Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 6709–6733.
DOI: 10.3934/math.2024327
Received: 21 December 2023
Revised: 23 January 2024
Accepted: 31 January 2024
Published: 19 February 2024

Research article

On a conjecture on transposed Poisson n-Lie algebras

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Abstract: The notion of a transposed Poisson n-Lie algebra has been developed as a natural generalization of a transposed Poisson algebra. It was conjectured that a transposed Poisson n-Lie algebra with a derivation gives rise to a transposed Poisson $(n+1)$-Lie algebra. In this paper, we focus on transposed Poisson n-Lie algebras. We have obtained a rich family of identities for these algebras. As an application of these formulas, we provide a construction of $(n+1)$-Lie algebras from transposed Poisson n-Lie algebras with derivations under a certain strong condition, and we prove the conjecture in these cases.

Keywords: Lie algebra; Poisson algebra; transposed Poisson algebra; n-Lie algebra; transposed Poisson n-Lie algebra
Mathematics Subject Classification: 17A30, 17B63

1. Introduction

A Poisson algebra is a triple, $(L, \cdot, [-,-])$, where $(L, \cdot)$ is a commutative associative algebra and $(L, [-,-])$ is a Lie algebra that satisfies the following Leibniz rule:

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z], \forall x, y, z \in L.$$

Poisson algebras appear naturally in the study of Hamiltonian mechanics and play a significant role in mathematics and physics, such as in applications of Poisson manifolds, integral systems, algebraic geometry, quantum groups, and quantum field theory (see [7, 11, 24, 25]). Poisson algebras can be viewed as the algebraic counterpart of Poisson manifolds. With the development of Poisson algebras, many other algebraic structures have been found, such as Jacobi algebras [1, 9], Poisson bialgebras

[20,23], Gerstenhaber algebras, Lie-Rinehart algebras [16,17,26], F-manifold algebras [12], Novikov-Poisson algebras [28], quasi-Poisson algebras [8] and Poisson n-Lie algebras [10].

As a dual notion of a Poisson algebra, the concept of a transposed Poisson algebra was recently introduced by Bai et al. [2]. A transposed Poisson algebra $(L, \cdot, [-,-])$ is defined by exchanging the roles of the two binary operations in the Leibniz rule defining the Poisson algebra:

$$2z \cdot [x, y] = [z \cdot x, y] + [x, z \cdot y], \forall x, y, z \in L,$$

where $(L, \cdot)$ is a commutative associative algebra and $(L, [-,-])$ is a Lie algebra.

It is shown that a transposed Poisson algebra possesses many important identities and properties and can be naturally obtained by taking the commutator in the Novikov-Poisson algebra [2]. There are many results on transposed Poisson algebras, such as those on transposed Hom-Poisson algebras [18], transposed BiHom-Poisson algebras [21], a bialgebra theory for transposed Poisson algebras [19], the relation between $\frac{1}{2}$-derivations of Lie algebras and transposed Poisson algebras [14], the relation between $\frac{1}{2}$-biderivations and transposed Poisson algebras [29], and the transposed Poisson structures with fixed Lie algebras (see [6] for more details).

The notion of an n-Lie algebra (see Definition 2.1), as introduced by Filippov [15], has found use in many fields in mathematics and physics [4, 5, 22, 27]. The explicit construction of n-Lie algebras has become one of the important problems in this theory. In [3], Bai et al. gave a construction of $(n + 1)$-Lie algebras through the use of n-Lie algebras and some linear functions. In [13], Dzhumadil′daev introduced the notion of a Poisson n-Lie algebra which can be used to construct an $(n + 1)$-Lie algebra under an additional strong condition. In [2], Bai et al. showed that this strong condition for $n = 2$ holds automatically for a transposed Poisson algebra, and they gave a construction of 3-Lie algebras from transposed Poisson algebras with derivations. They also found that this constructed 3-Lie algebra and the commutative associative algebra satisfy the analog of the compatibility condition for transposed Poisson algebras, which is called a transposed Poisson 3-Lie algebra. This motivated them to introduce the concept of a transposed Poisson n-Lie algebra (see Definition 2.2) and propose the following conjecture:

Conjecture 1.1. *[2] Let $n \geq 2$ be an integer and $(L, \cdot, \mu_n)$ a transposed Poisson n-Lie algebra. Let D be a derivation of $(L, \cdot)$ and (L, μ_n). Define an $(n + 1)$-ary operation:*

$$\mu_{n+1}(x_1, \cdots, x_{n+1}) := \sum_{i=1}^{n+1} (-1)^{i-1} D(x_i)\mu_n(x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}), \quad \forall x_1, \cdots, x_{n+1} \in L,$$

where $\hat{x}_i$ means that the i-th entry is omitted. Then, $(L, \cdot, \mu_{n+1})$ is a transposed Poisson $(n + 1)$-Lie algebra.

In this paper, based on the identities for transposed Poisson n-Lie algebras given in Section 2, we prove that Conjecture 1.1 holds under a certain strong condition described in Section 3 (see Definition 2.3 and Theorem 3.2).

Throughout the paper, all vector spaces are taken over a field of characteristic zero. To simplify notations, the commutative associative multiplication $(\cdot)$ will be omitted unless the emphasis is needed.

2. Identities in transposed Poisson n-Lie algebras

In this section, we first recall some definitions, and then we exhibit a class of identities for transposed Poisson n-Lie algebras.

Definition 2.1. *[15] Let $n \geq 2$ be an integer. An n-Lie algebra is a vector space L, together with a skew-symmetric linear map $[-, \cdots, -] : \otimes^n L \to L$, such that, for any $x_i, y_j \in L, 1 \leq i \leq n-1, 1 \leq j \leq n$, the following identity holds:*

$$[[y_1, \cdots, y_n], x_1, \cdots, x_{n-1}] = \sum_{i=1}^{n} (-1)^{i-1}[[y_i, x_1, \cdots, x_{n-1}], y_1, \cdots, \hat{y}_i, \cdots, y_n]. \tag{2.1}$$

Definition 2.2. *[2] Let $n \geq 2$ be an integer and L a vector space. The triple $(L, \cdot, [-, \cdots, -])$ is called a transposed Poisson n-Lie algebra if $(L, \cdot)$ is a commutative associative algebra and $(L, [-, \cdots, -])$ is an n-Lie algebra such that, for any $h, x_i \in L, 1 \leq i \leq n$, the following identity holds:*

$$nh\,[x_1, \cdots, x_n] = \sum_{i=1}^{n} [x_1, \cdots, hx_i, \cdots, x_n]. \tag{2.2}$$

Some identities for transposed Poisson algebras in [2] can be extended to the following theorem for transposed Poisson n-Lie algebras.

Theorem 2.1. *Let $(L, \cdot, [-, \cdots, -])$ be a transposed Poisson n-Lie algebra. Then, the following identities hold:*

(1) For any $x_i \in L, 1 \leq i \leq n+1$, we have

$$\sum_{i=1}^{n+1} (-1)^{i-1} x_i\,[x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}] = 0; \tag{2.3}$$

(2) For any $h, x_i, y_j \in L, 1 \leq i \leq n-1, 1 \leq j \leq n$, we have

$$\sum_{i=1}^{n} (-1)^{i-1} \left[h\,[y_i, x_1, \cdots, x_{n-1}], y_1, \cdots, \hat{y}_i, \cdots, y_n\right] = \left[h\,[y_1, \cdots, y_n], x_1, \cdots, x_{n-1}\right]; \tag{2.4}$$

(3) For any $x_i, y_j \in L, 1 \leq i \leq n-1, 1 \leq j \leq n+1$, we have

$$\sum_{i=1}^{n+1} (-1)^{i-1} [y_i, x_1, \cdots, x_{n-1}]\,[y_1, \cdots, \hat{y}_i, \cdots, y_{n+1}] = 0; \tag{2.5}$$

(4) For any $x_1, x_2, y_i \in L, 1 \leq i \leq n$, we have

$$\sum_{i=1}^{n} \sum_{j=1, j\neq i}^{n} \left[y_1, \cdots, y_i x_1, \cdots, y_j x_2, \cdots, y_n\right] = n(n-1) x_1 x_2\,[y_1, y_2, \cdots, y_n]. \tag{2.6}$$

Proof. (1) By Eq (2.2), for any $1 \leq i \leq n+1$, we have

$$nx_i\left[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_{n+1}\right] = \sum_{j\neq i}\left[x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_i x_j, \cdots, x_{n+1}\right].$$

Thus, we obtain

$$\sum_{i=1}^{n+1}(-1)^{i-1} n x_i\left[x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}\right] = \sum_{i=1}^{n+1}\sum_{j=1, j\neq i}^{n+1}(-1)^{i-1}\left[x_1, \cdots, \hat{x}_i, \cdots, x_i x_j, \cdots, x_{n+1}\right].$$

Note that, for any $i > j$, we have

$$\begin{aligned}
& (-1)^{i-1}\left[x_1, \cdots, x_{j-1}, x_i x_j, x_{j+1}, \cdots, \hat{x}_i, \cdots, x_n\right] \\
& +(-1)^{j-1}\left[x_1, \cdots, \hat{x}_j, \cdots, x_{i-1}, x_j x_i, x_{i+1}, \cdots, x_n\right] \\
= \; & (-1)^{i-1+(i-j-1)}\left[x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{i-1}, x_i x_j, x_{i+1}, \cdots, x_n\right] \\
& +(-1)^{j-1}\left[x_1, \cdots, \hat{x}_j, \cdots, x_{i-1}, x_j x_i, x_{i+1}, \cdots, x_n\right] \\
= \; & ((-1)^{-j-2} + (-1)^{j-1})\left[x_1, \cdots, x_{j-1}, x_{j+1}, \cdots, x_{i-1}, x_i x_j, x_{i+1}, \cdots, x_n\right] \\
= \; & 0,
\end{aligned}$$

which gives $\sum_{i=1}^{n+1}\sum_{j=1, j\neq i}^{n+1}(-1)^{i-1}\left[x_1, \cdots, \hat{x}_i, \cdots, x_i x_j, \cdots, x_{n+1}\right] = 0.$

Hence, we get

$$\sum_{i=1}^{n+1}(-1)^{i-1} n x_i\left[x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}\right] = 0.$$

(2) By Eq (2.2), we have

$$\begin{aligned}
& -\left[h\left[y_1, \cdots, y_n\right], x_1, \cdots, x_{n-1}\right] - \sum_{i=1}^{n-1}\left[\left[y_1, \cdots, y_n\right], x_1, \cdots, h x_i, \cdots, x_{n-1}\right] \\
= \; & -nh\left[\left[y_1, \cdots, y_n\right], x_1, \cdots, x_{n-1}\right],
\end{aligned}$$

and, for any $1 \leq j \leq n$,

$$\begin{aligned}
& (-1)^{j-1}(\left[h\left[y_j, x_1, \cdots, x_{n-1}\right], y_1, \cdots, \hat{y}_j, \cdots, y_{n-1}\right] \\
& + \sum_{i=1, i\neq j}^{n}\left[\left[y_j, x_1, \cdots, x_{n-1}\right], y_1, \cdots, h y_i, \cdots, \hat{y}_j, \cdots, y_{n-1}\right]) \\
= \; & (-1)^{j-1} nh\left[\left[y_j, x_1, \cdots, x_{n-1}\right], y_1, \cdots, \hat{y}_j, \cdots, y_{n-1}\right].
\end{aligned}$$

By taking the sum of the above $n+1$ identities and applying Eq (2.1), we get

$$-\left[h\left[y_1, \cdots, y_n\right], x_1, \cdots, x_{n-1}\right] - \sum_{i=1}^{n-1}\left[\left[y_1, \cdots, y_n\right], x_1, \cdots, h x_i, \cdots, x_{n-1}\right]$$

$$
\begin{aligned}
& + \sum_{j=1}^{n} (-1)^{j-1} \Big(\Big[h \Big[y_j, x_1, \cdots, x_{n-1} \Big], y_1, \cdots, \hat{y}_j, \cdots, y_{n-1} \Big] \\
& + \sum_{i=1, i \neq j}^{n} \Big[\Big[y_j, x_1, \cdots, x_{n-1} \Big], y_1, \cdots, h y_i, \cdots, \hat{y}_j, \cdots, y_{n-1} \Big] \Big) \\
= \; & -nh \left[[y_1, \cdots, y_n], x_1, \cdots, x_{n-1} \right] + \\
& nh \sum_{j=1}^{n} (-1)^{j-1} \Big[\Big[y_j, x_1, \cdots, x_{n-1} \Big], y_1, \cdots, \hat{y}_j, \cdots, y_{n-1} \Big] \\
= \; & 0.
\end{aligned}
$$

We denote

$$
\begin{aligned}
A_j &:= \sum_{i=1, i \neq j}^{n} (-1)^{i-1} \Big[[y_i, x_1, \cdots, x_{n-1}], y_1, \cdots, h y_j, \cdots, \hat{y}_i, \cdots, y_n \Big], 1 \leq j \leq n, \\
B_i &:= [[y_1, \cdots, y_n], x_1, \cdots, h x_i, \cdots, x_{n-1}], 1 \leq i \leq n-1.
\end{aligned}
$$

Then, the above equation can be rewritten as

$$
\begin{aligned}
& \sum_{i=1}^{n} (-1)^{i-1} \left[h [y_i, x_1, \cdots, x_{n-1}], y_1, \cdots, \hat{y}_i, \cdots, y_n \right] - \left[h [y_1, \cdots, y_n], x_1, \cdots, x_{n-1} \right] \\
& + \sum_{j=1}^{n} A_j - \sum_{i=1}^{n-1} B_i = 0.
\end{aligned} \tag{2.7}
$$

By applying Eq (2.1) to $A_j, 1 \leq j \leq n$, we have

$$
\begin{aligned}
A_j &= \sum_{i=1, i \neq j}^{n} (-1)^{i-1} \Big[[y_i, x_1, \cdots, x_{n-1}], y_1, \cdots, h y_j, \cdots, \hat{y}_i, \cdots, y_n \Big] \\
&= \Big[\Big[y_1, \cdots, h y_j, \cdots, y_n \Big], x_1, \cdots, x_{n-1} \Big] \\
& \quad + (-1)^j \Big[\Big[h y_j, x_1, \cdots, x_{n-1} \Big], y_1, \cdots, \hat{y}_j, \cdots, y_n \Big].
\end{aligned}
$$

Thus, we get

$$
\begin{aligned}
\sum_{j=1}^{n} A_j &= \sum_{j=1}^{n} \Big[\Big[y_1, \cdots, h y_j, \cdots, y_n \Big], x_1, \cdots, x_{n-1} \Big] \\
& \quad + \sum_{j=1}^{n} (-1)^j \Big[\Big[h y_j, x_1, \cdots, x_{n-1} \Big], y_1, \cdots, \hat{y}_j, \cdots, y_n \Big] \\
&= n \left[h [y_1, \cdots, y_n], x_1, \cdots, x_{n-1} \right] \\
& \quad + \sum_{j=1}^{n} (-1)^j \Big[\Big[h y_j, x_1, \cdots, x_{n-1} \Big], y_1, \cdots, \hat{y}_j, \cdots, y_n \Big].
\end{aligned}
$$

By applying Eq (2.1) to $B_i, 1 \leq i \leq n-1$, we have

$$
B_i = [[y_1, \cdots, y_n], x_1, \cdots, h x_i, \cdots, x_{n-1}]
$$

$$= \sum_{j=1}^{n} (-1)^{j-1} \left[\left[y_j, x_1, \cdots, hx_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right].$$

Thus, we get

$$\begin{aligned}
\sum_{i=1}^{n-1} B_i &= \sum_{i=1}^{n-1} \sum_{j=1}^{n} (-1)^{j-1} \left[\left[y_j, x_1, \cdots, hx_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right] \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n-1} (-1)^{j-1} \left[\left[y_j, x_1, \cdots, hx_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right].
\end{aligned}$$

Note that, by Eq (2.2), we have

$$\begin{aligned}
& \sum_{i=1}^{n-1} (-1)^{j-1} \left[\left[y_j, x_1, \cdots, hx_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right] \\
= \ & (-1)^{j-1} n \left[h\left[y_j, x_1, \cdots, x_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right] \\
& +(-1)^{j} \left[\left[hy_j, x_1, \cdots, x_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right].
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\sum_{i=1}^{n-1} B_i &= \sum_{j=1}^{n} (-1)^{j-1} n \left[h\left[y_j, x_1, \cdots, x_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right] \\
& \quad + \sum_{j=1}^{n} (-1)^{j} \left[\left[hy_j, x_1, \cdots, x_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right].
\end{aligned}$$

By substituting these equations into Eq (2.7), we have

$$\begin{aligned}
& \sum_{i=1}^{n} (-1)^{i-1} \left[h\left[y_i, x_1, \cdots, x_{n-1}\right], y_1, \cdots, \hat{y}_i, \cdots, y_n\right] - \left[h\left[y_1, \cdots, y_n\right], x_1, \cdots, x_{n-1}\right] \\
& +n\left[h\left[y_1, \cdots, y_n\right], x_1, \cdots, x_{n-1}\right] + \sum_{j=1}^{n} (-1)^{j} \left[\left[hy_j, x_1, \cdots, x_{n-1}\right], y_1, \cdots, \hat{y}_j, \cdots, y_n\right] \\
& - \sum_{j=1}^{n} (-1)^{j-1} n \left[h\left[y_j, x_1, \cdots, x_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right] \\
& - \sum_{j=1}^{n} (-1)^{j} \left[\left[hy_j, x_1, \cdots, x_i, \cdots, x_{n-1}\right], y_1, \cdots \hat{y}_j, \cdots, y_n\right] \\
= \ & 0,
\end{aligned}$$

which implies that

$$(n-1)\left(\sum_{i=1}^{n} (-1)^{i} \left[h\left[y_i, x_1, \cdots, x_{n-1}\right], y_1, \cdots, \hat{y}_i, \cdots, y_n\right] + \left[h\left[y_1, \cdots, y_n\right], x_1, \cdots, x_{n-1}\right]\right) = 0.$$

Therefore, the proof of Eq (2.4) is completed.

(3) By Eq (2.2), for any $1 \leq j \leq n+1$, we have

$$
\begin{aligned}
&(-1)^{j-1} n\left[y_j, x_1, \cdots, x_{n-1}\right]\left[y_1, \cdots, \hat{y}_j, \cdots, y_{n+1}\right] \\
=&\sum_{i=1, i \neq j}^{n+1}(-1)^{j-1}\left[y_1, \cdots, y_i\left[y_j, x_1, \cdots, x_{n-1}\right], \cdots, \hat{y}_j, \cdots, y_{n+1}\right].
\end{aligned}
$$

By taking the sum of the above $n+1$ identities, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{n+1}(-1)^{j-1} n\left[y_j, x_1, \cdots, x_{n-1}\right]\left[y_1, \cdots, \hat{y}_j, \cdots, y_{n+1}\right] \\
=&\sum_{j=1}^{n+1} \sum_{i=1, i \neq j}^{n+1}(-1)^{j-1}\left[y_1, \cdots, y_i\left[y_j, x_1, \cdots, x_{n-1}\right], \cdots, \hat{y}_j, \cdots, y_{n+1}\right].
\end{aligned}
$$

Thus, we only need to prove the following equation:

$$
\sum_{j=1}^{n+1} \sum_{i=1, i \neq j}^{n+1}(-1)^{j-1}\left[y_1, \cdots, y_i\left[y_j, x_1, \cdots, x_{n-1}\right], \cdots, \hat{y}_j, \cdots, y_{n+1}\right]=0.
$$

Note that

$$
\begin{aligned}
&\sum_{j=1}^{n+1} \sum_{i=1, i \neq j}^{n+1}(-1)^{j-1}\left[y_1, \cdots, y_i\left[y_j, x_1, \cdots, x_{n-1}\right], \cdots, \hat{y}_j, \cdots, y_{n+1}\right] \\
=&\sum_{i=1}^{n+1} \sum_{j=1, j \neq i}^{n+1}(-1)^{j-1}\left[y_1, \cdots, y_i\left[y_j, x_1, \cdots, x_{n-1}\right], \cdots, \hat{y}_j, \cdots, y_{n+1}\right] \\
=&\sum_{i=1}^{n+1} \sum_{j=1}^{i-1}(-1)^{i+j-1}\left[y_i\left[y_j, x_1, \cdots, x_{n-1}\right], y_1, \cdots, \hat{y}_j, \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\
&+\sum_{i=1}^{n+1} \sum_{j=i+1}^{n+1}(-1)^{i+j}\left[y_i\left[y_j, x_1, \cdots, x_{n-1}\right], y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_j, \cdots, y_{n+1}\right] \\
\overset{(2.4)}{=}&\sum_{i=1}^{n+1}(-1)^i\left[y_i\left[y_1, \cdots, \hat{y}_i, \cdots, y_{n+1}\right], x_1, \cdots, x_{n-1}\right] \\
\overset{(2.3)}{=}&\ 0.
\end{aligned}
$$

Hence, the conclusion holds.

(4) By applying Eq (2.2), we have

$$
\begin{aligned}
n^2 x_1 x_2\left[y_1, y_2, \cdots, y_n\right] &= n x_1 \sum_{j=1}^n\left[y_1, \cdots, y_j x_2, \cdots, y_n\right] \\
=&\sum_{i=1}^n \sum_{j=1, j \neq i}^n\left[y_1, \cdots, y_i x_1, \cdots, y_j x_2, \cdots, y_n\right]+\sum_{j=1}^n\left[y_1, \cdots, y_j x_1 x_2, \cdots, y_n\right]
\end{aligned}
$$

$$= \sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left[y_1,\cdots,y_ix_1,\cdots,y_jx_2,\cdots,y_n\right]+nx_1x_2\left[y_1,\cdots,y_n\right],$$

which gives

$$n(n-1)x_1x_2\left[y_1,y_2,\cdots,y_n\right]=\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\left[y_1,\cdots,y_ix_1,\cdots,y_jx_2,\cdots,y_n\right].$$

Hence, the proof is completed.

To prove Conjecture 1.1, we need the following extra condition.

Definition 2.3. *A transposed Poisson n-Lie algebra* $(L,\cdot,[-,\cdots,-])$ *is called strong if the following identity holds:*

$$\begin{aligned} &y_1\left[hy_2,x_1,\cdots,x_{n-1}\right]-y_2\left[hy_1,x_1,\cdots,x_{n-1}\right]+\sum_{i=1}^{n-1}(-1)^{i-1}hx_i\left[y_1,y_2,x_1,\cdots,\hat{x}_i,\cdots,x_{n-1}\right]\\ &=0\end{aligned} \tag{2.8}$$

for any $y_1,y_2,x_i\in L, 1\leq i\leq n-1$.

Remark 2.1. *When* $n=2$*, the identity is*

$$y_1\left[hy_2,x_1\right]+y_2\left[x_1,hy_1\right]+hx_1\left[y_1,y_2\right]=0,$$

which is exactly Theorem 2.5 (11) in [2]. Thus, in the case of a transposed Poisson algebra, the strong condition always holds. So far, we cannot prove that the strong condition fails to hold for $n\geq 3$.

Proposition 2.1. *Let* $(L,\cdot,[-,\cdots,-])$ *be a strong transposed Poisson n-Lie algebra. Then,*

$$y_1\left[hy_2,x_1,\cdots,x_{n-1}\right]-hy_1\left[y_2,x_1,\cdots,x_{n-1}\right]=y_2\left[hy_1,x_1,\cdots,x_{n-1}\right]-hy_2\left[y_1,x_1,\cdots,x_{n-1}\right] \tag{2.9}$$

for any $y_1,y_2,x_i\in L, 1\leq i\leq n-1$.

Proof. By Eq (2.3), we have

$$-hy_1\left[y_2,x_1,\cdots,x_{n-1}\right]+hy_2\left[y_1,x_1,\cdots,x_{n-1}\right]=\sum_{i=1}^{n-1}(-1)^{i-1}hx_i\left[y_1,y_2,x_1,\cdots,\hat{x}_i,\cdots,x_{n-1}\right].$$

Then, the statement follows from Eq (2.8).

3. Proof of the conjecture for strong transposed Poisson n-Lie algebras

In this section, we will prove Conjecture 1.1 for strong transposed Poisson n-Lie algebras. First, we recall the notion of derivations of transposed Poisson n-Lie algebras.

Definition 3.1. *Let* $(L,\cdot,[-,\cdots,-])$ *be a transposed Poisson n-Lie algebra. The linear operation* $D:L\to L$ *is called a derivation of* $(L,\cdot,[-,\cdots,-])$ *if the following holds for any* $u,v,x_i\in L, 1\leq i\leq n$:

(1) D is a derivation of $(L, \cdot)$, *i.e.,* $D(uv) = D(u)v + uD(v)$;
(2) D is a derivation of $(L, [-, \cdots, -])$, *i.e.,*

$$D([x_1, \cdots, x_n]) = \sum_{i=1}^{n} [x_1, \cdots, x_{i-1}, D(x_i), x_{i+1}, \cdots, x_n].$$

Lemma 3.1. *Let* $(L, \cdot, [-, \cdots, -])$ *be a transposed Poisson n-Lie algebra and D a derivation of* $(L, \cdot, [-, \cdots, -])$. *For any* $y_i \in L$, $1 \le i \le n+1$, *we have the following:*

(1)

$$\begin{aligned}&\sum_{i=1}^{n+1} (-1)^{i-1} D(y_i) D([y_1, \cdots, \hat{y}_i, \cdots, y_{n+1}])\\ =&\sum_{i=1}^{n+1} \sum_{j=1, j\neq i}^{n+1} (-1)^{i-1} D(y_i) \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_i, \cdots, y_{n+1}\right];\end{aligned} \tag{3.1}$$

(2)

$$\begin{aligned}&\sum_{i=1}^{n+1} (-1)^{i-1} D(y_i) D([y_1, \cdots, \hat{y}_i, \cdots, y_{n+1}])\\ =&\sum_{i=1}^{n+1} \sum_{j=1, j\neq i}^{n+1} \sum_{k=j+1, k\neq i}^{n+1} (-1)^{i} y_i \left[y_1, \cdots, D(y_j), \cdots, D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right],\end{aligned} \tag{3.2}$$

where, for any $i > j$, $\sum_i^j$ *denotes the empty sum, which is equal to zero.*

Proof. (1) The statement follows immediately from Definition 3.1.

(2) By applying Eq (3.1), we need to prove the following equation:

$$\begin{aligned}&\sum_{i=1}^{n+1} \sum_{j=1, j\neq i}^{n+1} (-1)^{i-1} nD(y_i) \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_i, \cdots, y_{n+1}\right]\\ =\ &\sum_{i=1}^{n+1} \sum_{j=1, j\neq i}^{n+1} \sum_{k=j+1, k\neq i}^{n+1} (-1)^{i} n y_i \left[y_1, \cdots, D(y_j), \cdots, D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right].\end{aligned}$$

For any $1 \le i \le n+1$, denote $A_i := n \sum_{j=1, j\neq i}^{n+1} (-1)^{i-1} D(y_i) \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_i, \cdots, y_{n+1}\right]$. Then, we have

$$\sum_{i=1}^{n+1} \sum_{j=1, j\neq i}^{n+1} (-1)^{i-1} nD(y_i) \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] = \sum_{i=1}^{n+1} A_i.$$

Note that

$$\begin{aligned}A_i =\ &(-1)^{i-1}(nD(y_i)[D(y_1), y_2, \cdots, \hat{y}_i, \cdots, y_{n+1}] + nD(y_i)[y_1, D(y_2), y_3, \cdots, \hat{y}_i, \cdots, y_{n+1}]\\ &+ \cdots + nD(y_i)[y_1, \cdots, \hat{y}_i, \cdots, y_n, D(y_{n+1})])\end{aligned}$$

$$
\begin{aligned}
&= (-1)^{i-1}([D(y_i)D(y_1), y_2, \cdots, \hat{y}_i, \cdots, y_{n+1}] \\
&\quad + \sum_{k=2,k\neq i}^{n+1} [D(y_1), y_2, \cdots, y_k D(y_i), \cdots, \hat{y}_i, \cdots, y_{n+1}] \\
&\quad + [y_1, D(y_i)D(y_2), y_3, \cdots, \hat{y}_i, \cdots, y_{n+1}] \\
&\quad + \sum_{k=1,k\neq 2,i}^{n+1} [y_1, D(y_2), y_3, \cdots, y_k D(y_i), \cdots, \hat{y}_i, \cdots, y_{n+1}] \\
&\quad + \cdots + [y_1, \cdots, \hat{y}_i, \cdots, y_n, D(y_i)D(y_{n+1})] \\
&\quad + \sum_{k=1,k\neq i}^{n} [y_1, \cdots, y_k D(y_i), \cdots, \hat{y}_i, \cdots, y_n, D(y_{n+1})]) \\
&= (-1)^{i-1} \sum_{j=1,j\neq i}^{n+1} \left[y_1, \cdots, D(y_i)D(y_j), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\
&\quad +(-1)^{i-1} \sum_{j=1,j\neq i}^{n+1} \sum_{k=1,k\neq j,i}^{n+1} \left[y_1, \cdots, D(y_j), \cdots, y_k D(y_i), \cdots, \hat{y}_i, \cdots, y_{n+1}\right].
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
\sum_{i=1}^{n+1} A_i &= \sum_{j=1}^{n+1} \sum_{i=1,i\neq j}^{n+1} (-1)^{j-1} \left[y_1, \cdots, D(y_j)D(y_i), \cdots, \hat{y}_j, \cdots, y_{n+1}\right] \\
&\quad + \sum_{i=1}^{n+1} \sum_{j=1,j\neq i}^{n+1} \sum_{k=1,k\neq i,j}^{n+1} (-1)^{i-1} \left[y_1, \cdots, D(y_j), \cdots, y_k D(y_i), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\
&= T_1 + T_2,
\end{aligned}
$$

where

$$
\begin{aligned}
T_1 &:= \sum_{j=1}^{n+1} \sum_{i=1,i\neq j}^{n+1} (-1)^{j-1} \left[y_1, \cdots, D(y_j)D(y_i), \cdots, \hat{y}_j, \cdots, y_{n+1}\right], \\
T_2 &:= \sum_{i=1}^{n+1} \sum_{j=1,j\neq i}^{n+1} \sum_{k=1,k\neq i,j}^{n+1} (-1)^{i-1} \left[y_1, \cdots, D(y_j), \cdots, y_k D(y_i), \cdots, \hat{y}_i, \cdots, y_{n+1}\right].
\end{aligned}
$$

Note that

$$
T_1 = \sum_{j,i=1}^{n+1} B_{ji},
$$

where $B_{ji} = (-1)^{j-1} \left[y_1, \cdots, D(y_j)D(y_i), \cdots, \hat{y}_j, \cdots, y_{n+1}\right]$ for any $1 \leq j \neq i \leq n+1$, and $B_{ii} = 0$ for any $1 \leq i \leq n+1$.

For any $1 \leq i, j \leq n+1$, without loss of generality, assume that $i < j$; then, we have

$$
B_{ji} + B_{ij}
$$

$$
\begin{aligned}
&= (-1)^{j-1}\left[y_1,\cdots,D(y_j)D(y_i),\cdots,\hat{y}_j,\cdots,y_{n+1}\right] \\
&\quad +(-1)^{i-1}\left[y_1,\cdots,\hat{y}_i,\cdots,D(y_i)D(y_j),\cdots,y_{n+1}\right] \\
&= (-1)^{j-1}\left[y_1,\cdots,D(y_j)D(y_i),\cdots,\hat{y}_j,\cdots,y_{n+1}\right] \\
&\quad +(-1)^{i-1+j-i+1}\left[y_1,\cdots,D(y_j)D(y_i),\cdots,\hat{y}_j,\cdots,y_{n+1}\right] \\
&= 0,
\end{aligned}
$$

which implies that $T_1 = \sum_{j,i=1}^{n+1} B_{ji} = 0$.

Thus, we get that $\sum_{i=1}^{n+1} A_i = T_2$.

We rewrite

$$
\begin{aligned}
&\sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=j+1,k\neq i}^{n+1}(-1)^i n y_i\left[y_1,\cdots,D(y_j),\cdots,D(y_k),\cdots,\hat{y}_i,\cdots,y_{n+1}\right] \\
=\ & \sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=j+1,k\neq i}^{n+1}\sum_{t=1,t\neq j,k,i}^{n+1}(-1)^i \\
&\cdot\left[y_1,\cdots,D(y_j),\cdots,D(y_k),\cdots,y_t y_i,\cdots,\hat{y}_i,\cdots,y_{n+1}\right] \\
&+\sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=j+1,k\neq i}^{n+1}(-1)^i\left[y_1,\cdots,y_i D(y_j),\cdots,D(y_k),\cdots,\hat{y}_i,\cdots,y_{n+1}\right] \\
&+\sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=j+1,k\neq i}^{n+1}(-1)^i\left[y_1,\cdots,D(y_j),\cdots,y_i D(y_k),\cdots,\hat{y}_i,\cdots,y_{n+1}\right] \\
=\ & M_1+M_2+M_3,
\end{aligned}
$$

where

$$
\begin{aligned}
M_1 &:= \sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=j+1,k\neq i}^{n+1}\sum_{t=1,t\neq j,k,i}^{n+1}(-1)^i \\
&\quad\cdot\left[y_1,\cdots,D(y_j),\cdots,D(y_k),\cdots,y_t y_i,\cdots,\hat{y}_i,\cdots,y_{n+1}\right], \\
M_2 &:= \sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=j+1,k\neq i}^{n+1}(-1)^i\left[y_1,\cdots,y_i D(y_j),\cdots,D(y_k),\cdots,\hat{y}_i,\cdots,y_{n+1}\right], \\
M_3 &:= \sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=j+1,k\neq i}^{n+1}(-1)^i\left[y_1,\cdots,D(y_j),\cdots,y_i D(y_k),\cdots,\hat{y}_i,\cdots,y_{n+1}\right].
\end{aligned}
$$

Note that

$$
\begin{aligned}
M_1 &= \sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=j+1,k\neq i}^{n+1}\sum_{t=1,t\neq j,k,i}^{n+1}(-1)^i \\
&\quad\cdot\left[y_1,\cdots,D(y_j),\cdots,D(y_k),\cdots,y_t y_i,\cdots,\hat{y}_i,\cdots,y_{n+1}\right]
\end{aligned}
$$

$$= \sum_{i,j,k,t=1}^{n+1} B_{ijkt},$$

where

$$B_{ijkt} = \begin{cases} 0, & \text{if any two indices are equal or } k < j; \\ (-1)^i \left[y_1, \cdots, D(y_j), \cdots, D(y_k), \cdots, y_t y_i, \cdots, \hat{y}_i, \cdots, y_{n+1}\right], & \text{otherwise.} \end{cases}$$

For any $1 \leq j, k \leq n+1$, without loss of generality, assume that $t < i$; then, we have

$$\begin{aligned} & B_{ijkt} + B_{tjki} \\ = & (-1)^i \left[y_1, \cdots, D(y_j), \cdots, D(y_k), \cdots, y_t y_i, \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\ & +(-1)^t \left[y_1, \cdots, D(y_j), \cdots, D(y_k), \cdots, \hat{y}_t, \cdots, y_t y_i, \cdots, y_{n+1}\right] \\ = & (-1)^i \left[y_1, \cdots, D(y_j), \cdots, D(y_k), \cdots, y_t y_i, \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\ & +(-1)^{t+i-t-1} \left[y_1, \cdots, D(y_j), \cdots, D(y_k), \cdots, y_t y_i, \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\ = & 0, \end{aligned}$$

which implies that $M_1 = 0$.

Therefore, we only need to prove the following equation:

$$\begin{aligned} & M_2 + M_3 \\ = & \sum_{i=1}^{n+1} \sum_{j=1, j\neq i}^{n+1} \sum_{k=1, k\neq i,j}^{n+1} (-1)^{i-1} \left[y_1, \cdots, D(y_j), \cdots, y_k D(y_i), \cdots, \hat{y}_i, \cdots, y_{n+1}\right]. \end{aligned}$$

First, we have

$$\begin{aligned} & \sum_{j=1, j\neq i}^{n+1} \sum_{k=j+1, k\neq i}^{n+1} (-1)^i \left[y_1, \cdots, y_i D(y_j), \cdots, D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\ & + \sum_{j=1, j\neq i}^{n+1} \sum_{k=j+1, k\neq i}^{n+1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\ = & \sum_{k=1, k\neq i}^{n+1} \sum_{j=1, j\neq i}^{k-1} (-1)^i \left[y_1, \cdots, y_i D(y_j), \cdots, D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\ & + \sum_{j=1, j\neq i}^{n+1} \sum_{k=j+1, k\neq i}^{n+1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\ = & \sum_{j=1, j\neq i}^{n+1} \sum_{k=1, k\neq i}^{j-1} (-1)^i \left[y_1, \cdots, y_i D(y_k), \cdots, D(y_j), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\ & + \sum_{j=1, j\neq i}^{n+1} \sum_{k=j+1, k\neq i}^{n+1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \end{aligned}$$

$$= \sum_{j=1,j\neq i}^{n+1} \sum_{k=1,k\neq i,j}^{n+1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right].$$

Thus,

$$\begin{aligned}
& M_2 + M_3 \\
= & \sum_{i=1}^{n+1} \sum_{j=1,j\neq i}^{n+1} \sum_{k=1,k\neq i,j}^{n+1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\
= & \sum_{j=1}^{n+1} \sum_{i=1,i\neq j}^{n+1} \sum_{k=1,k\neq i,j}^{n+1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\
= & \sum_{j=1}^{n+1} \sum_{i=1,i\neq j}^{n+1} \sum_{k=1,k\neq j}^{i-1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\
& + \sum_{j=1}^{n+1} \sum_{i=1,i\neq j}^{n+1} \sum_{k=i+1,k\neq j}^{n+1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_i, \cdots, y_i D(y_k), \cdots, y_{n+1}\right].
\end{aligned}$$

Note that, for any $1 \leq j \leq n+1$, we have

$$\begin{aligned}
& \sum_{i=1,i\neq j}^{n+1} \sum_{k=1,k\neq j}^{i-1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_i, \cdots, y_{n+1}\right] \\
= & \sum_{i=1,i\neq j}^{n+1} \sum_{k=1,k\neq j}^{i-1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, y_{k-1}, y_i D(y_k), y_{k+1}, \cdots, \hat{y}_i \cdots, y_{n+1}\right] \\
= & \sum_{i=1,i\neq j}^{n+1} \sum_{k=1,k\neq j}^{i-1} (-1)^{k-1} \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_k, \cdots, y_{i-1}, y_i D(y_k), y_{i+1}, \cdots, y_{n+1}\right] \\
= & \sum_{i=1,i\neq j}^{n+1} \sum_{k=1,k\neq j}^{i-1} (-1)^{k-1} \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_k, \cdots, y_i D(y_k), \cdots, y_{n+1}\right].
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \sum_{i=1,i\neq j}^{n+1} \sum_{k=i+1,k\neq j}^{n+1} (-1)^i \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_i, \cdots, y_i D(y_k), \cdots, y_{n+1}\right] \\
= & \sum_{i=1,i\neq j}^{n+1} \sum_{k=i+1,k\neq j}^{n+1} (-1)^{k-1} \left[y_1, \cdots, D(y_j), \cdots, y_i D(y_k), \cdots, \hat{y}_k, \cdots, y_{n+1}\right].
\end{aligned}$$

Thus,

$$\begin{aligned}
& M_2 + M_3 \\
= & \sum_{j=1}^{n+1} \sum_{i=1,i\neq j}^{n+1} \sum_{k=1,k\neq j}^{i-1} (-1)^{k-1} \left[y_1, \cdots, D(y_j), \cdots, \hat{y}_k, \cdots, y_i D(y_k), \cdots, y_{n+1}\right]
\end{aligned}$$

$$+\sum_{j=1}^{n+1}\sum_{i=1,i\neq j}^{n+1}\sum_{k=i+1,k\neq j}^{n+1}(-1)^{k-1}\left[y_1,\cdots,D(y_j),\cdots,y_iD(y_k),\cdots,\hat{y}_k,\cdots,y_{n+1}\right]$$
$$=\sum_{j=1}^{n+1}\sum_{i=1,i\neq j}^{n+1}\sum_{k=1,k\neq i,j}^{n+1}(-1)^{k-1}\left[y_1,\cdots,D(y_j),\cdots,y_iD(y_k),\cdots,\hat{y}_k,\cdots,y_{n+1}\right]$$
$$=\sum_{k=1}^{n+1}\sum_{j=1,j\neq k}^{n+1}\sum_{i=1,i\neq j,k}^{n+1}(-1)^{k-1}\left[y_1,\cdots,D(y_j),\cdots,y_iD(y_k),\cdots,\hat{y}_k,\cdots,y_{n+1}\right]$$
$$=\sum_{i=1}^{n+1}\sum_{j=1,j\neq i}^{n+1}\sum_{k=1,k\neq j,i}^{n+1}(-1)^{i-1}\left[y_1,\cdots,D(y_j),\cdots,y_kD(y_i),\cdots,\hat{y}_i,\cdots,y_{n+1}\right].$$

The proof is completed.

Theorem 3.1. *Let $(L,\cdot,[-,\cdots,-])$ be a strong transposed Poisson n-Lie algebra and D a derivation of $(L,\cdot,[-,\cdots,-])$. Define an $(n+1)$-ary operation:*

$$\mu_{n+1}(x_1,\cdots,x_{n+1}):=\sum_{i=1}^{n+1}(-1)^{i-1}D(x_i)[x_1,\cdots,\hat{x}_i,\cdots,x_{n+1}] \tag{3.3}$$

for any $x_i\in L, 1\leq i\leq n+1$. Then, (L,μ_{n+1}) is an $(n+1)$-Lie algebra.

Proof. For convenience, we denote

$$\mu_{n+1}(x_1,\cdots,x_{n+1}):=[x_1,\cdots,x_{n+1}].$$

On one hand, we have

$$\begin{aligned}
&[[y_1,\cdots,y_{n+1}],x_1,\cdots,x_n]\\
\overset{(3.3)}{=}\ &\sum_{i=1}^{n+1}(-1)^{i-1}\left[D(y_i)[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}],x_1,\cdots,x_n\right]\\
\overset{(3.3)}{=}\ &\sum_{i=1}^{n+1}(-1)^{i-1}D(D(y_i)[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}])[x_1,\cdots,x_n]\\
&+\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j-1}D(x_j)\left[D(y_i)[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}],x_1,\cdots,\hat{x}_j,\cdots,x_n\right]\\
=\ &\sum_{i=1}^{n+1}(-1)^{i-1}D^2(y_i)[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}][x_1,\cdots,x_n]\\
&+\sum_{i=1}^{n+1}(-1)^{i-1}D(y_i)D([y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}])[x_1,\cdots,x_n]\\
&+\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j-1}D(x_j)\left[D(y_i)[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}],x_1,\cdots,\hat{x}_j,\cdots,x_n\right]
\end{aligned}$$

$$
\begin{aligned}
&\overset{(3.1)}{=} \sum_{i=1}^{n+1}(-1)^{i-1}D^2(y_i)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right]\left[x_1,\cdots,x_n\right] \\
&\quad+\sum_{i=1}^{n+1}\sum_{k=1,k\neq i}^{n+1}(-1)^{i-1}D(y_i)\left[y_1,\cdots,D(y_k),\cdots,\hat{y}_i,\cdots,y_{n+1}\right]\left[x_1,\cdots,x_n\right] \\
&\quad+\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j-1}D(x_j)\Big[D(y_i)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right],x_1,\cdots,\hat{x}_j,\cdots,x_n\Big] \\
&= \sum_{i=1}^{n+1}(-1)^{i-1}D^2(y_i)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right]\left[x_1,\cdots,x_n\right] \\
&\quad+\sum_{k=1}^{n+1}\sum_{i=1}^{k-1}(-1)^{k+i-1}D(y_i)\left[D(y_k),y_1,\cdots,\hat{y}_i,\cdots,\hat{y}_k,\cdots,y_{n+1}\right]\left[x_1,\cdots,x_n\right] \\
&\quad+\sum_{k=1}^{n+1}\sum_{i=k+1}^{n+1}(-1)^{i+k}D(y_i)\left[D(y_k),y_1,\cdots,\hat{y}_k,\cdots,\hat{y}_i,\cdots,y_{n+1}\right]\left[x_1,\cdots,x_n\right] \\
&\quad+\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{i+j-1}D(x_j)\Big[D(y_i)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right],x_1,\cdots,\hat{x}_j,\cdots,x_n\Big].
\end{aligned}
$$

On the other hand, for any $1 \leq k \leq n$, we have

$$
\begin{aligned}
&(-1)^{k-1}\left[\left[y_k,x_1,\cdots,x_n\right],y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}\right] \\
\overset{(3.3)}{=}\;&(-1)^{k-1}\left[D(y_k)\left[x_1,\cdots,x_n\right],y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}\right] \\
&+\sum_{j=1}^{n}(-1)^{j+k-1}\Big[D(x_j)\Big[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\Big],y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}\Big] \\
\overset{(3.3)}{=}\;&(-1)^{k-1}D(D(y_k)\left[x_1,\cdots,x_n\right])\left[y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}\right] \\
&+\sum_{i=1}^{k-1}(-1)^{i+k-1}D(y_i)\left[D(y_k)\left[x_1,\cdots,x_n\right],y_1,\cdots,\hat{y}_i,\cdots,\hat{y}_k,\cdots,y_{n+1}\right] \\
&+\sum_{i=k+1}^{n+1}(-1)^{i+k}D(y_i)\left[D(y_k)\left[x_1,\cdots,x_n\right],y_1,\cdots,\hat{y}_k,\cdots,\hat{y}_i,\cdots,y_{n+1}\right] \\
&+\sum_{j=1}^{n}(-1)^{j+k-1}D(D(x_j)\Big[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\Big])\left[y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}\right] \\
&+\sum_{j=1}^{n}\sum_{i=k+1}^{n+1}((-1)^{i+j}D(y_i) \\
&\cdot\Big[D(x_j)\Big[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\Big],y_1,\cdots,\hat{y}_k,\cdots,\hat{y}_i,\cdots,y_{n+1}\Big]) \\
&+\sum_{j=1}^{n}\sum_{i=1}^{k-1}((-1)^{i+j-1}D(y_i) \\
&\cdot\Big[D(x_j)\Big[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\Big],y_1,\cdots,\hat{y}_i,\cdots,\hat{y}_k,\cdots,y_{n+1}\Big])
\end{aligned}
$$

$$
\begin{aligned}
= \quad & (-1)^{k-1} D^2(y_k)\,[x_1, \cdots, x_n]\,[y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}] \\
& + \sum_{i=1}^{k-1} (-1)^{i+k-1} D(y_i)\,[D(y_k)\,[x_1, \cdots, x_n], y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}] \\
& + \sum_{i=k+1}^{n+1} (-1)^{i+k} D(y_i)\,[D(y_k)\,[x_1, \cdots, x_n], y_1, \cdots, \hat{y}_k, \cdots, \hat{y}_i, \cdots, y_{n+1}] \\
& + \sum_{j=1}^{n} (-1)^{j+k-1} D^2(x_j)\left[y_k, x_1, \cdots, \hat{x}_j, \cdots, x_n\right][y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}] \\
& + \sum_{j=1}^{n} \sum_{i=k+1}^{n+1} ((-1)^{i+j} D(y_i) \\
& \cdot \left[D(x_j)\left[y_k, x_1, \cdots, \hat{x}_j, \cdots, x_n\right], y_1, \cdots, \hat{y}_k, \cdots, \hat{y}_i, \cdots, y_{n+1}\right]) \\
& + \sum_{j=1}^{n} \sum_{i=1}^{k-1} ((-1)^{i+j-1} D(y_i) \\
& \cdot \left[D(x_j)\left[y_k, x_1, \cdots, \hat{x}_j, \cdots, x_n\right], y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}\right]) \\
& + (-1)^{k-1} D(y_k) D([x_1, \cdots, x_n])\,[y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}] \\
& + \sum_{j=1}^{n} (-1)^{j+k-1} D(x_j) D(\left[y_k, x_1, \cdots, \hat{x}_j, \cdots, x_n\right])\,[y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}] \\
\overset{(3.2)}{=} \quad & (-1)^{k-1} D^2(y_k)\,[x_1, \cdots, x_n]\,[y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}] \\
& + \sum_{i=1}^{k-1} (-1)^{i+k-1} D(y_i)\,[D(y_k)\,[x_1, \cdots, x_n], y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}] \\
& + \sum_{i=k+1}^{n+1} (-1)^{i+k} D(y_i)\,[D(y_k)\,[x_1, \cdots, x_n], y_1, \cdots, \hat{y}_k, \cdots, \hat{y}_i, \cdots, y_{n+1}] \\
& + \sum_{j=1}^{n} (-1)^{j+k-1} D^2(x_j)\left[y_k, x_1, \cdots, \hat{x}_j, \cdots, x_n\right][y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}] \\
& + \sum_{j=1}^{n} \sum_{i=k+1}^{n+1} ((-1)^{i+j} D(y_i) \\
& \cdot \left[D(x_j)\left[y_k, x_1, \cdots, \hat{x}_j, \cdots, x_n\right], y_1, \cdots, \hat{y}_k, \cdots, \hat{y}_i, \cdots, y_{n+1}\right]) \\
& + \sum_{j=1}^{n} \sum_{i=1}^{k-1} ((-1)^{i+j-1} D(y_i) \\
& \cdot \left[D(x_j)\left[y_k, x_1, \cdots, \hat{x}_j, \cdots, x_n\right], y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}\right]) \\
& + \sum_{j=1}^{n} \sum_{t=j+1}^{n} (-1)^k y_k \left[x_1, \cdots, D(x_j), \cdots, D(x_t), \cdots, x_n\right][y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}]
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}(-1)^{k+i}x_i\left[D(y_k),x_1,\cdots,D(x_j),\cdots,\hat{x}_i,\cdots,x_n\right][y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}]\\
&+\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{t=j+1,t\neq i}^{n}(-1)^{k+i}x_i\left[y_k,x_1,\cdots,D(x_j),D(x_t),\cdots,\hat{x}_i,\cdots,x_n\right]\\
&\cdot[y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}].
\end{aligned}
$$

We denote

$$
\sum_{i=1}^{n+1}(-1)^{i-1}\left[[y_i,x_1,\cdots,x_n],y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right]=\sum_{i=1}^{7}A_i,
$$

where

$$
\begin{aligned}
A_1 &:= \sum_{i=1}^{n+1}(-1)^{i-1}D^2(y_i)[x_1,\cdots,x_n][y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}],\\
A_2 &:= \sum_{k=1}^{n+1}\sum_{j=1}^{n}(-1)^{k+j-1}D^2(x_j)\left[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\right][y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}],\\
A_3 &:= \sum_{i=1}^{n+1}\sum_{j=1}^{n}\sum_{k=j+1}^{n}(-1)^{i}y_i\left[x_1,\cdots,D(x_j),\cdots,D(x_k),\cdots,x_n\right][y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}],\\
A_4 &:= \sum_{k=1}^{n+1}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}\sum_{t=j+1,t\neq i}^{n}((-1)^{k+i}x_i\left[y_k,x_1,\cdots,D(x_j),D(x_t),\cdots,\hat{x}_i,\cdots,x_n\right]\\
&\quad\cdot[y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}]),\\
A_5 &:= \sum_{k=1}^{n+1}\sum_{i=1}^{k-1}(-1)^{k+i-1}D(y_i)\left[D(y_k)[x_1,\cdots,x_n],y_1,\cdots,\hat{y}_i,\cdots,\hat{y}_k,\cdots,y_{n+1}\right]\\
&\quad+\sum_{k=1}^{n+1}\sum_{i=k+1}^{n+1}(-1)^{i+k}D(y_i)\left[D(y_k)[x_1,\cdots,x_n],y_1,\cdots,\hat{y}_k,\cdots,\hat{y}_i,\cdots,y_{n+1}\right],\\
A_6 &:= \sum_{k=1}^{n+1}\sum_{i=1}^{n}\sum_{j=1,j\neq i}^{n}((-1)^{k+i}x_i\left[D(y_k),x_1,\cdots,D(x_j),\cdots,\hat{x}_i,\cdots,x_n\right]\\
&\quad\cdot[y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}]),\\
A_7 &:= \sum_{k=1}^{n+1}\sum_{j=1}^{n}\sum_{i=k+1}^{n+1}((-1)^{k+i+j}D(y_i)\\
&\quad\cdot\left[D(x_j)\left[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\right],y_1,\cdots,\hat{y}_k,\cdots,\hat{y}_i,\cdots,y_{n+1}\right])\\
&\quad+\sum_{k=1}^{n+1}\sum_{j=1}^{n}\sum_{i=1}^{k-1}((-1)^{k+i+j-1}D(y_i)\\
&\quad\cdot\left[D(x_j)\left[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\right],y_1,\cdots,\hat{y}_i,\cdots,\hat{y}_k,\cdots,y_{n+1}\right]).
\end{aligned}
$$

By Eq (2.5), for fixed j, we have

$$\sum_{k=1}^{n+1}(-1)^{k+j-1}D^2(x_j)\left[y_k, x_1, \cdots, \hat{x}_j, \cdots, x_n\right][y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}] = 0.$$

So, we obtain that $A_2 = 0$.

By Eq (2.3), for fixed j and k, we have

$$\sum_{i=1}^{n+1}(-1)^i y_i \left[x_1, \cdots, D(x_j), \cdots, D(x_k), \cdots, x_n\right][y_1, \cdots, \hat{y}_i, \cdots, y_{n+1}] = 0.$$

So, we obtain that $A_3 = 0$.

By Eq (2.5), for fixed j and t, we have

$$\sum_{k=1}^{n+1}(-1)^{k+i}x_i \left[y_k, x_1, \cdots, D(x_j), D(x_t), \cdots, \hat{x}_i, \cdots, x_n\right][y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}] = 0.$$

So, we obtain that $A_4 = 0$.

By Eq (2.9), for fixed i and k, we have

$$\begin{aligned}
&(-1)^{k+i-1}D(y_i)\left[D(y_k)\left[x_1, \cdots, x_n\right], y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}\right] \\
&+(-1)^{i+k}D(y_k)\left[D(y_i)\left[x_1, \cdots, x_n\right], y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}\right] \\
=\ &(-1)^{k+i-1}D(y_i)\left[D(y_k), y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}\right][x_1, \cdots, x_n] \\
&+(-1)^{i+k}D(y_k)\left[D(y_i), y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}\right][x_1, \cdots, x_n].
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
A_5 = &\sum_{k=1}^{n+1}\sum_{i=1}^{k-1}(-1)^{k+i-1}D(y_i)\left[D(y_k), y_1, \cdots, \hat{y}_i, \cdots, \hat{y}_k, \cdots, y_{n+1}\right][x_1, \cdots, x_n] \\
&+\sum_{k=1}^{n+1}\sum_{i=k+1}^{n+1}(-1)^{i+k}D(y_i)\left[D(y_k), y_1, \cdots, \hat{y}_k, \cdots, \hat{y}_i, \cdots, y_{n+1}\right][x_1, \cdots, x_n].
\end{aligned}$$

By Eq (2.3), for fixed j and k, we have

$$\begin{aligned}
&\sum_{i=1}^{n}(-1)^{k+i}x_i\left[D(y_k), x_1, \cdots, D(x_j), \cdots, \hat{x}_i, \cdots, x_n\right] \\
=\ &(-1)^{k-1}D(y_k)\left[x_1, \cdots, D(x_j), \cdots, x_n\right] + (-1)^{k+j-1}D(x_j)\left[D(y_k), x_1, \cdots, x_n\right] \\
=\ &(-1)^{k+j}D(y_k)\left[D(x_j), x_1, \cdots, \hat{x}_j, \cdots, x_n\right] + (-1)^{k+j-1}D(x_j)\left[D(y_k), x_1, \cdots, x_n\right].
\end{aligned}$$

Thus, we get

$$A_6 = \sum_{k=1}^{n+1}\sum_{j=1}^{n}(-1)^{k+j}D(y_k)\left[D(x_j), x_1, \cdots, \hat{x}_j, \cdots, x_n\right][y_1, \cdots, \hat{y}_k, \cdots, y_{n+1}]$$

$$+\sum_{k=1}^{n+1}\sum_{j=1}^{n}(-1)^{k+j-1}D(x_j)\left[D(y_k),x_1,\cdots,x_n\right]\left[y_1,\cdots,\hat{y}_k,\cdots,y_{n+1}\right].$$

By Eq (2.4), for fixed j and i, we have

$$\begin{aligned}
&\sum_{k=i+1}^{n+1}(-1)^{k+i+j-1}D(y_i)\left[D(x_j)\left[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\right],y_1,\cdots,\hat{y}_i,\cdots,\hat{y}_k,\cdots,y_{n+1}\right]\\
&+\sum_{k=1}^{i-1}(-1)^{k+i+j}D(y_i)\left[D(x_j)\left[y_k,x_1,\cdots,\hat{x}_j,\cdots,x_n\right],y_1,\cdots,\hat{y}_k,\cdots,\hat{y}_i,\cdots,y_{n+1}\right]\\
=\;&(-1)^{j+i-1}D(y_i)\left[D(x_j)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right],x_1,\cdots,\hat{x}_j,\cdots,x_n\right].
\end{aligned}$$

So, we obtain

$$A_7=\sum_{j=1}^{n}\sum_{i=1}^{n+1}(-1)^{j+i-1}D(y_i)\left[D(x_j)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right],x_1,\cdots,\hat{x}_j,\cdots,x_n\right].$$

By Eq (2.9), we have

$$\begin{aligned}
&(-1)^{i+j}D(y_i)\left[D(x_j),x_1,\cdots,\hat{x}_j,\cdots,x_n\right]\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right]\\
&+(-1)^{i+j-1}D(x_j)\left[D(y_i),x_1,\cdots,x_n\right]\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right]\\
&+(-1)^{j+i-1}D(y_i)\left[D(x_j)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right],x_1,\cdots,\hat{x}_j,\cdots,x_n\right]\\
=\;&(-1)^{j+i-1}D(x_j)\left[D(y_i)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right],x_1,\cdots,\hat{x}_j,\cdots,x_n\right].
\end{aligned}$$

So, we get

$$A_6+A_7=\sum_{i=1}^{n+1}\sum_{j=1}^{n}(-1)^{j+i-1}D(x_j)\left[D(y_i)\left[y_1,\cdots,\hat{y}_i,\cdots,y_{n+1}\right],x_1,\cdots,\hat{x}_j,\cdots,x_n\right].$$

Thus, we have

$$\sum_{i=1}^{7}A_i=A_1+A_5+A_6+A_7=\left[\left[y_1,\cdots,y_{n+1}\right],x_1,\cdots,x_n\right].$$

Therefore, (L,μ_{n+1}) is an $(n+1)$-Lie algebra.

Now, we can prove Conjecture 1.1 for strong transposed Poisson n-Lie algebras.

Theorem 3.2. *With the notations in Theorem 3.1,* $(L,\cdot,\mu_{n+1})$ *is a strong transposed Poisson* $(n+1)$*-Lie algebra.*

Proof. For convenience, we denote $\mu_{n+1}(x_1,\cdots,x_{n+1}):=\left[x_1,\cdots,x_{n+1}\right]$. According to Theorem 3.1, we only need to prove Eqs (2.2) and (2.8).

Proof of Eq (2.2). By Eq (3.3), we have

$$\sum_{i=1}^{n+1}\left[x_1,\cdots,hx_i,\cdots,x_{n+1}\right]$$

$$
\begin{aligned}
= \quad & D(hx_1)\,[x_2, \cdots, x_{n+1}] + \sum_{j=2}^{n+1} (-1)^{j-1} D(x_j) \left[hx_1, x_2, \cdots, \hat{x}_j, \cdots, x_{n+1}\right] \\
& - D(hx_2)\,[x_1, x_3, \cdots, x_{n+1}] + \sum_{j=1, j\neq 2}^{n+1} (-1)^{j-1} D(x_j) \left[x_1, hx_2, x_3, \cdots, \hat{x}_j, \cdots, x_{n+1}\right] \\
& + \cdots + (-1)^n D(hx_n)\,[x_1, \cdots, x_n] + \sum_{j=1}^{n} (-1)^{j-1} D(x_j) \left[x_1, \cdots, \hat{x}_j, \cdots, x_n, hx_{n+1}\right] \\
= \quad & \sum_{i=1}^{n+1} (-1)^{i-1} D(hx_i)\,[x_1, \cdots, \hat{x}_i, \cdots, x_n] \\
& + \sum_{i=1}^{n+1} \sum_{j=1, j\neq i}^{n+1} (-1)^{j-1} D(x_j) \left[x_1, , \cdots, hx_i, \cdots, \hat{x}_j, \cdots, x_{n+1}\right] \\
= \quad & \sum_{i=1}^{n+1} (-1)^{i-1} D(hx_i)\,[x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}] \\
& + \sum_{j=1}^{n+1} \sum_{i=1, i\neq j}^{n+1} (-1)^{j-1} D(x_j) \left[x_1, \cdots, hx_i, \cdots, \hat{x}_j, \cdots, x_{n+1}\right] \\
= \quad & \sum_{i=1}^{n+1} (-1)^{i-1} hD(x_i)\,[x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}] \\
& + \sum_{i=1}^{n+1} (-1)^{i-1} x_i D(h)\,[x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}] \\
& + \sum_{j=1}^{n+1} \sum_{i=1, i\neq j}^{n+1} (-1)^{j-1} D(x_j) \left[x_1, \cdots, hx_i, \cdots, \hat{x}_j, \cdots, x_{n+1}\right] \\
\overset{(2.3)}{=} \quad & \sum_{i=1}^{n+1} (-1)^{i-1} hD(x_i)\,[x_1, \cdots, \hat{x}_i, \cdots, x_{n+1}] \\
& + \sum_{j=1}^{n+1} \sum_{i=1, i\neq j}^{n+1} (-1)^{j-1} D(x_j) \left[x_1, \cdots, hx_i, \cdots, \hat{x}_j, \cdots, x_{n+1}\right] \\
\overset{(3.3)}{=} \quad & h\,[x_1, \cdots, x_{n+1}] + \sum_{j=1}^{n+1} \sum_{i=1, i\neq j}^{n+1} (-1)^{j-1} D(x_j) \left[x_1, \cdots, hx_i, \cdots, \hat{x}_j, \cdots, x_{n+1}\right] \\
\overset{(2.2)}{=} \quad & h\,[x_1, \cdots, x_{n+1}] + nh \sum_{j=1}^{n+1} (-1)^{j-1} D(x_j) \left[x_1, \cdots, \hat{x}_j, \cdots, x_{n+1}\right] \\
\overset{(3.3)}{=} \quad & h\,[x_1, \cdots, x_{n+1}] + nh\,[x_1, \cdots, x_{n+1}] \\
= \quad & (n+1)h\,[x_1, \cdots, x_{n+1}]\,.
\end{aligned}
$$

Proof of Eq (2.8). By Eq (3.3), we have

$$\begin{aligned}
& y_1\,[hy_2, x_1, \cdots, x_n] - y_2\,[hy_1, x_1, \cdots, x_n] + \sum_{i=1}^{n} (-1)^{i-1} hx_i\,[y_1, y_2, x_1, \cdots, \hat{x}_i, \cdots, x_n] \\
= \; & y_1 y_2 D(h)\,[x_1, \cdots, x_n] + y_1 hD(y_2)\,[x_1, \cdots, x_n] - y_1 D(x_1)\,[hy_2, x_2, \cdots, x_n] \\
& + y_1 D(x_2)\,[hy_2, x_1, x_3, \cdots, x_n] + \cdots + (-1)^n y_1 D(x_n)\,[hy_2, x_1, \cdots, x_{n-1}] \\
& - y_2 y_1 D(h)\,[x_1, \cdots, x_n] - y_2 hD(y_1)\,[x_1, \cdots, x_n] + y_2 D(x_1)\,[hy_1, x_2, \cdots, x_n] \\
& - y_2 D(x_2)\,[hy_1, x_1, x_3, \cdots, x_n] + \cdots + (-1)^{n-1} y_2 D(x_n)\,[hy_1, x_1, \cdots, x_{n-1}] \\
& + hx_1 D(y_1)\,[y_2, x_2, \cdots, x_n] - hx_1 D(y_2)\,[y_1, x_2, \cdots, x_n] \\
& + hx_1 D(x_2)\,[y_1, y_2, x_3, \cdots, x_n] + \cdots + (-1)^{n+1} hx_1 D(x_n)\,[y_1, y_2, x_2, \cdots, x_{n-1}] \\
& - hx_2 D(y_1)\,[y_2, x_1, x_3, \cdots, x_n] + hx_2 D(y_2)\,[y_1, x_1, x_3, \cdots, x_n] \\
& - hx_2 D(x_1)\,[y_1, y_2, x_3, \cdots, x_n] + \cdots + (-1)^{n+2} hx_2 D(x_n)\,[y_1, y_2, x_1, x_3, \cdots, x_{n-1}] \\
& + \cdots + (-1)^{n-1} hx_n D(y_1)\,[y_2, x_1, \cdots, x_{n-1}] + (-1)^n hx_n D(y_2)\,[y_1, x_1, \cdots, x_{n-1}] \\
& + (-1)^{n+1} hx_n D(x_1)\,[y_1, y_2, x_2, \cdots, x_{n-1}] \\
& + \cdots + (-1)^{2n-1} hx_n D(x_{n-1})\,[y_1, y_2, x_1, \cdots, x_{n-2}] \\
= \; & -y_2 hD(y_1)\,[x_1, \cdots, x_n] + hx_1 D(y_1)\,[y_2, x_2, \cdots, x_n] \\
& + \sum_{i=2}^{n} (-1)^{i-1} hx_i D(y_1)\,[y_2, x_1, \cdots, \hat{x}_i, \cdots, x_n] \\
& + y_1 hD(y_2)\,[x_1, \cdots, x_n] - hx_1 D(y_2)\,[y_1, x_2, \cdots, x_n] \\
& + \sum_{i=2}^{n} (-1)^{i} hx_i D(y_2)\,[y_1, x_1, \cdots, \hat{x}_i, \cdots, x_n] \\
& - y_1 D(x_1)\,[hy_2, x_2, \cdots, x_n] + y_2 D(x_1)\,[hy_1, x_2, \cdots, x_n] \\
& + \sum_{i=2}^{n} (-1)^{i-1} hx_i D(x_1)\,[y_1, y_2, x_2, \cdots, \hat{x}_i, \cdots, x_n] \\
& + y_1 D(x_2)\,[hy_2, x_1, x_3, \cdots, x_n] - y_2 D(x_2)\,[hy_1, x_1, x_3, \cdots, x_n] \\
& + hx_1 D(x_2)\,[y_1, y_2, x_3, \cdots, x_n] + \sum_{i=3}^{n} (-1)^{i} hx_i D(x_2)\,[y_1, y_2, x_1, x_3, \cdots, \hat{x}_i, \cdots, x_n] \\
& \cdots + (-1)^n y_1 D(x_n)\,[hy_2, x_1, \cdots, x_{n-1}] + (-1)^{n-1} y_2 D(x_n)\,[hy_1, x_1, \cdots, x_{n-1}] \\
& + \sum_{j=1}^{n-1} (-1)^{n+j-1} hx_j D(x_n)\left[y_1, y_2, x_1, \cdots, \hat{x}_j, \cdots, x_{n-1}\right] \\
= \; & A_1 + A_2 + \sum_{i=1}^{n} B_i,
\end{aligned}$$

where

$$A_1 \;:=\; -y_2 hD(y_1)\,[x_1, \cdots, x_n] + \sum_{i=1}^{n} (-1)^{i-1} hx_i D(y_1)\,[y_2, x_1, \cdots, \hat{x}_i, \cdots, x_n],$$

$$A_2 \quad := \quad y_1 hD(y_2)\,[x_1, \cdots, x_n] + \sum_{i=1}^{n} (-1)^i hx_i D(y_2)\,[y_1, x_1, \cdots, \hat{x}_i, \cdots, x_n],$$

and, for any $1 \le i \le n$,

$$\begin{aligned} B_i \quad := \quad & (-1)^i y_1 D(x_i)\,[hy_2, x_1, \cdots, \hat{x}_i, \cdots, x_n] + (-1)^{i-1} y_2 D(x_i)\,[hy_1, x_1, \cdots, \hat{x}_i, \cdots, x_n] \\ & + \sum_{j=1}^{i-1} (-1)^{i+j-1} hx_j D(x_i) \left[y_1, y_2, x_1, \cdots, \hat{x}_j, \cdots, \hat{x}_i, \cdots, x_n\right] \\ & + \sum_{j=i+1}^{n} (-1)^{i+j} hx_j D(x_i) \left[y_1, y_2, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n\right]. \end{aligned}$$

By Eq (2.3), we have

$$A_1 \quad = \quad hD(y_1)\left(-y_2\,[x_1, \cdots, x_n] + \sum_{i=1}^{n} (-1)^{i-1} x_i\,[y_2, x_1, \cdots, \hat{x}_i, \cdots, x_n]\right) = 0.$$

Similarly, we have that $A_2 = 0$.

By Eq (2.8), for any $1 \le i \le n$, we have

$$\begin{aligned} B_i \quad = \quad & (-1)^i D(x_i)(y_1\,[hy_2, x_1, \cdots, \hat{x}_i, \cdots, x_n] - y_2\,[hy_1, x_1, \cdots, \hat{x}_i, \cdots, x_n] \\ & + \sum_{j=1}^{i-1} (-1)^{j-1} hx_j \left[y_1, y_2, x_1, \cdots, \hat{x}_j, \cdots, \hat{x}_i, \cdots, x_n\right] \\ & + \sum_{j=i+1}^{n} (-1)^{j} hx_j \left[y_1, y_2, x_1, \cdots, \hat{x}_i, \cdots, \hat{x}_j, \cdots, x_n\right]) \\ = \quad & 0. \end{aligned}$$

Thus, we get

$$y_1\,[hy_2, x_1, \cdots, x_n] - y_2\,[hy_1, x_1, \cdots, x_n] + \sum_{i=1}^{n} (-1)^{i-1} hx_i\,[y_1, y_2, x_1, \cdots, \hat{x}_i, \cdots, x_n] = 0.$$

The proof is completed.

Example 3.1. *The commutative associative algebra $L = k[x_1, x_2, x_3]$, together with the bracket*

$$[x, y] := x \cdot D_1(y) - y \cdot D_1(x), \forall x, y \in L.$$

gives a transposed Poisson algebra $(L, \cdot, [-, -])$, where $D_1 = \partial_{x_1}$ ([2, Proposition 2.2]). Note that the transposed Poisson algebra $(L, \cdot, [-, -])$ is strong according to Remark 2.5. Now, let $D_2 = \partial_{x_2}$; one can check that D_2 is a derivation of $(L, \cdot, [-, -])$. Then, there exists a strong transposed Poisson 3-Lie algebra defined by

$$[x, y, z] := D_2(x)(yD_1(z) - zD_1(y)) + D_2(y)(zD_1(x) - xD_1(z)) + D_2(z)(xD_1(y) - yD_1(x)), \ \forall x, y, z \in L.$$

We note that $[x_1, x_2, x_3] = x_3$*, which is non-zero. The strong condition can be checked as follows: For any* $h, y_1, y_2, z_1, z_2 \in L$*, by a direct calculation, we have*

$$\begin{aligned} y_1[hy_2, z_1, z_2] &= y_1z_1hD_1(z_2)D_2(y_2) - y_1z_2hD_1(z_1)D_2(y_2) \\ &+ y_1y_2z_1D_1(z_2)D_2(h) - y_1y_2z_2D_1(z_1)D_2(h) \\ &- y_1y_2hD_1(z_2)D_2(z_1) + y_1z_2hD_1(y_2)D_2(z_1) + y_1y_2z_2D_1(h)D_2(z_1) \\ &+ y_1y_2hD_1(z_1)D_2(z_2) - y_1z_1hD_1(y_2)D_2(z_2) - y_1y_2z_1D_1(h)D_2(z_2), \end{aligned}$$

$$\begin{aligned} -y_2[hy_1, z_1, z_2] &= -y_2z_1hD_1(z_2)D_2(y_1) + y_2z_2hD_1(z_1)D_2(y_1) \\ &- y_1y_2z_1D_1(z_2)D_2(h) + y_1y_2z_2D_1(z_1)D_2(h) \\ &+ y_1y_2hD_1(z_2)D_2(z_1) - y_2z_2hD_1(y_1)D_2(z_1) - y_1y_2z_2D_1(h)D_2(z_1) \\ &- y_1y_2hD_1(z_1)D_2(z_2) + y_2z_1hD_1(y_1)D_2(z_2) + y_1y_2z_1D_1(h)D_2(z_2), \end{aligned}$$

$$\begin{aligned} hz_1[y_1, y_2, z_2] &= hy_2z_1D_1(z_2)D_2(y_1) - hz_1z_2D_1(y_2)D_2(y_1) - hy_1z_1D_1(z_2)D_2(y_2) \\ &+ hz_1z_2D_1(y_1)D_2(y_2) + hy_1z_1D_1(y_2)D_2(z_2) - hy_2z_1D_1(y_1)D_2(z_2), \end{aligned}$$

$$\begin{aligned} -hz_2[y_1, y_2, z_1] &= -hy_2z_2D_1(z_1)D_2(y_1) + hz_1z_2D_1(y_2)D_2(y_1) + hy_1z_2D_1(z_1)D_2(y_2) \\ &- hz_1z_2D_1(y_1)D_2(y_2) - hy_1z_2D_1(y_2)D_2(z_1) + hy_2z_2D_1(y_1)D_2(z_1). \end{aligned}$$

Thus, we get

$$y_1[hy_2, z_1, z_2] - y_2[hy_1, z_1, z_2] + hz_1[y_1, y_2, z_2] - hz_2[y_1, y_2, z_1] = 0.$$

4. Conclusions

We have studied transposed Poisson n-Lie algebras. We first established an important class of identities for transposed Poisson n-Lie algebras, which were subsequently used throughout the paper. We believe that the identities developed here will be useful in investigations of the structure of transposed Poisson n-Lie algebras in the future. Then, we introduced the notion of a strong transposed Poisson n-Lie algebra and derived an $(n + 1)$-Lie algebra from a strong transposed Poisson n-Lie algebra with a derivation. Finally, we proved the conjecture of Bai et al. [2] for strong transposed Poisson n-Lie algebras.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence tools in the creation of this article.

Acknowledgments

Ming Ding was supported by the Guangdong Basic and Applied Basic Research Foundation (2023A1515011739) and the Basic Research Joint Funding Project of University and Guangzhou City under grant number 202201020103.

Conflict of interest

The authors declare that there is no conflict of interest.

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