Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 6916–6932.
DOI: 10.3934/math.2024337
Received: 10 December 2023
Revised: 29 January 2024
Accepted: 08 February 2024
Published: 19 February 2024

Research article

Decay for thermoelastic laminated beam with nonlinear delay and nonlinear structural damping

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Abstract: This paper discussed the decay of a thermoelastic laminated beam subjected to nonlinear delay and nonlinear structural damping. We provided explicit and general energy decay rates of the solution by imposing suitable conditions on both weight delay and wave speeds. To achieve this, we leveraged the properties of convex functions and employed the multiplier technique as a specific approach to demonstrate our stability results.

Keywords: Laminated beam; Lyapunov functions; nonlinear damping; general decay; nonlinear delay; partial differential equations
Mathematics Subject Classification: 35B40, 35L56, 74F05, 93D15, 93D20

1. Introduction

In the current work, we study the following thermoelastic laminated beam along with nonlinear structural damping and nonlinear delay

$$\begin{cases} \varrho u_{tt} + G(\varphi - u_x)_x = 0, \\ I_\varrho(3v - \varphi)_{tt} - D(3v - \varphi)_{xx} - G(\varphi - u_x) = 0, \\ 3I_\varrho v_{tt} - 3Dv_{xx} + 3G(\varphi - u_x) + \gamma\theta_x + 4\delta v + \beta g_1(v_t(x,t)) + \mu g_2(v_t(x,t-\varsigma)) = 0, \\ \varrho_3\theta_t - k\theta_{xx} + \gamma v_{tx} = 0, \end{cases} \tag{1.1}$$

where

$$(x,t) \in (0,1) \times (0,\infty),$$

with the following initial and boundary conditions

$$\begin{cases} u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ \varphi(x,0) = \varphi_0(x),\ \theta(x,0) = \theta_0(x), \quad x \in (0,1), \\ u_t(x,0) = u_1(x),\ v_t(x,0) = v_1(x),\ \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,1), \\ u_x(0,t) = \varphi(0,t) = v(0,t) = \theta(0,t) = 0, \quad t > 0, \\ \varphi_x(1,t) = v_x(1,t) = u(1,t) = \theta(1,t) = 0, \quad t > 0, \\ v_t(x,t-\varsigma) = f_0(x,t-\varsigma), \quad (x,t) \in (0,1) \times (0,\varsigma). \end{cases} \tag{1.2}$$

Here, $u,\ \varphi,\ v$, and θ stand for the transverse displacement, the rotation angle, the amount of slip along the interface, and the difference temperature, respectively. $\varrho,\ G,\ I_\varrho,\ D,\ \delta$, and β are positive parameters representing the density, shear stiffness, mass moment of inertia, flexural rigidity, adhesive stiffness, and adhesive damping, respectively. We denote by $\varrho_3,\ k,\ \gamma$ the positive physical coefficients from thermoelasticity theory.

Herein $\varsigma > 0$ is the time delay, and the positive parameter μ is the delay weight.

The laminated beam is considered an interesting research subject owed to the broad business applicability of these materials across many industries, thus attracting the attention of researchers. Hansen and Spies in [1] proposed the following beam with two layers

$$\begin{cases} \varrho u_{tt} + G(\varphi - u_x)_x = 0, \\ I_\varrho(3v_{tt} - \varphi_{tt}) - D(3v_{xx} - \varphi_{xx}) - G(\varphi - u_x) = 0, \\ 3I_\varrho v_{tt} - 3Dv_{xx} + 3G(\varphi - u_x) + 4\delta v + 4\beta v_t = 0. \end{cases} \tag{1.3}$$

The model has a comparable character to the well-known classical Timoshenko system because its equations of movement were contrived based on the concepts of the Timoshenko beam theory. The impacts of the kinetics of interfacial slip are depicted by a third equation which interlocks with the first two ones. Due to their importance, these kind of problems are now highly regarded within the scientific community, and a revived resurgence of interest in examining the asymptotic behavior of the solution of diverse thermoelastic laminated beams has flourished nowadays, see [2–5]. For the readers, the background and the newest works on the qualitative properties related to this topic can be found, especially for the laminated beam, in [6, 7]. Recent studies have shown that delay may result in instability unless specific conditions are taken into account, and it can also lead to solutions that vary from those obtained in previous studies. Ensuring the stability of systems with delays is of utmost importance; hence, the studies of time delays has emerged as a critical and impactful field of research. Regarding the nonlinear delay, Mpungu and Apalara in [8] made a study worth mentioning, in which they took into account system (1.3) and incorporated nonlinear delay and nonlinear structural damping, specifically in the third equation. With the help of convenient conditions on both weight delay and wave speeds, the authors were able to establish a general energy decay rates of the solutions.

Djilali et al. [9] integrated a nonlinear delay into a viscoelastic Timoshenko beam problem and managed to demonstrate a global existence result, as well as asymptotic behavior of the solutions while presuming that a certain relation among the weight of the term with no delay and the weight of delay is maintained.

Concerning the researches on boundary stabilization. The work by Wang et al. in [10] was the first to present results and to prove an exponential decay result, the authors considered system (1.3) with mixed homogeneous, boundary conditions and unequal wave speeds. Many authors improved upon the work of [10], under the assumption that $\varrho G < I_\varrho$, to establish a similar exponential decay result [11,12].

Recently, Fayssal in [13], examined a thermoelastic laminated beam with structural damping and proved it to be exponentially stable when the condition below is valid:

$$\frac{\varrho}{G} = \frac{I_\varrho}{D}. \tag{1.4}$$

The remaining sections of the paper are organized as follows: In Section 2, we exhibit the study's major results after providing its necessary materials. In Section 3, we prove necessary lemmas that will support the proof of our results. In Section 4, once we go by the multiplier technique, our intended stability results are established.

2. Preliminaries

In this section, we give required assumptions and resources for our study, then we highlight our major results.

We start by setting the necessary assumptions as in [14]:

- ($\mathbf{A}_1$) The function $\mathfrak{g}_1 : \mathbb{R} \to \mathbb{R}$ is increasing and of class C^0. Moreover, there exist constants $\lambda_1,\ \lambda_2,\ \varepsilon > 0$ and a function $\mathfrak{X} \in C^1([0, +\infty))$, being strictly increasing, fulfilling $\mathfrak{X}(0) = 0$, and the latter is linear on $[0, \varepsilon]$ or strictly convex of class C^2 on $(0, \varepsilon]$, in a way that we have

$$\begin{cases} \mathfrak{r}^2 + \mathfrak{g}_1^2(\mathfrak{r}) \le \mathfrak{X}^{-1}(\mathfrak{r}\mathfrak{g}_1(\mathfrak{r})), & \text{for all } |\mathfrak{r}| \le \varepsilon, \\ \lambda_1|\mathfrak{r}| \le |\mathfrak{g}_1(\mathfrak{r})| \le \lambda_2|\mathfrak{r}|, & \text{for all } |\mathfrak{r}| \ge \varepsilon. \end{cases} \tag{2.1}$$

- ($\mathbf{A}_2$) The function $\mathfrak{g}_2 : \mathbb{R} \to \mathbb{R}$ is odd, increasing, and belongs to $C^1(\mathbb{R})$. In addition, there exist positive constants $\vartheta_*,\ \vartheta_1,\ \vartheta_2$, such that

$$|\mathfrak{g}_2'(\mathfrak{r})| \le \vartheta_*,$$

and

$$\vartheta_1 \mathfrak{r}\mathfrak{g}_2(\mathfrak{r}) \le \xi(\mathfrak{r}) \le \vartheta_2 \mathfrak{r}\mathfrak{g}_1(\mathfrak{r}), \tag{2.2}$$

where

$$\xi(\mathfrak{r}) = \int_0^{\mathfrak{r}} \mathfrak{g}_2(y)dy,$$

and

$$\vartheta_2\mu < \vartheta_1\beta. \tag{2.3}$$

Remark 2.1. *Exploiting assumption ($\mathbf{A}_1$), one can see that*

$$\mathfrak{r}\mathfrak{g}_1(\mathfrak{r}) > 0, \quad \forall \mathfrak{r} \ne 0.$$

It follows by ($\mathbf{A}_2$) and the monotonicity of $\mathfrak{g}_2$, *with the mean value theorem (for integrals) that*

$$\xi(\mathfrak{r}) \le \mathfrak{r}\mathfrak{g}_2(\mathfrak{r}), \tag{2.4}$$

therefore

$$\vartheta_1 \leq 1.$$

To deal with the nonlinearity of the delay, we shall present a constant κ that is positive and fulfilling

$$\frac{\mu(1-\vartheta_1)}{\vartheta_1} < \kappa < \frac{\beta - \vartheta_2 \mu}{\vartheta_2}. \tag{2.5}$$

As in [15], to begin, we introduce

$$\mathcal{S}(x,p,t) = v_t(x, t - \varsigma p) \quad \text{in } (0,1)\times(0,1)\times(0,\infty). \tag{2.6}$$

Thus, $\mathcal{S}$ satisfies

$$\varsigma \mathcal{S}_t(x,p,t) + \mathcal{S}_p(x,p,t) = 0. \tag{2.7}$$

Therefore, we obtain the following new system equivalent to the previous one (1.1)

$$\begin{cases} \varrho u_{tt} + G(\varphi - u_x)_x = 0, \\ I_\varrho(3v-\varphi)_{tt} - D(3v-\varphi)_{xx} - G(\varphi - u_x) = 0, \\ 3I_\varrho v_{tt} - 3Dv_{xx} + 3G(\varphi - u_x) + \gamma\theta_x + 4\delta v + \beta \mathrm{g}_1(v_t(x,t)) + \mu \mathrm{g}_2(\mathcal{S}(x,1,t)) = 0, \\ \varrho_3\theta_t - k\theta_{xx} + \gamma v_{tx} = 0, \\ \varsigma\mathcal{S}_t(x,p,t) + \mathcal{S}_p(x,p,t) = 0, \end{cases} \tag{2.8}$$

with

$$\begin{cases} u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ \varphi(x,0) = \varphi_0(x),\ \theta(x,0) = \theta_0(x), \quad x \in (0,1), \\ u_t(x,0) = u_1(x),\ v_t(x,0) = v_1(x),\ \varphi_t(x,0) = \varphi_1(x), \quad x \in (0,1), \\ u_x(0,t) = \varphi(0,t) = v(0,t) = \theta(0,t) = 0, \quad t > 0, \\ \varphi_x(1,t) = v_x(1,t) = u(1,t) = \theta(1,t) = 0, \quad t > 0, \\ \mathcal{S}(x,0,t) = v_t(x,t),\ \mathcal{S}(x,p,0) = f_0(x,-\varsigma p),\ (x,p) \in ((0,1))^2,\ t > 0. \end{cases} \tag{2.9}$$

Establishing the existence and uniqueness result is achievable by pursuing the reasoning behind the Faedo Galerkin approach, as expounded in [16]. To maintain simplicity, we will use $\mathcal{S}(p)$ to represent $\mathcal{S}(x,p,t)$.

Now, we shall present our energy of the system (2.8)-(2.9) by

$$\begin{aligned} \mathcal{E}(t) = {} & \frac{1}{2}\int_0^1 \left\{ \varrho u_t^2 + I_\varrho(3v_t - \varphi_t)^2 + D(3v_x - \varphi_x)^2 + 3I_\varrho v_t^2 + 3Dv_x^2 \right\} dx \\ & + \frac{1}{2}\int_0^1 \left\{ G(\varphi - u_x)^2 + 4\delta v^2 + \varrho_3\theta^2 \right\} dx + \int_0^1\int_0^1 \varsigma\kappa\xi(\mathcal{S}(p))\, dp dx, \end{aligned} \tag{2.10}$$

and right after, we exhibit the stability result.

Theorem 2.1. *Let $(u, \varphi, v, \theta, \mathcal{S})$ be the solution of (2.8)-(2.9). Suppose that ($\boldsymbol{A}_1$), ($\boldsymbol{A}_2$), and (1.4) hold, then, there exist positive constants $\alpha_0,\ \alpha_1,\ \alpha_2$, and ε_0 such that*

$$\mathcal{E}(t) \le \alpha_0 \mathfrak{X}_1^{-1}(\alpha_1 t + \alpha_2), \quad \forall t \ge 0, \tag{2.11}$$

where

$$\mathfrak{X}_1(t) = \int_t^1 \frac{1}{\mathfrak{X}_0(\mathfrak{r})} d\mathfrak{r},$$

and

$$\mathfrak{X}_0(t) = \begin{cases} t, & \text{if } \mathfrak{X} \text{ is linear on } [0, \varepsilon], \\ t\mathfrak{X}'(\varepsilon_0 t), & \text{if } \mathfrak{X}'(0) = 0 \text{ and } \mathfrak{X}'' > 0 \text{ on } (0, \varepsilon]. \end{cases}$$

Prior researches have given examples related to our stability result and assumptions; see [8].

3. Technical lemmas

The lemmas necessary to back up our proof of stability results will be established in this part. To achieve our stability result's proof, a specific method named the multiplier technique will be employed and a generic constant $K^* > 0$ will be used for the sake of simplicity. Note that K^* may change from line to line or in the same line.

Lemma 3.1. *Let $(u, \varphi, v, \theta, \mathcal{S})$ be the solution of (2.8)-(2.9), then, the energy functional satisfies*

$$\mathcal{E}'(t) \le -k \int_0^1 \theta_x^2 \, dx - \mathcal{M}_0 \int_0^1 v_t \mathfrak{g}_1(v_t) \, dx - \mathcal{M}_1 \int_0^1 \mathcal{S}(1) \mathfrak{g}_2(\mathcal{S}(1)) \, dx, \ \forall t \ge 0, \tag{3.1}$$

where $\mathcal{M}_0$ and $\mathcal{M}_1$ are positive constants.

Proof. To begin, let us multiply $(2.8)_1$–$(2.8)_4$ by u_t, $(3v_t - \varphi_t)$, v_t, and θ, respectively, then integrate over $(0, 1)$ and use integration by parts to get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_0^1 \left\{ \varrho u_t^2 + I_\varrho (3v_t - \varphi_t)^2 + D(3v_x - \varphi_x)^2 + 3I_\varrho v_t^2 + 3Dv_x^2 + 4\delta v^2 \right\} \, dx \\ &+ \frac{1}{2}\frac{d}{dt} \int_0^1 \left\{ G(\varphi - u_x)^2 + \varrho_3 \theta^2 \right\} \, dx \\ = &- k \int_0^1 \theta_x^2 \, dx - \beta \int_0^1 v_t \mathfrak{g}_1(v_t) \, dx - \mu \int_0^1 v_t \mathfrak{g}_2(\mathcal{S}(1)) \, dx. \end{aligned} \tag{3.2}$$

After that, we multiply Eq $(2.8)_5$ by $\kappa \mathfrak{g}_2(\mathcal{S}(p))$, integrate over $(0, 1) \times (0, 1)$, and notice that $\mathcal{S}(0) = v_t$, to find

$$\begin{aligned} \kappa\varsigma \int_0^1 \int_0^1 \mathfrak{g}_2(\mathcal{S}(p)) \mathcal{S}_t(p) \, dp dx &= -\kappa \int_0^1 \int_0^1 \partial_p \xi(\mathcal{S}(p)) \, dp dx \\ &= \kappa \int_0^1 \xi(\mathcal{S}(0)) dx - \kappa \int_0^1 \xi(\mathcal{S}(1)) dx \\ &= \kappa \int_0^1 \xi(v_t) dx - \kappa \int_0^1 \xi(\mathcal{S}(1)) dx, \end{aligned} \tag{3.3}$$

hence,

$$\kappa\varsigma\frac{d}{dt}\int_0^1\int_0^1 \xi\left(\mathcal{S}(p)\right)dpdx = \kappa\int_0^1 \xi(v_t)dx - \kappa\int_0^1 \xi(\mathcal{S}(1))dx,$$

which, together with both (3.2) and (2.2), gives us

$$\mathcal{E}'(t) \le -k\int_0^1 \theta_x^2 dx - (\beta - \vartheta_2\kappa)\int_0^1 v_t \mathrm{g}_1(v_t)dx - \kappa\int_0^1 \xi(\mathcal{S}(1))dx - \mu\int_0^1 v_t\mathrm{g}_2(\mathcal{S}(1))dx. \tag{3.4}$$

Let us now define the conjugate function of ξ by

$$\xi^*(\mathrm{r}) = \sup_{s\in\mathbb{R}^+}(\mathrm{r}s - \xi(\mathrm{r})),$$

thus, ξ^* is the Legendre transformation of ξ, and it is given as

$$\xi^*(\mathrm{r}) = \mathrm{r}(\xi')^{-1}(\mathrm{r}) - \xi\left[(\xi')^{-1}(\mathrm{r})\right], \quad \forall \mathrm{r} \ge 0. \tag{3.5}$$

In this way, the following relation is valid (see [14, 17])

$$\mathrm{r}s \le \xi^*(\mathrm{r}) + \xi(s), \quad \forall \mathrm{r}, s \ge 0. \tag{3.6}$$

Exploiting (3.5), along with the definition of ξ, leads to

$$\xi^*(\mathrm{r}) = \mathrm{r}\mathrm{g}_2^{-1}(\mathrm{r}) - \xi\left(\mathrm{g}_2^{-1}(\mathrm{r})\right). \tag{3.7}$$

The use of (3.7) together with (2.2) yields

$$\xi^*(\mathrm{g}_2(\mathcal{S}(1))) = \mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1)) - \xi(\mathcal{S}(1)) \le (1-\vartheta_1)\mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1)). \tag{3.8}$$

Therefore, taking advantage of (3.6), (3.8), and (2.2), we can write

$$\begin{aligned}
-\mu\int_0^1 v_t\mathrm{g}_2(\mathcal{S}(1))dx &\le \mu\int_0^1 \xi(v_t)dx + \mu\int_0^1 \xi^*(\mathrm{g}_2(\mathcal{S}(1)))dx \\
&\le \mu\int_0^1 \xi(v_t)dx + \mu(1-\vartheta_1)\int_0^1 \mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1))dx \\
&\le \vartheta_2\mu\int_0^1 v_t\mathrm{g}_1(v_t)\,dx + \mu(1-\vartheta_1)\int_0^1 \mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1))dx.
\end{aligned} \tag{3.9}$$

Finally, combining (3.9) and (3.4) with the help of (2.5) and (2.3), the estimate (3.1) is established. □

Lemma 3.2. *Consider the functional*

$$\mathcal{I}_1(t) = 3I_\varrho G\int_0^1 (3v-\varphi)v_t dx - \varrho D\int_0^1 (3v_x - \varphi_x)u_t dx - I_\varrho G\int_0^1 (3v_t - \varphi_t)u_x dx, \tag{3.10}$$

then, it satisfies

$$\begin{aligned}
\mathcal{I}_1'(t) \le &-\frac{GD}{2}\int_0^1 (3v_x-\varphi_x)^2 dx + \epsilon_1\int_0^1 (3v_t-\varphi_t)^2 dx + K^*\int_0^1 v_x^2 dx \\
&+\frac{K^*}{\epsilon_1}\int_0^1 v_t^2 dx + K^*\int_0^1 \mathrm{g}_1^2(v_t)dx + K^*\int_0^1 \mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1))dx \\
&+K^*\int_0^1 (\varphi-u_x)^2 dx + K^*\int_0^1 \theta_x^2 dx, \ \forall \epsilon_1 > 0.
\end{aligned} \tag{3.11}$$

Proof. We first take $\mathcal{I}_1'$, then exploit Eq $(2.8)_1$–$(2.8)_3$, integration by parts and $u_x = -(\varphi - u_x) + \varphi$, to reach

$$\begin{aligned}\mathcal{I}_1'(t) = &-GD\int_0^1 (3v_x-\varphi_x)^2dx + 3I_\varrho G\int_0^1 v_t(3v_t-\varphi_t)dx - 3G^2\int_0^1(\varphi-u_x)(3v-\varphi)dx\\ &-4\delta G\int_0^1(3v-\varphi)vdx - \gamma G\int_0^1(3v-\varphi)\theta_x dx - \beta G\int_0^1(3v-\varphi)\mathrm{g}_1(v_t)dx\\ &-\mu G\int_0^1(3v-\varphi)\mathrm{g}_2(\mathcal{S}(1))\,dx - G^2\int_0^1 u_x(\varphi-u_x)dx\\ &+(I_\varrho G-\varrho D)\int_0^1 u_t(3v-\varphi)_{xt}dx.\end{aligned} \tag{3.12}$$

Since $u_x = 3v - (\varphi - u_x) - (3v - \varphi)$ and hypothesis (1.4) holds, we find

$$\begin{aligned}\mathcal{I}_1'(t) = &-GD\int_0^1 (3v_x-\varphi_x)^2dx + 3I_\varrho G\int_0^1 v_t(3v_t-\varphi_t)dx - 2G^2\int_0^1(\varphi-u_x)(3v-\varphi)dx\\ &-4\delta G\int_0^1(3v-\varphi)vdx - \gamma G\int_0^1(3v-\varphi)\theta_x dx - \beta G\int_0^1(3v-\varphi)\mathrm{g}_1(v_t)dx\\ &-\mu G\int_0^1(3v-\varphi)\mathrm{g}_2(\mathcal{S}(1))\,dx + G^2\int_0^1 (\varphi-u_x)^2dx\\ &-3G^2\int_0^1 v(\varphi-u_x)dx.\end{aligned} \tag{3.13}$$

To continue, it is convenient to consider (2.4), along with (3.6) and (3.8), to obtain

$$\mathrm{g}_2^2(\mathcal{S}(1)) \le 2\mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1)). \tag{3.14}$$

We next apply (3.14) and Young and Poincaré's inequalities to finally get (3.11). □

Lemma 3.3. *Consider functional*

$$\mathcal{I}_2(t) := 3I_\varrho G\int_0^1 v_t(\varphi-u_x)dx - 3\varrho D\int_0^1 u_t v_x dx, \tag{3.15}$$

which satisfies, for any $\epsilon_2 > 0$,

$$\begin{aligned}\mathcal{I}_2'(t) \le &-G^2\int_0^1(\varphi-u_x)^2dx + \epsilon_2\int_0^1(3v_t-\varphi_t)^2dx + K^*\left(1+\frac{1}{\epsilon_2}\right)\int_0^1 v_t^2dx\\ &+K^*\int_0^1 v_x^2dx + K^*\int_0^1\theta_x^2dx + K^*\int_0^1 \mathrm{g}_1^2(v_t)dx + K^*\int_0^1\mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1))dx.\end{aligned} \tag{3.16}$$

Proof. We begin by differentiating $\mathcal{I}_2$, then we take advantage of both $(2.8)_{1,3}$, and integration by parts. We get

$$\begin{aligned}\mathcal{I}_2'(t) = &-3G^2\int_0^1(\varphi-u_x)^2dx + 3(\varrho D - I_\varrho G)\int_0^1 v_t u_{xt}dx + 3I_\varrho G\int_0^1 v_t\varphi_t dx\\ &-4\delta G\int_0^1(\varphi-u_x)vdx - \gamma G\int_0^1(\varphi-u_x)\theta_x d - \beta G\int_0^1(\varphi-u_x)\mathrm{g}_1(v_t)dx\\ &-\mu G\int_0^1(\varphi-u_x)\mathrm{g}_2(\mathcal{S}(1))\,dx.\end{aligned} \tag{3.17}$$

Since $\varphi_t = 3v_t - (3v_t - \varphi_t)$, and by (1.4), we have

$$\begin{aligned}\mathcal{I}_2'(t) = &-3G^2\int_0^1(\varphi - u_x)^2dx - 3I_\varrho G\int_0^1 v_t(3v_t - \varphi_t)dx - 4\delta G\int_0^1(\varphi - u_x)vdx\\ &+ 9I_\varrho G\int_0^1 v_t^2dx - \gamma G\int_0^1(\varphi - u_x)\theta_xdx - \beta G\int_0^1(\varphi - u_x)\mathrm{g}_1(v_t)dx\\ &- \mu G\int_0^1(\varphi - u_x)\mathrm{g}_2(\mathcal{S}(1))\,dx.\end{aligned}\tag{3.18}$$

Now, with the help of (3.14) and Young and Poincaré's inequalities, one can write

$$-4\delta G\int_0^1(\varphi - u_x)vdx \le \frac{\delta^2}{2}\int_0^1 v_x^2dx + \frac{G^2}{2}\int_0^1(\varphi - u_x)^2dx,\tag{3.19}$$

$$-\gamma G\int_0^1(\varphi - u_x)\theta_xdx \le \frac{\gamma^2}{2}\int_0^1\theta_x^2dx + \frac{G^2}{2}\int_0^1(\varphi - u_x)^2dx,\tag{3.20}$$

$$-\beta G\int_0^1(\varphi - u_x)\mathrm{g}_1(v_t)dx \le \frac{\beta^2}{2}\int_0^1\mathrm{g}_1^2(v_t)dx + \frac{G^2}{2}\int_0^1(\varphi - u_x)^2dx,\tag{3.21}$$

$$\begin{aligned}-\mu G\int_0^1(\varphi - u_x)\mathrm{g}_2(\mathcal{S}(1))dx &\le \frac{\mu^2}{2}\int_0^1\mathrm{g}_2^2(\mathcal{S}(1))dx + \frac{G^2}{2}\int_0^1(\varphi - u_x)^2dx,\\ &\le \mu^2\int_0^1\mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1))dx + \frac{G^2}{2}\int_0^1(\varphi - u_x)^2dx,\end{aligned}\tag{3.22}$$

and for any $\epsilon_2 > 0$,

$$-3I_\varrho G\int_0^1 v_t(3v_t - \varphi_t)dx \le \epsilon_2\int_0^1(3v_t - \varphi_t)^2dx + \frac{K^*}{\epsilon_2}\int_0^1 v_t^2dx.\tag{3.23}$$

The combination of (3.19)–(3.23) and (3.18) gives us (3.16). □

Lemma 3.4. *Consider the functional*

$$\mathcal{I}_3(t) := 3I_\varrho\int_0^1 vv_t\,dx - 3\varrho\int_0^1 v\int_0^x u_t(y)dydx,\tag{3.24}$$

then for any $\epsilon_3 > 0$*, it satisfies*

$$\begin{aligned}\mathcal{I}_3'(t) \le &-\delta\int_0^1 v^2dx - 3D\int_0^1 v_x^2dx + \epsilon_3\int_0^1 u_t^2dx + K^*\int_0^1\theta_x^2dx\\ &+ K^*\left(1 + \frac{1}{\epsilon_3}\right)\int_0^1 v_t^2dx + K^*\int_0^1\mathrm{g}_1^2(v_t)dx\\ &+ K^*\int_0^1\mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1))dx.\end{aligned}\tag{3.25}$$

Proof. We first find the derivative of $\mathcal{I}_3$, then exploit Eq $(2.8)_{1,3}$ along with integration by parts, to get

$$\begin{aligned}\mathcal{I}_3'(t) &= -4\delta \int_0^1 v^2 dx + 3I_\varrho \int_0^1 v_t^2 dx - 3D \int_0^1 v_x^2 dx - \gamma \int_0^1 v\theta_x dx \\ &- \beta \int_0^1 \mathrm{g}_1(v_t) v dx - \mu \int_0^1 \mathrm{g}_2\left(\mathcal{S}(1)\right) v dx - 3\varrho \int_0^1 v_t \int_0^x u_t(y) dy dx.\end{aligned}$$

By (3.14) and Young and Poincaré's inequalities, the proof is accomplished. □

Lemma 3.5. *Consider the functional*

$$\mathcal{I}_4(t) := -\varrho \int_0^1 u u_t \, dx, \tag{3.26}$$

then, it satisfies

$$\mathcal{I}_4'(t) \leq -\varrho \int_0^1 u_t^2 dx + K^* \int_0^1 v_x^2 dx + D \int_0^1 (3v_x - \varphi_x)^2 dx + K^* \int_0^1 (\varphi - u_x)^2 dx. \tag{3.27}$$

Proof. Taking $\mathcal{I}_4'$, Eq $(2.8)_1$, integration by parts and by $u_x = -(3v - \varphi) - (\varphi - u_x) + 3v$, we get

$$\begin{aligned}\mathcal{I}_4'(t) =& -\varrho \int_0^1 u_t^2 dx + G^2 \int_0^1 (\varphi - u_x)^2 dx - 3G \int_0^1 v(\varphi - u_x) \, dx \\ &+ G \int_0^1 (\varphi - u_x)(3v - \varphi) dx.\end{aligned} \tag{3.28}$$

By Young and Poincaré's inequalities, we establish (3.27).

□

Lemma 3.6. *Consider the functional*

$$\mathcal{I}_5(t) := -I_\varrho \int_0^1 (3v - \varphi)(3v - \varphi)_t dx, \tag{3.29}$$

then it satisfies

$$\mathcal{I}_5'(t) \leq 2D \int_0^1 (3v_x - \varphi_x)^2 dx - I_\varrho \int_0^1 (3v_t - \varphi_t)^2 dx + K^* \int_0^1 (\varphi - u_x)^2 dx. \tag{3.30}$$

Proof. By direct calculations, once we consider $(2.8)_2$ and integration by parts, we obtain

$$\mathcal{I}_5'(t) = D \int_0^1 (3v_x - \varphi_x)^2 dx - I_\varrho \int_0^1 (3v_t - \varphi_t)^2 d - G \int_0^1 (3v - \varphi)(\varphi - u_x) dx. \tag{3.31}$$

By Young and Poincaré's inequalities, we reach

$$-G \int_0^1 (3v - \varphi)(\varphi - u_x) dx \leq \frac{G^2}{4D} \int_0^1 (\varphi - u_x)^2 dx + D \int_0^1 (3v_x - \varphi_x)^2 dx. \tag{3.32}$$

Hence, the combination of (3.32) and (3.31) gives us (3.30). □

Lemma 3.7. *Consider the functional*

$$\mathcal{I}_6(t) := \varsigma \int_0^1 \int_0^1 e^{-p\varsigma} \xi(\mathcal{S}(p)) dp dx, \tag{3.33}$$

then it satisfies

$$\mathcal{I}_6'(t) \leq -\vartheta_1 e^{-\varsigma} \int_0^1 \mathcal{S}(1) \mathrm{g}_2(\mathcal{S}(1)) dx + \vartheta_2 \int_0^1 v_t \mathrm{g}_1(v_t) dx - \varsigma e^{-\varsigma} \int_0^1 \int_0^1 \xi(\mathcal{S}(p)) dp dx. \tag{3.34}$$

Proof. Taking both $\mathcal{I}_6'$ and Eq $(2.8)_5$, then exploiting $\mathcal{S}(0) = v_t$, we get

$$\begin{aligned}
\mathcal{I}_6'(t) &= \varsigma \int_0^1 \int_0^1 e^{-\varsigma p} \mathcal{S}_t(p) \mathrm{g}_2(\mathcal{S}(p))\, dp dx \\
&= -\int_0^1 \int_0^1 e^{-\varsigma p} \mathcal{S}_p(p) \mathrm{g}_2(\mathcal{S}(p))\, dp dx \\
&= -\int_0^1 \int_0^1 e^{-\varsigma p} \partial_p \xi(\mathcal{S}(p))\, dp dx \\
&= -\int_0^1 \int_0^1 \partial_p \left[e^{-\varsigma p} \xi(\mathcal{S}(p)) \right]\, dp dx - \varsigma \int_0^1 \int_0^1 e^{-\varsigma p} \xi(\mathcal{S}(p))\, dp dx \\
&= -e^{-\varsigma} \int_0^1 \xi(\mathcal{S}(1)) dx + \int_0^1 \xi(v_t) dx - \varsigma \int_0^1 \int_0^1 e^{-\varsigma p} \xi(\mathcal{S}(p))\, dp dx.
\end{aligned}$$

By using both (2.2) and $e^{-\varsigma} \leq e^{-p\varsigma} \leq 1,\ p \in (0,1)$, we then prove (3.34). □

4. Stability result

Our intended stability results are established here based on the previously stated lemmas.

Proof of Theorem 2.1. To begin, we consider a Lyapunov functional

$$\mathcal{K}(t) = N\mathcal{E}(t) + \sum_{i=1}^{6} N_i \mathcal{I}_i(t), \quad \forall t \geq 0, \tag{4.1}$$

where the constants $N,\ N_i > 0,\ i = 1 \cdots 6$, will be chosen later.

According to (4.1), we write

$$
\begin{aligned}
|\mathcal{K}(t)-N\mathcal{E}(t)| \le\; & N_1\varrho D\int_0^1 |u_t(3v_x-\varphi_x)|\,dx + 3N_1 I_\varrho G\int_0^1 |v_t(3v-\varphi)|\,dx \\
& + N_1 I_\varrho G\int_0^1 |u_x(3v_t-\varphi_t)|\,dx + 3N_2\varrho D\int_0^1 |u_t v_x|\,dx \\
& + 3N_2 I_\varrho G\int_0^1 |(\varphi-u_x)v_t|\,dx + 3N_3 I_\varrho\int_0^1 |v v_t|\,dx \\
& + 3N_3\varrho\int_0^1 \left|v\int_0^x u_t(y)dy\right| dx + N_4\varrho\int_0^1 |u_t u|\,dx \\
& + N_5 I_\varrho\int_0^1 |(3v-\varphi)_t(3v-\varphi)|\,dx \\
& + \varsigma N_6\int_0^1\int_0^1 e^{-p\varsigma}\,|\xi(\mathcal{S}(p))|\,dpdx.
\end{aligned}
$$

By Young, Cauchy-Schwarz, and Poincaré's inequalities, we have

$$
|\mathcal{K}(t)-N\mathcal{E}(t)| \le a\mathcal{E}(t), \quad \text{where } a>0,
$$

i.e.,

$$
(N-a)\mathcal{E}(t) \le \mathcal{K}(t) \le (N+a)\mathcal{E}(t). \tag{4.2}
$$

To continue, we take $\mathcal{K}'(t)$ and employ (3.1), (3.11), (3.16), (3.25), (3.27), (3.30), and (3.34), then we set

$$
N_1=\frac{8}{G},\ N_4=N_5=N_6=1, \quad \epsilon_1=\frac{I_\varrho}{4N_1},\ \epsilon_2=\frac{I_\varrho}{4N_2},\ \epsilon_3=\frac{\varrho}{2N_3},
$$

to get

$$
\begin{aligned}
\mathcal{K}'(t) \le & -\frac{I_\varrho}{2}\int_0^1 (3v_t-\varphi_t)^2dx - D\int_0^1 (3v_x-\varphi_x)^2dx - \left[G^2N_2-K^*\right]\int_0^1 (\varphi-u_x)^2dx \\
& -\frac{\varrho}{2}\int_0^1 u_t^2dx - \delta N_3\int_0^1 v^2dx - [3DN_3-K^*N_2-K^*]\int_0^1 v_x^2dx \\
& -[kN-K^*N_2-K^*N_3-K^*]\int_0^1 \theta_x^2dx - [\mathcal{M}_0N-\vartheta_2]\int_0^1 v_t\mathrm{g}_1(v_t)dx \\
& -[\mathcal{M}_1N-K^*N_2-K^*N_3-K^*+e^{-\varsigma}\vartheta_1]\int_0^1 \mathcal{S}(1)\mathrm{g}_2(\mathcal{S}(1))dx - \varsigma e^{-\varsigma}\int_0^1\int_0^1 \xi(\mathcal{S}(p))dpdx \\
& +[N_2K^*(1+N_2)+N_3K^*(1+N_3)+K^*]\int_0^1 v_t^2dx \\
& +[K^*N_2+K^*N_3+K^*]\int_0^1 \mathrm{g}_1^2(v_t)dx.
\end{aligned} \tag{4.3}
$$

We then select coefficients in (4.3), to make them all (with the exception of the last two) negative. By taking N_2 big enough such that

$$
G^2N_2-K^*>0,
$$

we can choose N_3 fairly large, so

$$3DN_3 - K^*N_2 - K^* > 0.$$

We set N big enough, to get (4.2) and

$$\begin{cases} \mathcal{M}_1 N - K^*N_2 - K^*N_3 - K^* + e^{-\varsigma}\vartheta_1 > 0, \\ kN - K^*N_2 - K^*N_3 - K^* > 0, \\ \mathcal{M}_0 N - \vartheta_2 > 0. \end{cases}$$

These choices, with Poincaré's inequality, lead to

$$\mathcal{K}'(t) \le -\vartheta_3 \mathcal{E}(t) + \vartheta_4 \int_0^1 \left(v_t^2 + \mathfrak{g}_1^2(v_t)\right) dx, \quad \vartheta_3, \vartheta_4 > 0, \ \forall t \ge 0. \tag{4.4}$$

In the context of our demonstration, we have two cases to treat:

Case 1. Suppose that $\mathfrak{X}$ is linear on $[0, \varepsilon]$. By hypothesis $(\mathbf{A}_1)$, we have

$$\begin{cases} \lambda_1 \mathfrak{r}^2 \le \mathfrak{r}\mathfrak{g}_1(\mathfrak{r}) \le \lambda_2 \mathfrak{r}^2, \\ \mathfrak{r}\lambda_1 \mathfrak{g}_1(\mathfrak{r}) \le \mathfrak{g}_1^2(\mathfrak{r}) \le \mathfrak{r}\lambda_2 \mathfrak{g}_1(\mathfrak{r}), \quad \forall \mathfrak{r} \in \mathbb{R}, \end{cases}$$

which, when combined with (4.4), results in

$$\mathcal{K}'(t) \le -\vartheta_3 \mathcal{E}(t) + \bar{\vartheta}_4 \int_0^1 v_t \mathfrak{g}_1(v_t) dx, \quad \bar{\vartheta}_4 > 0. \tag{4.5}$$

By merging (3.1) and (4.5), we find

$$\mathcal{K}'(t) \le -\vartheta_3 \mathcal{E}(t) - \vartheta_5 \mathcal{E}'(t), \quad \vartheta_5 > 0. \tag{4.6}$$

We will now proceed by presenting

$$\mathcal{K}_\star(t) := \mathcal{K}(t) + \vartheta_5 \mathcal{E}(t), \quad \forall t \ge 0. \tag{4.7}$$

Once considering (4.2), we see that

$$\bar{a}_1 \mathcal{E}(t) \le \mathcal{K}_\star(t) \le \bar{a}_2 \mathcal{E}(t), \quad \bar{a}_1, \bar{a}_2 > 0. \tag{4.8}$$

Consequently, when we consider (4.7) and (4.8), we get

$$\mathcal{K}_\star'(t) \le -\alpha_1 \mathcal{K}_\star(t), \quad \alpha_1 = \frac{\vartheta_3}{\bar{a}_2}. \tag{4.9}$$

Finally, we conclude by simply integrating (4.9) and employing (4.8), to prove that

$$\mathcal{E}(t) \le \alpha_0 e^{-\alpha_1 t}, \quad \text{where } \alpha_0 = \frac{\bar{a}_2 \mathcal{E}(0)}{\bar{a}_1}, \quad \forall t \ge 0. \tag{4.10}$$

Case 2. Suppose that $\mathfrak{X}$ is nonlinear on $(0, \varepsilon]$. We take as in [18], $0 < \varepsilon_1 \le \varepsilon$, to have

$$\mathfrak{r}\mathfrak{g}_1(\mathfrak{r}) \le \min\{\varepsilon, \mathfrak{X}(\varepsilon)\}, \quad \forall |\mathfrak{r}| \le \varepsilon_1.$$

It is helpful to consider the continuous function $\mathfrak{g}_1$, with $(\mathbf{A}_1)$ and to note that $|\mathfrak{g}_1(\mathfrak{r})| > 0,\ \mathfrak{r} \neq 0$, to have

$$\begin{cases} \mathfrak{r}^2 + \mathfrak{g}_1^2(\mathfrak{r}) \leq \mathfrak{X}^{-1}(\mathfrak{r}\mathfrak{g}_1(\mathfrak{r})), & |\mathfrak{r}| \leq \varepsilon_1, \\ \lambda_1|\mathfrak{r}| \leq |\mathfrak{g}_1(\mathfrak{r})| \leq \lambda_2|\mathfrak{r}|, & |\mathfrak{r}| \geq \varepsilon_1. \end{cases} \tag{4.11}$$

Now, we need to work on estimating

$$\int_0^1 \left(v_t^2 + \mathfrak{g}_1^2(v_t)\right) dx.$$

To this end, we consider, as in [19], the partitions below

$$B_1 = \{x \in (0,1) \ : \ |v_t| \leq \varepsilon_1\}, \quad B_2 = \{x \in (0,1) \ : \ |v_t| > \varepsilon_1\}.$$

The combination of the Jensen's inequality with the concavity of $\mathfrak{X}^{-1}$, results in

$$\mathfrak{X}^{-1}(B(t)) \geq \lambda_5 \int_{B_1} \mathfrak{X}^{-1}(v_t\mathfrak{g}_1(v_t))\, dx, \tag{4.12}$$

where

$$B(t) = \int_{B_1} v_t\mathfrak{g}_1(v_t)dx, \quad \text{and} \quad \lambda_5 > 0.$$

If we take (3.1), (4.11), and (4.12), we get

$$\begin{aligned} \int_0^1 \left(v_t^2 + \mathfrak{g}_1^2(v_t)\right)\, dx &= \int_{B_1} \left(v_t^2 + \mathfrak{g}_1^2(v_t)\right)\, dx + \int_{B_2} \left(v_t^2 + \mathfrak{g}_1^2(v_t)\right)\, dx \\ &\leq \int_{B_1} \mathfrak{X}^{-1}(v_t\mathfrak{g}_1(v_t))\, dx + \lambda_6 \int_{B_2} (v_t\mathfrak{g}_1(v_t))\, dx \\ &\leq \lambda_6\mathfrak{X}^{-1}(B(t)) - \lambda_6\mathcal{E}'(t), \quad \lambda_6 > 0. \end{aligned} \tag{4.13}$$

We then present the functional

$$\mathcal{K}_0(t) := \mathcal{K}(t) + \lambda_7\mathcal{E}(t), \quad \text{where } \lambda_7 > 0. \tag{4.14}$$

Relation (4.2) implies that

$$\eth_1\mathcal{E}(t) \leq \mathcal{K}_0(t) \leq \eth_2\mathcal{E}(t), \quad \eth_1, \eth_2 > 0. \tag{4.15}$$

Thus, once we merge (4.13) and (4.4) and exploit (4.14), we conclude that

$$\mathcal{K}_0'(t) \leq -\vartheta_3\mathcal{E}(t) + \lambda_7\mathfrak{X}^{-1}(B(t)), \quad \forall t \geq 0. \tag{4.16}$$

Let us now consider the functional below

$$\mathcal{K}_1(t) := \mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{K}_0(t) + \gamma_0\mathcal{E}(t), \quad \varepsilon_0 < \varepsilon, \ \ \gamma_0 > 0. \tag{4.17}$$

Combining (4.15) and the fact that

$$\mathcal{E}' \leq 0, \ \mathfrak{X}' > 0, \ \mathfrak{X}'' > 0, \quad \text{on } (0, \varepsilon],$$

we get

$$\bar{\eth}_1\mathcal{E}(t) \leq \mathcal{K}_1(t) \leq \bar{\eth}_2\mathcal{E}(t), \quad \bar{\eth}_1, \bar{\eth}_2 > 0. \tag{4.18}$$

Additionally, relation (4.16) yields

$$\begin{aligned}\mathcal{K}_1'(t) &= \varepsilon_0 \frac{\mathcal{E}'(t)}{\mathcal{E}(0)} \mathfrak{X}'' \left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) \mathcal{K}_0(t) + \mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{K}_0'(t) + \gamma_0\mathcal{E}'(t) \\ &\leq -\vartheta_3 \mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathcal{E}(t) + \lambda_7\mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathfrak{X}^{-1}(B(t)) + \gamma_0\mathcal{E}'(t).\end{aligned} \tag{4.19}$$

Let us set

$$Q = \lambda_7 \mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathfrak{X}^{-1}(B(t)).$$

Similar to what we did earlier with (3.5), we shall now estimate Q by letting $\mathfrak{X}^*$ be the convex conjugate of $\mathfrak{X}$ given by

$$\mathfrak{X}^*(\mathfrak{r}) = \mathfrak{r}(\mathfrak{X}')^{-1}(\mathfrak{r}) - \mathfrak{X}\left[(\mathfrak{X}')^{-1}(\mathfrak{r})\right] \leq \mathfrak{r}(\mathfrak{X}')^{-1}(\mathfrak{r}), \quad \text{where } \mathfrak{r} \in (0, \mathfrak{X}'(\varepsilon)). \tag{4.20}$$

Additionally, the use of the general Young's inequality, indicates

$$\mathfrak{r}s \leq \mathfrak{X}^*(\mathfrak{r}) + \mathfrak{X}(s), \quad \text{where } \mathfrak{r} \in (0, \mathfrak{X}'(\varepsilon)),\ s \in (0, \varepsilon]. \tag{4.21}$$

We set

$$\mathfrak{r} = \mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right), \quad \text{and } s = \mathfrak{X}^{-1}(B(t)),$$

By (4.20), (4.21), and

$$B(t) = \int_{B_1} v_t \mathfrak{g}_1(v_t)dx \leq \int_0^1 v_t\mathfrak{g}_1(v_t)dx \leq -\frac{1}{\mathcal{M}_0}\mathcal{E}'(t),$$

we have

$$Q = \lambda_7\mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right)\mathfrak{X}^{-1}(B(t)) \leq \lambda_7\varepsilon_0\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) - \lambda_8\mathcal{E}'(t),\ \lambda_8 > 0. \tag{4.22}$$

The replacement of (4.22) into (4.19) leads to

$$\mathcal{K}_1'(t) \leq -\left[\vartheta_3\mathcal{E}(0) - \lambda_7\varepsilon_0\right]\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) + (\gamma_0 - \lambda_8)\mathcal{E}'(t). \tag{4.23}$$

Now, selecting $\varepsilon_0 = \frac{\vartheta_3\mathcal{E}(0)}{2\lambda_7}$ and $\gamma_0 = 2\lambda_8$ gives us

$$\mathcal{K}_1'(t) \leq -\tilde{\vartheta}_3\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) + \lambda_8\mathcal{E}'(t); \quad \tilde{\vartheta}_3 = \frac{\vartheta_3\mathcal{E}(0)}{2},$$

and provided that $\mathcal{E}'(t) \leq 0$, we get

$$\mathcal{K}_1'(t) \leq -\tilde{\vartheta}_3\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\mathfrak{X}'\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\varepsilon_0\right) = -\tilde{\vartheta}_3\mathfrak{X}_0\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right), \tag{4.24}$$

where $\mathfrak{X}_0(\mathfrak{r}) = \mathfrak{r}\mathfrak{X}'(\varepsilon_0\mathfrak{r})$.

Now, $\mathfrak{X}$ being strictly convex on $(0, \varepsilon]$, implies that $\mathfrak{X}_0(\mathfrak{r}),\ \mathfrak{X}_0'(\mathfrak{r}) > 0$ on $(0, 1]$. Hence, letting

$$\mathcal{K}_{1*}(t) := \frac{\tilde{a}_1 \mathcal{K}_1(t)}{\mathcal{E}(0)}, \tag{4.25}$$

we find that

$$\tilde{a}_1 \mathcal{E}(t) \le \mathcal{K}_{1*}(t) \le \tilde{a}_2 \mathcal{E}(t), \quad \tilde{a}_1,\ \tilde{a}_2 > 0. \tag{4.26}$$

Furthermore, the employment of (4.24), results in

$$\mathcal{K}_{1*}'(t) \le -\frac{\tilde{a}_1 \tilde{\vartheta}_3}{\mathcal{E}(0)} \mathfrak{X}_0\left(\frac{\mathcal{E}(t)}{\mathcal{E}(0)}\right).$$

In addition, if we take into account (4.26) and that $\mathfrak{X}_0$ is increasing, we achieve

$$\mathcal{K}_{1*}'(t) \le -\alpha_1 \mathfrak{X}_0\left(\mathcal{K}_{1*}(t)\right), \quad \alpha_1 > 0,\ \forall t \ge 0. \tag{4.27}$$

According to (4.27), we have

$$\left[\mathfrak{X}_1\left(\mathcal{K}_{1*}(t)\right)\right]' \ge \alpha_1, \tag{4.28}$$

where

$$\mathfrak{X}_1(t) = \int_t^1 \frac{1}{\mathfrak{X}_0(\mathfrak{r})}\, d\mathfrak{r}.$$

If we integrate (4.28) over $(0, t)$, we get

$$\mathfrak{X}_1\left(\mathcal{K}_{1*}(t)\right) \ge \alpha_1 t + \alpha_2, \quad \alpha_2 = \mathfrak{X}_1\left(\mathcal{K}_{1*}(0)\right),\ \forall t \ge 0. \tag{4.29}$$

Since $\mathfrak{X}_1^{-1}$ is a decreasing function, we deduce

$$\mathcal{K}_{1*}(t) \le \mathfrak{X}_1^{-1}(\alpha_1 t + \alpha_2). \tag{4.30}$$

We exploit relation (4.26) to ultimately achieve

$$\mathcal{E}(t) \le \alpha_0 \mathfrak{X}_1^{-1}(\alpha_1 t + \alpha_2),\ \forall t \ge 0, \tag{4.31}$$

where $\alpha_0 = \frac{1}{\tilde{a}_1}$. The proof is then concluded.

□

5. Conclusions

A class of thermoelastic laminated beams is considered. In addition to the impact of thermoelasticity, we are interested here in the interaction between the weights of two terms with delay and without delay given in nonlinear forms. We have shown explicit and general energy decay rates of the solution by using the properties of convex functions and employing the multiplier technique.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was funded by King Khalid University through large research project under grant number R.G.P.2/252/44.

Conflict of interest

The authors declare that there are no conflicts of interest.

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