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Research article

Higher Jordan triple derivations on $*$-type trivial extension algebras

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Abstract: In this paper, we investigated the problem of describing the form of higher Jordan triple derivations on trivial extension algebras. We show that every higher Jordan triple derivation on a 2-torsion free $*$-type trivial extension algebra is a sum of a higher derivation and a higher anti-derivation. As for its applications, higher Jordan triple derivations on triangular algebras are characterized.

Keywords: derivation; anti-derivation; higher derivation; higher anti-derivation; higher Jordan triple derivation
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1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity, and $\mathcal{A}$ a unital algebra over $\mathcal{R}$. For any $X, Y \in \mathcal{A}$, denote the Jordan product of X, Y by $X \circ Y = XY + YX$. An additive mapping Δ from $\mathcal{A}$ into itself is called a derivation (resp., anti-derivation) if $\Delta(XY) = \Delta(X)Y + X\Delta(Y)$ (resp., $\Delta(XY) = \Delta(Y)X + Y\Delta(X)$) for all $X, Y \in \mathcal{A}$. It is called a Jordan derivation if $\Delta(X \circ Y) = \Delta(X) \circ Y + X \circ \Delta(Y)$ for all $X, Y \in \mathcal{A}$. It is called a Jordan triple derivation if $\Delta(X \circ Y \circ Z) = \Delta(X) \circ Y \circ Z + X \circ \Delta(Y) \circ Z + X \circ Y \circ \Delta(Z)$ for all $X, Y, Z \in \mathcal{A}$. Obviously, every derivation or anti-derivation is a Jordan derivation. However, the inverse statement is not true in general (see [1]). If a Jordan derivation or Jordan triple derivation is not a derivation, then it is said to be proper. Otherwise, it is said to be improper.

In the past few decades, the problem of characterizing the structure of Jordan derivations and Jordan triple derivations has attracted the attention of many mathematical workers and has achieved some important research results. For example, Herstein in [2] proved that every Jordan derivation on a prime ring not of characteristic 2 is a derivation. This result was extended by Cusack in [3] and Brešar in [4] to the case of semiprime. Zhang in [5, 6] showed that every Jordan derivation on a nest algebra or a 2-torsion free triangular algebra is a inner derivation or a derivation, respectively. Later, Hoger in [7] extended the result of Zhang in [6] and proved that, under certain conditions, each Jordan derivation on trivial extension algebras is a sum of a derivation and an anti-derivation. In addition, there have been many research results on Jordan triple derivations, as shown in references [8–11].

Definition 1.1. *Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ a unital algebra over $\mathcal{R}$, $\mathbb{N}_0$ be the set of all nonnegative integers, and $D = \{d_n\}_{n\in\mathbb{N}_0}$ be a family of additive maps on $\mathcal{A}$ such that $d_0 = id_{\mathcal{A}}$ (the identity map on $\mathcal{A}$). D is said to be:*

(*i*) *a higher derivation if for each $n \in \mathbb{N}_0$,*

$$d_n(XY) = \sum_{p+q=n} d_p(X)d_q(Y)$$

for all $X, Y \in \mathcal{A}$;

(*ii*) *a higher anti-derivation if for each $n \in \mathbb{N}_0$,*

$$d_n(XY) = \sum_{p+q=n} d_p(Y)d_q(X)$$

for all $X, Y \in \mathcal{A}$;

(*iii*) *a higher Jordan derivation if for each $n \in \mathbb{N}_0$,*

$$d_n(X \circ Y) = \sum_{p+q=n} d_p(X) \circ d_q(Y)$$

for all $X, Y \in \mathcal{A}$;

(*iv*) *a higher Jordan triple derivation if for each $n \in \mathbb{N}_0$,*

$$d_n(X \circ Y \circ Z) = \sum_{p+q+r=n} d_p(X) \circ d_q(Y) \circ d_r(Z)$$

for all $X, Y, Z \in \mathcal{A}$.

If a higher Jordan derivation or a higher Jordan triple derivation is not a higher derivation, then it is said to be proper. Otherwise, it is said to be improper. With the deepening of research on this topic, many research achievements have been obtained about higher Jordan derivations and higher Jordan triple derivations. For example, Xiao and Wei in [12] proved that every higher Jordan derivation on triangular algebras is a higher derivation; Fu, Xiao, and Du in [13] extended this conclusion, and proved that every nonlinear higher Jordan derivation on triangular algebras is a higher derivation. Later, Vishki, Mirzavaziri, and Moafian in [14] proved that, under certain conditions, every higher Jordan derivation on trivial extension algebras is a higher derivation, and this conclusion further extended the works of the authors of references [12, 13]. Salih and Haetinger in [15] proved that, under certain conditions, every higher Jordan triple derivation on prime rings is a higher derivation. Ashraf and Jabeenin [16] proved that every nonlinear higher Jordan triple derivable mapping on triangular algebras is a higher derivation.

In this paper, we are interested in describing the form of higher Jordan triple derivation on trivial extension algebras. As a main result, we give conditions under which each higher Jordan triple derivation on trivial extension algebras is a sum of a higher derivation and a higher anti-derivation. This result extends the study of Jordan derivation on trivial extension algebras [7], Jordan triple derivations on $*$-type trivial extension algebras [17], and Jordan higher derivations on trivial extension algebras [14].

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ a unital algebra over $\mathcal{R}$ and $\mathcal{M}$ be an $\mathcal{A}$-bimodule. Then the direct product $\mathcal{A} \oplus \mathcal{M}$ together with the pairwise addition, scalar product, and algebra multiplication defined by

$$(a, m)(b, n) = (ab, an + mb)(\forall a, b \in \mathcal{A}, m, n \in \mathcal{M})$$

is an $\mathcal{R}$-algebra with a unity $(1, 0)$ denoted by

$$\mathcal{T} = \mathcal{A} \oplus \mathcal{M} = \{(a, m) : a \in \mathcal{A}, m \in \mathcal{M}\}$$

and $\mathcal{T}$ is called a trivial extension algebra.

An important example of trivial extension algebra is the triangular algebra which was introduced by Cheung in [18]. Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$, and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module. Then, the $\mathcal{R}$-algebra

$$\mathcal{U} = \text{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra. Basic examples of triangular algebras are upper triangular matrix algebras and nest algebras.

It is well-known that every triangular algebra can be viewed as a trivial extension algebra. Indeed, denote by $\mathcal{A} \oplus \mathcal{B}$ the direct product as an $\mathcal{R}$-algebra, and then $\mathcal{M}$ is viewed as an $\mathcal{A} \oplus \mathcal{B}$-bimodule with the module action given by $(a, b)m = am$ and $m(a, b) = mb$ for all $(a, b) \in \mathcal{A} \oplus \mathcal{B}$ and $m \in \mathcal{M}$. Then triangular algebra $\mathcal{U}$ is isomorphic to trivial extensions algebra $\mathcal{T} = (\mathcal{A} \oplus \mathcal{B}) \oplus \mathcal{M}$. However, a trivial extension algebra is not necessarily a triangular algebra. For more details about trivial extension algebras, we refer the readers to [19–21].

The following notations will be used in our paper: Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ a unital algebra over $\mathcal{R}$, $\mathcal{M}$ an $\mathcal{A}$-bimodule, $\mathcal{T} = \mathcal{A} \oplus \mathcal{M}$ be a 2-torsion free trivial extension algebra (i.e., for any $X \in \mathcal{T}$, $2X = \{0\}$ implies $X = 0$), and denote by 1 and 0 are the unity and zero of $\mathcal{T} = \mathcal{A} \oplus \mathcal{M}$, respectively.

We say $\mathcal{T} = \mathcal{A} \oplus \mathcal{M}$ is a $*$-type trivial extension algebra if $\mathcal{A}$ has a non-trivial idempotent element e and $f = 1 - e$ such that

(i) $e\mathcal{M}f = \mathcal{M}$;

(ii) $exe\mathcal{M} = \{0\}$ implies $exe = 0, \forall x \in \mathcal{A}$;

(iii) $\mathcal{M}fxf = \{0\}$ implies $fxf = 0, \forall x \in \mathcal{A}$;

(iv) $exfye = 0 = fxeyf = 0, \forall x, y \in \mathcal{A}$.

For convenience, in the following we let $P_1 = (e, 0)$, $P_2 = (f, 0)$, and

$$\mathcal{T}_{ij} = P_i \mathcal{T} P_j(1 \leq i \leq j \leq 2).$$

It is not hard to see that the trivial extension algebra $\mathcal{T}$ may be represented as

$$\mathcal{T} = P_1 \mathcal{T} P_1 + P_1 \mathcal{T} P_2 + P_2 \mathcal{T} P_1 + P_2 \mathcal{T} P_2 = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{21} + \mathcal{T}_{22}.$$

Then every element $A \in \mathcal{T}$ may be represented as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{T}_{ij}(1 \leq i \leq j \leq 2)$. In the following, we give a property of $*$-type trivial extension algebras (see Lemma 1.1).

Lemma 1.1. *[17] Let $\mathcal{T}$ be a $*$-type trivial extension algebra and $1 \leq i \neq j \leq 2$. Then,*

(i) *for any $A_{11} \in \mathcal{T}_{11}$, if $A_{11}\mathcal{T}_{12} = 0$, then $A_{11} = 0$;*

(ii) *for any $A_{22} \in \mathcal{T}_{22}$, if $\mathcal{T}_{12}A_{22} = 0$, then $A_{22} = 0$;*

(iii) $A_{ij}B_{ji} = A_{ii}B_{ji} = A_{ij}B_{ii} = 0$, $\forall A_{ii}, B_{ii} \in \mathcal{T}_{ii}, \forall A_{ij} \in \mathcal{T}_{ij}, \forall B_{ji} \in \mathcal{T}_{ji}$.

For ease of reading, we provide the main conclusions of reference [17] as follows:

Theorem 1.1. *[17] Let $\mathcal{T} = \mathcal{A} \oplus \mathcal{M}$ be a 2-torsion free $*$-type trivial extension algebra and Δ be a Jordan triple derivation on $\mathcal{T}$. Then, there exists a derivation D and an anti-derivation φ on $\mathcal{T}$, respectively, such that*

$$\Delta(A) = D(A) + \varphi(A)$$

for all $A \in \mathcal{T}$.

2. Main results

The main result of this paper is the following theorem:

Theorem 2.1. *Let $\mathcal{T} = \mathcal{A} \oplus \mathcal{M}$ be a 2-torsion free $*$-type trivial extension algebra, and $D = \{d_n\}_{n\in\mathbb{N}_0}$ be a higher Jordan triple derivation on $\mathcal{T}$. Then, there exists a higher derivation $G = \{g_n\}_{n\in\mathbb{N}_0}$ and a higher anti-derivation $F = \{f_n\}_{n\in\mathbb{N}_0}$ on $\mathcal{T}$, respectively, such that*

$$d_n(X) = g_n(X) + f_n(X)$$

for any $n \geq 1$ and $X \in \mathcal{T}$.

In order to prove Theorem 2.1, we shall establish Theorems 2.2 and 2.3 in the following. We assume that $\mathcal{T}$ is a $*$-type trivial extension algebra, $\mathbb{N}_0$ is the set of all nonnegative integers, and $D = \{d_n\}_{n\in\mathbb{N}_0}$ is a higher Jordan triple derivation on $\mathcal{T}$.

In [17], it is proved that if d_1 is a Jordan triple derivation on $\mathcal{T}$, then for all $A_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i, j \leq 2)$, d_1 satisfies the following properties ($\mathcal{L}$):

(i) $d_1(P_1) = -d_1(P_2)$;

(ii) $d_1(P_1) = P_1d_1(P_1)P_2 + P_2d_1(P_1)P_1$ and $d_1(P_2) = P_1d_1(P_2)P_2 + P_2d_1(P_2)P_1$;

(iii) $P_2d_1(A_{11})P_2 = 0$, $P_1d_1(A_{11})P_2 = A_{11}d_1(P_1)$ and $P_2d_1(A_{11})P_1 = d_1(P_1)A_{11}$;

(iv) $P_1d_1(A_{22})P_1 = 0, P_1d_1(A_{22})P_2 = d_1(P_2)A_{22}$ and $P_2d_1(A_{22})P_1 = A_{22}d_1(P_2)$;

(v) $d_1(A_{12}) = P_1d_1(A_{12})P_2 + P_2d_1(A_{12})P_1$ and $d_1(A_{21}) = P_1d_1(A_{21})P_2 + P_2d_1(A_{21})P_1$;

(vi) $d_1(P_1) \circ d_1(P_2) = d_1(P_1) \circ d_1(A_{12}) = d_1(P_1) \circ d_1(A_{21}) = d_1(P_2) \circ d_1(A_{12}) = d_1(P_2) \circ d_1(A_{21}) = 0$;

(vii) $d_1(A_{12}) \circ d_1(A_{12}) = d_1(A_{21}) \circ d_1(A_{21}) = d_1(A_{12}) \circ d_1(A_{21}) = 0$.

Now, for all $A_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i, j \leq 2)$, we assume that d_k $(1 \leqslant k < n)$ satisfy the properties $\mathcal{L}$. In the following, we show that d_n satisfies the properties $\mathcal{L}$.

Lemma 2.1. *Let $D = \{d_n\}_{n\in\mathbb{N}_0}$ be a higher Jordan triple derivation on $\mathcal{T}$. Then, for each $n \geq 1$, and for any $A_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}, A_{21} \in \mathcal{T}_{21}$, $A_{22} \in \mathcal{T}_{22}$,*

(i) *$d_n(P_1) = P_1d_n(P_1)P_2 + P_2d_n(P_1)P_1$ and $d_n(P_2) = P_1d_n(P_2)P_2 + P_2d_n(P_2)P_1$;*

(ii) *$d_n(P_1) = -d_n(P_2)$;*

(iii) *$P_2d_n(A_{11})P_2 = 0$, $P_1d_n(A_{11})P_2 = A_{11}d_n(P_1)$ and $P_2d_n(A_{11})P_1 = d_n(P_1)A_{11}$;*

(iv) *$P_1d_n(A_{22})P_1 = 0, P_1d_n(A_{22})P_2 = d_n(P_2)A_{22}$ and $P_2d_n(A_{22})P_1 = A_{22}d_n(P_2)$;*

(v) $d_n(A_{12}) = P_1 d_n(A_{12})P_2 + P_2 d_n(A_{12})P_1$ *and* $d_n(A_{21}) = P_1 d_n(A_{21})P_2 + P_2 d_n(A_{21})P_1$;
(vi) $d_n(P_1) \circ d_n(P_2) = d_n(P_1) \circ d_n(A_{12}) = d_n(P_1) \circ d_n(A_{21}) = d_n(P_2) \circ d_n(A_{12}) = d_n(P_2) \circ d_n(A_{21}) = 0$;
(vii) $d_n(A_{12}) \circ d_n(A_{12}) = d_n(A_{21}) \circ d_n(A_{21}) = d_n(A_{12}) \circ d_n(A_{21}) = 0$.

Proof. (*i*) For each $n \geq 1$ and for any $X, Y, Z \in \mathcal{T}$, by the definition of $D = \{d_n\}_{n\in\mathbb{N}_0}$, we get

$$d_n(X \circ Y \circ Z) = \sum_{p+q+r=n} d_p(X) \circ d_q(Y) \circ d_r(Z). \tag{2.1}$$

Taking $X = Y = Z = P_1$ in Eq (2.1), we assume that d_k $(1 \leqslant k < n)$ satisfy the properties $\mathcal{L}$, and then it follows from Lemma 1.1 (*iii*) that

$$\begin{aligned}
4d_n(P_1) &= \sum_{p+q+r=n} d_p(P_1) \circ d_q(P_1) \circ d_r(P_1) \\
&= \sum_{p+q+r=n,1\leq p,q,r} d_p(P_1) \circ d_q(P_1) \circ d_r(P_1) + \sum_{q+r=n,1\leq q,r} P_1 \circ d_q(P_1) \circ d_r(P_1) \\
&+ \sum_{p+r=n,1\leq p,r} d_p(P_1) \circ P_1 \circ d_r(P_1) + \sum_{p+q=n,1\leq p,q} d_p(P_1) \circ d_q(P_1) \circ P_1 \\
&+ d_n(P_1) \circ P_1 \circ P_1 + P_1 \circ d_n(P_1) \circ P_1 + P_1 \circ P_1 \circ d_n(P_1) \\
&= d_n(P_1) \circ P_1 \circ P_1 + P_1 \circ d_n(P_1) \circ P_1 + P_1 \circ P_1 \circ d_n(P_1) \\
&= 4P_1 d_n(P_1)P_1 + 4P_1 d_n(P_1) + 4d_n(P_1)P_1.
\end{aligned}$$

This yields from the 2-torsion freeness of $\mathcal{T}$ that

$$P_1 d_n(P_1)P_1 = P_2 d_n(P_1)P_2 = 0.$$

Similarly, we get that

$$P_1 d_n(P_2)P_1 = P_2 d_n(P_2)P_2 = 0.$$

Therefore, $d_n(P_1) = P_1 d_n(P_1)P_2 + P_2 d_n(P_1)P_1$ and $d_n(P_2) = P_1 d_n(P_2)P_2 + P_2 d_n(P_2)P_1$.

(*ii*) For each $n \geq 1$, taking $X = P_1, Y = P_2, Z = P_1$ in Eq (2.1), we assume that d_k $(1 \leqslant k < n)$ satisfy the properties $\mathcal{L}$, then by Lemma 1.1 (*iii*) and Lemma 2.1 (*i*), we get that

$$\begin{aligned}
0 &= \sum_{p+q+r=n} d_p(P_1) \circ d_q(P_2) \circ d_r(P_1) \\
&= \sum_{p+q+r=n,1\leq p,q,r} d_p(P_1) \circ d_q(P_2) \circ d_r(P_1) + \sum_{q+r=n,1\leq q,r} P_1 \circ d_q(P_2) \circ d_r(P_1) \\
&+ \sum_{p+r=n,1\leq p,r} d_p(P_1) \circ P_2 \circ d_r(P_1) + \sum_{p+q=n,1\leq p,q} d_p(P_1) \circ d_q(P_2) \circ P_1 \\
&+ d_n(P_1) \circ P_2 \circ P_1 + P_1 \circ d_n(P_2) \circ P_1 + P_1 \circ P_2 \circ d_n(P_2) \\
&= d_n(P_1) \circ P_2 \circ P_1 + P_1 \circ d_n(P_2) \circ P_1 + P_1 \circ P_2 \circ d_n(P_2) \\
&= \{d_n(P_1)P_2 + P_2 d_n(P_1)\} \circ P_1 + \{P_1 d_n(P_2) + d_n(P_2)P_1\} \circ P_1 \\
&= P_1 d_n(P_1)P_2 + P_2 d_n(P_1)P_1 + P_1 d_n(P_2) + d_n(P_2)P_1 + 2P_1 d_n(P_2)P_1 \\
&= P_1 d_n(P_1)P_2 + P_2 d_n(P_1)P_1 + P_1 d_n(P_2)P_2 + P_2 d_n(P_2)P_1
\end{aligned}$$

$$= \quad d_n(P_1) + d_n(P_2).$$

(iii)–(iv) For each $n \geq 1$ and for any $A_{11} \in \mathcal{T}_{11}$, taking $X = A_{11}, Y = Z = P_2$ in Eq (2.1), we assume that d_k $(1 \leqslant k < n)$ satisfy the properties $\mathcal{L}$, and then by Lemma 1.1 (iii) and Lemma 2.1 (i, ii), we get that

$$\begin{aligned}
0 &= \sum_{p+q+r=n} d_p(A_{11}) \circ d_q(P_2) \circ d_r(P_2) \\
&= \sum_{p+q+r=n, 1\leq p,q,r} d_p(A_{11}) \circ d_q(P_2) \circ d_r(P_2) + \sum_{q+r=n, 1\leq q,r} A_{11} \circ d_q(P_2) \circ d_r(P_2) \\
&+ \sum_{p+r=n, 1\leq p,r} d_p(A_{11}) \circ P_2 \circ d_r(P_2) + \sum_{p+q=n, 1\leq p,q} d_p(A_{11}) \circ d_q(P_2) \circ P_2 \\
&+ d_n(A_{11}) \circ P_2 \circ P_2 + A_{11} \circ d_n(P_2) \circ P_2 + A_{11} \circ P_2 \circ d_n(P_2) \\
&= d_n(A_{11}) \circ P_2 \circ P_2 + A_{11} \circ d_n(P_2) \circ P_2 + A_{11} \circ P_2 \circ d_n(P_2) \\
&= \{d_n(A_{11})P_2 + P_2 d_n(A_{11})\} \circ P_2 + \{A_{11} d_n(P_2) + d_n(P_2) A_{11}\} \circ P_2 \\
&= \{d_n(A_{11})P_2 + P_2 d_n(A_{11}) + 2P_2 d_n(A_{11})P_2\} + \{A_{11} d_n(P_2) + P_2 d_n(P_2) A_{11}\} \\
&= d_n(A_{11})P_2 + P_2 d_n(A_{11}) + A_{11} d_n(P_2) + d_n(P_2) A_{11}.
\end{aligned}$$

This implies that $P_2 d_n(A_{11}) P_2 = 0.$ and

$$P_1 d_n(A_{11}) P_2 = A_{11} d_n(P_1) \text{ and } P_2 d_n(A_{11}) P_1 = d_n(P_1) A_{11}.$$

Similarly, for each $n \geq 1$ and for any $A_{22} \in \mathcal{T}_{22}$, we get that $P_1 d_n(A_{22}) P_1 = 0$, $P_1 d_n(A_{22}) P_2 = d_n(P_2) A_{22}$ and $P_2 d_n(A_{22}) P_1 = A_{22} d_n(P_2)$.

(v) For each $n \geq 1$ and for any $A_{12} \in \mathcal{T}_{12}$, taking $X = P_1, Y = A_{12}, Z = P_2$ in Eq (2.1), we assume that d_k $(1 \leqslant k < n)$ satisfy the properties $\mathcal{L}$, and then by Lemma 1.1 (iii) and Lemma 2.1 (i, ii), we get that

$$\begin{aligned}
d_n(A_{12}) &= \sum_{p+q+r=n} d_p(P_1) \circ d_q(A_{12}) \circ d_r(P_2) \\
&= \sum_{p+q+r=n, 1\leq p,q,r} d_p(P_1) \circ d_q(A_{12}) \circ d_r(P_2) + \sum_{q+r=n, 1\leq q,r} P_1 \circ d_q(A_{12}) \circ d_r(P_2) \\
&+ \sum_{p+r=n, 1\leq p,r} d_p(P_1) \circ A_{12} \circ d_r(P_2) + \sum_{p+q=n, 1\leq p,q} d_p(P_1) \circ d_q(A_{12}) \circ P_2 \\
&+ d_n(P_1) \circ A_{12} \circ P_2 + P_1 \circ d_n(A_{12}) \circ P_2 + P_1 \circ A_{12} \circ d_n(P_2) \\
&= P_1 \circ d_n(A_{12}) \circ P_2 \\
&= P_1 d_n(A_{12}) P_2 + P_2 d_n(A_{12}) P_1.
\end{aligned}$$

Similarly, for each $n \geq 1$ and for any $A_{21} \in \mathcal{T}_{21}$, we get that $d_n(A_{21}) = P_1 d_n(A_{21}) P_2 + P_2 d_n(A_{21}) P_1$.

(vi) For each $n \geq 1$ and for any $A_{12} \in \mathcal{T}_{12}$, $A_{21} \in \mathcal{T}_{21}$, by Lemma 1.1 (iii) and Lemma 2.1 (i, ii, v), we can easily check that (vi) holds. Similarly, we show (vii) holds. The proof is complete. □

Theorem 2.2. *Let $F = \{f_n\}_{n\in\mathbb{N}_0}$ be a sequence of mappings on $\mathcal{T}$ (with $f_0 = if_{\mathcal{T}}$). For each $n \geq 1$ and $X \in \mathcal{T}$, define*

$$f_n(X) = P_2 d_n(P_1 X P_2) P_1 + P_1 d_n(P_2 X P_1) P_2.$$

Then, F is a higher anti-derivation on $\mathcal{T}$.

It is clear that $f_n(A_{ii}) = 0$ and $f_n(A_{ij}) = P_j f_n(A_{ij})P_i$ for each $n \geq 1$, and for any $A_{ii} \in \mathcal{T}_{ii}, A_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$.

In the following, we show that $F = \{f_n\}_{n\in\mathbb{N}_0}$ is a higher anti-derivation, i.e., for each $n \geq 1$ and for any $X, Y \in \mathcal{T}$, f_n satisfies $f_n(XY) = \sum_{p+q=n} f_p(Y)f_q(X)$. For this, we introduce Lemmas 2.2 and 2.3, and prove Lemmas 2.2 and 2.3.

Lemma 2.2. *Let $f_n : \mathcal{T} \to \mathcal{T}$ be defined as in Theorem 2.2. Then, for each $n \geq 1$ and for any $A_{ii}, B_{ii} \in \mathcal{T}_{ii}, A_{ij}, B_{ij} \in \mathcal{T}_{ij}, B_{ji} \in \mathcal{T}_{ji}, B_{jj} \in \mathcal{T}_{jj}$ $(1 \leq i \neq j \leq 2)$,*

(i) $f_n(A_{ii}B_{ii}) = \sum_{p+q=n} f_p(B_{ii})f_q(A_{ii})$;

(ii) $f_n(A_{ii}B_{jj}) = \sum_{p+q=n} f_p(B_{jj})f_q(A_{ii})$;

(iii) $f_n(A_{ii}B_{ji}) = \sum_{p+q=n} f_p(B_{ji})f_q(A_{ii})$;

(iv) $f_n(A_{ij}B_{ii}) = \sum_{p+q=n} f_p(B_{ii})f_q(A_{ij})$;

(v) $f_n(A_{ij}B_{ij}) = \sum_{p+q=n} f_p(B_{ij})f_q(A_{ij})$;

(vi) $f_n(A_{ij}B_{ji}) = \sum_{p+q=n} f_p(B_{ji})f_q(A_{ij})$.

Proof. (*i*) For any $n \geq 1$ and $A_{ii}, B_{ii} \in \mathcal{T}_{ii}$ $(1 \leq i \leq 2)$, we get from $f_n(A_{ii}B_{ii}) = f_n(A_{ii}) = f_n(B_{ii}) = 0$ that

$$f_n(A_{ii}B_{ii}) = \sum_{p+q=n} f_p(B_{ii})f_q(A_{ii}).$$

Similarly, we can show (*ii*) holds.

(*iii*) For each $n \geq 1$ and for any $A_{ii} \in \mathcal{T}_{ii}, B_{ji} \in \mathcal{T}_{ji}$ $(1 \leq i \neq j \leq 2)$, on the one hand, we have $f_n(A_{ii}B_{ji}) = f_n(0) = 0$. On the other hand, it follows from $f_n(A_{ii}) = 0$ and $f_n(B_{ji}) = P_i f_n(B_{ji})P_j$ that

$$\begin{aligned}\sum_{p+q=n} f_p(B_{ji})f_q(A_{ii}) &= \sum_{p+q=n,1\leq p,q} f_p(B_{ji})f_q(A_{ii}) + f_n(B_{ji})A_{ii} + B_{ji}f_n(A_{ii})\\ &= f_n(B_{ji})A_{ii}\\ &= (P_i f_n(B_{ji})P_j)A_{ii}\\ &= 0.\end{aligned}$$

Therefore, $f_n(A_{ii}B_{ji}) = \sum_{p+q=n} f_p(B_{ji})f_q(A_{ii})$. Similarly, we get (*iv*).

(*v*) For each $n \geq 1$ and for any $A_{ij}, B_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$, on the one hand, we have $f_n(A_{ij}B_{ij}) = f_n(0) = 0$. On the other hand, we get from $f_n(B_{ij})f_n(A_{ij}) = \{P_j f_n(B_{ij})P_i\}\{P_j f_n(A_{ij})P_i\} = 0$ and Lemma 1.1 (*iii*) that

$$\begin{aligned}\sum_{p+q=n} f_p(B_{ij})f_q(A_{ij}) &= \sum_{p+q=n,1\leq p,q} f_p(B_{ij})f_q(A_{ij}) + f_n(B_{ij})A_{ij} + B_{ij}f_n(A_{ij})\\ &= f_n(B_{ij})A_{ij} + B_{ij}f_n(A_{ij})\\ &= \{P_j f_n(B_{ij})P_i\}A_{ij} + B_{ij}\{P_j f_n(A_{ij})P_i\}\\ &= 0.\end{aligned}$$

Therefore, $f_n(A_{ij}B_{ij}) = \sum_{p+q=n} f_p(B_{ij})f_q(A_{ij})$. Similarly, we get (*vi*). The proof is complete. □

Lemma 2.3. *Let $f_n : \mathcal{T} \to \mathcal{T}$ be defined as in Theorem 2.2. Then, for each $n \geq 1$ and for any $A_{ii} \in \mathcal{T}_{ii}, B_{ij} \in \mathcal{T}_{ij}, B_{jj} \in \mathcal{T}_{jj}$ ($1 \leq i \neq j \leq 2$),*
(*i*) $f_n(A_{ii}B_{ij}) = \sum_{p+q=n} f_p(B_{ij})f_q(A_{ii})$;
(*ii*) $f_n(A_{ij}B_{jj}) = \sum_{p+q=n} f_p(B_{jj})f_q(A_{ij})$.

Proof. (*i*) For each $n \geq 1$ and for any $A_{ii} \in \mathcal{T}_{ii}, B_{ij} \in \mathcal{T}_{ij}$ ($1 \leq i \neq j \leq 2$), it follows from $A_{ii}B_{ij} = A_{ii} \circ B_{ij} \circ P_j$ and Lemma 2.1 that

$$\begin{aligned}
f_n(A_{ii}B_{ij}) &= P_jd_n(A_{ii}B_{ij})P_i \\
&= P_jd_n(A_{ii} \circ B_{ij} \circ P_j)P_i \\
&= P_j\{\sum_{p+q+r=n} d_p(A_{ii}) \circ d_q(B_{ij}) \circ d_r(P_j)\}P_i \\
&= P_j\{\sum_{p+q+r=n,1\leq p,q,r} d_p(A_{ii}) \circ d_q(B_{ij}) \circ d_r(P_j)\}P_i \\
&+ P_j\{\sum_{q+r=n,1\leq q,r} A_{ii} \circ d_q(B_{ij}) \circ d_r(P_j)\}P_i \\
&+ P_j\{\sum_{p+r=n,1\leq r,t} d_p(A_{ii}) \circ B_{ij} \circ d_r(P_j)\}P_i \\
&+ P_j\{\sum_{p+q=n,1\leq r,s,t} d_p(A_{ii}) \circ d_q(B_{ij}) \circ P_j\}P_i \\
&+ P_j\{d_n(A_{ii}) \circ B_{ij} \circ P_j\}P_i + P_j\{A_{ii} \circ d_n(B_{ij}) \circ P_j\}P_i \\
&+ P_j\{A_{ii} \circ B_{ij} \circ d_n(P_j)\}P_i \\
&= P_j\{d_n(A_{ii}) \circ B_{ij} \circ P_j\}P_i + P_j\{A_{ii} \circ d_n(B_{ij}) \circ P_j\}P_i \\
&+ P_j\{A_{ii} \circ B_{ij} \circ d_n(P_j)\}P_i \\
&= P_j\{d_n(A_{ii})B_{ij}P_j + P_jd_n(A_{ii})B_{ij} + B_{ij}d_n(A_{ii})P_j\}P_i \\
&+ P_j\{A_{ii}d_n(B_{ij})P_j + P_jd_n(B_{ij})A_{ii}\}P_i \\
&+ P_j\{A_{ii}B_{ij} \circ d_n(P_j)\}P_i \\
&= P_jd_n(B_{ij})A_{ii}P_i \\
&= f_n(B_{ij})A_{ii}. \qquad (2.2)
\end{aligned}$$

On the other hand, it is follows from $f_n(A_{ii}) = 0$ ($n \geq 1$) that $f_{n-k}(B_{ij})d_k(A_{ii}) = 0$, so we get

$$\begin{aligned}
f_n(A_{ii}B_{ij}) &= f_n(B_{ij})A_{ii} + f_{n-1}(B_{ij})f_1(A_{ii}) + f_{n-2}(B_{ij})f_2(A_{ii}) + \ldots + B_{ij}f_n(A_{ii}) \\
&= \sum_{p+q=n} f_p(B_{ij})f_q(A_{ii}).
\end{aligned}$$

Similarly, we get (*ii*). The proof is complete. □

In the following, we give the completed proof of Theorem 2.2:

Proof of Theorem 2.2. For each $n \geq 1$, let $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$ be arbitrary elements of $\mathcal{T}$, where $A_{ij}, B_{ij} \in \mathcal{T}_{ij}$ ($1 \leq i, j \leq 2$). Then, it follows from Lemmas 2.2 and 2.3 that

$$f_n(AB) = f_n((A_{11} + A_{12} + A_{21} + A_{22})(B_{11} + B_{12} + B_{21} + B_{22}))$$

$$
\begin{aligned}
&= f_n(A_{11}B_{11}) + f_n(A_{11}B_{12}) + f_n(A_{11}B_{21}) + f_n(A_{11}B_{22}) \\
&+ f_n(A_{12}B_{11}) + f_n(A_{12}B_{12}) + f_n(A_{12}B_{21}) + f_n(A_{12}B_{22}) \\
&+ f_n(A_{21}B_{11}) + f_n(A_{21}B_{12}) + f_n(A_{21}B_{21}) + f_n(A_{21}B_{22}) \\
&+ f_n(A_{22}B_{11}) + f_n(A_{22}B_{12}) + f_n(A_{22}B_{21}) + f_n(A_{22}B_{22}) \\
&= \sum_{p+q=n} f_p(B_{11})f_q(A_{11}) + \sum_{p+q=n} f_p(B_{11})f_q(A_{12}) \\
&+ \sum_{p+q=n} f_p(B_{11})f_q(A_{21}) + \sum_{p+q=n} f_p(B_{11})f_q(A_{22}) \\
&+ \sum_{p+q=n} f_p(B_{12})f_q(A_{11}) + \sum_{p+q=n} f_p(B_{12})f_q(A_{12}) \\
&+ \sum_{p+q=n} f_p(B_{12})f_q(A_{21}) + \sum_{p+q=n} f_p(B_{12})f_q(A_{22}) \\
&+ \sum_{p+q=n} f_p(B_{21})f_q(A_{11}) + \sum_{p+q=n} f_p(B_{21})f_q(A_{12}) \\
&+ \sum_{p+q=n} f_p(B_{21})f_q(A_{21}) + \sum_{p+q=n} f_p(B_{21})f_q(A_{22}) \\
&+ \sum_{p+q=n} f_p(B_{22})f_q(A_{11}) + \sum_{p+q=n} f_p(B_{22})f_q(A_{12}) \\
&+ \sum_{p+q=n} f_p(B_{22})f_q(A_{21}) + \sum_{p+q=n} f_p(B_{22})f_q(A_{22}) \\
&= \sum_{p+q=n} f_p(B)f_q(A).
\end{aligned}
$$

Therefore, $F = \{f_n\}_{n\in\mathbb{N}_0}$ is a higher anti-derivation on $\mathcal{T}$. The proof is complete. □

Theorem 2.3. *Let $G = \{g_n\}_{n\in\mathbb{N}_0}$ be a sequence of mappings on $\mathcal{T}$ (with $g_0 = ig_{\mathcal{T}}$). For each $n \geq 1$ and for any $X \in \mathcal{T}$, define*

$$g_n(X) = d_n(X) - f_n(X).$$

Then, G is a higher derivation on $\mathcal{T}$.

Next, we show that $G = \{g_n\}_{n\in\mathbb{N}_0}$ is a higher derivation on $\mathcal{T}$. In order to prove G is a higher derivation, we introduce Lemmas 2.4–2.6, and then, using the mathematical induction, we prove Lemmas 2.4–2.6.

In [12] Theorem 1.3, we have proved that if $g_1 = d_1 - f_1$, then g_1 is a derivation on $\mathcal{T}$, i.e., for any $X, Y \in \mathcal{T}$, g_1 satisfies

$$g_1(XY) = g_1(X)Y + Xg_1(Y) = \sum_{p+q=1} g_p(X)g_q(Y).$$

Therefore, in the following, we assume that

$$g_k(XY) = \sum_{p+q=k} g_p(X)g_q(Y) \tag{2.3}$$

for each $1 \leqslant k < n$ and $X, Y \in \mathcal{T}$. Next, we prove that Lemmas 2.4–2.6 hold.

By the definitions of $F = \{f_n\}_{n\in\mathbb{N}_0}$ and $G = \{g_n\}_{n\in\mathbb{N}_0}$, and by Lemma 2.1, we can easily check that the following Lemma holds:

Lemma 2.4. *Let $g_n : \mathcal{T} \to \mathcal{T}$ be defined as in Theorem 2.3. Then, for each $n \geq 1$ and for any $A_{ii} \in \mathcal{T}_{ii}, A_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$,*

(i) $g_n(P_i) = -g_n(P_j)$ and $g_n(P_i) = P_i g_n(P_i)P_j + P_j g_n(P_i)P_i$;

(ii) $P_j g_n(A_{ii})P_j = 0, P_i g_n(A_{ii})P_j = A_{ii} g_n(P_i)$ and $P_j g_n(A_{ii})P_i = g_n(P_i)A_{ii}$;

(iii) $g_n(A_{ij}) = P_i g_n(A_{ij})P_j$.

Lemma 2.5. *Let $g_n : \mathcal{T} \to \mathcal{T}$ be defined as in Theorem 2.3. Then, for each $n \geq 1$, and for any $A_{ii}, B_{ii} \in \mathcal{T}_{ii}$, $B_{jj} \in \mathcal{T}_{jj}, A_{ij}, B_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$,*

(i) $g_n(A_{ii}B_{ij}) = \sum_{p+q=n} g_p(A_{ii})g_q(B_{ij})$;

(ii) $g_n(A_{ij}B_{jj}) = \sum_{p+q=n} g_p(A_{ij})g_q(B_{jj})$;

(iii) $g_n(A_{ii}B_{ii}) = \sum_{p+q=n} g_p(A_{ii})g_q(B_{ii})$;

(iv) $g_n(A_{ii}B_{jj}) = \sum_{p+q=n} g_p(A_{ii})g_q B_{jj})$.

Proof. (i) For each $n \geq 1$ and for any $A_{ii} \in \mathcal{T}_{ii}, B_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$, taking $X = A_{ii}, Y = B_{ij}, Z = P_j$ in Eq (2.1), and by Lemma 1.1 (iii) and Lemma 2.1, we get

$$\begin{aligned}
d_n(A_{ii}B_{ij}) &= d_n(A_{ii} \circ B_{ij} \circ P_j) \\
&= \sum_{p+q+r=n} d_p(A_{ii}) \circ d_q(B_{ij}) \circ d_r(P_j) \\
&= \sum_{p+q+r=n,1\leq p,q,r} d_p(A_{ii}) \circ d_q(B_{ij}) \circ d_r(P_j) + \sum_{q+r=n,1\leq q,r} A_{ii} \circ d_q(B_{ij}) \circ d_r(P_j) \\
&+ \sum_{p+r=n,1\leq p,r} d_p(A_{ii}) \circ B_{ij} \circ d_r(P_j) + \sum_{p+q=n,1\leq p,q} d_p(A_{ii}) \circ d_q(B_{ij}) \circ P_j \\
&+ d_n(A_{ii}) \circ B_{ij} \circ P_j + A_{ii} \circ d_n(B_{ij}) \circ P_j + A_{ii} \circ B_{ij} \circ d_n(P_j) \\
&= \sum_{p+q=n,1\leq p,q} d_p(A_{ii}) \circ d_q(B_{ij}) \circ P_j + d_n(A_{ii}) \circ B_{ij} \circ P_j + A_{ii} \circ d_n(B_{ij}) \circ P_j \\
&= \sum_{p+q=n,1\leq p,q} d_p(A_{ii}) \circ d_q(B_{ij}) + d_n(A_{ii})B_{ij} + A_{ii}d_n(B_{ij}) + d_n(B_{ij})A_{ii} \\
&= \sum_{p+q=n} d_p(A_{ii})d_q(B_{ij}) + d_n(B_{ij})A_{ii}.
\end{aligned}$$

Therefore, it follows from Eq (2.2), with $f_n(A_{ii}) = 0$ and $f_n(A_{ij}) = P_j f_n(A_{ij})P_i$ $(n \geq 1)$, that

$$\begin{aligned}
g_n(A_{ii}B_{ij}) &= d_n(A_{ii}B_{ij}) - f_n(A_{ii}B_{ij}) \\
&= \sum_{p+q=n} d_p(A_{ii})d_q(B_{ij}) + d_n(B_{ij})A_{ii} - d_n(B_{ij})A_{ii} \\
&= \sum_{p+q=n,1\leq p,q} d_p(A_{ii})d_q(B_{ij}) + d_n(A_{ii})B_{ij} + A_{ii}d_n(B_{ij}) \\
&= \sum_{p+q=n,1\leq p,q} \{d_p(A_{ii}) - f_p(A_{ii})\}d_q(B_{ij}) + \{d_n(A_{ii}) - f_n(A_{ii})\}B_{ij} + A_{ii}\{d_n(B_{ij}) - f_n(B_{ij})\}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{p+q=n,1\le p,q} g_p(A_{ii})d_q(B_{ij}) + g_n(A_{ii})B_{ij} + A_{ii}g_n(B_{ij})\\
&= \sum_{p+q=n,1\le p,q} g_p(A_{ii})\{d_q(B_{ij}) - f_q(B_{ij})\} + g_n(A_{ii})B_{ij} + A_{ii}g_n(B_{ij})\\
&= \sum_{p+q=n,1\le p,q} g_p(A_{ii})g_q(B_{ij}) + g_n(A_{ii})B_{ij} + A_{ii}g_n(B_{ij})\\
&= \sum_{p+q=n} g_p(A_{ii})g_q(B_{ij}).
\end{aligned}
$$

Similarly, we get that (ii) holds.

(iii) For each $n \geq 1$ and for any $A_{ii}, B_{ii} \in \mathcal{T}_{ii}, X_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$, by Lemma 2.5 (i) and Eq (2.3), on the one hand, we get

$$
\begin{aligned}
g_n(A_{ii}B_{ii}X_{ij}) &= g_n((A_{ii}B_{ii})X_{ij})\\
&= \sum_{p+q=n,1\le q} g_p(A_{ii}B_{ii})g_q(X_{ij}) + g_n(A_{ii}B_{ii})X_{ij}\\
&= \sum_{p+q=n,1\le q} \{\sum_{r+s=p} g_r(A_{ii})g_s(B_{ii})\}g_q(X_{ij}) + g_n(A_{ii}B_{ii})X_{ij}\\
&= \sum_{r+s+q=n,1\le q} g_r(A_{ii})g_s(B_{ii})g_q(X_{ij}) + g_n(A_{ii}B_{ii})X_{ij}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
g_n(A_{ii}B_{ii}X_{ij}) &= g_n(A_{ii}(B_{ii}X_{ij}))\\
&= \sum_{p+q=n} g_p(A_{ii})g_q(B_{ii}X_{ij})\\
&= \sum_{p+q=n} g_p(A_{ii}) \sum_{r+s=q} g_r(B_{ii})g_s(X_{ij})\\
&= \sum_{p+r+s=n} g_p(A_{ii})g_r(B_{ii})g_s(X_{ij})\\
&= \sum_{p+r+s=n,1\le s} g_p(A_{ii})g_r(B_{ii})g_s(X_{ij}) + \sum_{p+r=n} g_p(A_{ii})g_r(B_{ii})X_{ij}.
\end{aligned}
$$

Comparing the above two equations, we get

$$
\{g_n(A_{ii}B_{ii}) - \sum_{p+r=n} g_p(A_{ii})g_r(B_{ii})\}X_{ij} = 0, \forall X_{ij} \in \mathcal{T}_{ij}(1 \leq i \neq j \leq 2).
$$

This yields from Lemma 1.1 (i) that

$$
P_i g_n(A_{ii}B_{ii})P_i = P_i\{\sum_{p+r=n} g_p(A_{ii})g_r(B_{ii})\}P_i. \tag{2.4}
$$

Next, we show that

$$
P_i g_n(A_{ii}B_{ii})P_j = P_i\{\sum_{p+r=n} g_p(A_{ii})g_r(B_{ii})\}P_j \text{ and } P_j g_n(A_{ii}B_{ii})P_i = P_j\{\sum_{p+r=n} g_p(A_{ii})g_r(B_{ii})\}P_i.
$$

Indeed, for each $n \geq 1$ and for any $A_{ii}, B_{ii} \in \mathcal{T}_{ii}$ $(1 \leq i \neq j \leq 2)$, taking $X = A_{ii}, Y = Z = P_j$ in Eq (2.1), by Lemma 2.1, we get

$$\begin{aligned}
0 &= d_n(A_{ii} \circ P_j \circ P_j) \\
&= \sum_{p+q+r=n} d_p(A_{ii}) \circ d_q(P_j) \circ d_r(P_j) \\
&= \sum_{p+q+r=n, 1\leq p,q,r} d_p(A_{ii}) \circ d_q(P_j) \circ d_r(P_j) + \sum_{q+r=n, 1\leq q,r} A_{ii} \circ d_q(P_j) \circ d_r(P_j) \\
&+ \sum_{p+r=n, 1\leq p,r} d_p(A_{ii}) \circ P_j \circ d_r(P_j) + \sum_{p+q=n, 1\leq p,q} d_p(A_{ii}) \circ d_q(P_j) \circ P_j \\
&+ d_n(A_{ii}) \circ P_j \circ P_j + A_{ii} \circ d_n(P_j) \circ P_j + A_{ii} \circ P_j \circ d_n(P_j) \\
&= \sum_{p+q=n, 1\leq p,q} d_p(A_{ii}) \circ d_q(P_j) \circ P_j + d_n(A_{ii}) \circ P_j \circ P_j + A_{ii} \circ d_n(P_j) \circ P_j \\
&= \sum_{p+q=n, 1\leq p,q} \{d_p(A_{ii})d_q(P_j) + d_q(P_j)d_p(A_{ii})\} \circ P_j \\
&+ d_n(A_{ii})P_j + P_j d_n(A_{ii}) + A_{ii}d_n(P_j) + d_n(P_j)A_{ii} \\
&= \sum_{p+q=n} d_p(A_{ii})d_q(P_j) + \sum_{p+q=n} d_q(P_j)d_p(A_{ii}).
\end{aligned}$$

Therefore, we get from $\sum_{p+q=n} d_p(A_{ii})d_q(P_j) \in \mathcal{T}_{ij}$ and $\sum_{p+q=n} d_q(P_j)d_p(A_{ii}) \in \mathcal{T}_{ji}$ that

$$\sum_{p+q=n} d_p(A_{ii})d_q(P_j) = 0 \text{ and } \sum_{p+q=n} d_q(P_j)d_p(A_{ii}) = 0.$$

So we get from $f_k(A_{ii}) = 0$ and $f_k(P_j) = 0$ $(k \geq 1)$ that

$$\begin{aligned}
0 = \sum_{p+q=n} d_p(A_{ii})d_q(P_j) &= \sum_{p+q=n, 1\leq p,q} d_p(A_{ii})d_q(P_j) + d_n(A_{ii})P_j + A_{ii}d_n(P_j) \\
&= \sum_{p+q=n, 1\leq p,q} (d_p(A_{ii}) - f_p(A_{ii}))(d_q(P_j) - f_q(P_j)) \\
&+ (d_n(A_{ii}) - f_n(A_{ii}))P_j + A_{ii}(d_n(P_j) - f_n(P_j)) \\
&= \sum_{p+q=n, 1\leq p,q} g_p(A_{ii})g_q(P_j) + g_n(A_{ii})P_j + A_{ii}g_n(P_j) \\
&= \sum_{p+q=n} g_p(A_{ii})g_q(P_j) \\
&= \sum_{p+q=n, 1\leq q} g_p(A_{ii})g_q(P_j) + g_n(A_{ii})P_j \\
&= -\sum_{p+q=n, 1\leq q} g_p(A_{ii})g_q(P_i) + g_n(A_{ii})P_j \\
&= -\sum_{p+q=n} g_p(A_{ii})g_q(P_i) + g_n(A_{ii})P_i + g_n(A_{ii})P_j \\
&= -\sum_{p+q=n} g_p(A_{ii})g_q(P_i) + g_n(A_{ii}).
\end{aligned}$$

Therefore,

$$g_n(A_{ii}) = \sum_{p+q=n} g_p(A_{ii})g_q(P_i). \tag{2.5}$$

For each $n \geq 1$ and for any $A_{ii}, B_{ii} \in \mathcal{T}_{ii}$, by Eq (2.5), we get

$$\begin{aligned}
g_n(A_{ii}B_{ii}) &= \sum_{p+q=n} g_p(A_{ii}B_{ii})g_q(P_i) \\
&= \sum_{p+q=n,1\leq q} g_p(A_{ii}B_{ii})g_q(P_i) + g_n(A_{ii}B_{ii})P_i \\
&= \sum_{p+q=n,1\leq q} \{\sum_{r+s=p} g_r(A_{ii})g_s(B_{ii})\}g_q(P_i) + g_n(A_{ii}B_{ii})P_i \\
&= \sum_{r+s+q=n,1\leq q} g_r(A_{ii})g_s(B_{ii})g_q(P_i) + g_n(A_{ii}B_{ii})P_i \\
&= \sum_{r=0}^{n} g_r(A_{ii})\{\sum_{s+q=n-r,1\leq q} g_s(B_{ii})g_q(P_i)\} + g_n(A_{ii}B_{ii})P_i \\
&= \sum_{r=0}^{n} g_r(A_{ii})\{\sum_{s+q=n-r} g_s(B_{ii})g_q(P_i) - g_{n-r}(B_{ii})P_i\} + g_n(A_{ii}B_{ii})P_i \\
&= \sum_{r=0}^{n} g_r(A_{ii})\{g_{n-r}(B_{ii}) - g_{n-r}(B_{ii})P_i\} + g_n(A_{ii}B_{ii})P_i \\
&= \sum_{r=0}^{n} g_r(A_{ii})g_{n-r}(B_{ii}) - \sum_{r=0}^{n} g_r(A_{ii})g_{n-r}(B_{ii})P_i + g_n(A_{ii}B_{ii})P_i \\
&= \sum_{p+q=n} g_p(A_{ii})g_q(B_{ii}) + \sum_{p+q=n} g_p(A_{ii})g_q(B_{ii})P_i + g_n(A_{ii}B_{ii})P_i.
\end{aligned}$$

This implies that

$$P_i g_n(A_{ii}B_{ii})P_j = P_i\{\sum_{p+q=n} g_p(A_{ii})g_q(B_{ii})\}P_j. \tag{2.6}$$

Similarly, we get

$$P_j g_n(A_{ii}B_{ii})P_i = P_j\{\sum_{p+q=n} g_p(A_{ii})g_q(B_{ii})\}P_i. \tag{2.7}$$

Therefore, by Eqs (2.4), (2.6), (2.7) and Lemma 2.4 (ii), we get that

$$\begin{aligned}
g_n(A_{ii}B_{ii}) &= P_i\{\sum_{p+q=n} g_p(A_{ii})g_q(B_{ii})\}P_i + P_i\{\sum_{p+q=n} g_p(A_{ii})g_q(B_{ii})\}P_j \\
&+ P_j\{\sum_{p+q=n} g_p(A_{ii})g_q(B_{ii})\}P_i \\
&= \sum_{p+q=n} g_p(A_{ii})g_q(B_{ii}).
\end{aligned}$$

(iv) For each $n \geq 1$ and for any $A_{ii} \in \mathcal{T}_{ii}, B_{jj} \in \mathcal{T}_{jj}$ $(1 \leq i \neq j \leq 2)$, taking $X = A_{ii}, Y = B_{jj}, Z = P_j$ $(1 \leq i \neq j \leq 2)$ in Eq (2.1) , we get from Lemma 2.1 that

$$
\begin{aligned}
0 &= d_n(A_{ii} \circ B_{jj} \circ P_j) \\
&= \sum_{p+q+r=n} d_p(A_{ii}) \circ d_q(B_{jj}) \circ d_r(P_j) \\
&= \sum_{p+q+r=n, 1\leq p,q,r} d_p(A_{ii}) \circ d_q(B_{jj}) \circ d_r(P_j) + \sum_{q+r=n, 1\leq q,r} A_{ii} \circ d_q(B_{jj}) \circ d_r(P_j) \\
&+ \sum_{p+r=n, 1\leq p,r} d_p(A_{ii}) \circ B_{jj} \circ d_r(P_j) + \sum_{p+q=n, 1\leq p,q} d_p(A_{ii}) \circ d_q(B_{jj}) \circ P_j \\
&+ d_n(A_{ii}) \circ B_{jj} \circ P_j + A_{ii} \circ d_n(B_{jj}) \circ P_j + A_{ii} \circ B_{jj} \circ d_n(P_j) \\
&= \sum_{p+q=n, 1\leq p,q} d_p(A_{ii}) \circ d_q(B_{jj}) \circ P_j + d_n(A_{ii}) \circ B_{jj} \circ P_j + A_{ii} \circ d_n(B_{jj}) \circ P_j \\
&= \sum_{p+q=n, 1\leq p,q} d_p(A_{ii}) d_q(B_{jj}) \circ P_j + \sum_{p+q=n, 1\leq p,q} d_p(B_{jj}) d_q(A_{ii}) \circ P_j \\
&+ d_n(A_{ii}) B_{jj} + B_{jj} d_n(A_{ii}) + A_{ii} d_n(B_{jj}) + d_n(B_{jj}) A_{ii} \\
&= \sum_{p+q=n, 1\leq p,q} d_p(A_{ii}) d_q(B_{jj}) + \sum_{p+q=n, 1\leq p,q} d_p(B_{jj}) d_q(A_{ii}) \\
&+ d_n(A_{ii}) B_{jj} + B_{jj} d_n(A_{ii}) + A_{ii} d_n(B_{jj}) + d_n(B_{jj}) A_{ii} \\
&= \sum_{p+q=n} d_p(A_{ii}) d_q(B_{jj}) + \sum_{p+q=n} d_p(B_{jj}) d_q(A_{ii}).
\end{aligned}
$$

Hence, we get from $\sum_{p+q=n} d_p(A_{ii}) d_q(B_{jj}) \in \mathcal{T}_{ij}$ and $\sum_{p+q=n} d_p(B_{jj}) d_q(A_{ii}) \in \mathcal{T}_{ji}$ $(1 \leq i \neq j \leq 2)$ that

$$\sum_{p+q=n} d_p(A_{ii}) d_q(B_{jj}) = \sum_{p+q=n} d_p(B_{jj}) d_q(A_{ii}) = 0.$$

Therefore, it follows from $g_k(A_{ii}) = d_k(A_{ii})$ and $g_k(B_{jj}) = d_k(B_{jj})$ $(k \geq 1)$ that

$$g_n(A_{ii} B_{jj}) = 0 = \sum_{p+q=n} d_p(A_{ii}) d_q(B_{jj}) = \sum_{p+q=n} g_p(A_{ii}) g_q(B_{jj}).$$

The proof is complete. □

Lemma 2.6. *Let $g_n : \mathcal{T} \to \mathcal{T}$ be defined as in Theorem 2.3. Then for each $n \geq 1$ and for any $A_{ii} \in \mathcal{T}_{ii}, B_{jj} \in \mathcal{T}_{jj}, A_{ij}, B_{ij} \in \mathcal{T}_{ij}, A_{ji}, B_{ji} \in \mathcal{T}_{ji}$ $(1 \leq i \neq j \leq 2)$,*

(i) $g_n(A_{ij} B_{ji}) = \sum_{p+q} g_p(A_{ij}) g_q(B_{ji})$;

(ii) $g_n(A_{ij} B_{ij}) = \sum_{p+q=n} g_p(A_{ij}) g_q(B_{ij})$;

(iii) $g_n(A_{ii} B_{ji}) = \sum_{p+q=n} g_p(A_{ii}) g_q(B_{ji})$;

(iv) $g_n(A_{ji} B_{jj}) = \sum_{p+q=n} g_p(A_{ji}) g_q(B_{jj})$.

Proof. (i) For each $n \geq 1$ and for any $A_{ij} \in \mathcal{T}_{ij}, B_{ji} \in \mathcal{T}_{ji}$ $(1 \leq i \neq j \leq 2)$, it follows from Lemma 1 (iii) that $A_{ij} B_{ji} = 0$, and therefore we get

$$g_n(A_{ij} B_{ji}) = g_n(0) = 0.$$

On the other hand, by Lemma 1 (*iii*) and Lemma 2.4 (*iii*), we have $g_p(A_{ij})g_q(B_{ji}) = 0$, therefore we get

$$g_n(A_{ij}B_{ji}) = 0 = \sum_{p+q=n} g_p(A_{ij})g_q(B_{ji}).$$

Similarly, we get that (*ii*) holds.

(*iii*) For each $n \geq 1$ and for any $A_{ii} \in \mathcal{T}_{ii}, B_{ji} \in \mathcal{T}_{ji}$ $(1 \leq i \neq j \leq 2)$, by Lemma 1 (*iii*) and Lemma 2.4 (*ii, iii*), we get $g_p(A_{ii})g_q(B_{ji}) = 0$, and therefore we get

$$g_n(A_{ii}B_{ji}) = 0 = \sum_{p+q=n} g_p(A_{ii})g_q(B_{ji}).$$

Similarly, we get (*iv*) holds. The proof is complete. □

In the following, we complete the proof of Theorem 2.3.

Proof of Theorem 2.3. For any $n \geq 1$, let $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$ be arbitrary elements of $\mathcal{T}$, where $A_{ij}, B_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i, j \leq 2)$. It follows from Lemmas 2.4–2.6 that

$$\begin{aligned}
g_n(AB) &= g_n((A_{11} + A_{12} + A_{21} + A_{22})(B_{11} + B_{12} + B_{21} + B_{22})) \\
&= g_n(A_{11}B_{11}) + g_n(A_{11}B_{12}) + g_n(A_{11}B_{21}) + g_n(A_{11}B_{22}) \\
&+ g_n(A_{12}B_{11}) + g_n(A_{12}B_{12}) + g_n(A_{12}B_{21}) + g_n(A_{12}B_{22}) \\
&+ g_n(A_{21}B_{11}) + g_n(A_{21}B_{12}) + g_n(A_{21}B_{21}) + g_n(A_{21}B_{22}) \\
&+ g_n(A_{22}B_{11}) + g_n(A_{22}B_{12}) + g_n(A_{22}B_{21}) + g_n(A_{22}B_{22}) \\
&= \sum_{p+q=n} g_p(A_{11})g_q(B_{11}) + \sum_{p+q=n} g_p(A_{11})g_q(B_{12}) \\
&+ \sum_{p+q=n} g_p(A_{11})g_q(B_{21}) + \sum_{p+q=n} g_p(A_{11})g_q(B_{22}) \\
&+ \sum_{p+q=n} g_p(A_{12})g_q(B_{11}) + \sum_{p+q=n} g_p(A_{12})g_q(B_{12}) \\
&+ \sum_{p+q=n} g_p(A_{12})g_q(B_{21}) + \sum_{p+q=n} g_p(A_{12})g_q(B_{22}) \\
&+ \sum_{p+q=n} g_p(A_{21})g_q(B_{11}) + \sum_{p+q=n} g_p(A_{21})g_q(B_{12}) \\
&+ \sum_{p+q=n} g_p(A_{21})g_q(B_{21}) + \sum_{p+q=n} g_p(A_{21})g_q(B_{22}) \\
&+ \sum_{p+q=n} g_p(A_{22})g_q(B_{11}) + \sum_{p+q=n} g_p(A_{22})g_q(B_{21}) \\
&+ \sum_{p+q=n} g_p(A_{22})g_q(B_{21}) + \sum_{p+q=n} g_p(A_{22})g_q(B_{22}) \\
&= \sum_{p+q=n} g_p(A_{11} + A_{12} + A_{21} + A_{22})g_q(B_{11} + B_{12} + B_{21} + B_{22}) \\
&= \sum_{p+q=n} g_p(A)g_q(B).
\end{aligned}$$

Therefore, $G = \{g_n\}_{n\in\mathbb{N}_0}$ is a higher derivation on $\mathcal{T}$. The proof is complete. □

Next, we show that Theorem 2.1 holds.

Proof of Theorem 2.1. For each $n \geq 1$ and for any $A, B \in \mathcal{T}$, by Theorems 2.2 and 2.3, we obtain that

$$d_n(A) = g_n(A) + f_n(A),$$

where $G = \{g_n\}_{n\in\mathbb{N}_0}$ is a higher derivation and $F = \{f_n\}_{n\in\mathbb{N}_0}$ is a higher anti-derivation from $\mathcal{T}$ into itself such that $f_n(A_{ii}) = 0$ for all $A_{ii} \in \mathcal{T}_{ii}$ $(1 \leq i \leq 2)$. The proof is complete. □

Remark 2.1. *Let $D = \{d_n\}_{n\in\mathbb{N}_0}$ be a higher Jordan triple derivation from $\mathcal{T}$ into itself. Then, by Theorems 2.1 and 2.2, we obtain that the following statements are equivalent.*

(i) $D = \{d_n\}_{n\in\mathbb{N}_0}$ is a higher derivation;

(ii) $P_j d_n(A_{ij})P_i = 0$ for each $n \geq 1$ and for any $A_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$;

(iii) $d_n(A_{ij}) \in \mathcal{T}_{ij}$ for each $n \geq 1$ and for any $A_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$.

In the following, we show that every higher Jordan triple derivation on triangular algebras is a higher derivation.

Corollary 2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module, and $\mathcal{U}$ be the 2-torsion free triangular algebra, and $D = \{d_n\}_{n\in\mathbb{N}_0}$ be a higher Jordan triple derivation on $\mathcal{U}$. Then $D = \{d_n\}_{n\in\mathbb{N}_0}$ is a higher derivation.*

Proof of Corollary 2.1. Let $1_{\mathcal{A}}$ and $1_{\mathcal{B}}$ be the identities of the algebras $\mathcal{A}$ and $\mathcal{B}$, respectively, and let 1 be the identity of the triangular algebra $\mathcal{U}$. We denote

$$P_1 = \begin{pmatrix} 1_{\mathcal{A}} & 0 \\ 0 & 0 \end{pmatrix} \text{ by the standard idempotent of } \mathcal{U}, \quad P_2 = 1 - P_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1_{\mathcal{B}} \end{pmatrix}$$

and

$$\mathcal{U}_{ij} = P_i \mathcal{U} P_j \text{ for } 1 \leq i \leq j \leq 2.$$

It is clear that the triangular algebra $\mathcal{U}$ may be represented as

$$\mathcal{U} = P_1\mathcal{U}P_1 + P_1\mathcal{U}P_2 + P_2\mathcal{U}P_2 = \mathcal{A} + \mathcal{M} + \mathcal{B}.$$

Here $P_1\mathcal{U}P_1$ and $P_2\mathcal{U}P_2$ are subalgebras of $\mathcal{U}$ isomorphic to $\mathcal{A}$ and $\mathcal{B}$, respectively, and $P_1\mathcal{U}P_2$ is a $(P_1\mathcal{U}P_1, P_2\mathcal{U}P_2)$-bimodule isomorphic to the $(\mathcal{A}, \mathcal{B})$-bimodule $\mathcal{M}$.

By the definition of triangular algebra $\mathcal{U}$, we can easily check that $\mathcal{U}$ is a $*$-type trivial extension algebra, and so if $\mathcal{U}$ is a 2-torsion free triangular algebra, then for any $n \geq 1$, $A = A_{11} + A_{12} + A_{22} \in \mathcal{U}$, where $A_{ij} \in \mathcal{U}_{ij}$ $(1 \leq i, j \leq 2)$, we get from Theorem 2.1 that

$$d_n(A) = g_n(A) + f_n(A).$$

Where $G = \{g_n\}_{n\in\mathbb{N}_0}$ is a higher derivation and $F = \{f_n\}_{n\in\mathbb{N}_0}$ is a higher anti-derivation from $\mathcal{U}$ into itself such that $f_n(A_{ii}) = 0$ for all $A_{ii} \in \mathcal{U}_{ii}$ $(1 \leq i \leq 2)$. Next, we show that $f_n(A_{12}) = 0$ for each $n \geq 1$ and for any $A_{12} \in \mathcal{U}_{12}$.

Indeed, for any $A_{12} \in \mathcal{U}_{12}$, it follows from Lemma 2.1 (v) and $\mathcal{U}_{21} = \{0\}$ that

$$d_n(A_{12}) = P_1 d_n(A_{12})P_2 + P_2 d_n(A_{12})P_1 = P_1 d_n(A_{12})P_2.$$

And then we obtain from the definition of f_n in Theorem 2.2 that $f_n(A_{12}) = P_2 d_n(A_{12})P_1 = 0$. Therefore, for any $A \in \mathcal{U}$, $f_n(A) = 0$, so $D = \{g_n\}_{n\in\mathbb{N}_0}$ is a higher derivation. The proof is complete. □

Next, we give an application of Corollary 2.1 to certain special classes of triangular algebras, such as block upper triangular matrix algebras and nest algebras.

Let $\mathcal{R}$ be a commutative ring with identity and let $M_{n\times k}(\mathcal{R})$ be the set of all $n \times k$ matrices over $\mathcal{R}$. For $n \geq 2$ and $m \leq n$, the block upper triangular matrix algebra $T_n^{\bar{k}}(\mathcal{R})$ is a subalgebra of $M_n(\mathcal{R})$ with the form

$$\begin{pmatrix} M_{k_1}(\mathcal{R}) & M_{k_1\times k_2}(\mathcal{R}) & \cdots & M_{k_1\times k_m}(\mathcal{R}) \\ 0 & M_{k_2}(\mathcal{R}) & \cdots & M_{k_2\times k_m}(\mathcal{R}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_m}(\mathcal{R}) \end{pmatrix},$$

where $\bar{k} = (k_1, k_2, \cdots, k_m)$ is an ordered m-vector of positive integers such that $k_1 + k_2 + \cdots + k_m = n$.

A nest of a complex Hilbert space $\mathcal{H}$ is a chain $\mathcal{N}$ of closed subspaces of $\mathcal{H}$ containing $\{0\}$ and $\mathcal{H}$, which is closed under arbitrary intersections and closed linear span, and $\mathcal{B}(\mathcal{H})$ is the algebra of all bounded linear operators on $\mathcal{H}$. The nest algebra associated with $\mathcal{N}$ is the algebra

$$\mathrm{Alg}\mathcal{N} = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq \mathcal{N}, \text{ for all } N \in \mathcal{N}\}.$$

A nest $\mathcal{N}$ is called trivial if $\mathcal{N} = \{0, \mathcal{H}\}$. It is clear that every nontrivial nest algebra is a triangular algebra and every finite dimensional nest algebra is isomorphic to a complex block upper triangular matrix algebra.

Corollary 2.2. *Let $T_n^{\bar{k}}(\mathcal{R})$ be a 2-torsion free block upper triangular matrix algebra, and $D = \{d_n\}_{n\in\mathbb{N}_0}$ be a higher Jordan triple derivation on $T_n^{\bar{k}}(\mathcal{R})$. Then, $D = \{d_n\}_{n\in\mathbb{N}_0}$ is a higher derivation.*

Corollary 2.3. *Let $\mathcal{N}$ be a nontrivial nest of a complex Hilbert space $\mathcal{H}$,* Alg$\mathcal{N}$ *a nest algebra, and $D = \{d_n\}_{n\in\mathbb{N}_0}$ a higher Jordan triple derivation on* Alg$\mathcal{N}$*. Then, $D = \{d_n\}_{n\in\mathbb{N}_0}$ is a higher derivation.*

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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