AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 6993–7016.
DOI: 10.3934/math.2024341
Received: 06 January 2024
Revised: 01 February 2024
Accepted: 05 February 2024
Published: 19 February 2024

Research article

An approach to the global well-posedness of a coupled 3-dimensional Navier-Stokes-Darcy model with Beavers-Joseph-Saffman-Jones interface boundary condition

Linlin Tan[1], **Meiying Cui**[1,2] **and Bianru Cheng**[1,*]

[1] School of Mathematics and Center for Nonlinear Studies, Northwest University, Xi'an, 710127, China

[2] School of Mathematics and Statistics, Yulin University, Yulin, 719000, China

* **Correspondence:** Email: brcheng@nwu.edu.cn.

Abstract: This study focused on investigating the global well-posedness of a coupled Navier-Stokes-Darcy model with the Beavers-Joseph-Saffman-Jones interface boundary condition in the three-dimensional Euclidean space. By utilizing this approach, we successfully obtained the global strong solution of the system in the three-dimensional space. Furthermore, we demonstrated the exponential stability of this strong solution. The significance of such coupled systems lies in their pivotal role in the analysis of subsurface flow problems, particularly in the context of karst aquifers.

Keywords: global well-posedness; Navier-Stokes-Darcy model; Beavers-Joseph-Saffman-Jones interface boundary condition; large time behavior; exponential stability
Mathematics Subject Classification: 35A07, 74F10, 76D03, 76N10

1. Introduction

The study of the coupling between free flow and porous media flow has garnered widespread attention in recent years, owing to its diverse applications in geosciences (e.g., karst aquifers, hyporheic flow, contaminant transport), health sciences (e.g., blood flow), and industrial processes; see [1–5] and the references therein. Insights derived from a comprehensive understanding of the Navier-Stokes-Darcy equations can be readily employed to tackle various engineering challenges. The typical mathematical analysis on the well-posedness of the associated initial boundary value problem has been done by Layton et al. [6] and Discacciati et al. [7]. The mathematical analysis of the miscible displacement problem in the subsurface was done in a seminal paper by Alt-Luckhaus [8] and by others such as Fabrie-Langlais [9], Fabrie-Gallouët [10], and Marpeau-Saad [11].

We select a microelement in the fluid-structure coupling system to consider a plane as the research object, which means that the fluid flow on the cross-section of the interface is isotropic. The schematic diagram is shown in Figure 1:

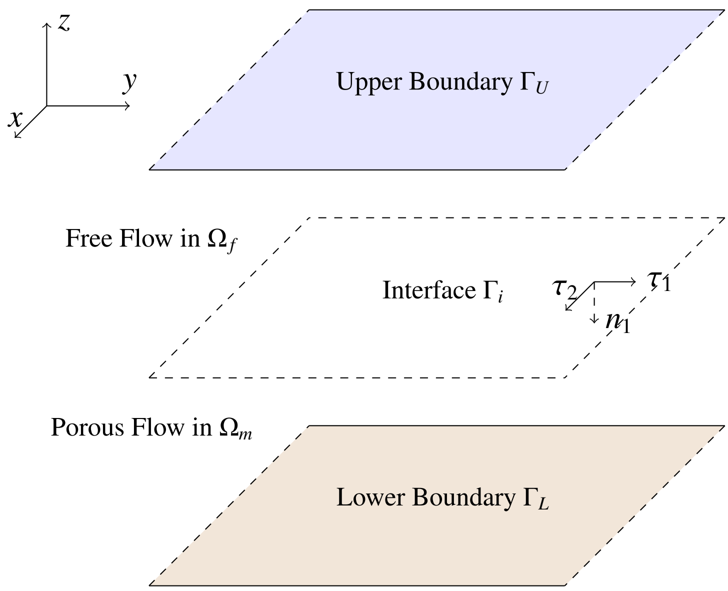

Figure 1. Domain $\Omega = \Omega_f \cup \Omega_m$.

For the fluid flow in the porous medium, we employ the mass conversation law of the porosity medium and Darcy's law [12, 13] to describe the system as follows:

$$\begin{cases} \mathbf{v} = -\dfrac{\Pi}{\mu_2}\nabla P_2, \text{ in } \Omega_m, \\ \nabla \cdot \mathbf{v} = 0, \text{ in } \Omega_m, \end{cases} \tag{1.1}$$

where $\mathbf{v} = \mathbf{v}(\mathbf{x}, t) \in \mathbb{R}^3$ denotes the velocity of the flow in the porosity medium. Obviously, we can get that

$$-\nabla \cdot (\frac{\Pi}{\mu_2}\nabla P_2) = 0, \text{ in } \Omega_m, \tag{1.2}$$

where $P_2 = P_2(\mathbf{x}, t) \in \mathbb{R}^3$, $\mu_2 > 0$ represents the pressure and the viscosity of the flow in Ω_m, respectively, and Π denotes the permeability tensor. We use the incompressible Navier-Stokes equations with constant viscosity $\mu_1 > 0$ to describe the flow in Ω_f as the following equations:

$$\begin{cases} \mathbf{u}_t - \nabla \cdot (2\mu_1 D(\mathbf{u}) - P_1 I) + \mathbf{u} \cdot \nabla \mathbf{u} = 0, \text{ in } \Omega_f, \\ \nabla \cdot \mathbf{u} = 0, \text{ in } \Omega_f, \end{cases} \tag{1.3}$$

where $\Omega_f \subset \mathbb{R}^3$ covers the domain of the free flow, $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u_1, u_2, u_3) \in \mathbb{R}^3$ is the velocity of the free flow, $P_1 = P_1(\mathbf{x}, t)$ represents the pressure of the flow in Ω_f, and μ_1 is the viscosity of the free flow.

The interface-boundary and initial conditions are given by

$$\begin{cases} -\mathbf{n}_1 \cdot (2\mu_1 D(\mathbf{u}) - P_1 I)\mathbf{n}_1 = P_2, \text{ on } \Gamma_i, \\ -\tau_i \cdot (2D(\mathbf{u}))\mathbf{n}_1 = \dfrac{\alpha}{\sqrt{tr\Pi}}\tau_i \cdot \mathbf{u}, \text{ on } \Gamma_i, \\ \mathbf{u} = \mathbf{0}, \text{ on } \Gamma_U, \\ P_2 = 0, \text{ on } \Gamma_L, \\ \mathbf{u} \cdot \mathbf{n}_1 = \mathbf{v} \cdot \mathbf{n}_1, \text{ on } \Gamma_i, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0, \\ P_i(\mathbf{x}, 0) = P_i^0, \ i = 1, \ 2, \end{cases} \tag{1.4}$$

where α is an empirically determined coefficient, $\tau_i, \ i = \{1, \ 2\}$ represents two orthogonal tangent vectors in the horizontal direction, and $\mathbf{n}_1$ denotes the exterior unit vector normal of $\partial\Omega_f$ satisfying

$$\mathbf{u} = (\mathbf{u} \cdot \ \mathbf{n}_1)\mathbf{n}_1 + (\mathbf{u} \cdot \ \tau_1)\tau_1 + (\mathbf{u} \cdot \ \tau_2)\tau_2,$$

$(1.4)_1$ is derived by the balance of force in the normal direction and Beavers-Joseph-Saffman-Jones interface boundary condition $(1.4)_2$ states the shear force to the tangential stress of the fluid velocity along Γ_i.

We acknowledge the pioneering work of researchers who have contributed to the fields of fluid dynamics in porous media, computational methods for solving coupled equations, and the development of interface-boundary conditions [14–18]. For the conditions at the sharp interface, a comprehensive review of these interface selections is provided in [19]. It is known that there are three options for the shear stress conditions in the tangential velocity: The BJ (Beavers-Joseph) condition [14], the BJJ (Beavers-Joseph-Jones) condition [17], and the BJSJ (Beavers-Joseph-Saffman-Jones) condition [18], which is equivalent to what is known as BJS interface condition in some other literature such as [20]. Additionally, two choices are available for the balance of force in the normal direction at the interface: The Lions interface condition and the Rankine-Hudoniot condition.

Research in this domain primarily centers on the classical Navier-Stokes-Darcy equations with sharp interface conditions (refer to [3, 6, 7, 21–34] and the related references), the Navier-Stokes-Darcy-Boussinesq equations involving temperature variations (refer to [35, 36] and related references), and the more complex Cahn-Hilliard-Navier-Stokes-Darcy equations with interface mixing (refer to [20, 37–40] and related references).

However, there have been few achievements in the mathematical analysis, especially in the case of well-posedness of strong solutions [41, 42]. This is mainly due to the strong coupling of the interface, which makes it difficult to obtain high-order estimates of the system. The convection phenomenon under consideration is notably more intricate than that in a single fluid (see [43] for the free-flow and [44, 45] for fluids in a porous medium).

In recent years, researchers have obtained results on non-stationary weak solutions [30, 46–48]. Cui, Dong, and Guo [41] have studied the strong solutions and exponential decay in the two-dimensional case, and we have extended these results to the problem in the 3-D Euclidean space in this paper. Our primary objective is to conduct an initial analysis on the global well-posedness of a coupled Navier-Stokes-Darcy model in the context of the Beavers-Joseph-Saffman-Jones interface boundary condition and establishing their uniqueness property.

Definition 1. *For any $T \in (0, +\infty]$, we first define a function space $X(0, T)$ as*

$$\begin{aligned} X(0,T) = \Big\{(\mathbf{u}, P_2) \,\Big|\, &\mathbf{u} \in L^\infty(0,T; H^2(\Omega_1)) \cap L^\infty(\tau, T; H^4(\Omega_1)), \\ \mathbf{u}_t \in L^\infty(0,T;L^2(\Omega_1)) &\cap L^2(0,T;H^1(\Omega_1)) \cap L^\infty(\tau,T;H^2(\Omega_1)), \\ &\mathbf{u}_{t,t} \in L^\infty(\tau,T;L^2(\Omega_1)) \cap L^2(\tau,T;H^1(\Omega_1)), \\ &P_2 \in L^\infty(0,T;H^3(\Omega_2)) \cap L^\infty(\tau,T;H^5(\Omega_2)), \\ &\partial_t P_2 \in L^2(0,T;H^1(\Omega_2)) \cap L^\infty(\tau,T;H^1(\Omega_2)), \\ &\partial_t^2 P_2 \in L^2(\tau,T;H^1(\Omega_2)), \forall \tau \in (0,T)\Big\}. \end{aligned}$$

$(\mathbf{u}, P_2) \in X(0, T)$ *is called the strong solution of* (1.2)–(1.4)*, if it satisfies systems* (1.2) *and* (1.3) *a.e. in* $\Omega \times (0, T)$*, and fulfills the conditions* (1.2)–(1.4).

Next, we present the main result of this paper.

Proposition 1.1. *(Local well-posedness) Let μ_1, μ_2 both be positive constants and assume that*

$$\lambda \mid \xi \mid^2 \leq \Pi\xi \cdot \xi \leq \Lambda \mid \xi \mid^2, \forall\, \xi \in \mathbb{R}^3, \tag{1.5}$$

the initial data $\mathbf{u}_0 \in H^2(\Omega_f)$ is divergence free, and the compatibility condition holds as follows

$$u_t^0 = \nabla \cdot (2\mu_1 D(\mathbf{u}_0) - P_1^0 I) - \mathbf{u}_0 \cdot \nabla \mathbf{u}_0, \tag{1.6}$$

where $u_t^0 = u_t \mid_{t=0}$ and $P_1^0 = P_1 \mid_{t=0}$. There exists a time $T > 0$ such that the 3-D coupled Navier-Stokes-Darcy systems (1.2)–(1.4) *have a unique strong solution* $(\mathbf{u}, P_2) \in X(0, T)$.

Theorem 1.2. *(Global well-posedness) The permeability tensor Π satisfies* (1.5) *and the initial divergence free velocity field $\mathbf{u}_0 \in H^2(\Omega_f)$ satisfies the compatibility condition* (1.6)*, then there exists a positive constant C depending only on μ_1, Ω_f, and λ, such that if*

$$\|\mathbf{u}_0\|_{L^2(\Omega_f)} \leq \epsilon_0 = \min\{\frac{\mu_1^2}{C^2M}, \frac{1}{16C^2M}, \frac{1}{8C}\}, \quad M = \max\{1, \sqrt{C}\|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}\},$$

the three-dimensional coupled Navier-Stokes-Darcy systems (1.2)–(1.4) *have a unique global strong solution* $(\mathbf{u}, P_2) \in X(0, +\infty)$ *as described in Definition 1.*

Moreover, the solution $(\mathbf{u}, P_2)$ has a decay rate

$$\|\mathbf{u}\|_{H^k(\Omega_f)} + \|P_2\|_{H^k(\Omega_m)} \leq Ce^{-ct}, \ \forall\, k \geq 0, \tag{1.7}$$

where the positive constants are $C = C(\mu_1,\ \lambda,\ \Omega_f,\ \|\mathbf{u}_0\|_{H^2(\Omega_f)})$ and $c = c(\mu_1, \Omega_f)$.

Remark 1. *In divergence from the findings of [41, 42], this paper makes two primary contributions. First, in terms of analytical techniques, the involvement of more intricate directional derivatives under the three-dimensional model renders estimations challenging and convoluted. Second, regarding outcomes, our research is centered on a strip domain, breaking away from the conventional assumption of periodicity. This marks a pioneering achievement as the first three-dimensional outcome in the rigorous examination of robust solutions for the Navier-Stokes-Darcy system. Notably, our results extend beyond, demonstrating applicability, even in the context of periodic domains.*

Furthermore, ϵ_0 in Theorem 1.2 will depend on $\|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}$.

Remark 2. *The decay rate obtained in* (1.7) *indicates that after time* $t > 0$*, the solution* $(\mathbf{u}, P_2)$ *is smooth, and all its derivatives decay for any order.*

The existence and uniqueness of local solutions to this problem can be obtained similarly to the approach in [41]. Therefore, our subsequent focus will be on the a priori estimates of the global solution.

2. A priori estimates

As is well known, the global strong solution to the nonlinear partial differential equations can be obtained by combining local solutions with global a priori estimates. The local solution is proved similarly to that in [41] and is omitted here. Instead, we present the crucial a priori estimates pivotal to establishing global well-posedness below. Note that $K = \frac{\Pi}{\mu_2}$ and $\mathcal{W} = \frac{\mu_1\alpha}{\sqrt{\operatorname{tr}\Pi}}$ for convenience in the subsequent discussion.

2.1. A priori estimates (I): Lower order estimates

Proposition 2.1. *If* (u, P_2) *is a smooth solution of the Navier-Stokes-Darcy systems* (1.3) *and* (1.4) *satisfying*

$$\sup_{0\le t\le T} \|\mathbf{u}\|_{L^2(\Omega_f)} \le 2\|\mathbf{u}_0\|_{L^2(\Omega_f)}, \quad \sup_{0\le t\le T} \|\nabla\mathbf{u}\|_{L^2(\Omega_f)} \le 2M, \tag{2.1}$$

then the following estimates hold:

$$\sup_{0\le t\le T} \|\mathbf{u}\|_{L^2(\Omega_f)} \le \|\mathbf{u}_0\|_{L^2(\Omega_f)}, \ \sup_{0\le t\le T} \|\nabla\mathbf{u}\|_{L^2(\Omega_f)} \le M, \tag{2.2}$$

and

$$\begin{aligned}
&\sup_{0<t<T} (\|\mathbf{u}\|^2_{H^2(\Omega_f)} + \|\mathbf{u}_t\|^2_{H^2(\Omega_f)} + \|P_2\|^2_{H^3(\Omega_m)}) + \int_0^T (\|\mathbf{u}_t\|^2_{H^1(\Omega_f)} + \|\nabla\mathbf{u}\|^2_{H^1(\Omega_f)} + \|\nabla\partial_t P_2\|^2_{L^2(\Omega_m)})\, dt \\
\le& C(\|\mathbf{u}_0\|^2_{H^2(\Omega_f)} + 1) \exp\{\|\mathbf{u}_0\|^2_{H^1(\Omega_f)} + \|\mathbf{u}_0\|^4_{H^1(\Omega_f)}\},
\end{aligned} \tag{2.3}$$

provided

$$\|\mathbf{u}_0\|_{L^2(\Omega_f)} \le \epsilon_0 = \min\{\frac{\mu_1^2}{C^2M}, \frac{1}{16C^2M}, \frac{1}{8C}\}, \quad M = \max\{1, \sqrt{C}\|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}\},$$

where C *depends only on* μ_1, Ω_f*, and* λ*.*

The proof of Proposition 2.1 can be successfully summarized by the following Lemmas 2.1–2.4.

Lemma 2.1. *Under the conditions of Proposition 2.1, it holds that*

$$\sup_{0\le t\le T} \|\mathbf{u}\|^2_{L^2(\Omega_f)} + 2\mu_1 \int_0^T \|D(\mathbf{u})\|^2_{L^2(\Omega_f)}\, dt \le \|\mathbf{u}_0\|^2_{L^2(\Omega_f)}. \tag{2.4}$$

Proof. Multiplying $(1.3)_1$ by $\mathbf{u}$ and integrating the result equation on Ω_f, then multiplying (1.2) by P_2 and integrating it on Ω_m, we add up the two resulting equations to have

$$2\mu_1\|D(\mathbf{u})\|^2_{L^2(\Omega_f)} + \mathcal{W}\sum_{i=1}^{2} \|\mathbf{u}\cdot\tau_i\|^2_{L^2(\Gamma_i)} + \frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega_f)} + \frac{\lambda}{\mu_2}\|\nabla P_2\|^2_{L^2(\Omega_m)}$$

$$
\begin{aligned}
&\leq -\int_{\Omega_f} (\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}\,d\mathbf{x}\\
&\leq \|\mathbf{u}\|_{L^3(\Omega_f)}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\|\mathbf{u}\|_{L^6(\Omega_f)}\\
&\leq C\|\mathbf{u}\|_{L^2(\Omega_f)}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}^{2}\\
&\leq C\|\mathbf{u}_0\|_{L^2(\Omega_f)}^{\frac{1}{2}}M^{\frac{1}{2}}\|D(\mathbf{u})\|_{L^2(\Omega_f)}^{2}\\
&\leq \mu_1\|D(\mathbf{u})\|_{L^2(\Omega_f)}^{2},
\end{aligned} \tag{2.5}
$$

where we have used Hölder's inequality, Young's inequality, Gagliardo-Nirenberg inequality, Korn's inequality, and (2.1). Here C depends on the domain Ω_f. By integrating (2.5) over $(0,t)$, one gets (2.4) with $\epsilon_0 \leq \frac{\mu_1^2}{C^2 M}$ and (2.1). □

Lemma 2.2. *Under the conditions of Proposition 2.1, it holds that*

$$
\sup_{0\leq t\leq T}\|D(\mathbf{u})\|_{L^2(\Omega_f)}^2 + \int_0^T \left(\|\mathbf{u}_t\|_{L^2(\Omega_f)}^2 + \|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2\right)dt \leq C\|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}^2. \tag{2.6}
$$

Proof. Multiply $(1.3)_1$ by $\mathbf{u}_t$ and integrate it over Ω_f, while differentiating (1.2) with respect to t, then multiply by P_2 and integrate it over Ω_m. Now, summing up the two resulting equations yields with Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality, Korn's inequality, and (2.1):

$$
\begin{aligned}
&\|\mathbf{u}_t\|_{L^2(\Omega_f)}^2 + \frac{d}{dt}\Big(\mu_1\|D(\mathbf{u})\|_{L^2(\Omega_f)}^2 + \frac{\mathcal{W}}{2}\sum_{i=1}^{2}\|\mathbf{u}\cdot\tau_i\|_{L^2(\Gamma_i)}^2 + \frac{1}{2}\int_{\Omega_m} K\nabla P_2\cdot\nabla P_2\,d\mathbf{x}\Big)\\
&\leq -\int_{\Omega_f}(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_t\,d\mathbf{x}\\
&\leq \|\mathbf{u}\|_{L^3(\Omega_f)}\|\mathbf{u}_t\|_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|_{L^6(\Omega_f)}\\
&\leq C\|\mathbf{u}\|_{L^2(\Omega_f)}^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}^{\frac{1}{2}}(\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)} + \|\nabla\mathbf{u}\|_{L^2(\Omega_f)})\|\mathbf{u}_t\|_{L^2(\Omega_f)}\\
&\leq C\|\mathbf{u}_0\|_{L^2(\Omega_f)}^{\frac{1}{2}}M^{\frac{1}{2}}(\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)} + \|\nabla\mathbf{u}\|_{L^2(\Omega_f)})\|\mathbf{u}_t\|_{L^2(\Omega_f)}\\
&\leq \frac{1}{4}\|\mathbf{u}_t\|_{L^2(\Omega_f)}^2 + C\|\mathbf{u}_0\|_{L^2(\Omega_f)}M\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2 + C\|\mathbf{u}_0\|_{L^2(\Omega_f)}M\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}^2\\
&\leq \frac{1}{4}\|\mathbf{u}_t\|_{L^2(\Omega_f)}^2 + C\|D(\mathbf{u})\|_{L^2(\Omega_f)}^2 + C\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2,
\end{aligned} \tag{2.7}
$$

where C depends on μ_1 and Ω_f.

Next, we come to estimate $\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}$. The fact that

$$
\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)} \leq C(\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_y)\|_{L^2(\Omega_f)} + \|\mathbf{u}_{z,z}\|_{L^2(\Omega_f)}) \tag{2.8}
$$

tells us to deduce the estimations on $\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)}$ minutely, we omit the details of the estimations on $\|D(\mathbf{u}_y)\|_{L^2(\Omega_f)}$ due to the symmetry in the horizontal direction. We know from (2.8) and (2.23)–(2.28) that

$$
\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)} \leq C(\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_y)\|_{L^2(\Omega_f)} + \|\mathbf{u}_t\|_{L^2(\Omega_f)} + \|D(\mathbf{u})\|_{L^2(\Omega_f)} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2(\Omega_f)})
$$

$$
\begin{aligned}
&\le C(\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_y)\|_{L^2(\Omega_f)} + \|\mathbf{u}_t\|_{L^2(\Omega_f)} + \|D(\mathbf{u})\|_{L^2(\Omega_f)} + \|\mathbf{u}\|_{L^3(\Omega_f)}\|\nabla\mathbf{u}\|_{L^6(\Omega_f)})\\
&\le C(\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_y)\|_{L^2(\Omega_f)} + \|\mathbf{u}_t\|_{L^2(\Omega_f)} + \|D(\mathbf{u})\|_{L^2(\Omega_f)} + \|\mathbf{u}\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|_{H^1(\Omega_f)})\\
&\le C\|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}}\|\nabla\mathbf{u}\|_{H^1(\Omega_f)} + C(\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_y)\|_{L^2(\Omega_f)} + \|\mathbf{u}_t\|_{L^2(\Omega_f)} + \|D(\mathbf{u})\|_{L^2(\Omega_f)}),
\end{aligned}
$$

thus, we can get from $C\|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}} \le \frac{1}{2}$ that

$$
\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)} \le C(\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_y)\|_{L^2(\Omega_f)} + \|\mathbf{u}_t\|_{L^2(\Omega_f)} + \|D(\mathbf{u})\|_{L^2(\Omega_f)}). \tag{2.9}
$$

Next, taking the partial derivative of $(1.3)_1$ with respect to x, we get

$$
\mathbf{u}_{t,x} - \nabla\cdot(2\mu_1 D(\mathbf{u}_x) - \partial_x P_1 I) + \partial_x(\mathbf{u}\cdot\nabla\mathbf{u}) = 0 \text{ in } \Omega_f \times (0,T). \tag{2.10}
$$

Multiplying (2.10) by $\mathbf{u}_x$ and integrating the result inequality with respect to $\mathbf{x}$ over Ω_m, then using Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality, Korn's inequality, and (2.1), one can get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_x\|^2_{L^2(\Omega_f)} + 2\mu_1\|D(\mathbf{u}_x)\|^2_{L^2(\Omega_f)} + \mathcal{W}\sum_{i=1}^{2}\|\mathbf{u}_x\cdot\tau_i\|^2_{L^2(\Gamma_i)} + \frac{\lambda}{\mu_2}\|\nabla\partial_x P_2\|^2_{L^2(\Omega_m)}\\
&\quad\le -\int_{\Omega_f}\partial_x(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_x\,d\mathbf{x}\\
&\quad\le -\int_{\Omega_f}((\mathbf{u}_x\cdot\nabla\mathbf{u})\cdot\mathbf{u}_x + (\mathbf{u}\cdot\nabla\mathbf{u}_x)\cdot\mathbf{u}_x)\,d\mathbf{x}\\
&\quad\le\|\mathbf{u}\|_{L^3(\Omega_f)}\|\mathbf{u}_x\|_{L^6(\Omega_f)}\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)} + \int_{\Gamma_i} -K\nabla P_2\cdot\mathbf{n}_1(\mathbf{u}\cdot\mathbf{u}_x)\,dS\\
&\quad\le C\|\mathbf{u}\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}_x\|^2_{L^2(\Omega_f)} + C\|\nabla\partial_x P_2\|_{L^4(\Gamma_i)}\|\mathbf{u}\|_{L^2(\Gamma_i)}\|\mathbf{u}_x\|_{L^4(\Gamma_i)}\\
&\quad\le C\|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}}\|D(\mathbf{u}_x)\|^2_{L^2(\Omega_f)} + C\|\nabla\mathbf{u}\|_{H^1(\Omega_f)}(\|\mathbf{u}\|_{L^2(\Omega_f)} + \|\mathbf{u}\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|^{\frac{1}{2}}_{L^2(\Omega_f)})\|\mathbf{u}_x\|_{H^1(\Omega_f)}\\
&\quad\le C\|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}}\|D(\mathbf{u}_x)\|^2_{L^2(\Omega_f)} + C\|\nabla\mathbf{u}\|_{H^1(\Omega_f)}(\|\mathbf{u}_0\|_{L^2(\Omega_f)} + \|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}})\\
&\quad\le C\|\nabla\mathbf{u}\|^2_{H^1(\Omega_f)}(\|\mathbf{u}_0\|_{L^2(\Omega_f)} + \|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}}),
\end{aligned} \tag{2.11}
$$

and, similarly, we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_y\|^2_{L^2(\Omega_f)} + 2\mu_1\|D(\mathbf{u}_y)\|^2_{L^2(\Omega_f)} + \mathcal{W}\sum_{i=1}^{2}\|\mathbf{u}_y\cdot\tau_i\|^2_{L^2(\Gamma_i)} + \frac{\lambda}{\mu_2}\|\nabla\partial_y P_2\|^2_{L^2(\Omega_m)}\\
&\quad\le C\|\nabla\mathbf{u}\|^2_{H^1(\Omega_f)}(\|\mathbf{u}_0\|_{L^2(\Omega_f)} + \|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}}).
\end{aligned} \tag{2.12}
$$

Therefore, plugging (2.9) into (2.7) and combining (2.1), (2.11), and (2.12), one gets

$$
\frac{d}{dt}(\mu_1\|D(\mathbf{u})\|^2_{L^2(\Omega_f)} + \frac{\mathcal{W}}{2}\sum_{i=1}^{2}\|\mathbf{u}\cdot\tau_i\|^2_{L^2(\Gamma_i)} + \frac{1}{2}\int_{\Omega_m}K\nabla P_2\cdot\nabla P_2\,d\mathbf{x} + \frac{1}{2}\|\mathbf{u}_x\|^2_{L^2(\Omega_f)} + \frac{1}{2}\|\mathbf{u}_y\|^2_{L^2(\Omega_f)})
$$

$$
\begin{aligned}
&+\frac{3}{4}\|\mathbf{u}_t\|^2_{L^2(\Omega_f)}+2\mu_1\|D(\mathbf{u}_x)\|^2_{L^2(\Omega_f)}+2\mu_1\|D(\mathbf{u}_y)\|^2_{L^2(\Omega_f)}\\
\le& C(\|\mathbf{u}_0\|_{L^2(\Omega_f)}+\|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}}+\|\mathbf{u}_0\|_{L^2(\Omega_f)}M)\|\nabla^2\mathbf{u}\|^2_{L^2(\Omega_f)}+C\|D(\mathbf{u})\|^2_{L^2(\Omega_f)}\\
\le& \frac{1}{2}(\frac{3}{4}\|\mathbf{u}_t\|^2_{L^2(\Omega_f)}+2\mu_1\|D(\mathbf{u}_x)\|^2_{L^2(\Omega_f)}+2\mu_1\|D(\mathbf{u}_y)\|^2_{L^2(\Omega_f)})+C\|D(\mathbf{u})\|^2_{L^2(\Omega_f)},
\end{aligned}
\tag{2.13}
$$

where we have used $C(\|\mathbf{u}_0\|_{L^2(\Omega_f)}+\|\mathbf{u}_0\|^{\frac{1}{2}}_{L^2(\Omega_f)}M^{\frac{1}{2}}+\|\mathbf{u}_0\|_{L^2(\Omega_f)}M)\le\frac{1}{2}$. Integrating (2.13) over $(0,t)$, we can obtain (2.6) with Lemma 2.1, Young's inequality, and

$$
\epsilon_0=\min\{\frac{\mu_1^2}{C^2M},\frac{1}{16C^2M},\frac{1}{8C}\},\ \ M=\max\{1,\ \sqrt{C}\|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}\}.
$$

Thus, the proof of Lemma 2.2 is completed. □

While combining Lemmas 2.1 and 2.2, we have

$$
\sup_{0\le t\le T}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\le\sqrt{C}\|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}\le M.
$$

Therefore, the proof of (2.2) is complete.

Lemma 2.3. *Under the conditions of Proposition 2.1, it holds that*

$$
\sup_{0\le t\le T}\|\mathbf{u}_t\|^2_{L^2(\Omega_f)}+\int_0^T(\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)}+\|\nabla\partial_tP_2\|^2_{L^2(\Omega_m)})\,dt\le C\|\mathbf{u}_0\|^2_{H^2(\Omega_f)}\exp\{\|\mathbf{u}_0\|^4_{H^1(\Omega_f)}\}.
\tag{2.14}
$$

Proof. Differentiate $(1.3)_1$ with respect to t, multiply by $\mathbf{u}_t$, then integrate over Ω_f. Differentiate (1.2) with respect to t, multiply by ∂_tP_2, then integrate over Ω_m. Adding up the two result equations, we obtain that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{u}_t\|^2_{L^2(\Omega_f)}+2\mu_1\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)}+\mathcal{W}\sum_{i=1}^2\|\mathbf{u}_t\cdot\tau_i\|^2_{L^2(\Gamma_i)}+\frac{\lambda}{\mu_2}\|\nabla\partial_tP_2\|^2_{L^2(\Omega_m)}\\
&\le-\int_{\Omega_f}\partial_t(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_t\,d\mathbf{x}\\
&\le C\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\|\mathbf{u}_t\|^2_{L^4(\Omega_f)}+C\|\mathbf{u}\|_{L^6(\Omega_f)}\|\mathbf{u}_t\|_{L^3(\Omega_f)}\|\nabla\mathbf{u}_t\|_{L^2(\Omega_f)}\\
&\le C\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\|\mathbf{u}_t\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}_t\|^{\frac{3}{2}}_{L^2(\Omega_f)}\\
&\le\mu_1\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)}+C\|D(\mathbf{u})\|^4_{L^2(\Omega_f)}\|\mathbf{u}_t\|^2_{L^2(\Omega_f)},
\end{aligned}
\tag{2.15}
$$

where we have used Hölder's inequality, Young's inequality, Gagliardo-Nirenberg inequality, and Korn's inequality, then we can obtain by Grönwall's inequality that

$$
\begin{aligned}
&\sup_{0\le t\le T}\|\mathbf{u}_t\|^2_{L^2(\Omega_f)}+\int_0^T(\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)}+\|\nabla\partial_tP_2\|^2_{L^2(\Omega_m)})\,ds\\
&\le\|\mathbf{u}_{t,0}\|^2_{L^2(\Omega_f)}\exp\{C\int_0^T\|D(\mathbf{u})\|^4_{L^2(\Omega_f)}\,ds\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2 \exp\{\sup_{0\leq t\leq T}\|D(\mathbf{u})\|_{L^2(\Omega_f)}^2\int_0^T\|D(\mathbf{u})\|_{L^2(\Omega_f)}^2\,ds\}\\
&\leq C\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2 \exp\{C\|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}^2\|\mathbf{u}_0\|_{L^2(\Omega_f)}^2\}\\
&\leq C\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2 \exp\{\|\mathbf{u}_0\|_{H^1(\Omega_f)}^4\}.
\end{aligned}
$$

The proof is completed with the supports of Lemmas 2.1 and 2.2. □

Lemma 2.4. *Under the conditions of Proposition 2.1, it holds that*

$$
\begin{aligned}
\sup_{0\leq t\leq T}(\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2+\|P_2\|_{H^3(\Omega_m)}^2)+\int_0^T(\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2+\|\nabla^3\mathbf{u}\|_{L^2(\Omega_m)}^2)\,dt&\\
\leq C(\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2+1)\exp\{\|\mathbf{u}_0\|_{H^1(\Omega_f)}^2\}.&
\end{aligned}
\tag{2.16}
$$

Proof. Put ∂_x on $(1.3)_1$ and multiply $\mathbf{u}_{t,x}$ by both sides of it, then integrate it on Ω_f; meanwhile, apply $\partial_x\partial_t$ to (1.2), then multiply by $\partial_x P_2$ and integrate it on Ω_m. Adding up the two resulting equations gives that

$$
\begin{aligned}
&\frac{d}{dt}(\mu_1\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)}^2+\frac{\mathcal{W}}{2}\sum_{i=1}^2\|\mathbf{u}_x\cdot\tau_i\|_{L^2(\Gamma_i)}^2+\frac{1}{2}\int_{\Omega_m}K\nabla P_{2,x}\cdot\nabla P_{2,x}\,d\mathbf{x})+\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2\\
&\leq-\int_{\Omega_f}\partial_x(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_{t,x}\,d\mathbf{x}\\
&\leq\frac{1}{2}\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2+\frac{1}{2}\|\mathbf{u}_x\|_{L^6(\Omega_f)}^2\|\nabla\mathbf{u}\|_{L^3(\Omega_f)}^2+C\|\mathbf{u}\|_{L^\infty(\Omega_f)}^2\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}^2\\
&\leq\frac{1}{2}\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2+C\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}^2\|\nabla\mathbf{u}\|_{L^3(\Omega_f)}^2+C\|\mathbf{u}\|_{L^\infty(\Omega_f)}^2\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}^2\\
&\leq\frac{1}{2}\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2+C\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}^2(\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}+\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}^2)\\
&\quad+C(\|\mathbf{u}\|_{L^2(\Omega_f)}^{\frac{1}{2}}\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^{\frac{3}{2}}+\|\mathbf{u}\|_{L^2(\Omega_f)}^2)\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}^2\\
&\leq\frac{1}{2}\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2+C\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}^2(\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}^2+\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2+\|\mathbf{u}\|_{L^2(\Omega_f)}^2)\\
&\leq\frac{1}{2}\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2+C\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)}^2(\|D(\mathbf{u})\|_{L^2(\Omega_f)}^2+\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2),
\end{aligned}
\tag{2.17}
$$

where we have used Poincaré's inequality, Hölder's inequality, Young's inequality, Gagliardo-Nirenberg inequality, and Korn's inequality. Similarly, we can derive that

$$
\begin{aligned}
&\frac{d}{dt}(\mu_1\|D(\mathbf{u}_y)\|_{L^2(\Omega_f)}^2+\frac{\mathcal{W}}{2}\sum_{i=1}^2\|\mathbf{u}_x\cdot\tau_i\|_{L^2(\Gamma_i)}^2+\frac{1}{2}\int_{\Omega_m}K\nabla P_{2,y}\cdot\nabla P_{2,y}\,d\mathbf{x})+\|\mathbf{u}_{t,y}\|_{L^2(\Omega_f)}^2\\
&\leq\frac{1}{2}\|\mathbf{u}_{t,y}\|_{L^2(\Omega_f)}^2+C\|D(\mathbf{u}_y)\|_{L^2(\Omega_f)}^2(\|D(\mathbf{u})\|_{L^2(\Omega_f)}^2+\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2),
\end{aligned}
\tag{2.18}
$$

then we add up (2.17) and (2.18), and use Grönwall's inequality, Lemma 2.1, and Lemma 2.2 to get

$$
\sup_{0\leq t\leq T}(\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)}^2+\|D(\mathbf{u}_y)\|_{L^2(\Omega_f)}^2)+\int_0^T(\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2+\|\mathbf{u}_{t,y}\|_{L^2(\Omega_f)}^2)\,ds
$$

$$\leq C\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2 \exp\{\int_0^T (\|D(\mathbf{u})\|_{L^2(\Omega_f)}^2 + \|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2)\, ds\}$$
$$\leq C\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2 \exp\{\|\mathbf{u}_0\|_{L^2(\Omega_f)}^2 + \|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}^2\}. \tag{2.19}$$

Therefore, know from (2.9) and (2.19) that

$$\begin{aligned}\sup_{0\leq t\leq T} \|\mathbf{u}_{z,z}\|_{L^2(\Omega_f)}^2 \leq & C\int_0^T (\|D(\mathbf{u}_x)\|_{L^2(\Omega_f)}^2 + \|D(\mathbf{u}_y)\|_{L^2(\Omega_f)}^2 + \|\mathbf{u}_t\|_{L^2(\Omega_f)}^2 + \|D(\mathbf{u})\|_{L^2(\Omega_f)}^2)\, ds\\ \leq & C\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2 \exp\{\|\mathbf{u}_0\|_{L^2(\Omega_f)}^2 + \|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}^2\} + \|\mathbf{u}_0\|_{L^2(\Omega_f)}^2 + \|\nabla\mathbf{u}_0\|_{L^2(\Omega_f)}^2\\ \leq & C(\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2 + 1)\exp\{\|\mathbf{u}_0\|_{H^1(\Omega_f)}^2\},\end{aligned} \tag{2.20}$$

and it is easy to get that (see [42])

$$\|\nabla P_2\|_{H^k(\Omega_m)}^2 \leq C\|\mathbf{u}\|_{H^{k-1}(\Omega_f)}^2,\ \forall\, k\geq 2, \tag{2.21}$$

thus, the proof of Lemma 2.4 is complete with (2.4), (2.6), (2.9), (2.14), (2.19), and (2.20). □

The proof of Proposition 2.1 has been finished. Next, we will proceed with the higher-order estimation involving the time weighting.

2.2. *A priori estimates (II): Higher order estimates*

Let $\sigma(t) = \min\{1, t\}$ and, from now on, the generic positive constant is defined by the right term of (2.3) as $N \triangleq C(\|\mathbf{u}_0\|_{H^2(\Omega_f)}^2 + 1)\exp\{\|\mathbf{u}_0\|_{H^1(\Omega_f)}^2 + \|\mathbf{u}_0\|_{H^1(\Omega_f)}^4\}$.

We have the following third-order estimates.

Lemma 2.5. *It holds that*

$$\begin{aligned}&\sup_{0\leq t\leq T}\sigma(t)(\|\nabla^3\mathbf{u}\|_{L^2(\Omega_f)}^2 + \|D(\mathbf{u}_t)\|_{L^2(\Omega_f)}^2 + \|\nabla\partial_t P_2\|_{L^2(\Omega_m)}^2 + \|\nabla^4 P_2\|_{L^2(\Omega_m)}^2)\\ &+\int_0^T \sigma(t)(\|D(\mathbf{u}_{t,x})\|_{L^2(\Omega_f)}^2 + \|D(\mathbf{u}_{t,y})\|_{L^2(\Omega_f)}^2 + \|\mathbf{u}_{t,t}\|_{L^2(\Omega_f)}^2 + \|D(\mathbf{u}_{x,x,x})\|_{L^2(\Omega_f)}^2 + \|D(\mathbf{u}_{x,x,y})\|_{L^2(\Omega_f)}^2\\ &+\|D(\mathbf{u}_{x,y,y})\|_{L^2(\Omega_f)}^2 + \|D(\mathbf{u}_{y,y,y})\|_{L^2(\Omega_f)}^2 + \|\nabla\partial_t\partial_x P_2\|_{L^2(\Omega_m)}^2 + \|\nabla\partial_t\partial_y P_2\|_{L^2(\Omega_m)}^2)\, dt\\ \leq & C(\|\mathbf{u}_0\|_{H^2(\Omega_f)}^{10} + 1)\exp\{\|\mathbf{u}_0\|_{H^1(\Omega_f)}^2 + \|\mathbf{u}_0\|_{H^1(\Omega_f)}^4\}.\end{aligned} \tag{2.22}$$

Proof. It follows from the fact that

$$\|\nabla^3\mathbf{u}\|_{L^2(\Omega_f)} \leq C(\|D(\mathbf{u}_{x,x})\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y})\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y})\|_{L^2(\Omega_f)} + \|\nabla\mathbf{u}_{z,z}\|_{L^2(\Omega_f)}).$$

First, we focus on $\mathbf{u}_{z,z}$. By $(1.3)_1$, we have

$$\begin{aligned}\|\mathbf{u}_{z,z}\|_{L^2(\Omega_f)}^2 \leq & \|\frac{1}{\mu_1}(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P_1 - \mathbf{u}_{x,x} - \mathbf{u}_{y,y})\|_{L^2(\Omega_f)}^2\\ \leq & C(\|\mathbf{u}_t\|_{L^2(\Omega_f)}^2 + \|\mathbf{u}\cdot\nabla\mathbf{u}\|_{L^2(\Omega_f)}^2 + \|\nabla P_1\|_{L^2(\Omega_f)}^2 + \|\mathbf{u}_{x,x}\|_{L^2(\Omega_f)}^2 + \|\mathbf{u}_{y,y}\|_{L^2(\Omega_f)}^2),\end{aligned} \tag{2.23}$$

and taking divergence to $(1.3)_1$, the elliptic problem can be obtained as follows:

$$-\Delta P_1 = \nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u}) \text{ in } \Omega_f\times(0,T). \tag{2.24}$$

With the boundary condition obtained by $(1.3)_2$ and $(1.4)_1$, we have

$$\begin{cases} P_1 \cdot \mathbf{n}_1 = 0 \quad \text{on } \Gamma_U, \\ P_1 = P_2 + 2\mu_1 \partial_z u_3 = P_2 - 2\mu_1 \partial_x u_1 - 2\mu_1 \partial_y u_2 \quad \text{on } \Gamma_i. \end{cases} \tag{2.25}$$

It is clear to show by Lemma 2.5 [41] and the Trace theorem that

$$\begin{aligned} \|\nabla P_1\|^2_{L^2(\Omega_f)} &\leq C\|P_1\|^2_{H^1(\Omega_f)} \leq C(\|\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})\|^2_{H^{-1}(\Omega_f)} + \|P_2 - 2\mu_1 \partial_x u_1 - 2\mu_1 \partial_y u_2\|^2_{H^{\frac{1}{2}}(\Gamma_i)}) \\ &\leq C(\|\mathbf{u} \cdot \nabla \mathbf{u}\|^2_{L^2(\Omega_f)} + \|\nabla P_2\|^2_{L^2(\Omega_m)} + \|\partial_x u_1\|^2_{H^1(\Omega_f)} + \|\partial_y u_2\|^2_{H^1(\Omega_f)}) \\ &\leq C(\|\mathbf{u} \cdot \nabla \mathbf{u}\|^2_{L^2(\Omega_f)} + \|\nabla P_2\|^2_{L^2(\Omega_m)} + \|D(\partial_x u_1)\|^2_{L^2(\Omega_f)} + \|D(\partial_y u_2)\|^2_{L^2(\Omega_f)}) \\ &\leq C(\|\mathbf{u} \cdot \nabla \mathbf{u}\|^2_{L^2(\Omega_f)} + \|\nabla P_2\|^2_{L^2(\Omega_m)} + \|D(\mathbf{u}_x)\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_y)\|^2_{L^2(\Omega_f)}). \end{aligned} \tag{2.26}$$

The estimation of $\|\mathbf{u} \cdot \nabla \mathbf{u}\|^2_{L^2(\Omega_f)}$ can be estimated by Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality, and Korn's inequality as follows:

$$\begin{aligned} \|\mathbf{u} \cdot \nabla \mathbf{u}\|^2_{L^2(\Omega_f)} \leq& \|\mathbf{u}\|^2_{L^\infty(\Omega_f)} \|\nabla \mathbf{u}\|^2_{L^2(\Omega_f)} \leq C\|\nabla \mathbf{u}\|_{L^2(\Omega_f)} \|\nabla^2 \mathbf{u}\|_{L^2(\Omega_f)} \|\nabla \mathbf{u}\|^2_{L^2(\Omega_f)} \\ \leq& C(\epsilon)\|D(\mathbf{u})\|^6_{L^2(\Omega_f)} + \epsilon\|\nabla^2 \mathbf{u}\|^2_{L^2(\Omega_f)}. \end{aligned} \tag{2.27}$$

Thus, we can derive with (2.23), (2.27), and (2.21), by Young's inequality, that

$$\|\nabla \mathbf{u}\|^2_{H^1(\Omega_f)} \leq C(\|\mathbf{u}_t\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_x)\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_y)\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u})\|^6_{L^2(\Omega_f)} + 1). \tag{2.28}$$

Obviously, from (2.23) with Hölder's inequality, Young's inequality, Gagliardo-Nirenberg inequality, Poincáre's inequality, Sobolev inequality, and Korn's inequality we have

$$\begin{aligned} \|\nabla \mathbf{u}_{z,z}\|_{L^2(\Omega_f)} \leq& C(\|\nabla \mathbf{u}_t\|_{L^2(\Omega_f)} + \|\nabla(\mathbf{u} \cdot \nabla \mathbf{u})\|_{L^2(\Omega_f)} + \|\nabla^2 P_1\|_{L^2(\Omega_f)} + \|\nabla \mathbf{u}_{x,x}\|_{L^2(\Omega_f)} + \|\nabla \mathbf{u}_{y,y}\|_{L^2(\Omega_f)}) \\ \leq& C(\|D(\mathbf{u}_t)\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x})\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y})\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y})\|_{L^2(\Omega_f)} \\ &+ \|\mathbf{u}_{x,z,z}\|_{L^2(\Omega_f)} + \|\mathbf{u}_{y,z,z}\|_{L^2(\Omega_f)} + \|\mathbf{u}\|^2_{H^2(\Omega_f)}), \end{aligned}$$

where we have used the estimation obtained based on (2.27) as follows:

$$\begin{aligned} \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{H^1(\Omega_f)} \leq& \|\mathbf{u}\|_{L^\infty(\Omega_f)} \|\nabla^2 \mathbf{u}\|_{L^2(\Omega_f)} + \|\nabla \mathbf{u}\|^2_{L^4(\Omega_f)} + \|\mathbf{u} \cdot \nabla \mathbf{u}\|_{L^2(\Omega_f)} \\ \leq& \|\nabla \mathbf{u}\|^{\frac{1}{2}}_{L^2(\Omega_f)} \|\nabla^2 \mathbf{u}\|^{\frac{3}{2}}_{L^2(\Omega_f)} + \|\nabla \mathbf{u}\|^2_{H^1(\Omega_f)} \\ \leq& C\|\mathbf{u}\|^2_{H^2(\Omega_f)}, \end{aligned} \tag{2.29}$$

and, again, using Lemma 2.5 [41] together with the Trace theorem to get that

$$\begin{aligned} &\|\nabla^2 P_1\|_{L^2(\Omega_f)} \leq \|P_1\|_{H^2(\Omega_f)} \leq \|\nabla \cdot (\mathbf{u} \cdot \nabla \mathbf{u})\|_{L^2(\Omega_f)} + \|P_2 - 2\mu_1 \partial_x u_1 - 2\mu_1 \partial_y u_2\|_{H^{\frac{3}{2}}(\Gamma_i)} \\ &\leq C(\|\mathbf{u}\|^2_{H^2(\Omega_f)} + \|P_2\|^2_{H^2(\Omega_m)} + \|\nabla \partial_x u_1\|_{H^1(\Omega_f)} + \|\nabla \partial_y u_2\|_{H^1(\Omega_f)}) \\ &\leq C(\|D(\mathbf{u}_{x,x})\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y})\|_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y})\|_{L^2(\Omega_f)} + \|\mathbf{u}_{x,z,z}\|_{L^2(\Omega_f)} + \|\mathbf{u}_{y,z,z}\|_{L^2(\Omega_f)} + \|\mathbf{u}\|^2_{H^2(\Omega_f)}), \end{aligned} \tag{2.30}$$

such that we have

$$\begin{aligned} \|\nabla^3 \mathbf{u}\|^2_{L^2(\Omega_f)} \leq& C(\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y})\|^2_{L^2(\Omega_f)} \\ &+ \|\mathbf{u}_{x,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{y,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\|^4_{H^2(\Omega_f)}). \end{aligned} \tag{2.31}$$

Next, we will get the bound of each term at the righthand of (2.31) step by step.

Step 1. Next, we apply $\partial_x\partial_y$ to $(1.3)_1$ and multiply $\mathbf{u}_{t,x,y}$ before integrating the result equations over Ω_f; meanwhile, we apply $\partial_x\partial_y$ to (1.2) and integrate it over Ω_m after multiplying $P_{2,x,y}$ on it. Finally, summing up the two resulting equations would come to (2.32), with the help of Hölder's inequality, Gagliardo-Nirenberg inequality, Young's inequality, and Korn's inequality,

$$\begin{aligned}
&\frac{d}{dt}(\mu_1\|D(\mathbf{u}_{x,y})\|^2_{L^2(\Omega_f)} + \frac{\mathcal{W}}{2}\sum_{i=1}^{2}\|\mathbf{u}_{x,y}\cdot\tau_i\|^2_{L^2(\Gamma_i)} + \frac{1}{2}\int_{\Omega_m} K\nabla P_{2,x,y}\cdot\nabla P_{2,x,y}\,d\mathbf{x}) + \|\mathbf{u}_{t,x,y}\|^2_{L^2(\Omega_f)}\\
\le& -\int_{\Omega_f}\partial_x\partial_y(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_{t,x,y}\,d\mathbf{x}\\
\le& \frac{1}{2}\|\mathbf{u}_{t,x,y}\|^2_{L^2(\Omega_f)} + \|\nabla\mathbf{u}\|^2_{L^\infty(\Omega_f)}\|\mathbf{u}_{x,y}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_x\|^2_{L^\infty(\Omega_f)}\|\nabla\mathbf{u}_y\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_y\|^2_{L^\infty(\Omega_f)}\|\nabla\mathbf{u}_x\|^2_{L^2(\Omega_f)}\\
&+ \|\mathbf{u}\|^2_{L^\infty(\Omega_f)}\|\nabla\mathbf{u}_{x,y}\|^2_{L^2(\Omega_f)}\\
\le& \frac{1}{2}\|\mathbf{u}_{t,x,y}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{x,y}\|^2_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|_{H^1(\Omega_f)}\|\nabla^2\mathbf{u}\|_{H^1(\Omega_f)} + \|\mathbf{u}_x\|_{H^1(\Omega_f)}\|\nabla\mathbf{u}_x\|_{H^1(\Omega_f)}\|\nabla\mathbf{u}_y\|^2_{L^2(\Omega_f)}\\
&+ \|\mathbf{u}_y\|_{H^1(\Omega_f)}\|\nabla\mathbf{u}_y\|_{H^1(\Omega_f)}\|\nabla\mathbf{u}_x\|^2_{L^2(\Omega_f)} + \|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}\|\nabla\mathbf{u}_{x,y}\|^2_{L^2(\Omega_f)}\\
\le& \frac{1}{2}\|\mathbf{u}_{t,x,y}\|^2_{L^2(\Omega_f)} + C(\|\mathbf{u}\|^3_{H^2(\Omega_f)}\|\nabla^2\mathbf{u}\|_{H^1(\Omega_f)} + \|\mathbf{u}\|^2_{H^2(\Omega_f)}\|D(\mathbf{u}_{x,y})\|^2_{L^2(\Omega_f)}).
\end{aligned}\tag{2.32}$$

Multiplying (2.32) by $\sigma(t)$ and integrating the result over $(0,t)$, with integration by parts and Young's inequality, obtains that

$$\begin{aligned}
&\sigma(t)\|D(\mathbf{u}_{x,y})\|^2_{L^2(\Omega_f)} + \frac{1}{2}\int_0^t\sigma(s)\|\mathbf{u}_{t,x,y}\|^2_{L^2(\Omega_f)}\,ds\\
\le& C\int_0^t\|\mathbf{u}\|^3_{H^2(\Omega_f)}\|\nabla^2\mathbf{u}\|_{H^1(\Omega_f)}\,ds + C\int_0^t(1+\|\mathbf{u}\|^2_{H^2(\Omega_f)})\|D(\mathbf{u}_{x,y})\|^2_{L^2(\Omega_f)}\,ds\\
\le& C(1+N^2).
\end{aligned}\tag{2.33}$$

Similarly, we can get

$$\sigma(t)\|D(\mathbf{u}_{x,x})\|^2_{L^2(\Omega_f)} + \frac{1}{2}\int_0^t\sigma(s)\|\mathbf{u}_{t,x,x}\|^2_{L^2(\Omega_f)}\,ds \le C(1+N^2),\tag{2.34}$$

and

$$\sigma(t)\|D(\mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)} + \frac{1}{2}\int_0^t\sigma(s)\|\mathbf{u}_{t,y,y}\|^2_{L^2(\Omega_f)}\,ds \le C(1+N^2).\tag{2.35}$$

Step 2. For $\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)}$, first differentiate $(1.3)_1$ with respect to t and multiply by $\partial_t^2\mathbf{u}$, then integrate it on Ω_f. Meanwhile, apply ∂_t^2 to (1.2) and multiply by $\partial_t^2 P_2$, then integrate it on Ω_m. At last, summing up the two resulting equations with Hölder's inequality, Gagliardo-Nirenberg inequality, and Young's inequality gains that

$$\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)} + \frac{d}{dt}(\mu_1\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)} + \frac{\mathcal{W}}{2}\sum_{i=1}^{2}\|\mathbf{u}_t\cdot\tau_i\|^2_{L^2(\Gamma_i)} + \frac{1}{2}\int_{\Omega_m}K\nabla\partial_t P_2\cdot\nabla\partial_t P_2\,d\mathbf{x})$$

$$
\begin{aligned}
&\leq -\int_{\Omega_f} \partial_t(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_{t,t}\,d\mathbf{x}\\
&\leq (\|\mathbf{u}_t\cdot\nabla\mathbf{u}\|_{L^2(\Omega_f)}+\|\mathbf{u}\cdot\nabla\mathbf{u}_t\|_{L^2(\Omega_f)})\|\mathbf{u}_{t,t}\|_{L^2(\Omega_f)}\\
&\leq \|\mathbf{u}_t\|^2_{L^4(\Omega_f)}\|\nabla\mathbf{u}\|^2_{L^4(\Omega_f)}+\|\mathbf{u}\|^2_{L^\infty(\Omega_f)}\|\nabla\mathbf{u}_t\|^2_{L^2(\Omega_f)}+\frac{1}{2}\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)}\\
&\leq C\|\nabla\mathbf{u}_t\|^2_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|^2_{H^1(\Omega_f)}+\frac{1}{2}\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)}.
\end{aligned}
\tag{2.36}
$$

Multiplying (2.36) by $\sigma(t)$ and integrating the result over $(0,t)$, with Lemma 2.3 and Young's inequality, obtains that

$$
\begin{aligned}
&\sigma(t)(\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)}+\|\nabla\partial_t P_2\|^2_{L^2(\Omega_f)})+\int_0^t\sigma(s)\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)}\,ds\\
&\qquad\leq C\int_0^t(1+\|\mathbf{u}\|^2_{H^2(\Omega_f)})\|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)}\,ds+C\int_0^t\|\nabla\partial_t P_2\|^2_{L^2(\Omega_f)}\,ds\\
&\qquad\leq C(1+N^2).
\end{aligned}
\tag{2.37}
$$

Step 3. Undoubtedly, from (2.25), (2.29), and (2.30), we get that

$$
\begin{aligned}
&\|\mathbf{u}_{x,z,z}\|^2_{L^2(\Omega_f)}+\|\mathbf{u}_{y,z,z}\|^2_{L^2(\Omega_f)}\\
\leq&\|\frac{1}{\mu_1}\partial_x(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla P_1-\mathbf{u}_{x,x}-\mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)}+\|\frac{1}{\mu_1}\partial_y(\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla P_1-\mathbf{u}_{x,x}-\mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)}\\
\leq&C(\|\mathbf{u}_{t,x}\|^2_{L^2(\Omega_f)}+\|\mathbf{u}_{t,y}\|^2_{L^2(\Omega_f)}+\|\partial_x(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)}+\|\partial_y(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)}+\|\nabla\partial_x P_1\|^2_{L^2(\Omega_f)}+\|\nabla\partial_y P_1\|^2_{L^2(\Omega_f)}\\
&+\|D(\mathbf{u}_{x,x})\|^2_{L^2(\Omega_f)}+\|D(\mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)}\\
\leq&C(\|\mathbf{u}_{t,x}\|^2_{L^2(\Omega_f)}+\|\mathbf{u}_{t,y}\|^2_{L^2(\Omega_f)}+\|\mathbf{u}\|^4_{H^2(\Omega_f)}+\|D(\mathbf{u}_{x,x})\|^2_{L^2(\Omega_f)}+\|D(\mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)}+\|D(\mathbf{u}_{x,y})\|^2_{L^2(\Omega_f)}+1),
\end{aligned}
\tag{2.38}
$$

where we have used the estimation:

$$
\begin{aligned}
&\|\partial_x(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)}+\|\partial_y(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)}\\
\leq&\|\mathbf{u}_x\cdot\nabla\mathbf{u}\|^2_{L^2(\Omega_f)}+\|\mathbf{u}\cdot\nabla\mathbf{u}_x\|^2_{L^2(\Omega_f)}+\|\mathbf{u}_y\cdot\nabla\mathbf{u}\|^2_{L^2(\Omega_f)}+\|\mathbf{u}\cdot\nabla\mathbf{u}_y\|^2_{L^2(\Omega_f)}\\
\leq&C(\|\mathbf{u}_x\|^2_{L^4(\Omega_f)}\|\nabla\mathbf{u}\|^2_{L^4(\Omega_f)}+\|\mathbf{u}\|^2_{L^\infty(\Omega_f)}\|\nabla\mathbf{u}_x\|^2_{L^2(\Omega_f)}+\|\mathbf{u}_y\|^2_{L^4(\Omega_f)}\|\nabla\mathbf{u}\|^2_{L^4(\Omega_f)}+\|\mathbf{u}\|^2_{L^\infty(\Omega_f)}\|\nabla\mathbf{u}_y\|^2_{L^2(\Omega_f)})\\
\leq&C(\|\mathbf{u}_x\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\mathbf{u}_x\|^{\frac{3}{2}}_{H^1(\Omega_f)}\|\nabla\mathbf{u}\|^2_{H^1(\Omega_f)}+\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}\|\nabla\mathbf{u}_x\|^2_{L^2(\Omega_f)}\\
&+\|\mathbf{u}_y\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\mathbf{u}_y\|^{\frac{3}{2}}_{H^1(\Omega_f)}\|\nabla\mathbf{u}\|^2_{H^1(\Omega_f)}+\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}\|\nabla\mathbf{u}_y\|^2_{L^2(\Omega_f)})\\
\leq&C\|\mathbf{u}\|^4_{H^2(\Omega_f)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\nabla\partial_x P_1\|^2_{L^2(\Omega_f)}+\|\nabla\partial_y P_1\|^2_{L^2(\Omega_f)}\leq C(\|\partial_x(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)}+\|\partial_x P_2\|^2_{H^1(\Omega_m)}+\|\partial_x^2 u_1\|^2_{H^1(\Omega_f)}+\|\partial_x\partial_y u_1\|^2_{H^1(\Omega_f)})\\
&\quad+C(\|\partial_y(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)}+\|\partial_y P_2\|^2_{H^1(\Omega_m)}+\|\partial_y^2 u_2\|^2_{H^1(\Omega_f)}+\|\partial_x\partial_y u_2\|^2_{H^1(\Omega_f)})\\
\leq&C(\|\mathbf{u}\|^4_{H^2(\Omega_f)}+\|D(\mathbf{u}_{x,x})\|^2_{L^2(\Omega_f)}+\|D(\mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)}+\|D(\mathbf{u}_{x,y})\|^2_{L^2(\Omega_f)}),
\end{aligned}
\tag{2.39}
$$

which are derived by Hölder's inequality, Gagliardo-Nirenberg inequality, and Young's inequality. Thus, from (2.38), we could derive that

$$\sigma(t)(\|\mathbf{u}_{x,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{y,z,z}\|^2_{L^2(\Omega_f)}) \leq C(1+N^2). \tag{2.40}$$

Step 4. Apply $\partial_t\partial_x$ to $(1.3)_1$, then multiply $\sigma(t)\mathbf{u}_{t,x}$ and integrate the result equation over Ω_f. Meanwhile, apply $\partial_t\partial_x$ to (1.2), then multiply $\sigma(t)\partial_t\partial_x P_2$ and integrate the result equation over Ω_m. Summing up the two resulting equations, we obtain with Hölder's inequality, Gagliardo-Nirenberg inequality, and Young's inequality that

$$\begin{aligned}
&\frac{d}{dt}(\sigma(t)\|\mathbf{u}_{t,x}\|^2_{L^2(\Omega_f)}) + \sigma(t)\mathcal{W}\sum_{i=1}^{2}\|\mathbf{u}_{t,x}\cdot\tau_i\|^2_{L^2(\Gamma_i)} + \sigma(t)\|D(\mathbf{u}_{t,x})\|^2_{L^2(\Omega_f)} + \sigma(t)\|\nabla\partial_t\partial_x P_2\|^2_{L^2(\Omega_m)}\\
\leq& C\|\mathbf{u}_{t,x}\|^2_{L^2(\Omega_f)} + C\sigma(t)\int_{\Omega_f}\partial_t\partial_x(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_{t,x}\,d\mathbf{x}\\
\leq& C\|\mathbf{u}_{t,x}\|^2_{L^2(\Omega_f)} + C\sigma(t)\|\mathbf{u}_{t,x}\|^2_{L^4(\Omega_f)}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)} + C\sigma(t)\|\mathbf{u}_x\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_t\|_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|_{L^4(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_t\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|_{L^4(\Omega_f)} + C\sigma(t)\|\mathbf{u}\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|_{L^4(\Omega_f)}\\
\leq& C\|\mathbf{u}_{t,x}\|^2_{L^2(\Omega_f)} + C\sigma(t)\|\mathbf{u}_{t,x}\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|^{\frac{3}{2}}_{H^1(\Omega_f)}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_x\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\mathbf{u}_x\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\nabla\mathbf{u}_t\|_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_t\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\mathbf{u}_t\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|^{\frac{3}{4}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\mathbf{u}_{t,x}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\\
\leq& C\|\mathbf{u}_{t,x}\|^2_{L^2(\Omega_f)} + \frac{1}{2}\sigma(t)\|D(\mathbf{u}_{t,x})\|^2_{L^2(\Omega_f)} + C\sigma(t)(1+\|\mathbf{u}\|^8_{H^2(\Omega_f)})\|\mathbf{u}_t\|^2_{H^1(\Omega_f)},
\end{aligned}$$

which means that

$$\int_0^t \sigma(s)(\|D(\mathbf{u}_{t,x})\|^2_{L^2(\Omega_f)} + \|\nabla\partial_t\partial_x P_2\|^2_{L^2(\Omega_m)})\,ds \leq C(1+N^5), \tag{2.41}$$

and, similarly, we have

$$\int_0^t \sigma(s)(\|D(\mathbf{u}_{t,y})\|^2_{L^2(\Omega_f)} + \|\nabla\partial_t\partial_y P_2\|^2_{L^2(\Omega_m)})\,ds \leq C(1+N^5). \tag{2.42}$$

Finally, conclusions can be derived from Proposition 2.1, (2.21), (2.31), (2.33), (2.34), (2.35), (2.37), (2.40), (2.41), and (2.42) that

$$\begin{aligned}
&\sigma(t)(\|\nabla^3\mathbf{u}\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_t)\|^2_{L^2(\Omega_f)} + \|\nabla\partial_t P_2\|^2_{L^2(\Omega_m)} + \|\nabla^4 P_2\|^2_{L^2(\Omega_f)})\\
&\quad+\int_0^t \sigma(s)(\|D(\mathbf{u}_{t,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{t,y})\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)} + \|\nabla\partial_t\partial_x P_2\|^2_{L^2(\Omega_m)} + \|\nabla\partial_t\partial_y P_2\|^2_{L^2(\Omega_m)})\,ds\\
\leq& C(1+N^5).
\end{aligned}$$

So far, it's time to complete the 3-order estimates in the next step.

Step 5. We detailedly display the estimate of $\int_0^t \sigma(s)\|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_m)}\,ds$ in this step, then the estimates of $\int_0^t \sigma(s)\|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_m)}\,ds$, $\int_0^t \sigma(s)\|D(\mathbf{u}_{x,y,y})\|^2_{L^2(\Omega_m)}\,ds$, and $\int_0^t \sigma(s)\|D(\mathbf{u}_{y,y,y})\|^2_{L^2(\Omega_m)}\,ds$ can be obtained in a similar derivation.

Apply $\partial_x^2\partial_y$ to $(1.3)_1$ and multiply by $\sigma(t)\mathbf{u}_{x,x,y}$ before integrating the resulting equations over Ω_f. Meanwhile, apply $\partial_x^2\partial_y$ to (1.2) and multiply $\sigma(t)\partial_x^2\partial_y P_2$ on it after that, integrate over Ω_m. Finally, summing up the above two resulting equations could come to

$$
\begin{aligned}
&\frac{d}{dt}(\sigma(t)\|\mathbf{u}_{x,x,y}\|^2_{L^2(\Omega_f)}) + \sigma(t)\|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)} \le C\|\mathbf{u}_{x,x,y}\|^2_{L^2(\Omega_f)} + C\sigma(t)\int_{\Omega_f}\partial_x^2\partial_y(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_{x,x,y}\,d\mathbf{x}\\
\le&C\|\mathbf{u}_{x,x,y}\|^2_{L^2(\Omega_f)} + C\sigma(t)\|\mathbf{u}_{x,x,y}\|^2_{L^4(\Omega_f)}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)} + C\sigma(t)\|\mathbf{u}_{x,x}\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_y\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|_{L^4(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_{x,y}\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|_{L^4(\Omega_f)} + C\sigma(t)\|\mathbf{u}_x\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_{x,y}\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|_{L^4(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_y\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_{x,x}\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|_{L^4(\Omega_f)} + C\sigma(t)\|\mathbf{u}\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_{x,x,y}\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|_{L^4(\Omega_f)}\\
\le&C\|\mathbf{u}_{x,x,y}\|^2_{L^2(\Omega_f)} + C\sigma(t)\|\mathbf{u}_{x,x,y}\|^{\frac{3}{2}}_{H^1(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{1}{2}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|_{L^2(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_{x,x}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_{x,x}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}_y\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_{x,y}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_{x,y}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}_x\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_x\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_x\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}_{x,y}\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\\
&+ C\sigma(t)\|\mathbf{u}_y\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_y\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}_{x,x}\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\\
&+ C\sigma(t)\|\nabla\mathbf{u}\|^{\frac{3}{4}}_{L^2(\Omega_f)}\|\mathbf{u}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\|\nabla\mathbf{u}_{x,x,y}\|_{L^2(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{3}{4}}_{H^1(\Omega_f)}\|\mathbf{u}_{x,x,y}\|^{\frac{1}{4}}_{L^2(\Omega_f)}\\
\le&C\|\mathbf{u}_{x,x,y}\|^2_{L^2(\Omega_f)} + \frac{1}{2}\sigma(t)\|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)} + C\sigma(t)(1+\|D(\mathbf{u}_{x,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y})\|^2_{L^2(\Omega_f)})(1+\|\mathbf{u}\|^8_{H^2(\Omega_f)}),
\end{aligned}
$$

the proof of which is reckoned from the Gagliardo-Nirenberg inequality and Young's inequality. It is easy to find that

$$\int_0^t \sigma(s)\|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)}\,ds \le C(1+N^5),$$

then, similarly, we have

$$\int_0^t \sigma(s)\|D(\mathbf{u}_{x,y,y})\|^2_{L^2(\Omega_f)}\,ds \le C(1+N^5),$$

and it is more concisely to be derived that

$$\int_0^t \sigma(s)(\|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y,y})\|^2_{L^2(\Omega_f)})\,ds \le C(1+N^5),$$

so that the proof of Lemma 2.5 is completed with Young's inequality. □

Using a similar argument as that in the proof of Lemma 2.5, we can easily obtain the following fourth-order estimates.

Lemma 2.6. *It holds that*

$$\sup_{0\le t\le T}\sigma(t)^2(\|\nabla^4\mathbf{u}\|^2_{L^2(\Omega_f)} + \|\nabla^2\mathbf{u}_t\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)} + \|\nabla^5 P_2\|^2_{L^2(\Omega_m)} + \|\nabla\partial_t\partial_x P_2\|^2_{L^2(\Omega_m)} + \|\nabla\partial_t\partial_y P_2\|^2_{L^2(\Omega_m)})$$

$$+\int_0^t \sigma(t)^2(\|\mathbf{u}_{t,x,x,x}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{t,x,y,y}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{t,y,y,y}\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{t,t})\|^2_{L^2(\Omega_f)} + \|\nabla\partial_t^2 P_2\|^2_{L^2(\Omega_m)})\, ds$$
$$\le C(\|\mathbf{u}_0\|^{18}_{H^2(\Omega_f)} + 1)\exp\{\|\mathbf{u}_0\|^2_{H^1(\Omega_f)} + \|\mathbf{u}_0\|^4_{H^1(\Omega_f)}\}. \tag{2.43}$$

Proof. It follows from the fact that

$$\|\nabla^4\mathbf{u}\|^2_{L^2(\Omega_f)} \le C(\|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y,y})\|^2_{L^2(\Omega_f)} + \|\nabla^2\mathbf{u}_{z,z}\|^2_{L^2(\Omega_f)}).$$

Obviously, from (2.23) with Hölder's inequality, Young's inequality, Gagliardo-Nirenberg inequality, Poincáre's inequality, Sobolev inequality, and Korn's inequality we have that

$$\|\nabla^2\mathbf{u}_{z,z}\|^2_{L^2(\Omega_f)} \le C(\|\nabla^2\mathbf{u}_t\|^2_{L^2(\Omega_f)} + \|\nabla^2(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)} + \|\nabla^3 P_1\|^2_{L^2(\Omega_f)} + \|\nabla^2\mathbf{u}_{x,x}\|^2_{L^2(\Omega_f)} + \|\nabla^2\mathbf{u}_{y,y}\|^2_{L^2(\Omega_f)}),$$

where

$$\begin{aligned}
&\|\mathbf{u}\cdot\nabla\mathbf{u}\|^2_{H^2(\Omega_f)} \le \|\nabla^2(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|^2_{H^1(\Omega_f)}\\
\le& C\|\nabla^2\mathbf{u}\cdot\nabla\mathbf{u}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\cdot\nabla^3\mathbf{u}\|^2_{L^2(\Omega_f)} + C\|\mathbf{u}\|^4_{H^2(\Omega_f)},\\
\le& \|\nabla^2\mathbf{u}\|^2_{L^4(\Omega_f)}\|\nabla\mathbf{u}\|^2_{L^4(\Omega_f)} + \|\mathbf{u}\|^2_{L^\infty(\Omega_f)}\|\nabla^3\mathbf{u}\|^2_{L^2(\Omega_f)} + C\|\mathbf{u}\|^4_{H^2(\Omega_f)}\\
\le& C(\|\nabla^2\mathbf{u}\|^2_{H^1(\Omega_f)}\|\nabla\mathbf{u}\|^2_{H^1(\Omega_f)} + \|\mathbf{u}\|^2_{H^2(\Omega_f)}\|\nabla^3\mathbf{u}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\|^4_{H^2(\Omega_f)})\\
\le& C(\|\mathbf{u}\|^2_{H^2(\Omega_f)}\|\nabla^3\mathbf{u}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\|^4_{H^2(\Omega_f)}).
\end{aligned} \tag{2.44}$$

Additionally, we leverage Lemma 2.5 [41], coupled with the Trace theorem. The methodology employed here mirrors that of (2.26). We obtain

$$\begin{aligned}
&\|\nabla^3 P_1\|^2_{L^2(\Omega_f)} \le \|P_1\|^2_{H^3(\Omega_f)} \le \|\nabla\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{H^1(\Omega_f)} + \|P_2 - 2\mu_1\partial_x u_1 - 2\mu_1\partial_y u_2\|^2_{H^{\frac{5}{2}}(\Gamma_i)}\\
\le& C(\|\mathbf{u}\|^2_{H^2(\Omega_f)}\|\nabla^3\mathbf{u}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\|^4_{H^2(\Omega_f)} + \|\mathbf{u}\|^2_{H^1(\Omega_f)} + \|\partial_x u_1\|^2_{H^3(\Omega_f)} + \|\partial_y u_2\|^2_{H^3(\Omega_f)})\\
\le& C(\|\mathbf{u}\|^2_{H^2(\Omega_f)}\|\nabla^3\mathbf{u}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\|^4_{H^2(\Omega_f)} + \|\nabla\mathbf{u}_{x,z,z}\|^2_{L^2(\Omega_f)} + \|\nabla\mathbf{u}_{y,z,z}\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_f)}\\
&+ \|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y,y})\|^2_{L^2(\Omega_f)}),
\end{aligned}$$

such that one gets

$$\begin{aligned}
\|\nabla^4\mathbf{u}\|^2_{L^2(\Omega_f)} \le& C(\|\nabla^2\mathbf{u}_t\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\|^2_{H^2(\Omega_f)}\|\nabla^3\mathbf{u}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\|^4_{H^2(\Omega_f)} + \|\nabla\mathbf{u}_{x,z,z}\|^2_{L^2(\Omega_f)} + \|\nabla\mathbf{u}_{y,z,z}\|^2_{L^2(\Omega_f)}\\
&+ \|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y,y})\|^2_{L^2(\Omega_f)}).
\end{aligned} \tag{2.45}$$

Step 1. We detail work on $\sigma(t)^2\|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)}$ in this step to get $\sigma(t)^2\|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_f)}$, $\sigma(t)^2\|D(\mathbf{u}_{x,y,y})\|^2_{L^2(\Omega_f)}$, and $\sigma(t)^2\|D(\mathbf{u}_{y,y,y})\|^2_{L^2(\Omega_f)}$.

Apply $\partial_x^2\partial_y$ to the Eq $(1.3)_1$ and multiply by $\partial_x^2\partial_y\mathbf{u}_t$. Meanwhile, apply $\partial_x^2\partial_y$ to (1.2) and multiply by $\partial_x^2\partial_y P_2$, integrate the two equations with respect to $\mathbf{x}$ by parts, then add up the resulting formulas to get

$$\|\mathbf{u}_{t,x,x,y}\|^2_{L^2(\Omega_f)} + \frac{d}{dt}(\mu_1\|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)} + \frac{\mathcal{W}}{2}\sum_{i=1}^{2}\|\mathbf{u}_{x,x,y}\cdot\tau_i\|^2_{L^2(\Gamma_i)} + \frac{1}{2}\int_{\Omega_m} K\nabla P_{2,x,x,y}\cdot\nabla P_{2,x,x,y}\, d\mathbf{x})$$

$$
\begin{aligned}
&\leq -\int_{\Omega_f} \partial_x^2\partial_y(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\partial_x^2\partial_y\mathbf{u}_t\ d\mathbf{x}\\
&\leq \frac{1}{2}\|\mathbf{u}_{t,x,x,y}\|_{L^2(\Omega_f)}^2+\|\mathbf{u}_{x,x,y}\|_{L^4(\Omega_f)}^2\|\nabla\mathbf{u}\|_{L^4(\Omega_f)}^2+\|\mathbf{u}_{x,x}\|_{L^4(\Omega_f)}^2\|\nabla\mathbf{u}_y\|_{L^4(\Omega_f)}^2+\|\mathbf{u}_{x,y}\|_{L^4(\Omega_f)}^2\|\nabla\mathbf{u}_x\|_{L^4(\Omega_f)}^2\\
&\quad+\|\mathbf{u}_y\|_{L^\infty(\Omega_f)}^2\|\nabla\mathbf{u}_{x,x}\|_{L^2(\Omega_f)}^2+\|\mathbf{u}_x\|_{L^\infty(\Omega_f)}^2\|\nabla\mathbf{u}_{x,y}\|_{L^2(\Omega_f)}^2+\|\mathbf{u}\|_{L^\infty(\Omega_f)}^2\|\nabla\mathbf{u}_{x,x,y}\|_{L^2(\Omega_f)}^2\\
&\leq \frac{1}{2}\|\mathbf{u}_{t,x,x,y}\|_{L^2(\Omega_f)}^2+\|\mathbf{u}_{x,x,y}\|_{L^2(\Omega_f)}^{\frac{1}{2}}\|\mathbf{u}_{x,x,y}\|_{H^1(\Omega_f)}^{\frac{3}{2}}\|\mathbf{u}\|_{H^2(\Omega_f)}^2+\|\nabla^2\mathbf{u}\|_{H^1(\Omega_f)}^2\|\nabla^2\mathbf{u}\|_{L^2(\Omega_f)}^2\\
&\quad+\|\mathbf{u}\|_{H^2(\Omega_f)}^2(\|\nabla\mathbf{u}_{x,x}\|_{L^2(\Omega_f)}^2+\|\nabla\mathbf{u}_{x,y}\|_{L^2(\Omega_f)}^2)+\|\mathbf{u}\|_{H^2(\Omega_f)}^2\|\nabla\mathbf{u}_{x,x,y}\|_{L^2(\Omega_f)}^2,
\end{aligned}
\tag{2.46}
$$

then by multiplying by $\sigma(s)^2$ and integrating over $(0,t)$, one has

$$
\sigma(t)^2\|D(\mathbf{u}_{x,x,y})\|_{L^2(\Omega_f)}^2+\frac{1}{2}\int_0^t\sigma(s)^2\|\mathbf{u}_{t,x,x,y}\|_{L^2(\Omega_f)}^2\,ds\leq C(1+N^6).
\tag{2.47}
$$

We can get the similar results on $\sigma(t)^2\|D(\mathbf{u}_{x,x,x})\|_{L^2(\Omega_f)}^2$, $\sigma(t)^2\|D(\mathbf{u}_{x,y,y})\|_{L^2(\Omega_f)}^2$, and $\sigma(t)^2\|D(\mathbf{u}_{y,y,y})\|_{L^2(\Omega_f)}^2$; thus, we have

$$
\begin{aligned}
&\sigma(t)^2(\|D(\mathbf{u}_{x,x,x})\|_{L^2(\Omega_f)}^2+\|D(\mathbf{u}_{x,y,y})\|_{L^2(\Omega_f)}^2+\|D(\mathbf{u}_{y,y,y})\|_{L^2(\Omega_f)}^2)\\
&\quad+\frac{1}{2}\int_0^t\sigma(s)^2(\|\mathbf{u}_{t,x,x,x}\|_{L^2(\Omega_f)}^2+\|\mathbf{u}_{t,x,y,y}\|_{L^2(\Omega_f)}^2+\|\mathbf{u}_{t,y,y,y}\|_{L^2(\Omega_f)}^2)\,ds\\
&\leq C(1+N^6).
\end{aligned}
\tag{2.48}
$$

Step 2. For $\|\nabla^2\mathbf{u}_t\|_{L^2(\Omega_f)}$, we have

$$
\|\nabla^2\mathbf{u}_t\|_{L^2(\Omega_f)}\leq C(\|\nabla\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}+\|\nabla\mathbf{u}_{t,y}\|_{L^2(\Omega_f)}+\|\mathbf{u}_{t,z,z}\|_{L^2(\Omega_f)}).
\tag{2.49}
$$

First, we apply $\partial_x\partial_t$ to the Eq $(1.3)_1$ and multiply by $\mathbf{u}_{t,t,x}$. Meanwhile, apply $\partial_x\partial_t$ to (1.2) and multiply by $\partial_x\partial_t P_2$, integrate the two equations with respect to x by parts, then we add up the resulting formulas to get

$$
\begin{aligned}
&\|\mathbf{u}_{t,t,x}\|_{L^2(\Omega_f)}^2+\frac{d}{dt}(\mu_1\|D(\mathbf{u}_{t,x})\|_{L^2(\Omega_f)}^2+\frac{\mathcal{W}}{2}\sum_{i=1}^2\|\mathbf{u}_{t,x}\cdot\tau_i\|_{L^2(\Gamma_i)}^2)+\frac{1}{2}\int_{\Omega_m}K\nabla\partial_t\partial_xP_2\cdot\nabla\partial_t\partial_xP_2\,d\mathbf{x}\\
&\qquad\leq -\int_{\Omega_f}\partial_t\partial_x(\mathbf{u}\cdot\nabla\mathbf{u})\cdot\mathbf{u}_{t,t,x}\,d\mathbf{x}\\
&\qquad\leq -\int_{\Omega_f}(\mathbf{u}_{t,x}\cdot\nabla\mathbf{u}+\mathbf{u}_t\cdot\nabla\mathbf{u}_x+\mathbf{u}_x\cdot\nabla\mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}_{t,x})\cdot\mathbf{u}_{t,t,x}\,d\mathbf{x}\\
&\qquad\leq\|\mathbf{u}_{t,x}\|_{L^4(\Omega_f)}^2\|\nabla\mathbf{u}\|_{L^4(\Omega_f)}^2+\|\mathbf{u}_t\|_{L^4(\Omega_f)}^2\|\nabla\mathbf{u}_x\|_{L^4(\Omega_f)}^2+\|\mathbf{u}_x\|_{L^\infty(\Omega_f)}^2\|\nabla\mathbf{u}_t\|_{L^2(\Omega_f)}^2+\|\mathbf{u}\|_{L^\infty(\Omega_f)}^2\|\nabla\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2\\
&\qquad\quad+\frac{1}{2}\|\mathbf{u}_{t,t,x}\|_{L^2(\Omega_f)}^2\\
&\qquad\leq C(\|\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^{\frac{1}{2}}\|\mathbf{u}_{t,x}\|_{H^1(\Omega_f)}^{\frac{3}{2}}\|\mathbf{u}\|_{H^2(\Omega_f)}^2+\|\mathbf{u}_t\|_{H^1(\Omega_f)}^2\|\mathbf{u}\|_{H^3(\Omega_f)}^2+\|\nabla\mathbf{u}_{t,x}\|_{L^2(\Omega_f)}^2\|\mathbf{u}\|_{H^2(\Omega_f)}^2)\\
&\qquad\quad+\frac{1}{2}\|\mathbf{u}_{t,t,x}\|_{L^2(\Omega_f)}^2.
\end{aligned}
\tag{2.50}
$$

Multiplying by $\sigma(s)^2$ on (2.50) and integrating over $(0,t)$ with (2.3) and (2.22) yields:

$$\sigma(t)^2(\|D(\mathbf{u}_{t,x})\|^2_{L^2(\Omega_f)}+\|\nabla\partial_t\partial_x P_2\|^2_{L^2(\Omega_m)})+\frac{1}{2}\int_0^t\sigma(s)^2\|\mathbf{u}_{t,t,x}\|^2_{L^2(\Omega_f)}\,ds\le C(1+N^6),\tag{2.51}$$

then, similarly, we have

$$\sigma(t)^2(\|D(\mathbf{u}_{t,y})\|^2_{L^2(\Omega_f)}+\|\nabla\partial_t\partial_y P_2\|^2_{L^2(\Omega_m)})+\frac{1}{2}\int_0^t\sigma(s)^2\|\mathbf{u}_{t,t,y}\|^2_{L^2(\Omega_f)}\,ds\le C(1+N^6).\tag{2.52}$$

Second, we try to get a bound of $\|\mathbf{u}_{t,z,z}\|_{L^2(\Omega_f)}$, and differentiating (2.23) with respect to t shows that

$$\begin{aligned}\|\mathbf{u}_{t,z,z}\|^2_{L^2(\Omega_f)}\le&\|\frac{1}{\mu_1}(\mathbf{u}_{t,t}+\partial_t(\mathbf{u}\cdot\nabla\mathbf{u})+\nabla\partial_t P_1-\mathbf{u}_{t,x,x}-\mathbf{u}_{t,y,y})\|^2_{L^2(\Omega_f)}\\ \le&C(\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)}+\|\mathbf{u}_t\|^2_{H^1(\Omega_f)}\|\mathbf{u}\|^2_{H^2(\Omega_f)}+\|\nabla\partial_t P_1\|^2_{L^2(\Omega_f)}+\|\mathbf{u}_{t,x,x}\|^2_{L^2(\Omega_f)}+\|\mathbf{u}_{t,y,y}\|^2_{L^2(\Omega_f)}).\end{aligned}\tag{2.53}$$

For $\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)}$, first apply ∂_t^2 to $(1.3)_1$ and multiply by $\sigma(t)^2\partial_t^2\mathbf{u}$. Meanwhile, apply ∂_t^2 to (1.2) and multiply by $\sigma(t)^2\partial_t^2 P_2$, then integrate it on Ω_m. At last, integrate the two resulting equations with respect to $\mathbf{x}$ and t and sum them up to arrive at

$$\begin{aligned}&\sigma(t)^2\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)}+\int_0^t\sigma(s)^2\int_{\Omega_f}(\|D(\mathbf{u}_{t,t})\|^2_{L^2(\Omega_f)}+\|\nabla\partial_t^2P_2\|^2_{L^2(\Omega_m)})\,d\mathbf{x}ds\\ &\le C\int_0^t\sigma(s)^2\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)}\,ds+C\int_0^t\sigma(s)^2(\mathbf{u}\cdot\nabla\mathbf{u})_{tt}\cdot\mathbf{u}_{t,t}\,ds\\ &\le C(1+N^5)+C\int_0^t\sigma(s)^2\|\mathbf{u}_{t,t}\|_{L^2(\Omega_f)}\|\nabla\mathbf{u}\|_{L^4(\Omega_f)}\|\mathbf{u}_{t,t}\|_{L^4(\Omega_f)}\,ds\\ &\quad+C\int_0^t\sigma(s)^2(\|\mathbf{u}_t\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_t\|_{L^2(\Omega_f)}+\|\mathbf{u}\|_{L^4(\Omega_f)}\|\nabla\mathbf{u}_{t,t}\|_{L^2(\Omega_f)})\|\mathbf{u}_{t,t}\|_{L^4(\Omega_f)}\,ds\\ &\le C(1+N^5)+\frac{1}{2}\int_0^t\sigma(s)^2\|D(\mathbf{u}_{t,t})\|^2_{L^2(\Omega_f)}\,ds+C\int_0^t\sigma(s)^2(\|\mathbf{u}_{t,t}\|^2_{L^2(\Omega_f)}(1+\|\mathbf{u}\|^8_{H^2(\Omega_f)})+\|\mathbf{u}_t\|^4_{H^1(\Omega_f)})\,ds\\ &\le C(1+N^9)+\frac{1}{2}\int_0^t\sigma(s)^2\|D(\mathbf{u}_{t,t})\|^2_{L^2(\Omega_f)}\,ds,\end{aligned}\tag{2.54}$$

where we have used Lemma 2.5. Next, it's time to deal with $\|\nabla\partial_t P_1\|_{L^2(\Omega_f)}$. We differentiate (2.24) and (2.25) with respect to t, and then using the standard L^2-estimate for the elliptic system with (2.3), (2.51), and (2.52), we have

$$\begin{aligned}\sigma(t)^2\|\nabla\partial_t P_1\|^2_{L^2(\Omega_f)}\le&C\sigma(t)^2(\|\mathbf{u}_t\|^2_{H^1(\Omega_f)}\|\mathbf{u}\|^2_{H^2(\Omega_f)}+\|\partial_t P_2\|^2_{H^1(\Omega_m)}+\|\mathbf{u}_{t,x}\|^2_{H^1(\Omega_f)}+\|\mathbf{u}_{t,y}\|^2_{H^1(\Omega_f)})\\ \le&C(1+N^6).\end{aligned}\tag{2.55}$$

Next, plugging (2.54) and (2.55) into (2.53), we can get that

$$\sigma(t)^2\|\mathbf{u}_{t,z,z}\|^2_{L^2(\Omega_f)}\le C(1+N^9),\tag{2.56}$$

Similarly, we can plug (2.56), (2.51), and (2.52) to (2.49) to obtain

$$\sigma(t)^2\|\nabla^2\mathbf{u}_t\|^2_{L^2(\Omega_f)}\le C(1+N^9).\tag{2.57}$$

Step 3. We work on $\|\nabla \mathbf{u}_{x,z,z}\|^2_{L^2(\Omega_f)} + \|\nabla \mathbf{u}_{y,z,z}\|^2_{L^2(\Omega_f)}$ in this step. Applying the gradient operator ∇ to (2.38), then multiplying by $\sigma(s)^2$ and integrating over $(0,t)$ yields

$$\begin{aligned}\sigma(t)^2(\|\nabla \mathbf{u}_{x,z,z}\|^2_{L^2(\Omega_f)} + \|\nabla \mathbf{u}_{y,z,z}\|^2_{L^2(\Omega_f)}) \le& \frac{1}{\mu_1}\sigma(t)^2\|\nabla\partial_x(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P_1 - \mathbf{u}_{x,x} - \mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)}\\ &+ \frac{1}{\mu_1}\sigma(t)^2\|\nabla\partial_y(\mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P_1 - \mathbf{u}_{x,x} - \mathbf{u}_{y,y})\|^2_{L^2(\Omega_f)}\\ \le& C(1+N^9) + \sigma(t)^2(\|\nabla^2\partial_x P_1\|^2_{L^2(\Omega_f)} + \|\nabla^2\partial_y P_1\|^2_{L^2(\Omega_f)},\end{aligned} \tag{2.58}$$

where we have used Proposition 2.1, (2.44), (2.47), (2.48), (2.51), and (2.52).

We now estimate $\|\nabla^2\partial_x P_1\|^2_{L^2(\Omega_f)} + \|\nabla^2\partial_y P_1\|^2_{L^2(\Omega_f)}$. We apply Lemma 2.5 in [41] together with the Trace theorem on (2.24) and (2.25), similarly as (2.39), to get that

$$\begin{aligned}&\sigma(t)^2(\|\nabla^2\partial_x P_1\|^2_{L^2(\Omega_f)} + \|\nabla^2\partial_y P_1\|^2_{L^2(\Omega_f)})\\ \le& C\sigma(t)^2(\|\mathrm{div}(\partial_x(\mathbf{u}\cdot\nabla\mathbf{u}))\|^2_{L^2(\Omega_f)} + \|\mathrm{div}(\partial_y(\mathbf{u}\cdot\nabla\mathbf{u}))\|^2_{L^2(\Omega_f)} + \|\partial_x P_2\|^2_{H^{\frac{3}{2}}(\Gamma_i)} + \|\partial_y P_2\|^2_{H^{\frac{3}{2}}(\Gamma_i)}\\ &+ \|\partial_x^2 u_1\|^2_{H^{\frac{3}{2}}(\Omega_f)} + \|\partial_x\partial_y u_1\|^2_{H^{\frac{3}{2}}(\Omega_f)} + \|\partial_y^2 u_2\|^2_{H^{\frac{3}{2}}(\Omega_f)} + \|\partial_x\partial_y u_2\|^2_{H^{\frac{3}{2}}(\Omega_f)})\\ \le& C\sigma(t)^2(\|\mathbf{u}\cdot\nabla\mathbf{u}\|^2_{H^2(\Omega_f)} + \|P_2\|^2_{H^3(\Omega_m)} + \|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,y,y})\|^2_{L^2(\Omega_f)}\\ &+ \|\mathbf{u}_{x,x,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{x,y,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{y,y,z,z}\|^2_{L^2(\Omega_f)})\\ \le& C(1+N^6) + C\sigma(t)^2(\|\mathbf{u}_{x,x,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{x,y,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{y,y,z,z}\|^2_{L^2(\Omega_f)}).\end{aligned} \tag{2.59}$$

Now, we respectively apply $\partial_x^2,\ \partial_x\partial_y,\ \partial_y^2$ to (2.23) and multiply by $\sigma(s)^2$ to get

$$\begin{aligned}&\sigma(t)^2(\|\mathbf{u}_{x,x,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{x,y,z,z}\|^2_{L^2(\Omega_f)} + \|\mathbf{u}_{y,y,z,z}\|^2_{L^2(\Omega_f)})\\ \le& C\sigma(t)^2(\|D(\mathbf{u}_{t,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{t,y})\|^2_{L^2(\Omega_f)} + \|\mathbf{u}\cdot\nabla\mathbf{u}\|^2_{H^2(\Omega_f)} + \|\nabla\partial_x^2 P_1\|^2_{L^2(\Omega_f)}\\ &+ \|\nabla\partial_x\partial_y P_1\|^2_{L^2(\Omega_f)} + \|\nabla\partial_y^2 P_1\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)})\\ \le& C(1+N^9) + C\sigma(t)^2(\|\nabla\partial_x^2 P_1\|^2_{L^2(\Omega_f)} + \|\nabla\partial_x\partial_y P_1\|^2_{L^2(\Omega_f)} + \|\nabla\partial_y^2 P_1\|^2_{L^2(\Omega_f)}),\end{aligned}$$

where we have used Lemma 2.5, (2.44), (2.51), and (2.52), and it remains to estimate the last three terms on the righthand side. Again, we apply Lemma 2.5 [41] together with the Trace theorem on (2.24) and (2.25), similarly as (2.39), to get that

$$\begin{aligned}&\sigma(t)^2(\|\nabla\partial_x^2 P_1\|^2_{L^2(\Omega_f)} + \|\nabla\partial_x\partial_y P_1\|^2_{L^2(\Omega_f)} + \|\nabla\partial_y^2 P_1\|^2_{L^2(\Omega_f)})\\ \le& C\sigma(t)^2(\|\partial_x^2(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)} + \|\partial_x\partial_y(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)} + \|\partial_y^2(\mathbf{u}\cdot\nabla\mathbf{u})\|^2_{L^2(\Omega_f)} + \|\partial_x^2 P_2\|^2_{H^1(\Omega_m)}\\ &+ \|\partial_y^2 P_2\|^2_{H^1(\Omega_m)} + \|\partial_x\partial_y P_2\|^2_{H^1(\Omega_m)} + \|\partial_x^3 u_1\|^2_{H^1(\Omega_f)} + \|\partial_x^2\partial_y u_1\|^2_{H^1(\Omega_f)}\\ &+ \|\partial_x\partial_y^2 u_1\|^2_{H^1(\Omega_f)} + \|\partial_x^2\partial_y u_2\|^2_{H^1(\Omega_f)} + \|\partial_x\partial_y^2 u_2\|^2_{H^1(\Omega_f)} + \|\partial_y^3 u_2\|^2_{H^1(\Omega_f)})\\ \le& C\sigma(t)^2(\|\mathbf{u}\cdot\nabla\mathbf{u}\|^2_{H^2(\Omega_f)} + \|P_2\|^2_{H^3(\Omega_m)} + \|D(\mathbf{u}_{x,x,x})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{x,x,y})\|^2_{L^2(\Omega_f)}\\ &+ \|D(\mathbf{u}_{x,y,y})\|^2_{L^2(\Omega_f)} + \|D(\mathbf{u}_{y,y,y})\|^2_{L^2(\Omega_f)})\\ \le& C(1+N^6),\end{aligned} \tag{2.60}$$

thus, combining the above steps, we can complete the proof of Lemma 2.6. □

3. Proof of Theorem 1.2

First of all, we know that the systems (1.3) and (1.4) have a unique local strong solution $(\mathbf{u}, P_2)$ on $\Omega \times (0, T_*]$ for some $T_* > 0$, so it's time to verify the continuity of the strong solution to extend it globally in time with counter-evidence.

It follows from the fact that $\mathbf{u}_0 \in H^2(\Omega_f)$ and Proposition 1.1 that there exists a $T_1 \in (0, T_*]$ such that (2.1) holds for $T = T_1$. Next, we set

$$T^* = \sup\{T \mid (\mathbf{u}, P_2) \text{ is a strong solution on } \Omega \times (0, T] \text{ and (2.1) holds}\}, \tag{3.1}$$

then $T^* \geq T_1 \geq 0$. Hence, for any $0 < \tau < T \leq T_*$ with T finite, we can get

$$\mathbf{u} \in L^\infty(\tau, T; H^4(\Omega_f)), \ \ \mathbf{u}_t \in L^\infty(\tau, T; H^2(\Omega_f)),$$

from Proposition 2.1, and Lemmas 2.5 and 2.6. Thus one can deduce that

$$\mathbf{u} \in C([\tau, T]; C^2(\Omega_f)) \cap C([\tau, T]; H^3(\Omega_f)) \tag{3.2}$$

from

$$L^\infty(\tau, T; H^4(\Omega_f)) \cap W^{1,\infty}(\tau, T; H^2(\Omega_f)) \hookrightarrow C([\tau, T]; C^2(\Omega_f)) \cap C([\tau, T]; H^3(\Omega_f)).$$

Now, we suppose that

$$T^* < \infty, \tag{3.3}$$

then $T = T^*$ holds by Proposition 2.1 and (2.2). It follows from (3.2) that

$$\mathbf{u}(x, T^*) := \lim_{t \to T^*} \mathbf{u}(x, t) \in H^2(\Omega_f), \tag{3.4}$$

thus, the initial data condition in Proposition 1.1 is satisfied, which gives that there exists a $T^{**} > T^*$ such that (2.1) holds for $T = T^{**}$. This contradicts the definition of T^* in (3.1), so $T^* = \infty$.

Finally, to finish the proof of Theorem 1.2, we have from (2.5) that

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega_f)} + \frac{\mu_1}{C_{\Omega_f}}\|\nabla \mathbf{u}\|^2_{L^2(\Omega_f)} \leq 0. \tag{3.5}$$

According to the Poincar inequality for three-dimensional cases, we have

$$\|\mathbf{u}\|^2_{L^2(\Omega_f)} \leq \frac{2}{\sqrt{3}} \mid \Omega_f \mid^{\frac{1}{3}} \|\nabla \mathbf{u}\|^2_{L^2(\Omega_f)}, \tag{3.6}$$

then substituting (3.6) into (3.5), we get

$$\frac{d}{dt}\|\mathbf{u}\|^2_{L^2(\Omega_f)} + \frac{3\mu_1}{2 \mid \Omega_f \mid^{\frac{2}{3}} C_{\Omega_f}}\|\mathbf{u}\|^2_{L^2(\Omega_f)} \leq 0. \tag{3.7}$$

Therefore, we have

$$\|\mathbf{u}\|^2_{L^2(\Omega_f)} \leq C\epsilon_0^2 e^{-ct}. \tag{3.8}$$

Here, the positive constants are $C = C(\mu_1, \ \lambda, \ \Omega_f, \| \mathbf{u}_0\|_{H^2(\Omega_f)})$ and $c = c(\mu_1, \Omega_f)$.

4. Conclusions

In conclusion, this study has made some progress in understanding the global well-posedness of a coupled Navier-Stokes-Darcy model with the Beavers-Joseph-Saffman-Jones interface boundary condition in three-dimensional Euclidean space. Through our investigation, we have achieved the establishment of a global strong solution for the system, marking a crucial advancement in the field. Moreover, we have demonstrated the exponential stability of this strong solution, further reinforcing its reliability. The implications of our findings extend to the analysis of subsurface flow problems, notably in the realm of karst aquifers, where such coupled systems play a pivotal role. By shedding light on the dynamics and behaviors of these systems, our research contributes to a deeper understanding of fluid flow phenomena in complex geological formations, offering valuable insights for both theoretical developments and practical applications in hydrogeology and related disciplines.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We wish to acknowledge the support of the National Natural Science Foundation of China (Grant No. 11931013, 12061081), the Natural Science Foundation of Shaanxi Province (Grant No. 2023-JC-QN-0073), the Natural Science Foundation of Yulin (Grant No. CXY-2022-76), and the Natural Science Foundation of Yulin University (Grant No. 21GK07) .

Conflict of interest

The authors have no conflicts to disclose.

References

1. J. R. Fanchi, *Principles of applied reservoir simulation*, Elsevier, 2005. http://dx.doi.org/10.1016/B978-075067933-6/50014-3
2. J. Bear, *Dynamics of fluids in porous media*, Courier Corporation, 1972. http://dx.doi.org/10.1097/00010694-197508000-00022
3. M. Discacciati, A. Quarteroni, Navier-Stokes/Darcy coupling: Modeling, analysis, and numerical approximation, *Rev. Mat. Comput.*, **2** (2009), 315–426. http://dx.doi.org/10.5209/rev_REMA.2009.v22.n2.16263
4. D. Han, D. Sun, X. Wang, Two-phase flows in karstic geometry, *Math. Method. Appl. Sci.*, **37** (2014), 3048–3063. http://dx.doi.org/10.1002/mma.3043
5. M. Alam, H. M. Byrne, G. P. R. Sekhar, Existence and uniqueness results on biphasic mixture model for an in-vivo tumor, *Appl. Anal.*, **101** (2022), 5442–5468. http://dx.doi.org/10.1080/00036811.2021.1895122

6. W. Layton, F. Schieweck, I. Yotov, Coupling fluid flow with porous media flow, *SIAM J. Numer. Anal.*, **40** (2003), 2195–2218. http://dx.doi.org/10.1137/S0036142901392766

7. M. Discacciati, E. Miglio, A. Quarteroni, Mathematical and numerical models for coupling surface and groundwater flows, *Appl. Numer. Math.*, **43** (2001), 57–74. http://dx.doi.org/10.1016/S0168-9274(02)00125-3

8. H. W. Alt, S. Luckhaus, Quasilinear elliptic-parabolic differential equation, *Math. Z.*, **3** (1983), 311–341. http://dx.doi.org/10.1070/SM1968v006n03ABEH001065

9. P. Fabrie, M. Langlais, Modelling wells in porous media flows, *SIAM J. Math. Anal.*, **23** (1992), 1375–1392. http://dx.doi.org/10.1142/S0218202500000367

10. P. Fabrie, T. Gallouët, Modelling wells in porous media flows, *Math. Mod. Method. Appl. Sci.*, **10** (2000), 673–709. http://dx.doi.org/10.1142/S0218202500000367

11. F. Marpeau, M. Saad, Mathematical analysis of radionuclides displacement in porous media with nonlinear adsorption, *J. Differ. Equations*, **228** (2006), 412–439. http://dx.doi.org/10.1016/j.jde.2006.03.023

12. K. Hans, N. Masmoudi, Well-posedness and uniform bounds for a nonlocal third order evolution operator on an infinite wedge, *Commun. Math. Phys.*, **320** (2013), 395–424. http://dx.doi.org/10.1007/s00220-013-1708-z

13. D. A. Nield, A. Bejan, *Convection in porous media*, 1 Eds., New York: Springer, 1992. http://dx.doi.org/10.1007/978-3-319-49562-0

14. G. S. Beavers, D. D. Joseph, Boundary conditions at a naturally permeable wall, *J. Fluid Mech.*, **30** (1967), 197–207. http://dx.doi.org/10.1017/s0022112067001375

15. H. K. Versteeg, W. Malalasekera, *An introduction to computational fluid dynamics: The finite volume method*, Pearson Education, 2007.

16. S. Whitaker, Flow in porous media I: A theoretical derivation of Darcy's law, *Transport Porous Med.*, **1** (1986), 3–25. http://dx.doi.org/10.1007/BF01036523

17. I. P. Jones, Low Reynolds number flow past a porous spherical shell, *Math. Proc. Cambridge*, **73** (1973), 231–238. http://dx.doi.org/10.1017/S0305004100047642

18. P. G. Saffman, On the boundary condition at the surface of a porous medium, *Stud. Appl. Math.*, **50** (1971), 93–101. http://dx.doi.org/10.1002/sapm197150293

19. M. McCurdy, N. Moore, X. Wang, Convection in a coupled free flow-porous media system, *SIAM J. Appl. Math.*, **6** (2019), 2313–2339. http://dx.doi.org/10.1137/19M1238095

20. D. Han, X. He, Q. Wang, Y. Wu, Existence and weak-strong uniqueness of solutions to the Cahn-Hilliard-Navier-Stokes-Darcy system in superposed free flow and porous media, *Nonlinear Anal.*, **211** (2021), 112411. http://dx.doi.org/10.1016/j.na.2021.112411

21. A. Çeşmelioğlu, B. Rivière, Primal discontinuous Galerkin methods for time-dependent coupled surface and subsurface flow, *J. Sci. Comput.*, **40** (2009), 115–140. http://dx.doi.org/10.1007/s10915-009-9274-4

22. V. Girault, B. Rivière, DG Approximation of coupled Navier-Stokes and Darcy equations by Beaver-Joseph-Saffman Interface condition, *SIAM J. Numer. Anal.*, 47 (2009), 2052–2089. http://dx.doi.org/10.1137/070686081

23. Y. Cao, Y. Chu, X. He, M. Wei, Decoupling the stationary Navier-Stokes-Darcy system with the Beavers-Joseph-Saffman interface condition, *Abstr. Appl. Anal.*, 2013. http://dx.doi.org/10.1155/2013/136483

24. M. Cai, M. Mu, J. Xu, Numerical solution to a mixed Navier-Stokes/Darcy model by the two-grid approach, *SIAM J. Numer. Anal.*, **47** (2009), 3325–3338. http://dx.doi.org/10.1137/080721868

25. G. Du, L. Zuo, Local and parallel finite element method for the mixed Navier-Stokes/Darcy model with Beavers-Joseph interface conditions, *Acta Math. Sci.*, **37** (2017), 1331–1347. http://dx.doi.org/10.1016/S0252-9602(17)30076-0

26. C. Qiu, X. He, J. Li, Y. Lin, A domain decomposition method for the time-dependent Navier-Stokes-Darcy model with Beavers-Joseph interface condition and defective boundary condition, *J. Comput. Phys.*, **411** (2020). http://dx.doi.org/10.1016/j.jcp.2020.109400

27. Y, Qin, Y. Hou, Optimal error estimates of a decoupled scheme based on two-grid finite element for mixed Navier-Stokes/Darcy model, *Acta Math. Sci.*, **38** (2018), 1361–1369.

28. X. He, J. Li, Y. Lin, J. Ming, A domain decomposition method for the steady-state Navier-Stokes-Darcy model with Beavers-Joseph interface condition, *SIAM J. Sci. Comput.*, **37** (2015), S264–S290. http://dx.doi.org/10.1137/140965776

29. L. Zuo, G. Du, A parallel two-grid linearized method for the coupled Navier-Stokes-Darcy problem, *Numer. Algorithms*, **77** (2018), 151–165, http://dx.doi.org/10.1007/s11075-017-0308-y

30. M. Discacciati, A. Quarteroni, *Analysis of a domain decomposition method for the coupling of Stokes and Darcy equations*, In: F. Brezzi, A. Buffa, S. Corsaro, A. Murli, Numerical mathematics and advanced applications, Milan: Springer, 2003. http://dx.doi.org/10.1007/978-88-470-2089-4-1

31. B. Rivière, I. Yotov, Locally conservative coupling of Stokes and Darcy flows, *SIAM J. Numer. Anal.*, **42** (2004), 1959–1977. http://dx.doi.org/10.1137/S0036142903427640

32. B. Rivière, Analysis of a discontinuous finite element method for the coupled Stokes and Darcy problems, *J. Sci. Comput.*, **22** (2005), 479–500. http://dx.doi.org/10.1007/s10915-004-4147-3

33. B. Rivière, *Analysis of a multi-numerics/multi-physics problem*, In: M. Feistauer, V. Doleší , P. Knobloch, K. Najzar, Numerical Mathematics and Advanced Applications, Berlin: Springer, 2004, 726–735. http://dx.doi.org/10.1007/978-3-642-18775-9_71

34. M. Discacciati, A. Quarteroni, A. Valli, Robin-Robin domain decomposition methods for the Stokes-Darcy coupling, *SIAM J. Numer. Anal.*, **45** (2007), 1246–1268. http://dx.doi.org/10.1137/06065091X

35. D. Han, Q. Wang, X. Wang, Dynamic transitions and bifurcations for thermal convection in the superposed free flow and porous media, *Physica D*, **414** (2020). http://dx.doi.org/10.1016/j.physd.2020.132687

36. X. Wang, H. Wu, Global weak solutions to the Navier-Stokes-Darcy-Boussinesq system for thermal convection in coupled free and porous media flows, *Adv. Differ. Equations*, 2021. http://dx.doi.org/10.57262/ade/1610420433

37. Y. Gao, D. Han, X. He, U. Rüde, Unconditionally stable numerical methods for Cahn-Hilliard-Navier-Stokes-Darcy system with different densities and viscosities, *J. Comput. Phys.*, **454** (2022). http://dx.doi.org/10.1016/j.jcp.2022.110968

38. W. Chen, D. Han, X. Wang, Y. Zhang, Uniquely solvable and energy stable decoupled numerical schemes for the Cahn-Hilliard-Navier-Stokes-Darcy-Boussinesq system, *J. Sci. Comput.*, **85** (2020). http://dx.doi.org/10.1007/s00211-017-0870-1

39. Y. Gao, X. He, L. Mei, X. Yang, Decoupled, linear, and energy stable finite element method for the Cahn-Hilliard-Navier-Stokes-Darcy phase field model, *SIAM J. Sci. Comput.*, **40** (2018), B110–B137. http://dx.doi.org/10.1137/16M1100885

40. D. Han, X. Wang, H. Wu, Existence and uniqueness of global weak solutions to a Cahn-Hilliard-Stokes-Darcy system for two phase incompressible flows in karstic geometry, *J. Differ. Equations*, **257** (2014), 3887–3933. http://dx.doi.org/10.1016/j.jde.2014.07.013

41. M. Cui, W. Dong, Z. Guo, Global well-posedness of coupled Navier-Stokes and Darcy equations, *J. Differ. Equations*, **388** (2024), 82–111. http://dx.doi.org/10.1016/j.jde.2023.12.044

42. P. Liu, W. Liu, *Global well-posedness of an initial-boundary value problem of the 2-D incompressible Navier-Stokes-Darcy system*, Netherland: Springer, **1** (2019). http://dx.doi.org/10.1007/s10440-018-0197-7

43. C. Foias, O. Manley, R. Temam, Attractors for the Bénard problem: Existence and physical bounds on their fractal dimension, *Nonlinear Anal.*, **11** (1987), 939–967. http://dx.doi.org/10.1016/0362-546X(87)90061-7

44. P. Fabrie, Solutions fortes et comportment asymtotique pour un modèle de convection naturelle en milieu poreux, *Acta Appl. Math.*, **7** (1986), 49–77. http://dx.doi.org/10.1007/BF00046977

45. H. V. Ly, E. S. Titi, Global Gevrey regularity for the Bénard convection in a porous medium with zero Darcy-Prandtl number, *J. Nonlinear Sci.*, **9** (1999), 333–362. http://dx.doi.org/10.1007/s003329900073

46. A. Çeşmelioğlu, B. Rivière, Existence of a weak solution for the fully coupled Navier-Stokes/Darcy-transport problem, *J. Differ. Equations*, **252** (2012), 4138–4175. http://dx.doi.org/10.1016/j.jde.2011.12.001

47. A. Çeşmelioğlu, V. Girault, B. Rivière, Time-dependent coupling of Navier-Stokes and Darcy flows, *ESAIM Math. Model. Num.*, **47** (2013), 540–555. http://dx.doi.org/10.1051/m2an/2012034

48. Y. Hou, D. Xue, Y. Jiang, On the weak solutions to steady-state mixed Navier-Stokes/Darcy model, *Acta Math. Sin.*, **39** (2023), 939–951. http://dx.doi.org/10.1007/s10114-022-9134-9

AIMS Press

© 2024 the Author(s), licensee AIMS Press. This is an open access article distributed under the terms of the Creative Commons Attribution License (http://creativecommons.org/licenses/by/4.0)