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Research article

Metrization of soft metric spaces and its application to fixed point theory

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Abstract: Soft set theory has attracted many researchers from several different branches. Sound theoretical improvements are accompanied with successful applications to practical solutions of daily life problems. However, some of the attempts of generalizing crisp concepts into soft settings end up with completely equivalent structures. This paper deals with such a case. The paper mainly presents the metrizability of the soft topology induced by a soft metric. The soft topology induced by a soft metric is known to be homeomorphic to a classical topology. In this work, it is shown that this classical topology is metrizable. Moreover, the explicit construction of an ordinary metric that induces the classical topology is given. On the other hand, it is also shown that soft metrics are actually cone metrics. Cone metrics are already proven to be an unsuccessful attempt of generalizing metrics. These results clarify that most, if not all, properties of soft metric spaces could be directly imported from the related classical theory. The paper concludes with an application of the findings, i.e., a new soft fixed point theorem is stated and proven with the help of the obtained homemorphism.

Keywords: soft metric space; soft topology; topology; metrizability; fixed point
Mathematics Subject Classification: 47H10, 54A05, 54E35

1. Introduction

The problem of dealing with uncertainties and imprecise data has led to several different types of sets. Among these alternatives, soft sets were introduced by Molodtsov in 1999 [1]. Soft set theory has rapidly widened its scope in various directions, including practical and theoretical fields. Generalizations of soft sets followed soonafter. These include fuzzy soft sets [2], soft rough sets [3], N-soft sets [4] and neutrosophic soft sets [5]. The applications to decision making problems start with Maji et al. [6], followed by many papers such the recent examples [7–12]. A synthesis of algebra with soft sets began with [13] and was followed by several investigations (see, for example, [14–17]). The relation between crisp algebras and soft algebras was recently studied in [18]. The interaction between soft sets and topology, called soft topology, was initiated by Shabir and Naz [19] and Çağman et al. [20] in 2011. A huge amount of research, including some further extensions of soft topologies, has been added to the literature since then (for up-to-date studies among others readers may refer to [21–25]). Studies in the direction of soft topology, where a particular choice of soft point definition is adopted, came to an end, with the paper by Matejdes [26]. Matejdes proved that soft topological spaces actually inherit a topology in the ordinary sense. The two-way transition between soft topologies and standard topologies are later shown explicitly in the study by Alcantud [27].

Another research area dealing with the notion of soft metrics was introduced by Das and Samanta [28]. After defining soft metrics, the authors showed that every soft metric induced a soft topology. Many papers dealing with fixed-point theorems in this new setting followed. Some recent examples are [29, 30]. On the other hand, there have been many other attempts on generalizations of metric spaces. A survey can be found in [31]. Among these generalizations, the cone metrics [32] could not survive after the papers [33, 34], where the metrizability of cone metric spaces were discussed. Both works obtained the same result with different methods, i.e., cone metrics do not form a new family of metric spaces.

Briefly, we observe that the interest in softs sets has quickly grown and many productive theoretical and applicational progresses have been achieved in many different branches. However, we also observe that some attempts of generalizing crisp concepts into corresponding soft settings end up with completely equivalent theories. That is, a two way transformation between the crisp and soft setting exists. This paper is about a particular one of such cases. The main objective of this paper is to show that the soft topology induced by a soft metric is actually metrizable with a crisp metric. This will be done by explicitly constructing a metric that produces the topology that is homeomorphic to the soft topology, induced by the given soft metric. It will also be shown that every soft metric is a cone metric. These observations will yield a sound understanding of the topological anatomy of soft metric spaces. Moreover, the establishment of the connection between soft metrics, classical metrics and former generalizations of metrics, will enable the transformation of most fixed-point theorems into the soft setting, as is pointed out in [35, 36].

Finally, the findings will be applied to state and prove a new soft fixed-point theorem. This is to show, how classical fixed-point theorems can be easily transformed into a soft fixed-point theorem.

2. Preliminaries

Definition 1. *[1] Let X and E be two nonempty sets. Denote the power set of X with $P(X)$. The pair (F, E) is called a soft set over X, where F is a mapping from E to $P(X)$.*

In this definition, the set E is intended to represent the set of parameters. In this way, a soft set can be viewed as a family of subsets of X each connected to some (if any) of the parameters.

During the paper, it will be assumed that the parameter set E is finite and nonempty and we will denote the number of its elements by $s(E)$. This assumption should not be considered to be restrictive since an infinite parameter set is not expected for the practical reasons the soft sets are introduced for. Moreover, many results claimed in the literature do actually not hold in the case of infinite number of parameters.

The particular case where the function ϵ: $E \to X$ is defined into X instead of $P(X)$ is called a soft element in X.

It is clear that every soft set (A, E) for which $A(e)$ is a one point set, for all $e \in E$, is in a one-to-one correspondence with a soft element:

$$(A, E) = \epsilon \Leftrightarrow A(e) = \epsilon(e), \quad \forall e \in E.$$

Definition 2. *[37] Let $B(\mathbb{R})$ denote the set of all nonempty bounded subsets of the real numbers and E be a set of parameters. Then, a mapping $F\colon E \to B(\mathbb{R})$ is called a soft real set. In the particular case where the soft real set (F, E) is a one-point set for each $e \in E$, after identifying (F, E) with the corresponding soft element, it will be called a soft real number.*

Notations such as $\widetilde{m}, \widetilde{n}$, and $\widetilde{r}$ are used to denote soft reals whereas $\overline{m}, \overline{n}$, and $\bar{r}$ denote constant soft real numbers. For constant soft real numbers, we have for example, $\overline{m}(e) = m$, for all $e \in E$.

One can easily observe that for a set of parameters E with $s(E) = k$, a soft real number can be identified with a vector in $\mathbb{R}^k$ by

$$\widetilde{m} \longleftrightarrow (\widetilde{m}(e_1), \widetilde{m}(e_2), \cdots, \widetilde{m}(e_k)).$$

In the sequel we will consider the standard topology on $\mathbb{R}^k$, and $\|.\|_\infty$ will denote the maximum norm.

Definition 3. *[37] A partial ordering $\lesseqqgtr$ is defined on the set of soft reals as follows:*

(i) $\widetilde{m} \lesseqqgtr \widetilde{n}$ if $\widetilde{m}(e) \leq \widetilde{n}(e)$, for all $e \in E$,

(ii) $\widetilde{m} \lesssim \widetilde{n}$ if $\widetilde{m}(e) < \widetilde{n}(e)$, for all $e \in E$, where $\widetilde{m}, \widetilde{n}$ are soft real numbers.

The arithmetic operations on soft real numbers and concepts such as positivity can be defined in a natural way and those definitions are omitted.

Definition 4. *A soft set (P, E) over X is said to be a soft point if there are $\lambda \in E$ and $x \in X$, such that $P(\lambda) = \{x\}$ and $P(\mu) = \emptyset, \forall \mu \in E \backslash \{\lambda\}$. In this case (P, E) will be denoted by P^x_λ.*

Definition 5. *A soft point P^x_λ is in a soft set (A, E) if*

$$P(\lambda) = \{x\} \subset A(\lambda)$$

and this will be shown with $P^x_\lambda \widetilde{\in} (F, E)$.

It should be noted that a soft set is nothing but a collection of all soft points belonging to it, i.e.,

$$(F, E) = \cup_{P^x_\lambda \widetilde{\in} (F,E)} P^x_\lambda.$$

In the following, $\widetilde{X}$ will denote the absolute soft set defined by

$$F(\lambda) = X, \forall \lambda \in E.$$

$SP(\widetilde{X})$ will stand for the collection of all soft points of $\widetilde{X}$. By $\mathbb{R}(E)^*$, we will show those soft real numbers which are not negative. A soft metric on $SP(\widetilde{X})$ is defined as follows.

Definition 6. *A mapping*

$$d : SP(\widetilde{X}) \times SP(\widetilde{X}) \to \mathbb{R}(E)^*$$

is said to be a soft metric on the soft set $\widetilde{X}$ if d satisfies the following conditions for all $P^x_\lambda, P^y_\mu, P^z_\nu \widetilde{\in} \widetilde{X}$:

$(M1)$ $d(P^x_\lambda, P^y_\mu) = \overline{0}$ *if and only if* $P^x_\lambda = P^y_\mu$,

$(M2)$ $d(P^x_\lambda, P^y_\mu) = d(P^y_\mu, P^x_\lambda)$,

$(M3)$ $d(P^x_\lambda, P^z_\nu) \lessapprox d(P^x_\lambda, P^y_\mu) + d(P^y_\mu, P^z_\nu)$.

The soft set $\widetilde{X}$ *with a soft metric d is called a soft metric space. This space is shown with the triple* $(\widetilde{X}, d, E)$ *or shortly with the pair* $(\widetilde{X}, d)$ *if there is no confusion about the parameter set E.*

The family of open balls defined by

$$B_d((x,\lambda),\widetilde{r}) = \{P^y_\mu \in SP(\widetilde{X}) : d(P^x_\lambda, P^y_\mu) \lesssim \widetilde{r}\}$$

forms a basis for a soft topology on $\widetilde{X}$.

It should be noted that it is practically irrelevant whether the soft metric is defined on a soft space $\widetilde{X}$ or on an ordinary space X. The notion of a soft metric deviates from a metric in its range rather than its domain.

Definition 7. *Let B be a real Banach space and P a subset of B. P is called a cone under the following conditions:*

(i) *P has an open complement, is nonempty and* $P \neq \{0\}$.

(ii) $ax + by \in P$ *for all* $x, y \in P$ *and nonnegative real numbers* a, b.

(iii) $P \cap (-P) = \{0\}$.

Starting with a cone $P \subset B$, a partial order $\leq$ with respect to P can be defined by, $x \leq y$ whenever $y - x \in P$. $x \prec y$ will stand for $x \leq y$ and $x \neq y$, while $x << y$ indicates that $y - x \in intP$, where $intP$ denotes the interior of P.

Definition 8. *[32] Let X be a nonempty set and P be a cone. Suppose the mapping d:* $X \times X \to P$ *satisfies:*

$(d1)$ $0 \leq d(x,y)$ *for all* $x, y \in X$ *and* $d(x,y) = 0 \Leftrightarrow x = y$;

$(d2)$ *d is symmetric*;

$(d3)$ $d(x,y) \leq d(x,z) + d(y,z)$ *for all* $x, y, z \in X$.

Then, d is called a cone metric on X and (X, d) *is called a cone metric space.*

3. The topology of soft metric spaces

In [26] it was shown that any soft topological space is actually homeomorphic to a topological space on $(X \times E)$, (in the original paper it is $(E \times X)$ but for the sake of compatibility with the text we will use $(X \times E)$). On the other hand, in [28] it is shown that every soft metric induces a soft topology. To get to a compatible setting, all soft points $P^x_\lambda \widetilde{\in} \widetilde{X}$ will be identified with the corresponding ordered pairs $(x, \lambda) \in (X \times E)$. So, we can conclude that any soft metric induces a classical topology on $(X \times E)$. In the sequel we will show that this induced classical topology is metrizable. Furthermore, we will construct a metric from the underlying soft metric that is metrizing the mentioned classical topology. As a result, it will be clear that every soft metric is isomorphic to a classical metric. For further understanding of soft metrics, it will be also shown that a soft metric is a vector valued metric, more precisely a special case of cone metrics.

At this point we want to emphasize again that since a soft metric space $(\widetilde{X}, d)$ together with its concept of soft points $P^x_\lambda \widetilde{\in} \widetilde{X}$ is in a one-to-one correspondence with a classical topology on $(X \times E)$, we will hereafter employ the classical notions of second countability, regularity and metrizability freely in the space $(\widetilde{X}, d)$.

Proposition 1. *A soft metric space* $(\widetilde{X}, d)$ *is second countable.*

Proof. The standard base of the induced soft topology consists of the open balls $B_d((x, \lambda), \widetilde{r})$. If the soft real number $\widetilde{r}$ is restricted to take rational values only, say $\widetilde{r_q}$, the family of open balls $B_d((x, \lambda), \widetilde{r_q})$ is a countable basis for the topology. □

Proposition 2. *A soft metric space* $(\widetilde{X}, d)$ *is regular.*

Proof. Let $K \subset SP(\widetilde{X})$ be a nonempty closed set and $P^x_\lambda \in SP(\widetilde{X})$ be a point not in K. Clearly, $d(P^x_\lambda, K) > \overline{0}$ since $d(P^x_\lambda, K) = 0$ would imply that P^x_λ is a closure point of K, and therefore belongs to K. Let

$$2\widetilde{r} = d(P^x_\lambda, K).$$

Consider the open ball

$$V = B_d((x, \lambda), \widetilde{r})$$

and the open set

$$U = \cup_{P^y_\mu \in K} \{B_d((y, \mu), \widetilde{r}).$$

U and V are distinct open sets and can not intersect since that would violate the triangle inequality. Moreover, $P^x_\lambda \in V$ and $K \subset U$, so we arrive at the result that $(\widetilde{X}, d)$ is regular. □

Corollary 1. *By the Urysohn metrizability theorem every soft metric space is metrizable.*

The natural question is, can the metric corresponding to the soft topology be explicitly given? The answer is yes and familiar. But, before that, we will proceed showing another property of soft metrics, namely its relation with cone metrics.

Proposition 3. *The set of all nonnegative soft real numbers* $\mathbb{R}(E)^*$ *with the parameter set E is a cone on* $\mathbb{R}^{s(E)}$.

Proof. $\mathbb{R}(E)^*$ is clearly a nonempty subset of $\mathbb{R}^{s(E)}$ and $\mathbb{R}^{s(E)}$ is a Banach space. Moreover, $\mathbb{R}(E)^* \neq \{\overline{0}\}$.

(i) We will show that $\mathbb{R}(E)^*$ is closed with the standard topology on $\mathbb{R}^{s(E)}$. Let $x \in (\mathbb{R}(E)^*)^c$,

$$r = \min_{e \in E}\{|\widetilde{x}(e)|\}, \quad r > 0.$$

Thus, we have

$$B(\widetilde{x}, r) \subset (\mathbb{R}(E)^*)^c.$$

Therefore, $(\mathbb{R}(E)^*)^c$ is an open set and $\mathbb{R}(E)^*$ is closed.

(ii) Let $\widetilde{x}, \widetilde{y} \in \mathbb{R}(E)^*$ and a, b be two nonnegative real numbers. It is obvious that the linear combination $a\widetilde{x} + b\widetilde{y}$ is a nonnegative vector in $\mathbb{R}^{s(E)}$. Therefore, $a\widetilde{x} + b\widetilde{y} \in \mathbb{R}(E)^*$.

(*iii*) Let $\widetilde{x} \in \mathbb{R}(E)^*$. If $\widetilde{x} \neq \overline{0}$ then $-\widetilde{x} \notin \mathbb{R}(E)^*$. If $\widetilde{x} = \overline{0}$ then $-\widetilde{x} = \overline{0} \in \mathbb{R}(E)^*$. So, we observe that

$$\mathbb{R}(E)^* \cap (-\mathbb{R}(E)^*) = \{\overline{0}\}.$$

□

Proposition 4. *The ordering $\leq$ induced by the cone $\mathbb{R}(E)^*$ coincides with the ordering $\lesssim$ defined on $\mathbb{R}(E)^*$.*

Proof. $\widetilde{x} \lessapprox \widetilde{y} \Leftrightarrow \forall \lambda \in \{e_1, e_2, \cdots, e_{s(E)}\}, \quad \widetilde{x}(\lambda) \leq \widetilde{y}(\lambda) \Leftrightarrow \widetilde{x}(\lambda) - \widetilde{y}(\lambda) \leq 0 \Leftrightarrow \widetilde{y} - \widetilde{x} \in \mathbb{R}(E)^* \Leftrightarrow \widetilde{x} \leq \widetilde{y}.$ □

Theorem 1. *Let X be an initial universal set, E be a finite set of parameters and d be a soft metric on the soft set $\widetilde{X}$. Identifying the soft real number $d(P^x_\lambda, P^y_\mu)$ with the corresponding vector in $\mathbb{R}^{s(E)}$ the function*

$$d_c : (X \times E) \times (X \times E) \to \mathbb{R}^{s(E)},$$

defined by

$$d_c((x, \lambda), (y, \mu)) = \left[d(P^x_\lambda, P^y_\mu)\right]$$

is a cone metric.

Proof. The claim is a conclusion of Propositions 3 and 4. □

The observation that every soft metric is actually a cone metric enables us to carry all results obtained for cone metrics to soft metrics. In particular, the metrizability of cone metric spaces is investigated in [33, 34].

The theorem below explicitly states the metrizing metric of a soft metric space.

Theorem 2. *Let X be an initial universal set, E be a set of parameters and d be a soft metric on the soft set $\widetilde{X}$. The topology on $(X \times E)$ induced by the soft metric d is metrizable with*

$$d'((x, \lambda), (y, \mu)) = \|d_c((x, \lambda), (y, \mu))\|_\infty .$$

Proof. The function

$$D((x, \lambda), (y, \mu)) = \|d_c((x, \lambda), (y, \mu))\|_\infty$$

is obviously a metric on $(X \times E)$. We will show that D and d induce the same topology on $(X \times E)$. To do this, we will show that every open ball $B_d((x, \lambda), \widetilde{r})$ in $(X \times E, d)$ is an open set in $(X \times E, D)$ and every open ball $B_D((x, \lambda), r)$ in $(X \times E, D)$ is an open set in $(X \times E, d)$. Consider the open ball $B_d((x, \lambda), \widetilde{r})$ where $\overline{0} \lesssim \widetilde{r}$. Let

$$(y, \mu) \in B_d((x, \lambda), \widetilde{r})$$

with $(y, \mu) \neq (x, \lambda)$. Set

$$\widetilde{c} = \widetilde{r} - d(P^x_\lambda, P^y_\mu).$$

Clearly, $\overline{0} \lesssim \widetilde{c}$ and

$$B_d((y, \mu), \widetilde{c}) \subset B_d((x, \lambda), \widetilde{r}).$$

Choosing c' to be the smallest component of c, we observe that

$$B_D((y, \mu), c') \subset B_d((y, \mu), \widetilde{c}) \subset B_d((x, \lambda), \widetilde{r}).$$

And therefore, each point of the set $B_d((x,\lambda),\widetilde{r})$ is an interior point with respect to the metric D. So $B_d((x,\lambda),\widetilde{r})$ is an open set in $(X\times E, D)$. Conversely, let

$$(y,\mu)\in B_D((x,\lambda),r)$$

with $(y,\mu)\neq(x,\lambda)$ and

$$c = r - D((x,\lambda),(y,\mu))$$

and consider the soft real number $\overline{c}$. Since

$$D((x,\lambda),(y,\mu)) = \|d_c((x,\lambda),(y,\mu))\|_\infty$$

and

$$d_c((x,\lambda),(y,\mu)) = d(P^x_\lambda, P^y_\mu)\lnapprox \overline{c},$$

we see that

$$B_d((y,\mu),\overline{c}) = B_D((y,\mu),c)\subset B_D((x,\lambda),r).$$

And therefore, the set $B_D((x,\lambda),r)$ is open in the topological space induced by the soft metric. □

It should be noted that the metric

$$d' = \|d_c((x,\lambda),(y,\mu))\|_\infty$$

used in the foregoing theorem is called the compatible metric in [35, 36] where the authors used this metric to show that some fixed point theorems in soft metric spaces can be obtained from their classical counterparts. Here we have the main result:

Corollary 2. *A soft metric d on a space $\widetilde{X}$ is isomorphic to the metric*

$$d' = \|d_c((x,\lambda),(y,\mu))\|_\infty$$

on the space $(X\times E)$.

Figure 1 illustrates the complete relation.

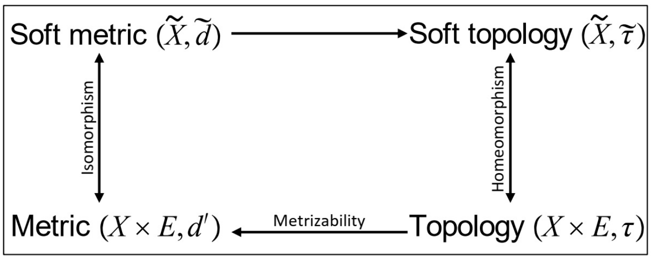

Figure 1. Complete relation between soft and classical metrics.

4. Application to a fixed point theorem

The purpose of this short section is to show an application of the isomorphism obtained in the foregoing section. With the help of the isomorphism, all known results about metrizable topologies can be transferred to soft metrics with proper modifications. As an example, the following soft fixed point theorem does not exist in the relevant literature to the best of the authors' knowledge.

By $[\overline{0}, \overline{\infty})$, we denote the set of constant soft reals $\{\overline{r}\colon\ 0 \leq r < \infty\}$ and we consider the Euclidian metric on this set.

Theorem 3. *Let $(\widetilde{X}, d)$ be a complete soft metric space, $\phi\colon SP(\widetilde{X}) \to [\overline{0}, \overline{\infty})$ be a lower semi-continuous function and $T\colon SP(\widetilde{X}) \to SP(\widetilde{X})$ be a mapping. Suppose that for any two positive real numbers a, b with $a \leq b$, there exists a soft real number $\widetilde{\gamma}(\widetilde{a}, \widetilde{b}) \in (\overline{0}, \overline{1})$ such that for all $P^x_\lambda, P^y_\mu \in SP(\widetilde{X})$,*

$$a \leq \left\|d(P^x_\lambda, P^y_\mu)\right\|_\infty + \left\|\phi(P^x_\lambda)\right\|_\infty + \left\|\phi(P^y_\mu)\right\|_\infty \leq b$$

implies that

$$\left[d(TP^x_\lambda, TP^y_\mu) + \phi(TP^x_\lambda) + \phi(TP^y_\mu)\right] \lessapprox \widetilde{\gamma}(a, b)\left[d(P^x_\lambda, P^y_\mu) + \phi(P^x_\lambda) + \phi(P^y_\mu)\right].$$

Then, T has a unique fixed point $P^{x^*}_{\lambda^*}$ satisfying $\phi(P^{x^*}_{\lambda^*}) = \overline{0}$.

Proof. We will prove the theorem with the help of the isomorphism we obtained in our main result.

Since $(\widetilde{X}, d)$ is a complete soft metric space, the isomorphic space $(X \times E, d')$ is also complete. Let us define $\varphi((x, \lambda)) = r$, where, $\overline{r} = \phi(P^x_\lambda)$.

Now, $\varphi\colon (X \times E) \to [0, \infty)$ is a lower semi-continuous function. We choose arbitrary numbers a, b such that $0 < a < b$. By assumption, we have that for a soft real number $\widetilde{\gamma}(a, b) \in (\overline{0}, \overline{1})$, the condition,

$$a \leq \left\|d(P^x_\lambda, P^y_\mu)\right\|_\infty + \left\|\phi(P^x_\lambda)\right\|_\infty + \left\|\phi(P^y_\mu)\right\|_\infty \leq b$$

implies that

$$\left[d(TP^x_\lambda, TP^y_\mu) + \phi(TP^x_\lambda) + \phi(TP^y_\mu)\right] \lessapprox \widetilde{\gamma}(a, b)\left[d(P^x_\lambda, P^y_\mu) + \phi(P^x_\lambda) + \phi(P^y_\mu)\right].$$

The condition here is equivalent to

$$a \leq d'((x, \lambda), (y, \mu)) + \varphi((x, \lambda)) + \varphi((y, \mu)) \leq b.$$

On the other hand, if

$$\left[d(TP^x_\lambda, TP^y_\mu) + \phi(TP^x_\lambda) + \phi(TP^y_\mu)\right] \lessapprox \widetilde{\gamma}(a, b)\left[d(P^x_\lambda, P^y_\mu) + \phi(P^x_\lambda) + \phi(P^y_\mu)\right],$$

we may observe that

$$\left[d'(T(x, \lambda), T(y, \mu)) + \varphi(T(x, \lambda)) + \varphi(T(y, \mu))\right] \leq \|\widetilde{\gamma}(a, b)\|_\infty \left[d'((x, \lambda), (y, \mu)) + \varphi((x, \lambda)) + \varphi((y, \mu))\right].$$

Setting

$$\gamma = \|\widetilde{\gamma}(a, b)\|_\infty\,,$$

we obtain following: For any two positive real numbers a, b with $a \leq b$, there exists a number $\gamma \in (0, 1)$ such that for all $(x, \lambda), (y, \mu) \in X \times E$,

$$a \leq d'((x, \lambda), (y, \mu)) + \varphi((x, \lambda)) + \varphi((y, \mu)) \leq b$$

implies that

$$\left[d'(T(x, \lambda), T(y, \mu)) + \varphi(T(x, \lambda)) + \varphi(T(y, \mu))\right] \leq \gamma \left[d'((x, \lambda), (y, \mu)) + \varphi((x, \lambda)) + \varphi((y, \mu))\right].$$

Keeping in mind that $(X \times E, d')$ is a complete metric space and $\varphi\colon (X \times E) \to [0, \infty)$ is a lower semi-continuous function, we conclude by [38, Theorem 2.1] that T has a unique fixed point (x^*, λ^*) (or equivalently $P_{\lambda^*}^{x^*}$) satisfying

$$\varphi((x^*, \lambda^*)) = 0.$$

Thus,

$$\phi(P_{\lambda^*}^{x^*}) = \overline{0}.$$

□

As it is shown in the proof, considering the corresponding classic metric on $(X \times E)$, the theorem is equivalent to its classical version in [38]. One can even go further by modifying the generalized version of this theorem in [39] to obtain its soft counterpart.

5. Conclusions

In this work, soft metrics are investigated closely and some important results are obtained. It is shown that soft metrics possess all properties of cone metrics. The metrizability is proven and an isomorphism with a classical metric space is observed. Results are used to demonstrate how fixed point theorems can be transformed into the soft setting. All findings help foster a better understanding of soft metrics and provide additional insight into them. It may be concluded that soft metrics, despite the fact that the concept is not introduced with the intention of a new generalization, do not form a new type of metrics. From the topological point of view, soft metrics do not yield any original results either since they induce a topology that can be obtained by a standard metric in a natural way.

It should be noted that there are widely used soft point definitions (e.g., [19, 40]) in the literature other than the one employed in soft metrics in the sense of Das and Samanta [28]. Quite naturally, different definitions of soft points as well as different definitions of other topological concepts such as compactness and separation axioms ([41–43]), lead to complete different soft topological structures. It would be interesting for future works to investigate new soft metrics in those different structures. Since some of those structures offer richer properties which are not compatible with the classical topological facts (see for instance [26, 42, 43]), it is natural to expect richer results of soft metrics in such spaces as well. Another related area for possible future works is soft fuzzy metric spaces [44] where we also observe some fixed point related results [45].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

All authors declare no conflicts of interest in this paper

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