Each node includes n sub-states. The communication constraint is that partial nodes can transmit information to each other or lose information between sub-states, which results in only partial sub-information being transmitted perfectly. Furthermore, one node may have different failed channels for different neighbors, that is, the received information of the node from different neighbor nodes may be different, increasing dramatically the complexity of synchronization analysis.

In this paper, we consider the following two generalized fractional complex network models with partial information losses:

$${}_C\mathscr{D}_{t_0,t}^{\beta,\psi} x_i(t) = f(x_i(t)) + c\,sig^p\left(\sum_{j=1}^{N} a_{ij} K_{ij} B\left(x_j(t) - x_i(t)\right)\right), \tag{4.1}$$

$${}_C\mathscr{D}_{t_0,t}^{\beta,\psi} x_i(t) = f(x_i(t)) + c \sum_{j=1}^{N} a_{ij} K_{ij}\, sig^p\left(B\left(x_j(t) - x_i(t)\right)\right), \tag{4.2}$$

in which $i \in \mathcal{V}, x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ indicates the status information, $x_{il}(t)$ is the l-th ($l = 1, 2, \ldots, n$) layer of the node $i, f(x_i(t)) = (f_1(x_{i1}(t)), f_2(x_{i2}(t)), \ldots, f_n(x_{in}(t)))^T \in \mathbb{R}^n$

is a nonlinear function, c denotes coupling strength, and $0 < p < 1$. For $y = (y_1, y_2, \ldots, y_n)^T$, let $sig^p(y_i) = |y_i|^p \, sign(y_i)$ and $sig^p(y) = (sig^p(y_1), sig^p(y_2), \ldots, sig^p(y_n))^T$. The diagonal matrix $B = diag\{b_1, b_2, \ldots, b_n\}$ represents the inner coupling matrix , with $b_i > 0, i = 1, 2, \ldots, n$. $K_{ij} = diag\left\{k_{ij}^1, k_{ij}^2, \ldots, k_{ij}^n\right\}, 0 \le k_{ij}^l \le 1$, indicates the channel matrix, and k_{ij}^l determines the information loss ratio of the l-th subchannel.

Remark 4.1. *Compared with existing network models with partial information transmission, it is easy to see that the main distinction is* $0 \le k_{ij}^l \le 1$. k_{ij}^l *can be used to determine the information loss ratio of the l-th subchannel between nodes i and j. Specifically,* $k_{ij}^l = 1$ *means that all sub-information of the nodes can be transmitted completely. When* $k_{ij}^l = 0$*, the l-th information transmission between i and j is a failure. In addition,* $0 < k_{ij}^l < 1$ *denotes the ratio of information loss. If* $K_{ij} = I_n$*, complex networks* (4.1) *and* (4.2) *are consistent with the networks in [41]. If* $K_{ij} = 0$ *or* 1*, and the generalized fractional derivative is replaced by an integer derivative, then the complex networks* (4.1) *and* (4.2) *are the same as the networks in [34]. Therefore, complex networks* (4.1) *and* (4.2) *can be considered as a generalization of the existing models.*

Definition 4.1. *[31] Generalized fractional complex network* (4.1) *or* (4.2) *is called synchronized provided*

$$\lim_{t\to\infty} \left\| x_i(t) - x_j(t) \right\| = 0$$

for any initial value $x(t_0) = \left(x_1^T(t_0), x_2^T(t_0), \ldots, x_N^T(t_0)\right)^T \in \mathbb{R}^{n\times N}$ *and any* $i, j \in \mathcal{V}$.

By Definition 4.1, it is not difficult to find that if the complex network can realize synchronization, then the sub-state of each node is also synchronized. However, because of the communication restriction, it becomes more difficult to recognize each sub-state from the channel matrix. To overcome this difficulty, the state layered method [34] is used here.

Setting $M_{ij} = a_{ij} K_{ij} = \mathrm{diag}\left\{m_{ij}^1, m_{ij}^2, \ldots, m_{ij}^n\right\}$, with $i, j \in \mathcal{V}$, it is obvious that $M_{ij} = \mathrm{diag}\left\{a_{ij}k_{ij}^1, a_{ij}k_{ij}^2 , \ldots, a_{ij}k_{ij}^n\right\}$. The state layered matrix and metric matrix are given as

$$M_l = \begin{pmatrix} 0 & m_{12}^l & \cdots & m_{1N}^l \\ m_{21}^l & 0 & \cdots & m_{2N}^l \\ \vdots & \vdots & \ddots & \vdots \\ m_{N1}^l & m_{N2}^l & \cdots & 0 \end{pmatrix}, D_l = \begin{pmatrix} d_1^l & 0 & \cdots & 0 \\ 0 & d_2^l & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_N^l \end{pmatrix},$$

respectively, in which $d_i^l = \sum\limits_{j=1, j\neq i}^{N} m_{ij}^l, l = 1, 2, \ldots, n$. Then, the Laplace matrix can be defined as $W_l = D_l - M_l$.

Based on the state layered matrix, systems (4.1) and (4.2) are rewritten as l-th layer sub-information:

$$_C\mathscr{D}_{t_0,t}^{\beta,\psi} x_{il}(t) = f_l\left(x_{il}(t)\right) + c\,sig^p\left(\sum_{j=1}^{N} m_{ij}^l b_l \left(x_{jl} - x_{il}\right)\right), \tag{4.3}$$

$$_C\mathscr{D}_{t_0,t}^{\beta,\psi} x_{il}(t) = f_l\left(x_{il}(t)\right) + c\sum_{j=1}^{N} m_{ij}^l sig^p\left(b_l\left(x_{jl} - x_{il}\right)\right). \tag{4.4}$$

Correspondingly, for any initial value $x_l(t_0) = (x_{1l}(t_0), x_{2l}(t_0), \ldots, x_{Nl}(t_0))^T \in \mathbb{R}^N (l = 1, 2, \ldots, n)$, complex network (4.3) or (4.4) can reach synchronization if the following conditions are true:

$$\lim_{t\to\infty} \left| x_{il}(t) - x_{jl}(t) \right| = 0, \quad i, j \in \mathcal{V}.$$

Hypothesis 4.1. *Suppose that $f_l(x) \in \mathbb{R}$ satisfies the Lipschitz condition, i.e.,*

$$|f_l(x_1) - f_l(x_2)| \leq \eta_l |x_1 - x_2|,$$

in which $\eta_l > 0, l = 1, 2, \ldots, n$.

Now, two synchronization criteria of generalized fractional complex networks (4.3) and (4.4) are investigated. First, the synchronization of model (4.3) will be studied.

Theorem 4.1. *Under Hypothesis 4.1, consider generalized Caputo fractional complex network* (4.3). *If $c > r_l$ and the undirected graph of matrix $W_l, l = 1, 2, \ldots, n$, is connected, then the synchronization of network* (4.3) *can be realized in region $\bigcap_{l=1}^{n} \mathcal{D}_r^l = \left\{ x_{il}(t) : \frac{2\eta_l \kappa^{\frac{1-p}{2}}}{\left(2b_{\min}\lambda_2^l\right)^{\frac{1+p}{2}}} \leq r_l, i \in \mathcal{V} \right\}$, where κ is a constant, $\kappa \geq \frac{1}{2} b_l \lambda_N^l \|x_l\|^2$, $0 < p < 1$, and λ_2^l and λ_N^l denote the minimum positive eigenvalue and maximum eigenvalue of W_l, respectively.*

Proof. Consider the following Lyapunov candidate function:

$$G_l(t) = \frac{1}{2} b_l x_l^T(t) W_l x_l(t),$$

in which $l = 1, 2, \ldots, n$, and $x_l(t) = (x_{1l}(t), x_{2l}(t), \ldots, x_{Nl}(t))^T$.

Let $F_l(x_l(t)) = (f_l(x_{1l}(t)), f_l(x_{2l}(t)), \ldots, f_l(x_{Nl}(t)))^T$. From Lemma 2.5, one derives

$$\begin{aligned}
&{}_C\mathscr{D}_{t_0,t}^{\beta,\psi} G_l(t) \\
\leq & b_l x_l^T(t) W_l \, {}_C\mathscr{D}_{t_0,t}^{\beta,\psi} x_l(t) \\
= & b_l x_l^T(t) W_l \left[F_l(x_l(t)) - c\,sig^p(b_l W_l x_l(t)) \right] \\
= & b_l x_l^T W_l F_l(x_l) - c b_l (W_l x_l)^T sig^p(b_l W_l x_l) \\
\leq & b_l x_l^T W_l F_l(x_l) - c b_l^{p+1} \left(\|W_l x_l\|^2 \right)^{\frac{p+1}{2}} \\
= & \frac{b_l}{2} \sum_{i,j=1}^{N} m_{ij}^l \left(x_{il} - x_{jl} \right) \left(f_l(x_{il}) - f_l\left(x_{jl}\right) \right) - c b_l^{p+1} \left(\|W_l x_l\|^2 \right)^{\frac{p+1}{2}} \\
\leq & \frac{b_l}{2} \sum_{i,j=1}^{N} m_{ij}^l \left| x_{il} - x_{jl} \right| \cdot \left| f_l(x_{il}) - f_l\left(x_{jl}\right) \right| - c b_l^{p+1} \left(\|W_l x_l\|^2 \right)^{\frac{p+1}{2}} \\
\leq & \frac{\eta_l}{2} b_l \sum_{i,j=1}^{N} m_{ij}^l \left(x_{il} - x_{jl} \right)^2 - c b_l^{p+1} \left(\|W_l x_l\|^2 \right)^{\frac{p+1}{2}} \\
= & \eta_l b_l x_l^T(t) W_l x_l(t) - c b_l^{p+1} \left(\|W_l x_l\|^2 \right)^{\frac{p+1}{2}} \\
= & 2\eta_l G_l(t) - c b_l^{p+1} \left(\|W_l x_l\|^2 \right)^{\frac{p+1}{2}}.
\end{aligned} \tag{4.5}$$

Suppose that λ_i^l is the eigenvalue of W_l and satisfies $0 = \lambda_1^l \le \lambda_2^l \le \ldots \le \lambda_N^l$ and v_i^l is the eigenvector corresponding to λ_i^l. Then, $\left\{v_i^l\right\}$ is the standard orthogonal basis in $\mathbb{R}^N$. Therefore, for some $\theta_i \in \mathbb{R}$, one has $x_l = \sum_{i=1}^N \theta_i v_i^l$. Accordingly, from $W_l x_l = \sum\limits_{i=1}^N \theta_i \lambda_i^l v_i^l$, one derives

$$\|W_l x_l\|^2 = \sum_{i=1}^N \theta_i^2 \left(\lambda_i^l\right)^2 \ge \lambda_2^l \sum_{i=1}^N \theta_i^2 \lambda_i^l = \lambda_2^l x_l^T W_l x_l. \tag{4.6}$$

Combining (4.5) and (4.6), one gets

$$\begin{aligned}
&{}_C\mathscr{D}_{t_0,t}^{\beta,\psi} G_l(t) \\
\le& 2\eta_l G_l(t) - c b_l^{p+1} \left(\|W_l x_l\|^2\right)^{\frac{p+1}{2}} \\
\le& 2\eta_l G_l(t) - c b_l^{p+1} \left(\lambda_2^l x_l^T W_l x_l\right)^{\frac{p+1}{2}} \\
\le& 2\eta_l G_l(t) - c \left(2b_{\min} \lambda_2^l\right)^{\frac{p+1}{2}} G_l^{\frac{p+1}{2}}(t) \\
=& -c \left(2b_{\min} \lambda_2^l\right)^{\frac{p+1}{2}} G_l^{\frac{p+1}{2}}(t) \left(1 - \frac{2\eta_l G_l^{\frac{1-p}{2}}(t)}{c \left(2b_{\min} \lambda_2^l\right)^{\frac{p+1}{2}}}\right) \\
\le& -c \left(2b_{\min} \lambda_2^l\right)^{\frac{p+1}{2}} \left(1 - \frac{2\eta_l \kappa^{\frac{1-p}{2}}}{c \left(2b_{\min} \lambda_2^l\right)^{\frac{p+1}{2}}}\right) G_l^{\frac{p+1}{2}}(t).
\end{aligned}$$

Let $\varsigma_1 = c\left(2b_{\min}\lambda_2^l\right)^{\frac{p+1}{2}} \left(1 - \frac{2\eta_l \kappa^{\frac{1-p}{2}}}{c\left(2b_{\min}\lambda_2^l\right)^{\frac{p+1}{2}}}\right)$. Then

$${}_C\mathscr{D}_{t_0,t}^{\beta,\psi} G_l(t) \le -\varsigma_1 G_l^{\frac{p+1}{2}}(t), \tag{4.7}$$

where $\varsigma_1 > 0$. According to Theorem 3.1, one gets

$$G_l(t) \le G_l\left((t_0)\right) E_\beta \left(-\varsigma_1 G_l^{\frac{p-1}{2}} \left(\psi(t) - \psi(t_0)\right)^\beta\right),$$

which implies

$$\lim_{t\to\infty} G_l(t) = 0.$$

From $G_l(t) = \frac{b_l}{4} \sum_{i,j=1}^N m_{ij}^l \left(x_{il}(t) - x_{jl}(t)\right)^2$, one obtains

$$\lim_{t\to\infty} \left|x_{il}(t) - x_{jl}(t)\right| = 0,$$

that is, when $c > \frac{2\eta_l \kappa^{\frac{1-p}{2}}}{\left(2b_{\min}\lambda_2^l\right)^{\frac{1+p}{2}}}$, the sub-information of the l-layer can realize synchronization. Therefore, the synchronization of all nodes can be realized, i.e., the synchronization of network (4.3) can be reached in region $\bigcap\limits_{l=1}^n \mathcal{D}_r^l$. □

Next, the synchronization problem of network (4.4) will be discussed.

Theorem 4.2. *Under Hypothesis 4.1, consider the generalized Caputo fractional complex network* (4.4). *If* $c > r_l$ *and the undirected graph of* $W_l, l = 1, 2, \ldots, n$, *is connected, then the synchronization of network* (4.4) *can be realized in region* $\bigcap_{l=1}^{n} \mathcal{D}_r^l = \left\{x_{il}(t) : \frac{2\eta_l}{z}\mu^{\frac{1-p}{2}} \le r_l, i \in \mathcal{V}\right\}$, *where* μ *is a constant,* $\mu \ge \frac{b_l}{2N}\|e_l\|^2, 0 < p < 1, z = \min_{\|e_l\|=1, e_l^T 1_N = 0} Z(e_l)$, *and* $e_{il}(t)$ *and* $Z(e_l)$ *will be defined in the subsequent proof.*

Proof. Suppose that $s^l(t) = \frac{1}{N}\sum_{i=1}^{N} x_{il}, l = 1, 2, \ldots, n$. Notice that $m_{ij}^l = m_{ji}^l$, $s^l(t)$ satisfies

$$ {}_{C}\mathscr{D}_{t_0,t}^{\beta,\psi} s^l(t) = \frac{1}{N}\sum_{i=1}^{N} {}_{C}\mathscr{D}_{t_0,t}^{\beta,\psi} x_{il}(t) = \frac{1}{N}\sum_{i=1}^{N} f_l\left(x_{il}(t)\right). $$

Let $e_{il}(t) = x_{il}(t) - s^l(t)$, and one gets the error system:

$$ {}_{C}\mathscr{D}_{t_0,t}^{\beta,\psi} e_{il}(t) = f_l\left(x_{il}(t)\right) + c\sum_{j=1}^{N} m_{ij}^l sig^p\left(b_l\left(x_{jl}(t) - x_{il}(t)\right)\right) - \frac{1}{N}\sum_{i=1}^{N} f_l\left(x_{il}(t)\right). $$

Consider the Lyapunov function

$$ G_l(t) = \frac{b_l}{2N}\sum_{i=1}^{N} e_{il}^2(t). $$

From Lemma 2.5, one gets

$$ \begin{aligned} {}_{C}\mathscr{D}_{t_0,t}^{\beta,\psi} G_l(t) \le& \frac{b_l}{N}\sum_{i=1}^{N} e_{il}(t)\, {}_{C}\mathscr{D}_{t_0,t}^{\beta,\psi} e_{il}(t) \\ =& \frac{b_l}{N}\sum_{i=1}^{N} e_{il}(t)\Big[f_l\left(x_{il}(t)\right) - f_l\left(s^l(t)\right) + f_l\left(s^l(t)\right) - \frac{1}{N}\sum_{i=1}^{N} f_l\left(x_{il}(t)\right) \\ &+ c\sum_{j=1}^{N} m_{ij}^l sig^p\left(b_l\left(x_{jl}(t) - x_{il}(t)\right)\right)\Big]. \end{aligned} $$

Set $R_1 = \frac{b_l}{N}\sum_{i=1}^{N} e_{il}(t)\left(f_l\left(x_{il}(t)\right) - f_l\left(s^l(t)\right)\right), R_2 = \frac{b_l}{N}\sum_{i=1}^{N} e_{il}(t)\left(f_l\left(s^l(t)\right) - \frac{1}{N}\sum_{j=1}^{N} f_l\left(x_{jl}(t)\right)\right), R_3 = \frac{cb_l}{N} \cdot \sum_{i=1}^{N} e_{il}(t)\sum_{j=1}^{N} m_{ij}^l sig^p\left(b_l\left(x_{jl}(t) - x_{il}(t)\right)\right)$. Hence ${}_{C}\mathscr{D}_{t_0,t}^{\beta,\psi} G_l(t) \le R_1 + R_2 + R_3$.

By Hypothesis 4.1, one obtains

$$ \begin{aligned} R_1 =& \frac{b_l}{N}\sum_{i=1}^{N} e_{il}(t)\left(f_l\left(x_{il}(t)\right) - f_l\left(s^l(t)\right)\right) \\ \le& \frac{b_l}{N}\sum_{i=1}^{N} |e_{il}(t)| \cdot \left|f_l\left(x_{il}(t)\right) - f_l\left(s^l(t)\right)\right| \end{aligned} $$

$$
\begin{aligned}
&\leq \frac{b_l}{N}\eta_l \sum_{i=1}^{N} |e_{il}(t)| \cdot |x_{il}(t) - s^l(t)| \\
&\leq \frac{b_l}{N}\eta_l \sum_{i=1}^{N} e_{il}^2(t) \\
&= 2\eta_l G_l(t).
\end{aligned}
$$

For the second item,

$$
\begin{aligned}
R_2 =& \frac{b_l}{N} \sum_{i=1}^{N} e_{il}(t) \left(f_l\left(s^l(t)\right) - \frac{1}{N} \sum_{j=1}^{N} f_l\left(x_{jl}(t)\right) \right) \\
=& b_l \left(\frac{1}{N} \sum_{i=1}^{N} e_{il}(t) \right) \cdot \left(f_l\left(s^l(t)\right) - \frac{1}{N} \sum_{j=1}^{N} f_l\left(x_{jl}(t)\right) \right).
\end{aligned}
$$

Notice that

$$
\frac{1}{N} \sum_{i=1}^{N} e_{il}(t) = \frac{1}{N} \sum_{i=1}^{N} \left(x_{il}(t) - s^l(t) \right) = \frac{1}{N} \sum_{i=1}^{N} x_{il}(t) - \frac{1}{N} \sum_{i=1}^{N} s^l(t) = s^l(t) - s^l(t) = 0,
$$

that is, $R_2 = 0$.

For the last item,

$$
\begin{aligned}
R_3 =& \frac{cb_l}{N} \cdot \sum_{i=1}^{N} e_{il}(t) \sum_{j=1}^{N} m_{ij}^l sig^p\left(b_l\left(x_{jl}(t) - x_{il}(t)\right)\right) \\
=& \frac{cb_l}{N} \sum_{i,j=1}^{N} m_{ij}^l e_{il}(t) sig^p\left(b_l\left(e_{jl}(t) - e_{il}(t)\right)\right) \\
=& -\frac{cb_l}{2N} \sum_{i,j=1}^{N} m_{ij}^l \left(e_{jl}(t) - e_{il}(t)\right) sig^p\left(b_l\left(e_{jl}(t) - e_{il}(t)\right)\right) \\
=& -\frac{c}{2N} \sum_{i,j=1}^{N} m_{ij}^l \left| b_l\left(e_{jl}(t) - e_{il}(t)\right) \right|^{p+1}.
\end{aligned}
$$

Setting

$$
Z_l(e_l) = \frac{1}{2N} \sum_{i,j=1}^{N} m_{ij}^l \left| b_l\left(e_{jl}(t) - e_{il}(t)\right) \right|^{p+1},
$$

in which $e_l = (e_{1l}, e_{2l}, \ldots, e_{Nl})^T$, and $Z(e_l) = Z_l(e_l) G_l^{-\frac{p+1}{2}}$. It is not difficult to find that $Z(ae_l) = Z(e_l)$ for all $a \in \mathbb{R}$ and $a \neq 0$. Let $z = \min_{\|e_l\|=1, e_l^T 1_N = 0} Z(e_l)$. Because $Z_l(e_l) \geq 0, G_l(t) \geq 0$, one gets $z \geq 0$. Suppose that $z = 0$. Then. there exists an e_l' satisfying $Z_l\left(e_l'\right) = 0$. From the connectivity of matrix W_l, one obtains $Z_l\left(e_l'\right) = 0 \Leftrightarrow e_l' = a'\mathbf{1}_N$. With $e_l'^T\mathbf{1}_N = 0$, one has $e_l' = 0$, which contradicts $\left\|e_l'\right\| = 1$. Thus, $z > 0$. We have $R_3 \leq -czG_l^{\frac{p+1}{2}}(t)$.

Through the above analysis, one gets

$$\begin{aligned} {}_C\mathscr{D}_{t_0,t}^{\beta,\psi}G_l(t) \leq& 2\eta_l G_l(t) - czG_l^{\frac{p+1}{2}}(t) \\ =& -czG_l^{\frac{p+1}{2}}(t)\left(1 - \frac{2\eta_l}{cz}G_l^{\frac{1-p}{2}}(t)\right) \\ \leq& -cz\left(1 - \frac{2\eta_l}{cz}\mu^{\frac{1-p}{2}}\right)G_l^{\frac{p+1}{2}}(t). \end{aligned}$$

Set $\varsigma_2 = cz\left(1 - \frac{2\eta_l}{cz}\mu^{\frac{1-p}{2}}\right)$,

$${}_C\mathscr{D}_{t_0,t}^{\beta,\psi}G_l(t) \leq -\varsigma_2 G_l^{\frac{p+1}{2}}(t),$$

in which $\varsigma_2 > 0$. By virtue of Theorem 3.1, one gets

$$\lim_{t\to\infty}|e_{il}(t)| = 0,$$

that is, when $c > \frac{2\eta_l}{z}\mu^{\frac{1-p}{2}}$, the synchronization of the l-th layer sub-information can be achieved, which implies that the synchronization of network (4.4) can be reached in region $\bigcap_{l=1}^{n}\mathcal{D}_r^l$. □

Remark 4.2. *At present, studies of synchronization in classic fractional complex networks have made rich achievements. However, the synchronization problem of the fractional complex network with partial communication channel losses has not been discussed until now. Therefore, the results of Theorems 4.1 and 4.2 advance the current research on synchronization problems in complex networks with fractional derivative, also holding for classic fractional complex networks, such as Caputo, Riemann-Liouville, Hadamard-type, and exponential networks.*

5. Numerical example

Two numerical examples with different kernel functions are given to show the effectiveness of the proposed theories.

Example 5.1. *Consider the networks* (4.3) *and* (4.4) *with seven* ($N = 7$) *nodes, and each node with three* ($n = 3$) *sub-states. The nonlinear function is described as follows:*

$$f(x_i(t)) = \begin{pmatrix} \frac{1}{2}(|x_{i1}(t)+1| - |x_{i1}(t)-1|) \\ \frac{1}{2}(|x_{i2}(t)+1| - |x_{i2}(t)-1|) \\ \frac{1}{2}(|x_{i3}(t)+1| - |x_{i3}(t)-1|) \end{pmatrix},$$

in which $x_i(t) = (x_{i1}(t), x_{i2}(t), x_{i3}(t))^T, i = 1, 2, \ldots, 7$. *It is not difficult to verify that f satisfies the Lipschitz condition about constants* $(\eta_1, \eta_2, \eta_3) = (1, 1, 1)$.

The inner and outer coupling matrices are represented by

$$B=\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathcal{A}=\begin{pmatrix} 0 & 2 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & 2 & 0 & 0 & 3 & 3 \\ 3 & 2 & 0 & 3 & 0 & 0 & 4 \\ 1 & 0 & 3 & 0 & 4 & 2 & 0 \\ 1 & 0 & 0 & 4 & 0 & 4 & 3 \\ 0 & 3 & 0 & 2 & 4 & 0 & 1 \\ 0 & 3 & 4 & 0 & 3 & 1 & 0 \end{pmatrix}.$$

Then, the channel matrices are provided to describe the channel states as follows:

$$\begin{aligned}
&K_{12}=K_{21}=\mathrm{diag}\{0.1,0,0.2\}, && K_{13}=K_{31}=\mathrm{diag}\{0.9,0.1,0\}, && K_{14}=K_{41}=\mathrm{diag}\{1,0,0.2\},\\
&K_{15}=K_{51}=\mathrm{diag}\{0,0.1,0.9\}, && K_{23}=K_{32}=\mathrm{diag}\{1,0.5,0\}, && K_{26}=K_{62}=\mathrm{diag}\{0.4,0,0.8\},\\
&K_{27}=K_{72}=\mathrm{diag}\{0,1,1\}, && K_{34}=K_{43}=\mathrm{diag}\{0,0.25,1\}, && K_{45}=K_{54}=\mathrm{diag}\{1,1,0\},\\
&K_{46}=K_{64}=\mathrm{diag}\{0,0,1\}, && K_{56}=K_{65}=\mathrm{diag}\{1,1,0\}, && K_{57}=K_{75}=\mathrm{diag}\{0,1,0.8\},\\
&K_{67}=K_{76}=\mathrm{diag}\{0.5,0,1\}.
\end{aligned}$$

Figure 1 displays the topological structure of the connections between each node in the network and gives an example to show subchannel losses (between sub-node x_{22} and sub-node x_{32}) and failure (between sub-node x_{23} and sub-node x_{33}). Figure 2 presents the topological structure diagram of the transmissions of the three sub-information layers between seven nodes.

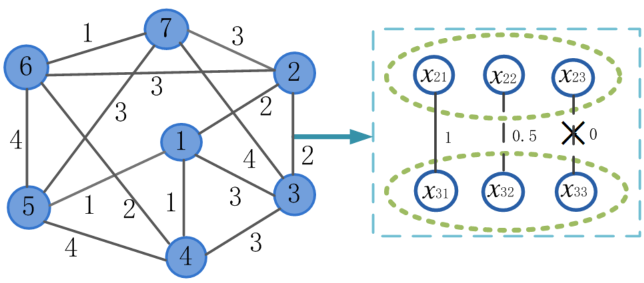

Figure 1. The topology structure of complex networks with partial information losses.

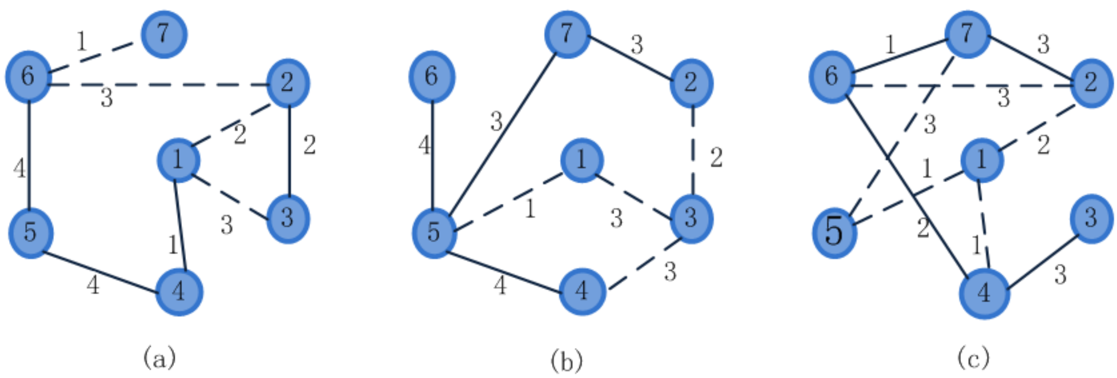

Figure 2. The topological structures of the three sub-information layers.

From the outer coupling matrix $\mathcal{A}$ and the channel matrix K_{ij}, the corresponding state layer matrices $M_l, l = 1, 2, 3$, can be attained. Only M_1 is given below, as it is easy to get M_2 and M_3 similarly.

$$M_1 = \begin{pmatrix} 0 & 0.2 & 2.7 & 1 & 0 & 0 & 0 \\ 0.2 & 0 & 2 & 0 & 0 & 1.2 & 0 \\ 2.7 & 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 4 & 0 \\ 0 & 1.2 & 0 & 0 & 4 & 0 & 0.5 \\ 0 & 0 & 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}.$$

The conditions in Theorems 4.1 and 4.2 are easy to verify. It is not difficult to find that the undirected graph of matrix G_l is connected. Then, the parameters are given as $c = 10, p = 0.2$, and initial values of the nodes are chosen randomly from the interval $[0, 10]$. *In order to show the generality of the proposed method, the logarithmic function* $\log(t)$ *and the inverse hyperbolic cosine function* $\text{arccosh}(t) = \log(t + \sqrt{t^2 - 1})$ *are chosen as kernel functions in the generalized Caputo fractional derivative for networks* (4.3) *and* (4.4).

Figures 3 and 4 show the synchronization of complex network (4.3) *with kernel functions* $\psi(t) = \log(t)$ *and* $\psi(t) = \text{arccosh}(t)$, *respectively. Similarly, Figures 5 and 6 show that the synchronization errors of network* (4.4) *with different kernel functions* $\psi(t)$ *can also tend to* 0. *These simulation results demonstrate the feasibility of the proposed theories.*

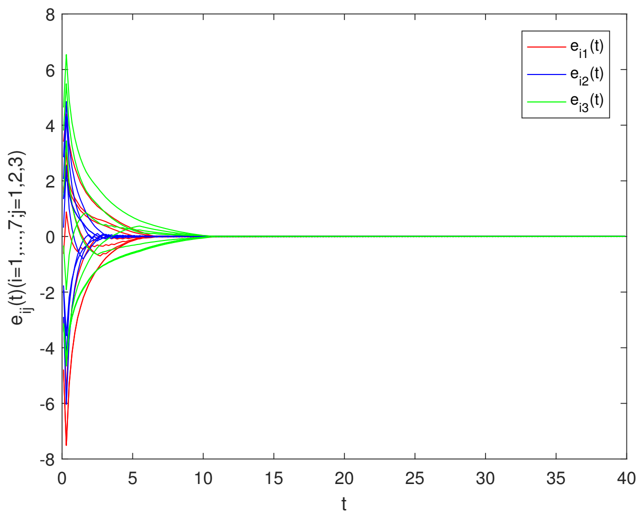

Figure 3. When $\psi(t) = \log(t), \beta = 0.99, t_0 = 0.1$, the synchronization errors of network (4.3).

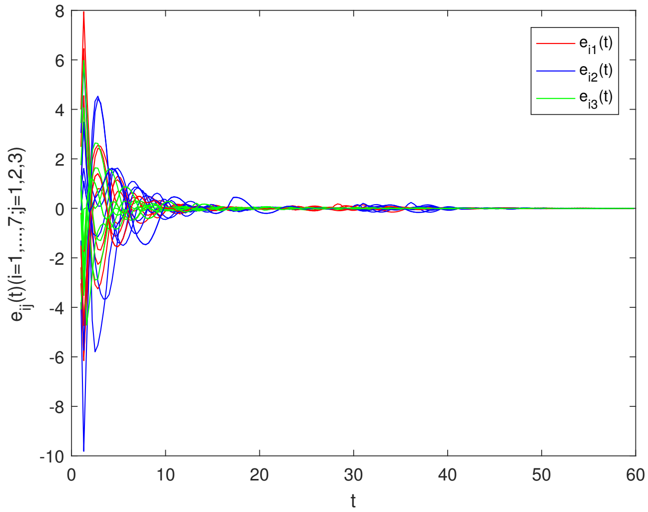

Figure 4. When $\psi(t) = \text{arccosh}(t), \beta = 0.97, t_0 = 1$, the synchronization errors of network (4.3).

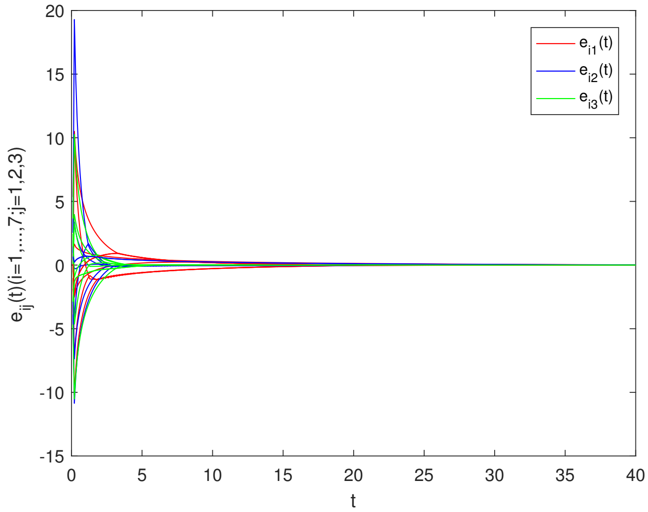

Figure 5. When $\psi(t) = \log(t), \beta = 0.98, t_0 = 0.1$, the synchronization errors of network (4.4).

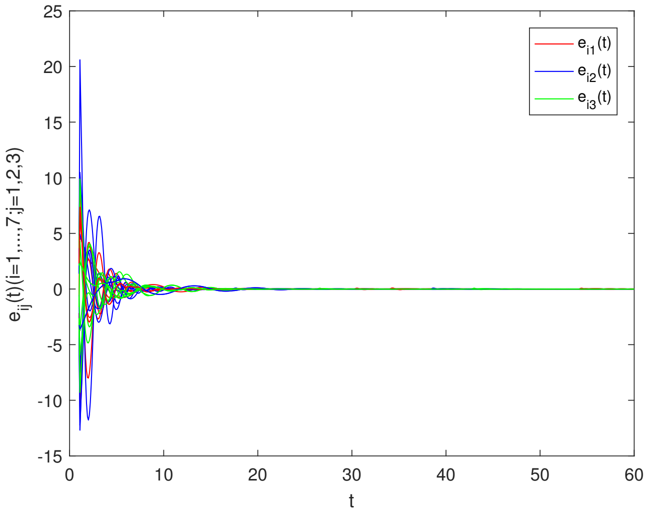

Figure 6. When $\psi(t) = \text{arccosh}(t), \beta = 0.94, t_0 = 1$, the synchronization errors of network (4.4).

Example 5.2. *Using the same parameters as in Example 5.1 except that the nonlinear function is given as*

$$f(x_i(t)) = \begin{pmatrix} 0.001x_{i1}(t)x_{i3}(t) + 0.01x_{i1}(t) - 1 \\ 0.02x_{i3}(t) - 0.001x_{i2}(t)x_{i3}(t) + 0.8 \\ 0.001x_{i1}x_{i3}(t) - 0.01x_{i3}(t) + 1.5 \end{pmatrix},$$

we can also prove the effectiveness of Theorems 4.1 and 4.2. Figures 7 and 8 show the trajectories of the synchronization errors for network (4.3) *with kernel functions* $\psi(t) = \log(t)$ *and* $\psi(t) = \text{arccosh}(t)$, *respectively. Figures 9 and 10 reflect the trajectories of the synchronization errors for network* (4.4) *with kernel functions* $\psi(t) = \log(t)$ *and* $\psi(t) = \text{arccosh}(t)$ *, respectively. From the simulation results and graphs, it is clear that* (4.3) *and* (4.4) *can achieve asymptotic synchronization.*

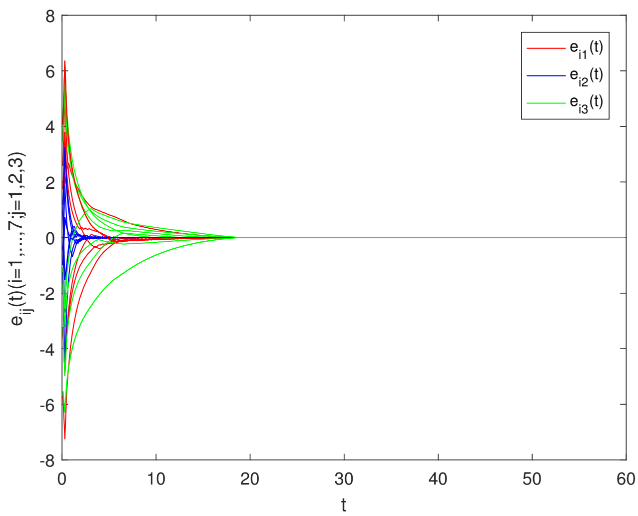

Figure 7. When $\psi(t) = \log(t), \beta = 0.97, t_0 = 0.1$, the synchronization errors of network (4.3).

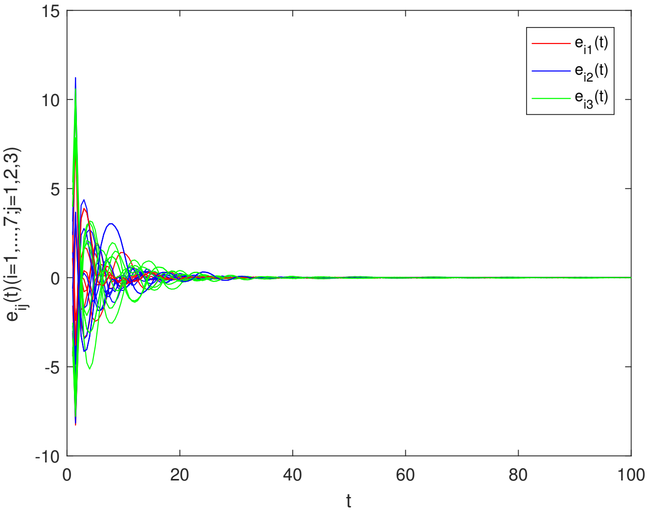

Figure 8. When $\psi(t) = \text{arccosh}(t), \beta = 0.98, t_0 = 1$, the synchronization errors of network (4.3).

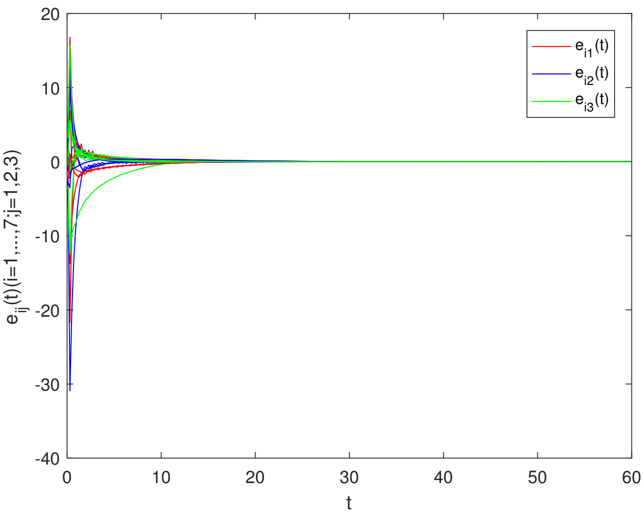

Figure 9. When $\psi(t) = \log(t), \beta = 0.96, t_0 = 0.1$, the synchronization errors of network (4.4).

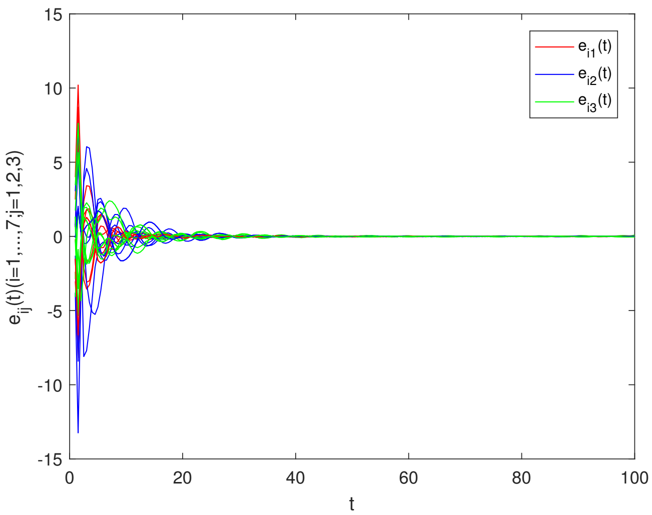

Figure 10. When $\psi(t) = \text{arccosh}(t), \beta = 0.99, t_0 = 1$, the synchronization errors of network (4.4).

6. Conclusions

The synchronization of two complex networks with generalized Caputo fractional derivative and communication constraint has been discussed. In this work, the considered networks are more realistic. By using the generalized Laplace transform , a new stability theorem for a nonlinear generalized fractional system is proved. By utilizing the new stability theorem and the state layered method, synchronization criteria of two nonlinear coupling models with partial information transmission are derived. Finally, two numerical examples with different kernel functions are given to illustrate the effectiveness of the proposed results.

There are some potential limitations to this study: a) The conditions in Theorems 4.1 and 4.2 are too strict, such that some practical networks have difficulty satisfying these conditions; b) The actual background of generalized fractional complex networks (4.3) and (4.4) remains unclear.

In the future, we may focus on the following meaningful topics: a) considering the stability of a generalized fractional system in incommensurate systems, switching systems, and time-delay systems; b) developing the synchronization of a complex network with generalized fractional derivative and communication constraint in impulse systems, stochastic systems, and time-delay systems.

Use of AI tools declaration

The authors declare they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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