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Research article

Inclusion properties for analytic functions of q-analogue multiplier-Ruscheweyh operator

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Abstract: The results of this work have a connection with the geometric function theory and they were obtained using methods based on subordination along with information on $\mathfrak{q}$-calculus operators. We defined the $\mathfrak{q}$-analogue of multiplier- Ruscheweyh operator of a certain family of linear operators $I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\varsigma)$ $(s \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \mathbb{N} = \{1,2,3,..\}\,; \ell,\lambda,\mu \geq 0, 0 < \mathfrak{q} < 1)$. Our major goal was to build some analytic function subclasses using $I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(\varsigma)$ and to look into various inclusion relationships that have integral preservation features.

Keywords: analytic function; $\mathfrak{q}$-difference operator; $\mathfrak{q}$-analogue Catas operator; $\mathfrak{q}$-analogue of Ruscheweyh operator
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1. Introduction

Denote **A** as the normalized analytical function $\mathfrak{f}(z)$ in the open unit disk $\mathbf{U} = \{z : |z| < 1\}$ such that

$$\mathfrak{f}(z) = z + \sum_{\kappa=2}^{\infty} a_\kappa z^\kappa. \tag{1.1}$$

Subordination of two functions $\mathfrak{f}$ and $\mathfrak{I}$ is denoted by $\mathfrak{f} \prec \mathfrak{I}$ and defined as $\mathfrak{f}(z) = \mathfrak{I}(\chi(z))$, where $\chi(z)$ is the Schwartz function in $\mathbf{U}$ (see [1–3]). Let S, S^*, and C stand for the respective univalent, starlike, and convex subclasses of **A**.

Here, we review the fundamental $\mathfrak{q}$-calculus definitions and information that is used in this paper.

The use of $\mathfrak{q}$-difference equations in the setting of the geometric function theory was pioneered by Jackson [4, 5], Carmichael [6], Mason [7], and Trijitzinsky [8]. Ismail et al. [9] introduced certain $\mathfrak{q}$-function theory-related characteristics for the first time. Additionally, various $\mathfrak{q}$-calculus applications related to generalized subclasses of analytic functions have been researched by numerous authors; see [10–19]. Motivated by these $\mathfrak{q}$-developments in the geometric function theory, many authors added their contributions in this direction, which has made this research area much more attractive in works like [20–22]. The Jackson's $\mathfrak{q}$-difference operator $\mathfrak{d}_{\mathfrak{q}} : \mathbf{A} \to \mathbf{A}$ is defined by

$$\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(z) := \begin{cases} \frac{\mathfrak{f}(z)-\mathfrak{f}(\mathfrak{q}z)}{(1-\mathfrak{q})z} & (z \neq 0;\ 0 < \mathfrak{q} < 1) \\ \\ \mathfrak{f}'(0) & (z = 0). \end{cases} \tag{1.2}$$

It comes to light that, for $\kappa \in \mathbb{N}$ and $z \in \mathbf{U}$,

$$\mathfrak{d}_{\mathfrak{q}} \left\{ \sum_{\kappa=1}^{\infty} a_\kappa z^\kappa \right\} = \sum_{\kappa=1}^{\infty} [\kappa]_{\mathfrak{q}}\, a_\kappa z^{\kappa-1}, \tag{1.3}$$

where

$$\begin{aligned} [\kappa]_{\mathfrak{q}} &= \frac{1-\mathfrak{q}^\kappa}{1-\mathfrak{q}} = 1 + \sum_{n=1}^{\kappa-1} \mathfrak{q}^n, \qquad [0]_{\mathfrak{q}} = 0, \\ [\kappa]_{\mathfrak{q}}! &= \begin{cases} [\kappa]_{\mathfrak{q}}\, [\kappa-1]_{\mathfrak{q}} \,......... [2]_{\mathfrak{q}}\, [1]_{\mathfrak{q}} & \kappa = 1, 2, 3, ... \\ 1 & \kappa = 0. \end{cases} \end{aligned} \tag{1.4}$$

The $\mathfrak{q}$-difference operator is subject to the following basic laws:

$$\mathfrak{d}_{\mathfrak{q}} \left(c\mathfrak{f}(z) \pm d\hbar(z) \right) = c\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}(z) \pm d\mathfrak{d}_{\mathfrak{q}}\hbar(z) \tag{1.5}$$

$$\mathfrak{d}_{\mathfrak{q}} \left(\mathfrak{f}(z)\, \hbar(z) \right) = \mathfrak{f}(\mathfrak{q}z)\, \mathfrak{d}_{\mathfrak{q}} \left(\hbar(z) \right) + \hbar(z)\mathfrak{d}_{\mathfrak{q}} \left(\mathfrak{f}(z) \right) \tag{1.6}$$

$$\mathfrak{d}_{\mathfrak{q}} \left(\frac{\mathfrak{f}(z)}{\hbar(z)} \right) = \frac{\mathfrak{d}_{\mathfrak{q}} \left(\mathfrak{f}(z) \right) \hbar(z) - \mathfrak{f}(z)\, \mathfrak{d}_{\mathfrak{q}} \left(\hbar(z) \right)}{\hbar(\mathfrak{q}z)\hbar(z)}, \quad \hbar(\mathfrak{q}z)\hbar(z) \neq 0 \tag{1.7}$$

$$\mathfrak{d}_{\mathfrak{q}} \left(\log \mathfrak{f}(z) \right) = \frac{\ln \mathfrak{q}}{\mathfrak{q}-1} \frac{\mathfrak{d}_{\mathfrak{q}} \left(\mathfrak{f}(z) \right)}{\mathfrak{f}(z)}, \tag{1.8}$$

where $\mathfrak{f}, \hbar \in \mathbf{A}$, and c and d are real or complex constants.

Jackson in [5] introduced the $\mathfrak{q}$-integral of $\mathfrak{f}$ as:

$$\int_0^z \mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \ = z(1-\mathfrak{q}) \sum_{\kappa=0}^{\infty} \mathfrak{q}^\kappa \mathfrak{f}(z\mathfrak{q}^\kappa)$$

and

$$\lim_{q \to 1-} \int_0^z \mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \ = \int_0^z \mathfrak{f}(t)\mathfrak{d}t,$$

where $\int_0^z \mathfrak{f}(t)\mathfrak{d}t$, is the ordinary integral.

The discipline of the geometric function theory has the great advantage of studying linear operators. The introduction and analysis of such linear operators with reference to $\mathfrak{q}$-analogues has recently piqued the interest of numerous renowned academics. The authors of [23] investigated the $\mathfrak{q}$-analogue of the Ruscheweyh derivative operator and looked at some of its characteristics. The $\mathfrak{q}$-Běrnardi integral operator was first introduced by Noor et al. [24].

In [25], Aouf and Madian investigate the $\mathfrak{q}$-analogue Ĉătas operator $I_{\mathfrak{q}}^{s}(\lambda, \ell) : \mathbf{A} \to \mathbf{A}$ ($s \in \mathbb{N}_0, \ell, \lambda \geq 0, 0 < \mathfrak{q} < 1$) as follows:

$$\begin{aligned} I_{\mathfrak{q}}^{s}(\lambda, \ell)\mathfrak{f}(z) &= z + \sum_{\kappa=2}^{\infty} \left(\frac{[1+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [1+\ell]_{\mathfrak{q}})}{[1+\ell]_{\mathfrak{q}}} \right)^{s} a_{\kappa} z^{\kappa} \\ (s &\in \mathbb{N}_0, \ell, \lambda \geq 0, 0 < \mathfrak{q} < 1). \end{aligned}$$

Also, in 2014, Aldweby and Darus [26] investigated the $\mathfrak{q}$-analogue of the Ruscheweyh operator $\mathfrak{R}_{\mathfrak{q}}^{\mu}\mathfrak{f}(z)$:

$$\mathfrak{R}_{\mathfrak{q}}^{\mu}\mathfrak{f}(z) = z + \sum_{\kappa=2}^{\infty} \frac{[\kappa+\mu-1]_{\mathfrak{q}}}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!} a_{\kappa} z^{\kappa}, \ (\mu \geq 0, 0 < \mathfrak{q} < 1),$$

where $[a]_{\mathfrak{q}}$ and $[a]_{\mathfrak{q}}!$ are defined in (1.4).

Set

$$\mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s}(z) = z + \sum_{\kappa=2}^{\infty} \left(\frac{[1+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [1+\ell]_{\mathfrak{q}})}{[1+\ell]_{\mathfrak{q}}} \right)^{s} z^{\kappa}.$$

Now, we define a new function $\mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s,\mu}(z)$ in terms of the Hadamard product (or convolution) by:

$$\mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s}(z) * \mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s,\mu}(z) = z + \sum_{\kappa=2}^{\infty} \frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!} z^{\kappa}.$$

Motivated essentially by the $\mathfrak{q}$-analogue of the Ŕuscheweyh operator and the $\mathfrak{q}$-analogue Cătas operator, we now introduce the operator $I_{\mathfrak{q},\mu}^{s}(\lambda, \ell) : \mathbf{A} \to \mathbf{A}$ defined by

$$I_{\mathfrak{q},\mu}^{s}(\lambda, \ell)\mathfrak{f}(z) = \mathfrak{f}_{\mathfrak{q},\lambda,\ell}^{s,\mu}(z) * \mathfrak{f}(z) \tag{1.9}$$

where $s \in \mathbb{N}_0, \ell, \lambda, \mu \geq 0, 0 < \mathfrak{q} < 1$. For $\mathfrak{f} \in \mathbf{A}$; and (1.9), it is clear that

$$I_{\mathfrak{q},\mu}^{s}(\lambda, \ell)\mathfrak{f}(z) = z + \sum_{\kappa=2}^{\infty} \left(\frac{[1+\ell]_{\mathfrak{q}}}{[1+\ell]_{\mathfrak{q}} + \lambda([\kappa+\ell]_{\mathfrak{q}} - [1+\ell]_{\mathfrak{q}})} \right)^{s} \frac{[\kappa+\mu-1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa-1]_{\mathfrak{q}}!} a_{\kappa} z^{\kappa}. \tag{1.10}$$

We use (1.10) to deduce the following:

$$\begin{aligned} z\eth_{\mathfrak{q}}\left(I_{\mathfrak{q},\mu}^{s+1}(\lambda, \ell)\mathfrak{f}(z)\right) &= \frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}} I_{\mathfrak{q},\mu}^{s}(\lambda, \ell)\mathfrak{f}(z) - \left(\frac{[\ell+1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}} - 1\right) I_{\mathfrak{q},\mu}^{s+1}(\lambda, \ell)\mathfrak{f}(z), \\ (\lambda &> 0), \end{aligned} \tag{1.11}$$

$$\mathfrak{q}^{\mu} z\eth_{\mathfrak{q}}\left(I_{\mathfrak{q},\mu}^{s}(\lambda, \ell)\mathfrak{f}(z)\right) = [\mu+1]_{\mathfrak{q}} I_{\mathfrak{q},\mu+1}^{s}(\lambda, \ell)\mathfrak{f}(z) - [\mu]_{\mathfrak{q}} I_{\mathfrak{q},\mu}^{s}(\lambda, \ell)\mathfrak{f}(z). \tag{1.12}$$

We note that :

(*i*) If $s = 0$ and $\mathfrak{q} \to 1^{-}$, we get $\mathfrak{R}^{\mu}\mathfrak{f}(z)$ as the Ŕusscheweyh differential operator [27], which has been investigated by numerous authors [28–30];

(ii) If we set $\mathfrak{q} \to 1^{-}$, we obtain $I^{m}_{\lambda,\ell,\mu}\mathfrak{f}(z)$, which was presented by Aouf and El-Ashwah [31];

(iii) If we set $\mu = 0$ and $\mathfrak{q} \to 1^{-}$, we obtain $J^{m}_{\mathrm{p}}(\lambda, \ell)\mathfrak{f}(z)$, which was presented by El-Ashwah and Aouf (with $p = 1$) [32];

(iv) If $\mu = 0$, $\ell = \lambda = 1$, and $\mathfrak{q} \to 1^{-}$, we obtain $I^{\alpha}\mathfrak{f}(z)$, which was investigated by Jung et al [33];

(v) If $\mu = 0$, $\lambda = 1$, $\ell = 0$, and $\mathfrak{q} \to 1^{-}$, we obtain $I^{s}\mathfrak{f}(z)$, which was presented by Salagean [34];

(vi) If we set $\mu = 0$ and $\lambda = 1$, we obtain $I^{\ell}_{\mathfrak{q},s}\mathfrak{f}(z)$, which was presented by Shah and Noor [35];

(vii) If we set $\mu = 0$, $\lambda = 1$, and $q \to 1^{-}$, we obtain $J^{s}_{\mathfrak{q},\ell}$, the Srivastava–Attiya operator; see [36, 37];

(vii) $I^{1}_{\mathfrak{q},0}(1, 0) = \int_0^z \frac{\mathfrak{f}(t)}{t}\mathfrak{d}_{\mathfrak{q}}t.$ ($\mathfrak{q}$-Alexander operator [35]);

($viii$) $I^{1}_{\mathfrak{q},0}(1, \ell) = \frac{[1+\varrho]_{\mathfrak{q}}}{z^{\varrho}} \int_0^z t^{\varrho-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t$ ($\mathfrak{q}$-Bernardi operator [24]);

(ix) $I^{1}_{\mathfrak{q},0}(1, 1) = \frac{[2]_{\mathfrak{q}}}{z} \int_0^z \mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t$ ($\mathfrak{q}$-Libera operator [24]).

We also observe that:

(i) $I^{s}_{\mathfrak{q},\mu}(1, 0)\mathfrak{f}(z) = I^{s}_{\mathfrak{q},\mu}\mathfrak{f}(z)$

$$\mathfrak{f}(z) \in \mathbf{A} : I^{s}_{\mathfrak{q},\mu}\mathfrak{f}(z) = z + \sum_{\kappa=2}^{\infty} \left(\frac{1}{[\kappa]_{\mathfrak{q}}}\right)^{s} \frac{[\kappa + \mu - 1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa - 1]_{\mathfrak{q}}!} a_{\kappa} z^{\kappa},\ (s \in \mathbb{N}_0, \mu \geq 0, 0 < \mathfrak{q} < 1, z \in \mathbf{U}).$$

(ii) $I^{s}_{\mathfrak{q},\mu}(1, \ell)\mathfrak{f}(z) = I^{s,\ell}_{\mathfrak{q},\mu}\mathfrak{f}(z)$

$$\begin{aligned} \mathfrak{f}(z) \in \mathbf{A} &: I^{s,\ell}_{\mathfrak{q},\mu}\mathfrak{f}(z) = z + \sum_{\kappa=2}^{\infty} \left(\frac{[1+\ell]_{\mathfrak{q}}}{[\kappa+\ell]_{\mathfrak{q}}}\right)^{s} \frac{[\kappa + \mu - 1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa - 1]_{\mathfrak{q}}!} a_{\kappa} z^{\kappa}, \\ (s \in \mathbb{N}_0, \ell &> 0, \mu \geq 0, 0 < \mathfrak{q} < 1, z \in \mathbf{U}). \end{aligned}$$

(iii) $I^{s}_{\mathfrak{q},\mu}(\lambda, 0)\mathfrak{f}(z) = I^{s,\lambda}_{\mathfrak{q},\mu}\mathfrak{f}(z)$

$$\begin{aligned} \mathfrak{f}(z) \in \mathbf{A} &: I^{s,\lambda}_{\mathfrak{q},\mu}\mathfrak{f}(z) = z + \sum_{\kappa=2}^{\infty} \left(\frac{1}{1+\lambda([\kappa]_{\mathfrak{q}}-1)}\right)^{s} \frac{[\kappa + \mu - 1]_{\mathfrak{q}}!}{[\mu]_{\mathfrak{q}}![\kappa - 1]_{\mathfrak{q}}!} a_{\kappa} z^{\kappa}, \\ (s \in \mathbb{N}_0, \lambda &> 0, \mu \geq 0, 0 < \mathfrak{q} < 1, z \in \mathbf{U}). \end{aligned}$$

With $\varphi(0) = 1$ and $\Re\varphi(z) > 0$ in $\mathbf{U}$, Φ is the class of analytic functions $\varphi(z)$ and is a set of univalent convex functions in $\mathbf{U}$.

Definition 1.1. $\mathfrak{f} \in \mathbf{A}$ is definitely in the class $ST_{\mathfrak{q}}(\varphi)$ if it satisfies:

$$\frac{z\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}(z)\right)}{\mathfrak{f}(z)} \prec \varphi(z),$$

where $\mathfrak{d}_{\mathfrak{q}}$ is the $\mathfrak{q}$-difference operator.

Analogously, $\mathfrak{f} \in \mathbf{A}$ is definitely in the class $CV_{\mathfrak{q}}(\varphi)$ if

$$z\mathfrak{d}_{\mathfrak{q}}\left(\mathfrak{f}(z)\right) \in ST_{\mathfrak{q}}(\varphi). \tag{1.13}$$

By using the operators defined above, we determine the next part:

Definition 1.2. Suppose that $\mathfrak{f} \in \mathbf{A}$, s is real, and $\ell > -1$, then

$$\mathfrak{f} \in ST^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi) \Leftrightarrow I^{s}_{\mathfrak{q},\mu}(\lambda, \ell)\mathfrak{f}(z) \in ST_{\mathfrak{q}}(\varphi),$$

and

$$\mathfrak{f} \in CV^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi) \Leftrightarrow I^{s}_{\mathfrak{q},\mu}(\lambda, \ell)\mathfrak{f}(z) \in CV_{\mathfrak{q}}(\varphi). \tag{1.14}$$

It is clear that

$$\mathfrak{f} \in CV^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi) \Leftrightarrow z(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in ST^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi). \tag{1.15}$$

Special cases:

(*i*) If $s = 0, \mu = 0$, and $\varphi(z) = \frac{1+Mz}{1+Nz}$ $(-1 \leq N < M \leq 1)$, then $ST^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi)$ decreases to the class $S^{*}_{\mathfrak{q}}(M, N)$, investigated by Noor et al. [24]. Moreover, if $\mathfrak{q} \to 1^{-}$, then $S^{*}_{\mathfrak{q}}(M, N)$ coincides with $S^{*}[M, N]$ (see [38]).

(*ii*) If $s = 0, \mu = 0$, and $\varphi(z) = \frac{1+Mz}{1+Nz}$ $(-1 \leq N < M \leq 1)$, then $CV^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi)$ decreases to the class $K_{\mathfrak{q}}(M, N)$, introduced by Seoudy and Aouf. [39]. Moreover, if $\mathfrak{q} \to 1^{-}$, then $CV^{*}_{\mathfrak{q}}(M, N)$ coincides with the class $CV^{*}[M, N]$ (see [38]).

(*iii*) If $s = 0, \mu = 0$, and $\varphi(z) = \frac{1}{1-\mathfrak{q}z}$, then $ST^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi)$ reduces to the class $ST_{\mathfrak{q}}$, investigated by Noor [40].

(*iv*) If $s = 0, \mu = 0$, and $\varphi(z) = \frac{1+z}{1-\mathfrak{q}z}$, then $ST^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi)$ decreases to the class $S^{*}_{\mathfrak{q}}$, investigated by Noor et al. [41].

2. Inclusion results

The next lemma is required to demonstrate our findings:

Lemma 2.1. *[42] Suppose that γ and δ are complex numbers with $\gamma \neq 0$ and let $\hbar(z)$ be analytic in $\mathbf{U}$ with $\hbar(0) = 1$ and $Re\{\gamma\hbar(z) + \delta\} > 0$. If $\omega(z) = 1 + \omega_1 z + \omega_2 z^2 + ...$is analytic in $\mathbf{U}$, then*

$$\omega(z) + \frac{z\mathfrak{d}_{\mathfrak{q}}\omega(z)}{\gamma\omega(z) + \delta} \prec \hbar(z),$$

and $\omega(z) \prec \hbar(z)$.

Theorem 2.1. *Assume that $\varphi(z)$ is an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(z)) > 0$ for $z \in \mathbf{U}$, then, for positive real s and $\ell, \mu \geq 0, \lambda > 0, 0 < \mathfrak{q} < 1$ with $[\ell + 1]_{\mathfrak{q}} > \lambda\mathfrak{q}^{\ell}$,*

$$ST^{s}_{\mathfrak{q},\mu+1}(\lambda, \ell)(\varphi) \subset ST^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi) \subset ST^{s+1}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi).$$

Proof. Let $\mathfrak{f} \in ST^{s}_{\mathfrak{q},\mu}(\lambda, \ell)(\varphi)$, and we set

$$\omega(z) = \frac{z\mathfrak{d}_{\mathfrak{q}}\left(I^{s+1}_{\mathfrak{q},\mu}(\lambda, \ell)\mathfrak{f}(z)\right)}{I^{s+1}_{\mathfrak{q},\mu}(\lambda, \ell)\mathfrak{f}(z)}, \tag{2.1}$$

where $\omega(z)$ is analytic in $\mathbf{U}$ $\omega(0) = 1$.

From identity (1.11) and (2.1), we can easily write

$$\frac{z\mathfrak{d}_{\mathfrak{q}}\left(I^{s+1}_{\mathfrak{q},\mu}(\lambda, \ell)\mathfrak{f}(z)\right)}{I^{s+1}_{\mathfrak{q},\mu}(\lambda, \ell)\mathfrak{f}(z)} = \frac{[\ell + 1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}}\frac{I^{s}_{\mathfrak{q},\mu}(\lambda, \ell)\mathfrak{f}(z)}{I^{s+1}_{\mathfrak{q},\mu}(\lambda, \ell)\mathfrak{f}(z)} - \left(\frac{[\ell + 1]_{\mathfrak{q}}}{\lambda\mathfrak{q}^{\ell}} - 1\right), \quad \lambda > 0,$$

or, equivalently,

$$\frac{[\ell+1]_{\mathfrak{q}}}{\lambda \mathfrak{q}^{\ell}} \frac{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)}{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)} = \omega(z) + \eta_{\mathfrak{q}} \tag{2.2}$$

where $\eta_{\mathfrak{q}} = \left(\frac{[\ell+1]_{\mathfrak{q}}}{\lambda \mathfrak{q}^{\ell}} - 1\right)$.

On the $\mathfrak{q}$-logarithmic differentiation of (2.2), we have

$$\frac{z\mathfrak{d}_{\mathfrak{q}}\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)\right)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)} = \omega(z) + \frac{z\mathfrak{d}_{\mathfrak{q}}\omega(z)}{\omega(z)+\eta_{\mathfrak{q}}}. \tag{2.3}$$

Since $\mathfrak{f} \in ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$, from (2.3) we have

$$\omega(z) + \frac{z\mathfrak{d}_{\mathfrak{q}}\omega(z)}{\omega(z)+\eta_{\mathfrak{q}}} \prec \varphi(z).$$

By applying Lemma 2.1, we conclude that $\omega(z) \prec \varphi(z)$. Consequently,

$$\frac{z\mathfrak{d}_{\mathfrak{q}}\left(I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)\right)}{I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)} \prec \varphi(z),$$

then $\mathfrak{f} \in ST^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$. To prove the first part, let $\mathfrak{f} \in ST^{s}_{\mathfrak{q},\mu+1}(\lambda,\ell)(\varphi)$ and set

$$\chi(z) = \frac{z\mathfrak{d}_{\mathfrak{q}}\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)\right)}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)},$$

where χ is analytic in $\mathbf{U}$ $\chi(0) = 1$. It follows $\chi \prec \varphi$ by applying the same arguments as those described before with (1.12). Theorem 2.1's proof is now complete. □

Theorem 2.2. *Suppose that $\varphi(z)$ is an analytic and convex univalent function with $\varphi(0) = 1$ and $Re(\varphi(z)) > 0$ for $z \in \mathbf{U}$, then, for positive real s and $\ell, \mu \geq 0, \lambda > 0, 0 < \mathfrak{q} < 1$ with $[\ell+1]_{\mathfrak{q}} > \lambda \mathfrak{q}^{\ell}$,*

$$CV^{s}_{\mathfrak{q},\mu+1}(\lambda,\ell)(\varphi) \subset CV^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi) \subset CV^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi).$$

Proof. Let $CV^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$. Applying (1.15), we show that

$$\begin{aligned}
\mathfrak{f} \in\ & CV^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi) \Leftrightarrow I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z) \in CV_{\mathfrak{q}}(\varphi) \\
\Leftrightarrow\ & z\mathfrak{d}_{\mathfrak{q}}\left(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)\right) \in ST_{\mathfrak{q}}(\varphi) \\
\Leftrightarrow\ & z(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi) \\
\Leftrightarrow\ & z(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f}) \in ST^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi) \\
\Leftrightarrow\ & z\mathfrak{d}_{\mathfrak{q}}\left(I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)\right) \in ST_{\mathfrak{q}}(\varphi) \\
\Leftrightarrow\ & I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)(z(\mathfrak{d}_{\mathfrak{q}}\mathfrak{f})) \in ST_{\mathfrak{q}}(\varphi) \\
\Leftrightarrow\ & I^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z) \in CV_{\mathfrak{q}}(\varphi) \\
\Leftrightarrow\ & \mathfrak{f} \in CV^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi).
\end{aligned}$$

We can demonstrate the first part using arguments similar to those described above. Theorem 2.2's proof is now complete. □

Corollary 2.1. *Suppose that s is a positive real and $\ell, \lambda, \mu \geq 0, 0 < \mathfrak{q} < 1$ with $[\ell+1]_{\mathfrak{q}} > \lambda \mathfrak{q}^{\ell}$, then, for* $\varphi(z) = \frac{1+Mz}{1+Nz}$ $(-1 \leq N < M \leq 1)$,

$$\begin{aligned} ST^{s}_{\mathfrak{q},\mu+1}(\lambda,\ell)\left(\frac{1+Mz}{1+Nz}\right) &\subset ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\left(\frac{1+Mz}{1+Nz}\right) \subset ST^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\left(\frac{1+Mz}{1+Nz}\right), \\ CV^{s}_{\mathfrak{q},\mu+1}(\lambda,\ell)(\varphi)\left(\frac{1+Mz}{1+Nz}\right) &\subset CV^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)\left(\frac{1+Mz}{1+Nz}\right) \subset CV^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)\left(\frac{1+Mz}{1+Nz}\right) \end{aligned}$$

Furthermore, for $M = 0$ and $N = -\mathfrak{q}$, and for $M = 1$ and $N = -\mathfrak{q}$,

$$\begin{aligned} ST^{s}_{\mathfrak{q},\mu+1}(\lambda,\ell)\left(\frac{1}{1-\mathfrak{q}z}\right) &\subset ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\left(\frac{1}{1-\mathfrak{q}z}\right) \subset ST^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\left(\frac{1}{1-\mathfrak{q}z}\right) \text{ and} \\ ST^{s}_{\mathfrak{q},\mu+1}(\lambda,\ell)\left(\frac{1+z}{1-\mathfrak{q}z}\right) &\subset ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\left(\frac{1+z}{1-\mathfrak{q}z}\right) \subset ST^{s+1}_{\mathfrak{q},\mu}(\lambda,\ell)\left(\frac{1+z}{1-\mathfrak{q}z}\right), \end{aligned}$$

respectively.

By employing the same justifications as before, the following conclusions can be demonstrated.

3. The classes uniformity under $\mathfrak{q}-$Bernardi integral operator

We introduce the $\mathfrak{q}$-Bernardi integral operator for analytic functions in this section by applying an aspect of $\mathfrak{q}$-calculus as stated by:

$$\begin{aligned} \Im_{\mathfrak{q},\varrho}\mathfrak{f}(z) &= \frac{[1+\varrho]_{\mathfrak{q}}}{z^{\varrho}} \int_0^z t^{\varrho-1}\mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \\ &= \sum_{\kappa=1}^{\infty} \left(\frac{[1+\varrho]_{\mathfrak{q}}}{[\kappa+\varrho]_{\mathfrak{q}}}\right) a_{\kappa} z^{\kappa}, \quad \varrho = 1,2,3,\ldots . \end{aligned} \tag{3.1}$$

We note that, for $\varrho = 1$ in (3.1), there is the $\mathfrak{q}$-Łibera integral operator defined as

$$\begin{aligned} \Im_{\mathfrak{q}}\mathfrak{f}(z) &= \frac{[2]_{\mathfrak{q}}}{z} \int_0^z \mathfrak{f}(t)\mathfrak{d}_{\mathfrak{q}}t \\ &= \sum_{\kappa=1}^{\infty} \left(\frac{[2]_{\mathfrak{q}}(1-\mathfrak{q})}{1-\mathfrak{q}^{\kappa+1}}\right) a_{\kappa} z^{\kappa}, \quad (0 < \mathfrak{q} < 1). \end{aligned}$$

For $0 < \mathfrak{q} < 1$, we have

$$\begin{aligned} \lim_{\mathfrak{q}\to 1^-} \Im_{\mathfrak{q},\varrho}\mathfrak{f}(z) &= \sum_{\kappa=1}^{\infty} \frac{(1+\varrho)}{(\kappa+\varrho)} a_{\kappa} z^{\kappa}, \\ \lim_{\mathfrak{q}\to 1^-} \Im_{\mathfrak{q}}\mathfrak{f}(z) &= \sum_{\kappa=1}^{\infty} \frac{2}{(\kappa+1)} a_{\kappa} z^{\kappa}, \end{aligned}$$

which are defined in [27].

Theorem 3.1. *Let* $\mathfrak{f} \in ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$, $\varphi(0) = 1$, $\varrho \geq -1$, *and* $Re(\varphi(z)) > 0$, *then* $\Im_{\mathfrak{q},\varrho}\mathfrak{f} \in ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$, *where* $\Im_{\mathfrak{q},\varrho}\mathfrak{f}(z)$ *is called a* $\mathfrak{q}$*-Bernardi integral operator defined in (3.1).*

Proof. Let $\mathfrak{f} \in ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$. If we put $\mathfrak{F}(z) = \Im_{\mathfrak{q},\varrho}\mathfrak{f}(z)$,

$$\frac{z\mathfrak{d}_{\mathfrak{q}}(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{F}(z))}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{F}(z)} = \aleph(z), \tag{3.2}$$

where $\aleph(z)$ is analytic in $\mathbf{U}$ with $\aleph(0) = 1$.

From (3.1), we show that

$$\frac{\mathfrak{d}_{\mathfrak{q}}(z^{\varrho}\mathfrak{F}(z))}{[1+\varrho]_{\mathfrak{q}}} = z^{\varrho-1}\mathfrak{f}(z).$$

Applying the $\mathfrak{q}$-difference operator's products, we get

$$z\mathfrak{d}_{\mathfrak{q}}\mathfrak{F}(z) = \left(1 + \frac{[\varrho]_{\mathfrak{q}}}{\mathfrak{q}^{\varrho}}\right)\mathfrak{f}(z) - [\varrho]_{\mathfrak{q}}\,\mathfrak{F}(z). \tag{3.3}$$

From (2.3), (3.3), and (1.10) there is

$$\aleph(z) = \left(1 + \frac{[\varrho]_{\mathfrak{q}}}{\mathfrak{q}^{\varrho}}\right)\frac{z\mathfrak{d}_{\mathfrak{q}}(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z))}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{F}(z)} - [\varrho]_{\mathfrak{q}}\,.$$

On $\mathfrak{q}$-logarithmic differentiation, we get

$$\frac{z\mathfrak{d}_{\mathfrak{q}}(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z))}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{f}(z)} = \aleph(z) + \frac{z\mathfrak{d}_{\mathfrak{q}}\aleph(z)}{\aleph(z) + [\varrho]_{\mathfrak{q}}}. \tag{3.4}$$

Since $\mathfrak{f} \in ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$, we can revise (3.4) as

$$\aleph(z) + \frac{z\mathfrak{d}_{\mathfrak{q}}\aleph(z)}{\aleph(z) + [\varrho]_{\mathfrak{q}}} \prec \varphi(z).$$

Now, by using Lemma 2.1, we conclude $\aleph(z) \prec \varphi(z)$. Consequently, $\frac{z\mathfrak{d}_{\mathfrak{q}}(I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{F}(z))}{I^{s}_{\mathfrak{q},\mu}(\lambda,\ell)\mathfrak{F}(z)} \prec \varphi(z)$. Hence, $\mathfrak{F}(z) \in ST^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$. □

The following conclusion can be demonstrated by employing reasons that are similar to those in Theorem 3.1.

Theorem 3.2. *Assume that* $\mathfrak{f} \in CV^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$, *then* $\Im_{\mathfrak{q},\varrho}\mathfrak{f}(z) \in CV^{s}_{\mathfrak{q},\mu}(\lambda,\ell)(\varphi)$, *where* $\Im_{\mathfrak{q},\varrho}\mathfrak{f}(z)$ *is defined by* (3.1).

Remark 3.1. (*i*) If we set $\mathfrak{q} \to 1^{-}$, we *can get the results investigated by* Aouf and El-Ashwah ([33]; *Theorems 1, 2 at* $\eta = 0$*);*

(*ii*) *If we put* $\mu = 0$ *and* $\lambda = 1$, *we can get the results investigated by Shah and Noor ([35]; Theorems 2.2, 2.3, 2.6);*

(*iii*) *Through the use of the specialization of the parameters* s, μ, λ, ℓ, *and* $\mathfrak{q}$, *we get all the results connecting with all the operators mentioned in the introduction.*

4. Conclusions

The novel findings in this study are connected to new classes of analytic normalized functions in $\mathbf{U}$. To introduce some subclasses of univalent functions, we develop the $\mathfrak{q}$-analogue multiplier-Ruscheweyh operator$I^{s}_{\mathfrak{q},\mu}(\lambda, \ell)$ using the notion of a $\mathfrak{q}$-difference operator. The $\mathfrak{q}$-analogue of the Ruscheweyh operator and the $\mathfrak{q}$-analogue of the Cătas operator are also used to introduce and study distinct subclasses. We looked into the integral preservation property and the inclusion outcomes for the newly defined classes. In the future, this work will motivate other authors to contribute in this direction for many generalized subclasses of $\mathfrak{q}$-close-to-convex, Quasi-convex univalent, and generalized operators for multivalent functions.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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