AIMS Mathematics

AIMS Mathematics, 9(3): 6951–6963.
DOI: 10.3934/math.2024339
Received: 28 November 2023
Revised: 15 January 2024
Accepted: 19 January 2024
Published: 19 February 2024

http://www.aimspress.com/journal/Math

Research article

Simultaneous blow-up of the solution for a singular parabolic system with concentrated sources

Wai Yuen Chan*

Department of MIS, Computer Science, and Mathematics, Texas A&M University-Texarkana, Texarkana, TX 75503, USA

* **Correspondence:** Email: wychan@tamut.edu.

Abstract: A singular parabolic system with homogeneous Dirichlet boundary conditions and concentrated nonlinear reaction sources was examined. This paper investigated the existence and uniqueness of the solution. The sufficient conditions for the solution being blown up simultaneously in a finite time were determined. In addition, it was shown that the solution blew up everywhere in the domain except boundary points.

Keywords: blow-up; Dirac delta function; Green's function; singular parabolic problem
Mathematics Subject Classification: 35K35, 35K55, 35K57, 35K58, 35K61

1. Introduction

Let b be a constant such that $b \in (0,1)$, L is the singular parabolic operator such that $Lu = u_t - u_{xx} - bu_x/x$, T is a positive real number, $p \geqslant 1$, $q \geqslant 1$, and $\delta(x)$ is the Dirac delta function. In this paper, we investigate the following system of singular parabolic initial-boundary value problems with concentrated nonlinear source functions:

$$\begin{cases} Lu = \delta(x-b)v^p, & Lv = \delta(x-b)u^q, & \text{in } (0,1)\times(0,T), \\ u(x,0) = u_0(x), & v(x,0) = v_0(x), & \text{for } x \in [0,1], \\ u(0,t) = u(1,t) = 0, & v(0,t) = v(1,t) = 0, & \text{for } t \in (0,T). \end{cases} \tag{1.1}$$

It is assumed that the initial conditions $u_0(x)$ and $v_0(x)$ are nonnegative functions on $[0,1]$ such that $u_0(b)$ and $v_0(b)$ are positive, and they satisfy conditions below:

$$\begin{cases}\frac{b(u_0)_x}{x}+(u_0)_{xx}+\delta(x-b)v_0^p\geqslant 0 \text{ in } (0,1),\\ \frac{b(v_0)_x}{x}+(v_0)_{xx}+\delta(x-b)u_0^q\geqslant 0 \text{ in } (0,1).\end{cases} \tag{1.2}$$

The singular operator L is associated with axially symmetric solutions to the heat equation [1]. This operator is also related to axially symmetric potentials of elliptic equations [2–5] when $u_t = 0$. If $b=(N-1)/2$, from [6], Lu represents the backward differential equation corresponding to the radial component of N-dimensional Brownian motion. In addition, Lu can describe a stochastic process of a chain of random walks [7]. Further, $Lu=0$ is interpreted as the Fokker-Planck equation of a diffusion problem in which b represents the drift [8]. Feller [8] discussed the existence and uniqueness of solution of $Lu=0$ over the domain $(0,\infty)$, Colton [9] investigated the Cauchy problem of $Lu=0$, and Brezis et al. [6] studied the differentiability of solution. In addition, Alexiades [10] investigated the existence and uniqueness of solution over a domain with moving boundary for $Lu=f(x,t)$. Chan and Chen [11] studied the singular quenching problem below:

$$Lu=\frac{1}{1-u}. \tag{1.3}$$

To a given b, they determined the critical length of u numerically through evaluating the approximated solution of the integral solution of (1.3).

For $x\in\mathbb{R}^N$, Escobedo and Herrero [12] studied the blow-up of the Cauchy problem of the following semilinear parabolic system:

$$u_t-\Delta u=v^p,$$
$$v_t-\Delta v=u^q.$$

They proved that u and v blow up in a finite time when $pq>1$ and $(\max\{p,q\}+1)/(pq-1)\geqslant N/2$. Levine [13] obtained a similar result for the homogeneous Dirichlet boundary-value problem.

In this paper, we prove the existence and uniqueness of solutions u and v. Under some assumptions on $u_0(x)$ and $v_0(x)$, we show that they blow up in a finite time if $p\geqslant 1$, $q\geqslant 1$, and $pq>1$. We further prove that u and v both blow up simultaneously at $x=b$, then we obtain the result that u and v both blow up totally in $(0,1)$.

2. Existence and uniqueness of the solution

Let $G(x,\xi,t-s)$ be the Green's function of $Lu=0$. For any x and $\xi\in[0,1]$, where t and s are variables and they belong to $(-\infty,\infty)$, $G(x,\xi,t-s)$ satisfies the problem below:

$$LG=\delta(t-s)\delta(x-\xi) \text{ for } s<t, x\in(0,1),$$
$$G(x,\xi,t-s)=0 \text{ for } t<s, G(0,\xi,t-s)=G(1,\xi,t-s)=0.$$

By Chan and Wong [14], the Green's function of the operator L is given by

$$G(x,\xi,t-s)=\sum_{n=1}^{\infty}\xi^b\,\phi_n(\xi)\phi_n(x)e^{-\lambda_n(t-s)}, \tag{2.1}$$

where λ_n for $n=1,2,3,\ldots$ are the eigenvalues of the Sturm-Liouville problem

$$\frac{d^2\Phi}{dx^2}+\frac{b}{x}\frac{d\Phi}{dx}+\mu\Phi=0,\ \Phi(0)=0=\Phi(1), \tag{2.2}$$

where μ is an unknown constant. Let $\phi_n(x)$ be the corresponding eigenfunction of λ_n. The mathematical formula of $\phi_n(x)$ is

$$\phi_n(x) = \frac{2^{1/2} x^\nu J_\nu\left(\lambda_n^{1/2} x\right)}{\left|J_{\nu+1}\left(\lambda_n^{1/2}\right)\right|},$$

where J_ν is the Bessel function of the first kind of order $\nu = (1-b)/2$. The first eigenvalue λ_1 of (2.2) is positive and $\phi_1(x)$ satisfies: $\phi_1(x) > 0$ in (0,1) and $\int_0^1 x^b \phi_1 dx = 1$. For ease of computation, let us state the result of Lemma 1 of Chan and Wong.

Lemma 2.1. *There exist positive constants* k_1 *and* k_2 *such that*
(a) $|\phi_n(x)| \leqslant k_1 x^{-b/2}$ *for* x *in* $(0,1]$;
(b) $|\phi_n(x)| \leqslant k_2 \lambda_n^{1/4}$ *for* x *on* $[0,1]$.

By theorems of the Bessel-Fourier series expansion of Watson [15, pp. 591 and 594], $\int_0^1 G(x,\xi,t) u_0(\xi) d\xi$ converges to $u_0(x)$ for $x \in [0,1]$ when $t \to 0$. By the result of Chan and Wong, the integral solution of (1.1) is given by

$$\begin{cases} u(x,t) = \int_0^1 G(x,\xi,t) u_0(\xi) d\xi + \int_0^t G(x,b,t-s) v^p(b,s) ds, \\ v(x,t) = \int_0^1 G(x,\xi,t) v_0(\xi) d\xi + \int_0^t G(x,b,t-s) u^q(b,s) ds. \end{cases} \tag{2.3}$$

We let

$$H(x,t,u_0,v) = \int_0^1 G(x,\xi,t) u_0(\xi) d\xi + \int_0^t G(x,b,t-s) v^p(b,s) ds,$$

$$I(x,t,v_0,u) = \int_0^1 G(x,\xi,t) v_0(\xi) d\xi + \int_0^t G(x,b,t-s) u^q(b,s) ds,$$

then the integral solution represented by (2.3) is equivalent to

$$\begin{cases} u(x,t) = H(x,t,u_0,v), \\ v(x,t) = I(x,t,v_0,u). \end{cases} \tag{2.4}$$

In the following, we show that the integral solution represented by (2.4) is a unique continuous solution of (1.1). To achieve this, let us construct two sequences: $\{u_m\}$ and $\{v_m\}$; they satisfy the iteration process

$$\begin{cases} Lu_m = \delta(x-b) v_{m-1}^p, & Lv_m = \delta(x-b) u_{m-1}^q, & \text{in } (0,1)\times(0,T), \\ u_m(x,0) = u_0(x), & v_m(x,0) = v_0(x), & \text{for } x \in [0,1], \\ u_m(0,t) = u_m(1,t) = 0, & v_m(0,t) = v_m(1,t) = 0, & \text{for } t \in (0,T), \end{cases} \tag{2.5}$$

for $m = 1,2,\ldots$ We assume the initial iteration: $u_0(x,t) = u_0(x)$ and $v_0(x,t) = v_0(x)$ on $[0,1]\times[0,T)$. By (2.4), the integral solution of $\{u_m\}$ and $\{v_m\}$ in (2.5) is given by

$$\begin{cases} u_m(x,t) = H(x,t,u_0,v_{m-1}), \\ v_m(x,t) = I(x,t,v_0,u_{m-1}). \end{cases} \tag{2.6}$$

By Lemma 4 of Chan and Wong, $G(x,\xi,t-s)$ is a positive function in the set

$\{(x,\xi;t,s)$: x and ξ are in $(0,1)$, and $t > s \geqslant 0\}$. It indicates that $u_m > 0$ and $v_m > 0$ in $(0,1) \times (0,T)$. The following lemma shows that $\{u_m\}$ and $\{v_m\}$ are nondecreasing sequences and they are continuous functions.

Lemma 2.2. *The sequences* $\{u_m\}$ *and* $\{v_m\}$ *described by (2.6) are monotone nondecreasing:* $u_0(x) \leqslant u_1(x,t) \leqslant \cdots \leqslant u_m(x,t) \leqslant \cdots, v_0(x) \leqslant v_1(x,t) \leqslant \cdots \leqslant v_m(x,t) \leqslant \cdots$, *and* $\{u_m\}$ *and* $\{v_m\}$ *are continuous on* $[0,1] \times [0,T)$.

Proof. From (1.2), we deduce $H(x,t,u_0,v_0) \geqslant 0$ and $I(x,t,v_0,u_0) \geqslant 0$, then by (2.6) with $m = 1$, we obtain

$$\begin{cases} u_1(x,t) - u_0(x) \geqslant H(x,t,u_0,v_0) - H(x,t,u_0,v_0) = 0, \\ v_1(x,t) - v_0(x) \geqslant I(x,t,v_0,u_0) - I(x,t,v_0,u_0) = 0, \end{cases}$$

for $x \in [0,1]$. Therefore, $u_1(x,t) - u_0(x) \geqslant 0$ and $v_1(x,t) - v_0(x) \geqslant 0$ for $x \in [0,1]$. Suppose that it is true for $m = k$; it gives $u_k(x,t) - u_{k-1}(x,t) \geqslant 0$ and $v_k(x,t) - v_{k-1}(x,t) \geqslant 0$ for $(x,t) \in [0,1] \times [0,T)$. Suppose $m = k + 1$, and $G\,(x,\xi,t-s)$ is a positive function for $t > s$

$$u_{k+1}(x,t) - u_k(x,t) = H(x,t,u_0,v_k) - H(x,t,u_0,v_{k-1}) \geqslant 0.$$

Similarly, we have $v_{k+1}(x,t) - v_k(x,t) \geqslant 0$ for $(x,t) \in [0,1] \times [0,T)$. By the mathematical induction, the sequences of solutions $\{u_m\}$ and $\{v_m\}$ are monotone nondecreasing.

To prove that $u_m(x,t)$ and $v_m(x,t)$ are continuous on $[0,1] \times [0,T)$, let us consider (2.6) with $m = 1$. It yields

$$u_1(x,t) = H(x,t,u_0,v_0) = \int_0^1 G(x,\xi,t)u_0(\xi)d\xi + \int_0^t G(x,b,t-s)v_0^p(b,s)ds.$$

We show that $G(x,\xi,t)u_0(\xi)$ and $G(x,b,t-s)v_0^p(b,s)$ are integrable. Let us prove that $\int_0^t G(x,b,t-s)v_0^p(b,s)ds$ is bounded for $x \in [0,1]$ and $t > 0$. By (2.1), we have

$$\int_0^t G(x,b,t-s)v_0^p(b,s)ds \leqslant \sum_{n=1}^{\infty} |\phi_n(b)||\phi_n(x)| \int_0^t e^{-\lambda_n(t-s)} v_0^p(b)ds.$$

Let $k_3 = \max\{max_{x\in[0,1]} v_0(x), max_{x\in[0,1]} u_0(x)\}$, then

$$\int_0^t G(x,b,t-s)v_0^p(b,s)ds \leqslant k_3^p \sum_{n=1}^{\infty} |\phi_n(b)||\phi_n(x)| \frac{\left(1 - e^{-\lambda_n t}\right)}{\lambda_n}.$$

By Lemma 2.1, it gives

$$\int_0^t G(x,b,t-s)v_0^p(b,s)ds \leqslant k_3^p k_1 k_2 b^{-b/2} \sum_{n=1}^{\infty} \frac{\left(1-e^{-\lambda_n t}\right)}{\lambda_n^{3/4}}. \tag{2.7}$$

By Watson [15, p. 506], $O(\lambda_n) = O(n^2)$ for n large. This shows that the above series converges for $t > 0$. The upper bound of the series is independent of x for each fixed $t > 0$. For each positive integer j greater than 1, the finite series $\sum_{n=1}^{j} \int_0^t b^b \phi_n(b)\phi_n(x)e^{-\lambda_n(t-s)}\, v_0^p(b,s)\, ds$ is continuous for $x \in [0,1]$ and $t > 0$. Further, by (2.7), this finite sum satisfies the following inequality:

$$\sum_{n=1}^{j}\int_0^t b^b\phi_n(b)\phi_n(x)e^{-\lambda_n(t-s)}\,v_0^p(b,s)\,ds \leqslant k_3^p k_1 k_2 b^{-b/2}\sum_{n=1}^{\infty}\frac{\left(1-e^{-\lambda_n t}\right)}{\lambda_n^{3/4}}.$$

Therefore, $\sum_{n=1}^{j}\int_0^t b^b\phi_n(b)\phi_n(x)e^{-\lambda_n(t-s)}\,v_0^p(b,s)\,ds$ converges uniformly. As $j\to\infty$, $\int_0^t G(x,b,t-s)v_0^p(b,s)ds$ is continuous for $x\in[0,1]$ and $t>0$. Through a similar computation, it yields

$$\int_0^1 G(x,\xi,t)u_0(\xi)d\xi \leqslant k_3\sum_{n=1}^{\infty}|\phi_n(x)|\,e^{-\lambda_n t}\int_0^1\phi_n(\xi)d\xi.$$

By Lemma 2.1(a) and $\phi_n(0)=\phi_n(1)=0$, the righthand series converges for $x\in[0,1]$ when $t>0$. By the similar computation above, we obtain $\int_0^1 G(x,\xi,t)u_0(\xi)d\xi$ as continuous for $x\in[0,1]$ and $t>0$. By (2.7), $\int_0^t G(x,b,t-s)v_0^p(b,s)ds\to 0$ as $t\to 0$. Since $\int_0^1 G(x,\xi,t)u_0(\xi)d\xi$ converges to $u_0(x)$ for $x\in[0,1]$ when $t\to 0$, it follows that $\lim_{t\to 0}u_1(x,t)=u_0(x)$. Therefore, $u_1(x,t)=H(x,t,u_0,v_0)$ is continuous on $[0,1]\times[0,T)$. By the mathematical induction and computation above, $u_m(x,t)$ is continuous on $[0,1]\times[0,T)$ for $m=1,2,\ldots$ A similar computation proves that $v_m(x,t)$ is continuous on $[0,1]\times[0,T)$ for $m=1,2,\ldots$ This completes the proof.

Let h be a positive real number less than T. We show that sequences $\{(u_m)_t\}$ and $\{(v_m)_t\}$ are nonnegative functions over $[0,1]$.

Lemma 2.3. *$(u_m)_t\geqslant 0$ and $(v_m)_t\geqslant 0$ on $[0,1]\times(0,T)$ for $m=1,2,\ldots$*

Proof. To establish this result, we prove that $u_m(x,t+h)\geqslant u_m(x,t)$ on $[0,1]\times[0,T-h]$ through the mathematical induction. When $m=1$, we follow Lemma 2.2 to get $u_1(x,h)\geqslant u_0(x)$ and $v_1(x,h)\geqslant v_0(x)$. From (2.5), we deduce

$$\begin{cases} Lu_1(x,t+h)=\delta(x-b)v_0^p(x) \text{ in } (0,1)\times(0,T-h],\\ u_1(x,h)=u_1(x,h) \text{ for } x\in[0,1],\\ u_1(0,t+h)=u_1(1,t+h)=0 \text{ for } t\in(0,T-h],\end{cases}$$

and

$$\begin{cases} Lv_1(x,t+h)=\delta(x-b)u_0^q(x) \text{ in } (0,1)\times(0,T-h],\\ v_1(x,h)=v_1(x,h) \text{ for } x\in[0,1],\\ v_1(0,t+h)=v_1(1,t+h)=0 \text{ for } t\in(0,T-h].\end{cases}$$

When (2.5) is subtracted from the above two problems, we deduce

$$\begin{cases} L(u_1(x,t+h)-u_1(x,t))=\delta(x-b)(v_0^p(x)-v_0^p(x)) \text{ in } (0,1)\times(0,T-h],\\ u_1(x,h)-u_1(x,0)=u_1(x,h)-u_0(x)\geqslant 0 \text{ for } x\in[0,1],\\ u_1(0,t+h)-u_1(0,t)=u_1(1,t+h)-u_1(1,t)=0 \text{ for } t\in(0,T-h],\end{cases}$$

and

$$\begin{cases} L(v_1(x,t+h)-v_1(x,t))=\delta(x-b)(u_0^q(x)-u_0^q(x)) \text{ in } (0,1)\times(0,T-h],\\ v_1(x,h)-v_1(x,0)=v_1(x,h)-v_0(x)\geqslant 0 \text{ for } x\in[0,1],\\ v_1(0,t+h)-v_1(0,t)=v_1(1,t+h)-v_1(1,t)=0 \text{ for } t\in(0,T-h].\end{cases}$$

By (2.3),

$$\begin{cases} u_1(x,t+h) - u_1(x,t) = \int_0^1 G(x,\xi,t)(u_1(\xi,h) - u_0(\xi))d\xi \geqslant 0, \\ v_1(x,t+h) - v_1(x,t) = \int_0^1 G(x,\xi,t)(v_1(\xi,h) - v_0(\xi))d\xi \geqslant 0. \end{cases}$$

Therefore, $u_1(x,t+h) \geqslant u_1(x,t)$ and $v_1(x,t+h) \geqslant v_1(x,t)$ on $[0,1] \times [0,T-h]$. As $h \to 0$, it implies $(u_1)_t \geqslant 0$ and $(v_1)_t \geqslant 0$ over $[0,1] \times (0,T)$. Assume that this statement is true for $m = k$. When $m = k+1$, it follows a similar computation and Lemma 2.2 to achieve

$$u_{k+1}(x,t+h) - u_{k+1}(x,t) = \int_0^1 G(x,\xi,t)\big(u_{k+1}(\xi,h) - u_0(\xi)\big)d\xi$$

$$+ \int_0^t G(x,b,t-s)\left(v_k^p(b,s+h) - v_k^p(b,s)\right)ds \geqslant 0.$$

Thus, $u_{k+1}(x,t+h) \geqslant u_{k+1}(x,t)$. Similarly, we obtain $v_{k+1}(x,t+h) \geqslant v_{k+1}(x,t)$. When $h \to 0$, $(u_{k+1})_t \geqslant 0$ and $(v_{k+1})_t \geqslant 0$ over $[0,1] \times (0,T)$. By the mathematical induction, we conclude that $(u_m)_t \geqslant 0$ and $(v_m)_t \geqslant 0$ on $[0,1] \times (0,T)$ for $m = 1,2,\ldots$ This completes the proof.

Lemma 2.4. *Given* $\bar{t} > 0$, *there exist positive constants* α_1 and α_2 *(depending on* $\bar{t}$*) both less than* 1 *such that*

$$max_{[0,1]\times[0,\bar{t}]}|H(x,t,u_0,v_m) - H(x,t,u_0,v_{m-1})| < \alpha_1\, max_{[0,1]\times[0,\bar{t}]}|v_m - v_{m-1}|,$$

and

$$max_{[0,1]\times[0,\bar{t}]}|I(x,t,v_0,u_m) - I(x,t,v_0,u_{m-1})| < \alpha_2\, max_{[0,1]\times[0,\bar{t}]}|u_m - u_{m-1}|.$$

Proof. By (2.6), we get

$$|H(x,t,u_0,v_m) - H(x,t,u_0,v_{m-1})| \leqslant \int_0^t G(x,b,t-s)\left|v_m^p(b,s) - v_{m-1}^p(b,s)\right|ds.$$

Since $p \geqslant 1$, the mean value theorem, and Lemma 2.2, there exists a positive constant k_4 such that

$$\left|v_m^p(b,s) - v_{m-1}^p(b,s)\right| \leqslant pk_4^{p-1}\max_{[0,1]\times[0,t]}|v_m - v_{m-1}|,$$

for $s \in [0,t]$. Based on the above inequality, we obtain the expression

$$|H(x,t,u_0,v_m) - H(x,t,u_0,v_{m-1})| \leqslant pk_4^{p-1}\max_{[0,1]\times[0,t]}|v_m - v_{m-1}| \int_0^t G(x,b,t-s)ds.$$

We follow a similar method of Lemma 2.2 to obtain an upper bound of the integral of the Green's function

$$\int_0^t G(x,b,t-s)ds \leqslant \sum_{n=1}^{\infty}|\phi_n(b)||\phi_n(x)|\frac{\left(1-e^{-\lambda_n t}\right)}{\lambda_n} \leqslant k_1k_2b^{-b/2}\sum_{n=1}^{\infty}\frac{\left(1-e^{-\lambda_n t}\right)}{\lambda_n^{3/4}}.$$

The righthand series converges and tends to zero for positive t being close to 0. Let us choose $t=\bar{t}$ sufficiently close to 0 such that

$$\alpha_1 = pk_4^{p-1}k_1k_2b^{-b/2}\sum_{n=1}^{\infty}\frac{\left(1-e^{-\lambda_n \bar{t}}\right)}{\lambda_n^{3/4}} < 1.$$

Hence, $\max_{[0,1]\times[0,\bar{t}]}|H(x,t,u_0,v_m)-H(x,t,u_0,v_{m-1})| < \alpha_1\max_{[0,1]\times[0,\bar{t}]}|v_m-v_{m-1}|$. Similarly, we have $\alpha_2<1$ such that $\max_{[0,1]\times[0,\bar{t}]}|I(x,t,v_0,u_m)-I(x,t,v_0,u_{m-1})| < \alpha_2\max_{[0,1]\times[0,\bar{t}]}|u_m-u_{m-1}|$. The proof is complete.

Let k_5 and k_6 be positive constants greater than k_3. We prove that $\{u_m\}$ and $\{v_m\}$ are bounded above by k_5 and k_6, respectively, on $[0,1]\times[0,\bar{t}]$.

Lemma 2.5. *For any* $k_5>k_3$ *and* $k_6>k_3$, *there exists* $\bar{t}>0$ *such that* $k_5>u_m$ *and* $k_6>v_m$ *on* $[0,1]\times[0,\bar{t}]$ *for* $m=1,2,...$

Proof. Based on Lemma 2.2, we know that $\int_0^1 G(x,\xi,t)u_0(\xi)d\xi$, $\int_0^t G(x,b,t-s)v_{m-1}^p(b,s)ds$, $\int_0^1 G(x,\xi,t)v_0(\xi)d\xi$, and $\int_0^t G(x,b,t-s)u_{m-1}^q(b,s)ds$ are bounded above when $m=1,2,...$ By Lemma 2.2 again, $\lim_{t\to 0}\int_0^1 G(x,\xi,t)v_0(\xi)d\xi = v_0(x)$ and $\lim_{t\to 0}\int_0^1 G(x,\xi,t)u_0(\xi)d\xi = u_0(x)$ on $[0,1]$, and $k_5>k_3$ and $k_6>k_3$. Thus, there exists $\bar{t}>0$ such that

$$k_5 > \int_0^1 G(x,\xi,\bar{t})k_3d\xi + \int_0^{\bar{t}} G(x,b,\bar{t}-s)k_6^p ds \geqslant H(x,\bar{t},u_0,v_0) = u_1(x,\bar{t}),$$

$$k_6 > \int_0^1 G(x,\xi,\bar{t})k_3d\xi + \int_0^{\bar{t}} G(x,b,\bar{t}-s)k_5^q ds \geqslant I(x,\bar{t},v_0,u_0) = v_1(x,\bar{t}),$$

for $x\in[0,1]$. By Lemma 2.3, $(u_m)_t\geqslant 0$ and $(v_m)_t\geqslant 0$, which leads to $k_5>u_1$ and $k_6>v_1$ on $[0,1]\times[0,\bar{t}]$. Assume that it is true for $m=i$, we follow a similar calculation to obtain $k_5>u_{i+1}$ and $k_6>v_{i+1}$ on $[0,1]\times[0,\bar{t}]$. By the mathematical induction, we have $k_5>u_m$ and $k_6>v_m$ on $[0,1]\times[0,\bar{t}]$ for $m=1,2,...$

Let $u(x,t)=\lim_{m\to\infty}u_m(x,t)$ and $v(x,t)=\lim_{m\to\infty}v_m(x,t)$. We have the result below.

Theorem 2.6. *The integral equation (2.3) has a unique continuous solution* $k_5>u(x,t)\geqslant u_0(x)$ *and* $k_6>v(x,t)\geqslant v_0(x)$ *on* $[0,1]\times[0,\bar{t}]$.

Proof. Based on Lemma 2.2 and Dini's theorem, the sequences $\{u_m\}$ and $\{v_m\}$ converge uniformly to continuous solutions $u(x,t)$ and $v(x,t)$ such that $k_5>u(x,t)\geqslant u_0(x)$ and $k_6>v(x,t)\geqslant v_0(x)$ on $[0,1]\times[0,\bar{t}]$. By Lemma 2.4, $u(x,t)$ and $v(x,t)$ are unique. The proof is complete.

Based on Lemma 2.3, we obtain that u_t and v_t are nonnegative.

Lemma 2.7. $u_t\geqslant 0$ *and* $v_t\geqslant 0$ *on* $[0,1]\times(0,\bar{t}]$.

Let t_b be the supremum of $\bar{t}$ such that the integral solution (2.3) has a unique continuous solution on $[0,1]\times[0,\bar{t}]$. We follow Theorem 3 of Chan [16] to obtain the result below.

Theorem 2.8. *If* t_b *is finite, then* $u(x,t)$ *and* $v(x,t)$ *are unbounded somewhere on* $[0,1]$ *when* $t\to t_b$.

3. Blow-up of the solution

In this section, let us assume that the initial data $u_0(x)$ and $v_0(x)$ both attain positive maximum at $x = b$ only. By Theorem 2.6, $u(b,t) > 0$ and $v(b,t) > 0$ for $t > 0$. Our first result is to prove that u and v both reach their maximum at $x = b$, then we show that u and v blow up in a finite time.

Lemma 3.1. *u and v both attain their maximum at $x = b$ for $t \geqslant 0$.*

Proof. We prove this result using contradiction. From (1.1), it gives

$$\begin{aligned} Lu &= \delta(x-b)v^p, && \text{in } (0,1)\times(0,T),\\ u(x,0) &= u_0(x), && \text{for } x \in [0,1],\\ u(0,t) &= u(1,t) = 0, && \text{for } t \in (0,T). \end{aligned}$$

Let $t = t_1$ be the infimum of t such that $u(x,t)$ attains another maximum at x_1 for some $x_1 \in (0,b) \cup (b,1)$. As $u(x,0) = u_0(x)$ reaches its positive maximum at $x = b$, $t_1 > 0$, then $u_x(x_1,t_1) = 0$ and $u_{xx}(x_1,t_1) < 0$. By Lemma 2.7, it yields $u_t(x_1,t_1) \geqslant 0$. Therefore, $Lu(x_1,t_1) > 0$. It contradicts $Lu = 0$ at $x \neq b$ for $t > 0$. Thus, $u(x,t)$ attains its maximum at $x = b$ for $t \geqslant 0$. Through a similar computation, we prove that $v(x,t)$ attains its maximum at $x = b$ for $t \geqslant 0$. This completes the proof.

We multiply (1.1) by $x^b\phi_1(x)$, and then integrate the expression with respect to x over $[0,1]$,

$$\int_0^1 x^b\phi_1 u_t\,dx - \int_0^1 x^b\phi_1\left(\frac{b}{x}u_x + u_{xx}\right)dx = \int_0^1 \delta(x-b)x^b\phi_1 v^p\,dx,$$

$$\int_0^1 x^b\phi_1 v_t\,dx - \int_0^1 x^b\phi_1\left(\frac{b}{x}v_x + v_{xx}\right)dx = \int_0^1 \delta(x-b)x^b\phi_1 u^q\,dx.$$

Using integration by parts and (2.2), it yields

$$\left(\int_0^1 x^b\phi_1 u\,dx\right)_t + \lambda_1\left(\int_0^1 x^b\phi_1 u\,dx\right) = b^b\phi_1(b)v^p(b,t), \tag{3.1}$$

$$\left(\int_0^1 x^b\phi_1 v\,dx\right)_t + \lambda_1\left(\int_0^1 x^b\phi_1 v\,dx\right) = b^b\phi_1(b)u^q(b,t). \tag{3.2}$$

By Lemma 3.1, $u(x,t) \leqslant u(b,t)$ and $v(x,t) \leqslant v(b,t)$ on $[0,1]$ for $t \geqslant 0$, we have

$$\left(\int_0^1 x^b\phi_1 u\,dx\right)^q \leqslant u^q(b,t) \text{ and } \left(\int_0^1 x^b\phi_1 v\,dx\right)^p \leqslant v^p(b,t).$$

Let $k_7 = b^b\phi_1(b)$. The system of differential equations of (3.1) and (3.2) becomes

$$\left(\int_0^1 x^b\phi_1 u\,dx\right)_t + \lambda_1\left(\int_0^1 x^b\phi_1 u\,dx\right) \geqslant k_7\left(\int_0^1 x^b\phi_1 v\,dx\right)^p, \tag{3.3}$$

$$\left(\int_0^1 x^b\phi_1 v\,dx\right)_t + \lambda_1\left(\int_0^1 x^b\phi_1 v\,dx\right) \geqslant k_7\left(\int_0^1 x^b\phi_1 u\,dx\right)^q. \tag{3.4}$$

Let $\Psi(t) = \int_0^1 x^b\phi_1 u\,dx$ and $\Lambda(t) = \int_0^1 x^b\phi_1 v\,dx$, then (3.3) and (3.4) become

$$\begin{cases}\Psi' + \lambda_1\Psi \geqslant k_7\Lambda^p,\\ \Lambda' + \lambda_1\Lambda \geqslant k_7\Psi^q.\end{cases}$$

Let $\Gamma > 0$ and B be a positive constant such that

$$B > \max\{[k_7^2(p-1)/(2\lambda_1)]^{1/(p-1)}, \Gamma^{-1/(p-1)}/\Lambda(0), \Gamma^{-p/(p-1)}/[\Lambda'(0)(p-1)]\}. \quad (3.5)$$

To obtain the result if either u or v blows up somewhere on $[0,1]$ in a finite time, we study the blow-up of system of differential inequalities (3.3) and (3.4) when $p \geqslant 1$, $q \geqslant 1$, and $pq > 1$. We examine (3.3) and (3.4) in two different cases: (I) $q = 1$, $p > 1$, or $p = 1$, $q > 1$; (II) q and $p > 1$. To case (I), we prove either u or v to blow up in a finite time through constructing a lower solution. We then compare the solutions of case (I) with case (II) to show either u or v to blow up in a finite time by the Picard iterates.

Lemma 3.2. *(I) Suppose that* $q = 1$ *and* $p > 1$, *or* $p = 1$ *and* $q > 1$, *then either* u *or* v *blows up somewhere on* $[0,1]$ *in a finite time.*

(II) Suppose that $p > 1$ *and* $q > 1$, *and assume that*

(a) $(\Lambda(0))^p k_7/\lambda_1 \geqslant \Psi(0)$ *and* $(\Psi(0))^q k_7/\lambda_1 \geqslant \Lambda(0)$,

(b) $\Psi(0) \geqslant 1$ *or* $\Lambda(0) \geqslant 1$,

then either u *or* v blows *up somewhere on* $[0,1]$ *in a finite time.*

Proof. Case (I): Suppose $q = 1$ and $p > 1$ and let $U(t) = \int_0^1 x^b\phi_1 u\, dx$ and $V(t) = \int_0^1 x^b\phi_1 v\, dx$. Let us consider the following system of ordinary differential equations:

$$\begin{cases}U' + \lambda_1 U = k_7 V^p,\\ V' + \lambda_1 V = k_7 U,\end{cases} \quad (3.6)$$

with the initial conditions: $U(0) = \Psi(0)$ and $V(0) = \Lambda(0)$. We differentiate the second equation of (3.6) with respect to t and then substitute it into the first one to obtain

$$\frac{1}{k_7}V'' + \frac{2\lambda_1}{k_7}V' + \frac{\lambda_1^2}{k_7}V = k_7V^p,$$

with the initial conditions: $V(0) = \Lambda(0) > 0$ and $V'(0) = \Lambda'(0) > 0$. For $t \in [0,\Gamma)$, we construct a lower solution of the above equation as $E(t) = (\Gamma - t)^{-1/(p-1)}/B$, where B satisfies $V(0) > \Gamma^{-1/(p-1)}/B = E(0)$ and $V'(0) > \Gamma^{-p/(p-1)}/[B\,(p-1)]$. The first and second derivatives of $E(t)$ are

$$E'(t) = \frac{1}{B(p-1)(\Gamma-t)^{p/(p-1)}} \text{ and } E''(t) = \frac{p}{B(p-1)^2(\Gamma-t)^{(2p-1)/(p-1)}}.$$

We evaluate the expression $E''/k_7 + 2\lambda_1E'/k_7 + \lambda_1^2E/k_7 - k_7E^p$ to have

$$\frac{E''}{k_7} + \frac{2\lambda_1}{k_7}E' + \frac{\lambda_1^2}{k_7}E - k_7E^p$$

$$= \frac{p}{k_7B(p-1)^2(\Gamma-t)^{\frac{2p-1}{p-1}}} + \frac{1}{B(\Gamma-t)^{\frac{p}{p-1}}}\left[\frac{2\lambda_1}{k_7(p-1)} - \frac{k_7}{B^{p-1}}\right] + \frac{\lambda_1^2}{Bk_7(\Gamma-t)^{\frac{1}{p-1}}}.$$

By (3.5), we have $E''/k_7 + 2\lambda_1 E'/k_7 + \lambda_1^2 E/k_7 - k_7 E^p > 0$, then by the comparison theorem [17, p. 96], we have $V(t) \geqslant E(t)$ for $t \in [0,\Gamma)$. As $E(t) \to \infty$ when $t \to \Gamma$, $V(t)$ blows up in a finite time less than or equal to Γ. Thus, $U + V$ blows up in a finite time. By the comparison theorem, $\Psi + \Lambda$ blows up in a finite time. Hence, either u or v in (3.3) and (3.4) blows up on $[0,1]$ in a finite time. We follow a similar computation to obtain the same result when $p = 1$ and $q > 1$, with p and $\Lambda(0)$ being replaced by q and $\Psi(0)$, respectively, in (3.5).

Case (II): Suppose that $p > 1$ and $q > 1$. Let $W(t) = \int_0^1 x^b \phi_1 u \, dx$ and $Y(t) = \int_0^1 x^b \phi_1 v \, dx$ be the solution of the following system of ordinary differential equations:

$$\begin{cases} W' + \lambda_1 W = k_7 Y^p, \\ Y' + \lambda_1 Y = k_7 W^q, \end{cases}$$

with initial conditions: $W(0) = \Psi(0)$ and $Y(0) = \Lambda(0)$. Multiplying the integrating factor $e^{\lambda_1 t}$ on both sides, this system becomes

$$\frac{d}{dt} e^{\lambda_1 t} W = k_7 e^{\lambda_1 t} Y^p \text{ and } \frac{d}{dt} e^{\lambda_1 t} Y = k_7 e^{\lambda_1 t} W^q.$$

Using the Picard iterates, the approximated solutions of W and Y are given by

$$W_m(t) = e^{-\lambda_1 t} \left[W(0) + k_7 \int_0^t e^{\lambda_1 s} (Y_{m-1})^p \, ds \right], \tag{3.7}$$

$$Y_m(t) = e^{-\lambda_1 t} \left[Y(0) + k_7 \int_0^t e^{\lambda_1 s} (W_{m-1})^q \, ds \right], \tag{3.8}$$

with $W_0 = W(0)$ and $Y_0 = Y(0)$, $m = 1, 2, \dots$ By assumption (a), it yields $W(0) \leqslant W_1(t) \leqslant W_2(t) \dots \leqslant W_m(t) \leqslant \cdots$ and $Y(0) \leqslant Y_1(t) \leqslant Y_2(t) \dots \leqslant Y_m(t) \leqslant \cdots$ Similarly, the representation of the approximated solutions of U and V are

$$U_m(t) = e^{-\lambda_1 t} \left[U(0) + k_7 \int_0^t e^{\lambda_1 s} (V_{m-1})^p \, ds \right], \tag{3.9}$$

$$V_m(t) = e^{-\lambda_1 t} \left[V(0) + k_7 \int_0^t e^{\lambda_1 s} U_{m-1} \, ds \right], \tag{3.10}$$

with $U_0 = U(0)$ and $V_0 = V(0)$, $m = 1, 2, \dots$ The system (3.9) and (3.10) is subtracted from (3.7) and (3.8) to yield

$$W_m(t) - U_m(t) = e^{-\lambda_1 t} k_7 \int_0^t e^{\lambda_1 s} [(Y_{m-1})^p - (V_{m-1})^p] \, ds,$$

$$Y_m(t) - V_m(t) = e^{-\lambda_1 t} k_7 \int_0^t e^{\lambda_1 s} [(W_{m-1})^q - U_{m-1}] \, ds.$$

We show that $W_m(t) \geqslant U_m(t)$ and $Y_m(t) \geqslant V_m(t)$ for $m = 1, 2, \dots$ through the mathematical induction. When $m = 1$,

$$W_1(t) - U_1(t) = e^{-\lambda_1 t} k_7 \int_0^t e^{\lambda_1 s} [(Y_0)^p - (V_0)^p] \, ds,$$

$$Y_1(t) - V_1(t) = e^{-\lambda_1 t} k_7 \int_0^t e^{\lambda_1 s}[(W_0)^q - U_0]\, ds = e^{-\lambda_1 t} k_7 \int_0^t e^{\lambda_1 s} U_0[(W_0)^{q-1} - 1]\, ds.$$

By assumption (b), $W_0 \geqslant 1$. It concludes that $W_1(t) = U_1(t)$ and $Y_1(t) \geqslant V_1(t)$. Assume that it is true for $m = i$, so that $W_i(t) \geqslant U_i(t)$ and $Y_i(t) \geqslant V_i(t)$. When $m = i + 1$, we have

$$W_{i+1}(t) - U_{i+1}(t) = e^{-\lambda_1 t} k_7 \int_0^t e^{\lambda_1 s}[(Y_i)^p - (V_i)^p]\, ds \geqslant 0,$$

$$Y_{i+1}(t) - V_{i+1}(t) = e^{-\lambda_1 t} k_7 \int_0^t e^{\lambda_1 s} U_i[(W_i)^{q-1} - 1]\, ds \geqslant 0.$$

Thus, $W_{i+1}(t) \geqslant U_{i+1}(t)$ and $Y_{i+1}(t) \geqslant V_{i+1}(t)$. By the mathematical induction, $W_m(t) \geqslant U_m(t)$ and $Y_m(t) \geqslant V_m(t)$ for $m = 1, 2, \ldots$ By the convergence of the Picard iterates [18, pp. 71–74], $U_m(t) \to U(t)$, $V_m(t) \to V(t)$, $W_m(t) \to W(t)$, and $Y_m(t) \to Y(t)$ as $m \to \infty$. Thus, $W + Y \geqslant U + V$. As $U + V$ blows up in a finite time, $W + Y$ blows up in a finite time. In case $\Lambda(0) \geqslant 1$ in assumption (b), we repeat the above procedure to compare the solution of (3.7) and (3.8) to Case (I) for $p = 1$ and $q > 1$ to obtain the same result. Hence, either u or v in (3.3) and (3.4) blows up on $[0,1]$ in a finite time. This completes the proof.

Theorem 3.3. *If either u or v blows up at $t = t_b$, then u and v blow up simultaneously at the point $x = b$ at $t = t_b$.*

Proof. Without loss of generality, we suppose that u blows up somewhere at x^* where $x^* \in (0,1)$ and $x^* \neq b$. Let a_1 and a_2 be positive real numbers such that x^* and $b \in [a_1, a_2] \subsetneq (0,1)$. By (2.3), the integral solution of $u(x,t)$ at $x = b$ is given by

$$u(b,t) = \int_0^1 G(b,\xi,t) u_0(\xi) d\xi + \int_0^t G(b,b,t-s) v^p(b,s) ds.$$

By the mean value theorem, there exists $\theta \in (0,t)$ such that

$$u(b,t) = \int_0^1 G(b,\xi,t) u_0(\xi) d\xi + G(b,b,t-\theta) \int_0^t v^p(b,s)\, ds.$$

As $G(b,b,t-\theta) > 0$ for $t > \theta$, there exists a positive constant k_8 (depending on b and t) such that $G(b,b,t-\theta) \geqslant k_8$ and

$$u(b,t) \geqslant \int_0^1 G(b,\xi,t) u_0(\xi) d\xi + k_8 \int_0^t v^p(b,s)\, ds.$$

Since u does not blow up at b, $\int_0^t v^p(b,s)\, ds$ is bounded when $t \to t_b$. Further, there exists a positive constant k_9 (depending on x and t) such that $G(x,\xi,t) \leqslant k_9$ for $x \in [a_1, a_2]$. Let us consider the integral solution of $u(x,t)$ at $x = x^*$, which satisfies the following inequality:

$$u(x^*,t) \leqslant \int_0^1 G(b,\xi,t)u_0(\xi)d\xi + k_9\int_0^t v^p(b,s)\,ds. \tag{3.11}$$

Because $\int_0^t v^p(b,s)\,ds$ is bounded when $t \to t_b$, we have $u(x^*,t)$ being bounded when $t \to t_b$. It leads to a contradiction. Thus, b is a blow-up point of $u(x,t)$. By (3.11), if we replace x^* by b, then we have $\int_0^t v^p(b,s)\,ds \to \infty$ as $t \to t_b$. Thus, $v(x,t)$ blows up at $x = b$. Hence, $u(x,t)$ and $v(x,t)$ blow up simultaneously at $x = b$.

Corollary 3.4. *If u and v blow up at the point* $x = b$ *at* $t = t_b$*, then the blow-up set of u and v is* $(0,1)$.

Proof. By Theorem 3.3, $u(b,t) \to \infty$ as $t \to t_b$. It implies $\int_0^t v^p(b,s)\,ds \to \infty$ as $t \to t_b$. For $x \in [a_1,a_2] \subsetneq (0,1)$, we know $\int_0^t G(x,b,t-s)v^p(b,s)ds \geqslant \int_0^t k_8 v^p(b,s)\,ds \to \infty$ as $t \to t_b$. By (2.3), we have $u(x,t) \to \infty$ as $t \to t_b$ for $x \in [a_1,a_2]$. Similarly, it yields $v(x,t) \to \infty$ as $t \to t_b$ for $x \in [a_1,a_2]$ if $v(b,t) \to \infty$ as $t \to t_b$. Since $[a_1,a_2]$ is any subset of $(0,1)$, the blow-up set of u and v is $(0,1)$. This completes the proof.

4. Conclusions

In this paper, the existence and uniqueness of the solution of a system of singular parabolic problems with concentrated nonlinear reaction sources and homogeneous Dirichlet boundary condition: $u_t - u_{xx} - bu_x/x = \delta(x-b)v^p$, $v_t - v_{xx} - bv_x/x = \delta(x-b)u^q$ are established. The result is obtained by investigating the corresponding integral solutions. If $p \geqslant 1$, $q \geqslant 1$, and $pq > 1$, we prove that either u or v blows up in a finite time through constructing a lower solution. Further, we show that u and v blow up simultaneously in a finite time at b, and they blow up everywhere in the domain except the boundary.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares no conflict of interest in this paper.

References

1. O. Arena, On a singular parabolic equation related to axially symmetric heat potentials, *Ann. Mat. Pure Appl.*, **105** (1975), 347–393. https://doi.org/10.1007/BF02414938
2. A. Huber, On the uniqueness of generalized axially symmetric potentials, *Ann. Math.*, **60** (1954), 351–358.
3. N. Ortner, P. Wagner, On Green's functions in generalized axially symmetric potential theory, *Appl. Anal.*, **99** (2020), 1171–1180. https://doi.org/10.1080/00036811.2018.1523394
4. A. Weinstein, Generalized axially symmetric potential theory, *Bull. Amer. Math. Soc.*, **59** (1953), 20–38.

5. J. Wittsten, Generalized axially symmetric potentials with distributional boundary values, *Bull. Sci. Math.*, **139** (2015), 892–922.
6. H. Brezis, W. Rosenkrantz, B. Singer, On a degenerate elliptic-parabolic equation occurring in the theory of probability, *Commun. Pure Appl. Math.*, **24** (1971), 395–416.
7. J. Lamperti, A new class of probability limit theorems, *J. Math. Mech.*, **11** (1962), 749–772.
8. W. Feller, Two singular diffusion problems, *Ann. Math.*, **54** (1951), 173–182.
9. D. Calton, Cauchy's problem for a singular parabolic partial differential equation, *J. Differ. Equ.*, **8** (1970), 250–257.
10. V. Alexiades, Generalized axially symmetric heat potentials and singular parabolic initial boundary value problems, *Arch. Ration. Mech. Anal.*, **79** (1982), 325–350.
11. C. Y. Chan, C. S. Chen, A numerical method for semilinear singular parabolic quenching problems, *Quart. Appl. Math.*, **47** (1989), 45–57.
12. M. Escobedo, M. Herrero, Boundedness and blow up for a semilinear reaction-diffusion system, *J. Differ. Equ.*, **89** (1991), 176–202. https://doi.org/10.1016/0022-0396(91)90118-S
13. H. A. Levine, A Fujita type global existence-global nonexistence theorem for a weakly coupled system of reaction-diffusion equations, *Z. Angew. Math. Phys.*, **42** (1991), 408–430.
14. C. Y. Chan, B. M. Wong, Existence of classical solutions for singular parabolic problems, *Quart. Appl. Math.*, **53** (1995), 201–213.
15. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1944.
16. W. Y. Chan, Blow-up of the solution for a semilinear parabolic problem with a mixed source, *Mathematics*, **10** (2022), 1–12. https://doi.org/10.3390/math10173156
17. C. V. Pao, *Nonlinear parabolic and elliptic equations*, New York: Plenum Press, 1992.
18. M. Braun, *Differential equations and their applications*, 4 Eds., New York: Springer, 1992.