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Research article

On a complete parametric Sturm-Liouville problem with sign changing coefficients

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Abstract: In this paper we study a complete second order differential equation of Sturm-Liouville type under Dirichlet boundary condition and where the variable coefficients are allowed to be sign changing. Through critical point theory, we obtain the existence of two nontrivial generalized solutions by requiring a specific growth on the nonlinearity. Moreover, the solutions turn out to be nonnegative and with opposite energy sign.

Keywords: critical point theory; Sturm-Liouville equations; nonlinear differential problems; ordinary differential equations; multiple results
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1. Introduction

Sturm-Liouville differential equations have been widely studied by many authors, because they are useful for describing many types of physical and chemical phenomena. Among the others, we mention the Boyd equation on the eddies in the atmosphere [9], the Laplace tidal wave equation [15] and the equation describing the gas dynamics in a fuel cell [5]. For a detailed overview of Sturm-Liouville differential equations we refer to the book [1].

In literature there are many existence and multiplicity results for Sturm-Liouville problems obtained through different techniques as critical point theory [6, 8, 11–14], fixed point theory [16], upper-lower solutions [17, 18] and so on, see also the references therein. Nevertheless, in the mentioned papers the variable coefficients of the differential equation are required to be positive. Moreover, in order to relate to real life phenomena, it is necessary to consider a more general and realistic assumption and to allow the variable coefficients to change their sign. Also, our approach is based on variational methods. Note that the variational formulation of the considered problem is not natural due to the presence of the term with the first derivative; indeed, these types of problems are often referred to as "non-variational problems" since there is no simple associated minimization problem. However, we consider a specific functional, which is different from the classical energy functional and that can be studied by variational methods.

In this paper we study the following nonlinear parameter-depending problem with a complete Sturm-Liouville second-order differential equation and Dirichlet boundary condition

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda f(x,u) & \text{in }]a,b[, \\ u(a) = u(b) = 0, \end{cases} \tag{D_λ}$$

where λ is a positive parameter, $f \in L^1([a,b]\times\mathbb{R})$ is a function that satisfies the Carathéodory hypotheses and the variable coefficients $\gamma, \delta \in L^\infty([a,b])$ are such that

$$\operatorname*{ess\,inf}_{x\in[a,b]} \delta(x) > -\left(\frac{\pi}{b-a}\right)^2.$$

Problem (D_λ) has already been addressed (see [3,4]). Most notably as an application, in [3] the authors obtain a nontrivial solution by providing information for the proportional voltage regulation of a DC-DC buck converter.

In our paper we establish the existence of two nontrivial and nonnegative solutions with opposite energy sign (see Theorem 3.1) using a two-critical point theorem due to Bonanno-D'Aguì [7], here recalled in Theorem 2.1. The main hypotheses of Theorem 3.1 are two. The first one is an algebraic condition on the nonlinear term (see (h_2)) that guarantees the existence of a first critical point, which actually is a local minimum for the energy functional associated to our problem. The second one is the classical Ambrosetti-Rabinowitz condition, which is needed to ensure the boundedness of the Palais-Smale sequences for the energy functional associated with the problem (D_λ) and so the existence of another critical point, that is a mountain pass point.

The paper is arranged as follows. In Section 2, we formulate the main definitions and tools needed to demonstrate our main results. In particular, we recall the abstract critical point theorem proposed by Bonanno-D'Aguì (Theorem 2.1). Section 3 begins with the introduction of a lemma (Lemma 3.3) that prove the connection between the behavior of the nonlinearity, specifically the Ambrosetti-Rabinowitz condition, and the Palais-Smale condition of the energy functional. Following, we establish our main result, providing an answer to the existence of solutions to problem (D_λ). More precisely, we prove the existence of two nontrivial nonnegative generalized solutions to (D_λ), see Theorem 3.1. By requiring stronger hypotheses on the nonlinearity, Theorem 3.2 ensures that the generalized solutions are positive. Finally, in Section 4 we study specific cases of the problem (D_λ), in particular when f has separable variables. We give corollaries and an example to underline the applicability of our results.

2. Preliminaries

In this section we remind some preliminaries in order to study problem (D_λ) and we also recall the main tool of our investigation (Theorem 2.1). To this end, $L^2([a,b])$ indicates the usual Lebesgue space equipped with the norm $\|u\|_2$ and we denote by $W^{1,2}([a,b])$ and $W_0^{1,2}([a,b])$ the Sobolev spaces endowed with the usual norms

$$\|u\|_{1,2} = \left(\int_a^b |u(x)|^2\,\mathrm{d}x + \int_a^b |u'(x)|^2\,\mathrm{d}x\right)^{\frac{1}{2}},$$

$$\|u\|_{1,2,0} = \|u'\|_2 = \left(\int_a^b |u'(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}},$$

respectively.

Consider the following boundary value problem

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda f(x,u) & \text{in }]a,b[, \\ u(a) = u(b) = 0, \end{cases} \tag{D_λ}$$

where λ is a positive parameter, $\gamma, \delta \in L^\infty([a,b])$ such that

$$\operatorname*{ess\,inf}_{x\in[a,b]} \delta(x) > -\left(\frac{\pi}{b-a}\right)^2, \tag{2.1}$$

and $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, namely

(i) $x \to f(x,t)$ is measurable for all $t \in \mathbb{R}$;

(ii) $t \to f(x,t)$ is continuous for a.a. $x \in [a,b]$;

(iii) for all $s > 0$ the function $\sup_{|t|\leq s} |f(\cdot,t)|$ belongs to $L^1([a,b])$.

Set

$$\Gamma(x) = \int_a^x \gamma(\xi) \, \mathrm{d}\xi \quad \forall\, x \in [a,b],$$

and equip the space $X = W_0^{1,2}([a,b])$ with the following norm

$$\|u\|_X = \left(\int_a^b e^{-\Gamma(x)} |u'(x)|^2 \, \mathrm{d}x + \int_a^b e^{-\Gamma(x)} \delta(x) |u(x)|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}.$$

This norm is equivalent to the norm $\|u\|_{1,2,0} = \|u'\|_2$ on X. In particular, thanks to assumption (2.1), in [3, Proposition 2.2] the authors prove that there exist two constants m and M, with $M \geq m > 0$, such that

$$m\|u'\|_2 \leq \|u\|_X \leq M\|u'\|_2\,, \tag{2.2}$$

for all $u \in X$, where

$$\begin{aligned} m &= \begin{cases} \left(\min\limits_{x\in[a,b]} e^{-\Gamma(x)} \right)^{\frac{1}{2}} & \text{if } \operatorname*{ess\,inf}\limits_{x\in[a,b]} \delta(x) \geq 0, \\ \left[\min\limits_{x\in[a,b]} e^{-\Gamma(x)} \left(1 + \operatorname*{ess\,inf}\limits_{x\in[a,b]} \delta(x) \left(\frac{b-a}{\pi}\right)^2 \right) \right]^{\frac{1}{2}} & \text{if } \operatorname*{ess\,inf}\limits_{x\in[a,b]} \delta(x) < 0, \end{cases} \\ M &= \begin{cases} \left[\max\limits_{x\in[a,b]} e^{-\Gamma(x)} \left(1 + \operatorname*{ess\,sup}\limits_{x\in[a,b]} \delta(x) \left(\frac{b-a}{\pi}\right)^2 \right) \right]^{\frac{1}{2}} & \text{if } \operatorname*{ess\,sup}\limits_{x\in[a,b]} \delta(x) \geq 0, \\ \left(\max\limits_{x\in[a,b]} e^{-\Gamma(x)} \right)^{\frac{1}{2}} & \text{if } \operatorname*{ess\,sup}\limits_{x\in[a,b]} \delta(x) < 0. \end{cases} \end{aligned} \tag{2.3}$$

Moreover, the following inequality holds

$$\max_{x\in[a,b]} |u(x)| \leq \frac{(b-a)^{\frac{1}{2}}}{2m}\|u\|_X \qquad \forall\, u \in X, \tag{2.4}$$

where m is given in (2.3), see [3, Remark 2.3] .
Now, we present the functionals useful to study problem (D_λ). To this aim, we set

$$F(x,t) = \int_0^t f(x,\xi)\,\mathrm{d}\xi \quad \forall\, x \in [a,b], \forall\, t \in \mathbb{R},$$

and we define $\Phi, \Psi : X \to \mathbb{R}$ as follows

$$\begin{aligned} \Phi(u) &= \frac{1}{2}\|u\|_X^2, \\ \Psi(u) &= \int_a^b e^{-\Gamma(x)} F(x,u(x))\,\mathrm{d}x, \end{aligned} \tag{2.5}$$

for all $u \in X$. Standard computations show that Φ and Ψ are C^1-functionals and one has

$$\begin{aligned} \Phi'(u)(v) &= \int_a^b e^{-\Gamma(x)} u'(x)v'(x)\,\mathrm{d}x + \int_a^b e^{-\Gamma(x)}\delta(x)u(x)v(x)\,\mathrm{d}x, \\ \Psi'(u)(v) &= \int_a^b e^{-\Gamma(x)} f(x,u(x))v(x)\,\mathrm{d}x, \end{aligned}$$

for all $u, v \in X$. Furthermore, for every $\lambda > 0$ we consider the so-called energy functional $I_\lambda : X \to \mathbb{R}$ associated to problem (D_λ), namely

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u) \qquad \forall\, u \in X. \tag{2.6}$$

Our purpose is to study the existence of generalized solutions for problem (D_λ), which are functions $u : [a,b] \to \mathbb{R}$ such that

- $u \in C^1([a,b])$;
- $u' \in AC([a,b])$, that is the set of the absolute functions;
- $u(a) = u(b) = 0$;
- $-u''(x) + \gamma(x)u'(x) + \delta(x)u(x) = \lambda f(x,u(x))$ for a.a. $x \in]a,b[$.

In particular, in [3, Proposition 2.3] the authors prove that u is a generalized solution of (D_λ) if and only if it is a critical point of the energy functional I_λ, i.e.

$$\int_a^b e^{-\Gamma(x)} u'(x)v'(x)\,\mathrm{d}x + \int_a^b e^{-\Gamma(x)}\delta(x)u(x)v(x)\,\mathrm{d}x - \lambda \int_a^b e^{-\Gamma(x)} f(x,u(x))v(x)\,\mathrm{d}x = 0,$$

for all $v \in X$.

Finally we recall the definition of (PS)-condition and the main tool of our investigation, which is a two critical points theorem due to Bonanno-D'Aguì, see [7, Theorem 2.1].

Definition 2.1. Let $(X, \|\cdot\|)$ be a Banach space, X^* its dual and $L : X \to \mathbb{R}$ a Gâteaux differentiable functional. We say that L satisfies the Palais-Smale condition (in short, (PS)-condition), if any sequence $\{u_n\} \subseteq X$ such that

(P1) $L(u_n)$ is bounded,

(P2) $\lim_{n\to\infty} \|L(u_n)\|_{X^*} = 0$,

has a convergent subsequence in X.

Theorem 2.1. *Let X be a real Banach space and let $\Phi, \Psi : X \to \mathbb{R}$ be two continuously Gâteaux differentiable functionals such that*

$$inf_X\Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there are $r \in \mathbb{R}$ and $\tilde{u} \in X$, with $0 < \Phi(\tilde{u}) < r$, such that

$$\frac{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}{r} < \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \tag{2.7}$$

and for all $\lambda \in \Lambda_r = \left]\frac{\Phi(\tilde{u})}{\Psi(\tilde{u})}, \frac{r}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r])} \Psi(u)}\right[$, the functional $I_\lambda = \Phi - \lambda\Psi$ satisfies (PS)*-condition and it is unbounded from below.*
Then, for each $\lambda \in \Lambda_r$, the functional $\Phi - \lambda\Psi$ admits at least two non-zero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

3. Main result

In this section we establish the existence of at least two nonnegative generalized solutions of problem (D_λ) with opposite energy sign. Since we are interested in nonnegative solutions, we truncate the nonlinear term in zero and without any loss of generality, we may assume

$$f(x,t) = f(x,0) \qquad \forall\, t \le 0,\ \forall\, x \in [a,b].$$

Indeed, we mention the following results which can be found in [3].

Lemma 3.1. (see [3, Lemma 2.1]) *Assume that $f(x,0) \ge 0$ for a.a. $x \in [a,b]$. Then, any generalized solution of problem (D_λ) is nonnegative.*

Lemma 3.2. (see [3, Lemma 2.2]) *Assume that $f(x,t) \ge 0$ for a.a. $x \in [a,b]$ and for all $t \ge 0$. Then, any non-zero generalized solution of problem (D_λ) is positive.*

Our aim is to apply Theorem 2.1 to the functionals Φ and Ψ defined in (2.5). Therefore, we first prove the following result on the properties of the energy functional related to problem (D_λ).

Lemma 3.3. *Assume that f fulfils the Ambrosetti–Rabinowitz condition, that is*

$$\textit{there exist } s \in \mathbb{R}^+, \eta > 2 :\ 0 < \eta F(x,t) \le t f(x,t) \qquad \forall\, x \in [a,b], \forall\, t \ge s. \tag{AR}$$

Then, for every $\lambda > 0$ the energy functional I_λ given in (2.6) *satisfies the (PS)-condition and is unbounded from below.*

Proof. First, we prove that I_λ fulfils the (PS)-condition for each $\lambda > 0$.
Fix $\lambda > 0$ and let $\{u_n\} \subseteq X$ be such that (P_1) and (P_2) hold. We observe that

$$\begin{aligned}\eta I_\lambda(u_n) - I'_\lambda(u_n)(u_n) &= \left(\frac{1}{2}\eta - 1\right)\|u_n\|_X^2 \\ &\quad - \lambda\left[\int_a^b e^{-\Gamma(x)}\left(\eta F(x,u_n(x)) - f(x,u_n(x))u_n(x)\right)\,\mathrm{d}x\right],\end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, by (AR)-condition and L^1-Carathéodory assumption, we obtain that

$$\int_a^b \eta F(x,t) - t f(x,t)\,\mathrm{d}x \leq s(\eta+1)\int_a^b \sup_{|t|\leq s}|f(x,t)|\,\mathrm{d}x = A \qquad \forall\, t \geq 0,$$

with $A \geq 0$. Then, it follows that

$$\eta I_\lambda(u_n) - I'_\lambda(u_n)(u_n) \geq \left(\frac{1}{2}\eta - 1\right)\|u_n\|_X^2 - \lambda A \qquad \forall\, n \in \mathbb{N}. \tag{3.1}$$

From (P1) and (P2), we have that there exist $M > 0$ and $\{\varepsilon_n\} \subseteq \mathbb{R}^+$, with $\varepsilon_n \to 0^+$, such that

$$I_\lambda(u_n) \geq M \quad \text{and} \quad |I'_\lambda(u_n)(v)| \leq \varepsilon_n \qquad \forall\, v \in X,\ \|v\| \leq 1, \tag{3.2}$$

for all $n \in \mathbb{N}$. Choosing $v = \dfrac{u_n}{\|u_n\|_X}$ and combining (3.1) with (3.2), we derive that

$$\left(\frac{1}{2}\eta - 1\right)\|u_n\|_X^2 \leq \eta M + \varepsilon_n\|u_n\|_X + \lambda A \qquad \forall\, n \in \mathbb{N}.$$

Thus, $\{u_n\}$ is bounded in X. Therefore, since X is reflexive, there exists a subsequence which is weakly convergent in X. Moreover, since the embedding of X into $C^0([a,b])$ is compact, it strongly converges in $C^0([a,b])$. Summing up and renaming the subsequence again with $\{u_n\}$, we have

$$u_n \rightharpoonup u \quad \text{in } X \qquad \text{and} \qquad u_n \to u \quad \text{in } C^0([a,b]).$$

Clearly, since Φ' is a linear operator and $u_n \rightharpoonup u$ in X, one has

$$\langle \Phi'(u), u_n - u\rangle \to 0,$$

and standard computations show that

$$\langle \Phi'(u_n), u_n - u\rangle \to 0.$$

Hence, it follows that

$$\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle \to 0. \tag{3.3}$$

Furthermore, we have

$$\begin{aligned}\langle \Phi'(u_n) - \Phi'(u), u_n - u\rangle &= \langle \Phi'(u_n), u_n\rangle + \langle \Phi'(u), u\rangle - \langle \Phi'(u_n), u\rangle - \langle \Phi'(u), u_n\rangle \\ &\leq \|u_n\|_X^2 + \|u\|_X^2 - \|u_n\|_X\|u\|_X - \|u\|_X\|u_n\|_X\end{aligned}$$

$$= (\|u_n\|_X - \|u\|_X)^2 .$$

Therefore, from (3.3), we obtain

$$\lim_{n\to+\infty} \|u_n\|_X = \|u\|_X.$$

Since X is uniformly convex, Proposition III.30 in [10] ensures that

$$\lim_{n\to+\infty} \|u_n - u\|_X = 0,$$

thus our claim is true.

Now, we prove that the energy functional is unbounded from below. To this aim, classical computations show that if f satisfies (AR)-condition, then there exist $B, C \in L^1([a,b])$, with $C(x) > 0$ and $B(x) \geq 0$ for every $x \in [a,b]$, such that

$$F(x,t) \geq C(x)t^\eta - B(x) \qquad \forall\, x \in [a,b], \forall\, t \geq 0.$$

Fixing $u \in X$, with $u \geq 0, \|u\|_X \neq 0$, and $h \in \mathbb{R}^+$, we have

$$\begin{aligned} I_\lambda(hu) &\leq \frac{1}{2}\|hu\|_X^2 - \lambda \left(\max_{x\in[a,b]} e^{-\Gamma(x)}\right)\left(h^\eta \int_a^b C(x)u^\eta \,\mathrm{d}x - \int_a^b B(x)\,\mathrm{d}x\right) \\ &\leq C_1h^2 - \lambda C_2 h^\eta + \lambda C_3, \end{aligned}$$

for any $\lambda > 0$ and for some $C_i \geq 0, i = 1,2,3$. Passing to the limit for $h \to +\infty$, we achieve our aim and the proof is complete. □

Now, we present our main result. Put

$$K = \frac{1}{2}\frac{m^2}{M^2}\frac{\min\limits_{x\in[a,b]} e^{-\Gamma(x)}}{\max\limits_{x\in[a,b]} e^{-\Gamma(x)}}, \tag{3.4}$$

$$\tilde{C} = \frac{2m^2}{(b-a)\max\limits_{x\in[a,b]} e^{-\Gamma(x)}}. \tag{3.5}$$

Theorem 3.1. *Suppose $f(x,0) \geq 0$ for a.a. $x \in [a,b]$. Assume that* (AR)*-condition holds and there exist two positive constants c, d, with $d < c$, such that*

(h$_1$) $F(x,t) \geq 0$ *for a.a.* $x \in \left[a, a + \frac{b-a}{4}\right] \cup \left[b - \frac{b-a}{4}, b\right]$ *and for all* $t \in [0,d]$,

(h$_2$) $$\frac{\int_a^b \max\limits_{|t|<c} F(x,t)\,\mathrm{d}x}{c^2} < K\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x}{d^2}.$$

Then, for each $\lambda \in \Lambda_{c,d}$, where

$$\Lambda_{c,d} := \left]\frac{\tilde{C}}{K}\frac{d^2}{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x}, \tilde{C}\frac{c^2}{\int_a^b \max\limits_{|t|\leq c} F(x,t)\,\mathrm{d}x}\right[, \tag{3.6}$$

problem (D_λ) admits at least two nontrivial and nonnegative generalized solutions $u_1, u_2 \in W_0^{1,2}([a,b])$ with opposite energy sign.

Proof. Take $(X, \|\cdot\|_X)$, Φ and Ψ as in Section 2. Our goal is to apply Theorem 2.1 to the functional $I_\lambda = \Phi - \lambda\Psi$. Clearly, Φ and Ψ satisfy the required regularity assumptions and from Lemma 3.3 we know that I_λ satisfies the (PS)-condition and is unbounded from below for any $\lambda > 0$. So, it remains to verify condition (2.7). To this aim, set

$$r = \frac{2m^2}{(b-a)}c^2.$$

For each $u \in X$ such that $\Phi(u) = \frac{1}{2}\|u\|_X^2 < r$, taking (2.4) into account, one has

$$|u(x)| \leq \frac{(b-a)^{\frac{1}{2}}}{2m})\|u\|_X \leq \frac{(b-a)^{\frac{1}{2}}}{2m}(2r)^{\frac{1}{2}} = \left(\frac{b-a}{2m^2}r\right)^{\frac{1}{2}} = c,$$

for all $x \in [a, b]$. Hence

$$\sup_{\Phi(u)<r} \Psi(u) \leq \max_{x\in[a,b]} e^{-\Gamma(x)} \int_a^b \max_{|t|\leq c} F(x,t)\,\mathrm{d}x.$$

Thus, it follows that

$$\frac{\sup\limits_{\Phi(u)<r} \Psi(u)}{r} \leq \frac{b-a}{2m^2} \max_{x\in[a,b]} e^{-\Gamma(x)} \frac{\int_a^b \max\limits_{|t|\leq c} F(x,t)\,\mathrm{d}x}{c^2}. \tag{3.7}$$

Now, we define

$$\tilde{u}(x) = \begin{cases} \frac{4d}{b-a}(x-a) & \text{if } x \in \left[a, a+\frac{b-a}{4}\right[, \\ d & \text{if } x \in \left[a+\frac{b-a}{4}, b-\frac{b-a}{4}\right], \\ \frac{4d}{b-a}(b-x) & \text{if } x \in \left]b-\frac{b-a}{4}, b\right]. \end{cases}$$

Clearly, $\tilde{u} \in X$ and $\|\tilde{u}\|_{1,2,0}^2 = \|\tilde{u}'\|_2^2 = \dfrac{8d^2}{b-a}$. Then, by (2.2) we get

$$\Phi(\tilde{u}) = \frac{1}{2}\|\tilde{u}\|_X^2 \leq \frac{1}{2}M^2\|\tilde{u}'\|_2^2 = \frac{4M^2d^2}{b-a}.$$

Moreover, taking (h_1) into account, one has

$$\Psi(\tilde{u}) = \int_a^b e^{-\Gamma(x)} F(x, \tilde{u}(x))\,\mathrm{d}x \geq \min_{x\in[a,b]} e^{-\Gamma(x)} \int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x.$$

Hence, we obtain

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{b-a}{4M^2} \min_{x\in[a,b]} e^{-\Gamma(x)} \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x}{d^2},$$

that is

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{b-a}{2m^2} \max_{x\in[a,b]} e^{-\Gamma(x)} K \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x}{d^2}. \tag{3.8}$$

Now, we verify that $\tilde{u} \in \Phi^{-1}(]0, r[)$. From ($h_2$) and $0 < d < c$, we obtain $\dfrac{\sqrt{2}M}{m}d < c$. Indeed, arguing by contradiction, if we assume that $c \leq \dfrac{\sqrt{2}Md}{m}$, then

$$\frac{\int_a^b \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}{c^2} \geq \frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x}{c^2} \geq \frac{1}{2}\frac{m^2}{M^2}\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x}{d^2}$$
$$\geq \frac{1}{2}\frac{m^2}{M^2}\frac{\min_{x\in[a,b]} e^{-\Gamma(x)}}{\max_{x\in[a,b]} e^{-\Gamma(x)}}\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x}{d^2},$$

which contradicts hypothesis (h_2). Hence, we have

$$0 < \Phi(\tilde{u}) \leq \frac{4M^2d^2}{b-a} < \frac{2m^2c^2}{b-a} = r.$$

Consequently, combining (3.7), (3.8) and (h_2), we get

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r])}\Psi(u)}{r} \leq \frac{1}{\tilde{C}}\frac{\int_a^b \max_{|t|\leq c} F(x,t)\,\mathrm{d}x}{c^2} < \frac{K}{\tilde{C}}\frac{\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} F(x,d)\,\mathrm{d}x}{d^2} \leq \frac{\Psi(\tilde{u})}{\Phi(\tilde{u})},$$

and assumption (2.7) of Theorem 2.1 is satisfied. Clearly, $\Lambda_{c,d}$ is non-empty because of (h_2) and the previous inequality implies that

$$\Lambda_{c,d} \subseteq \Lambda_r.$$

Therefore, Theorem 2.1 ensures that for each $\lambda \in \Lambda_{c,d}$ the functional I_λ admits at least two non-zero critical point $u_1, u_2 \in X$, which are nontrivial generalized solutions of problem (D_λ), such that $I_\lambda(u_1) < 0 < I_\lambda(u_2)$. Finally, from Lemma 3.1 it follows that $u_1 \geq 0$ and $u_2 \geq 0$. □

The following result deals with positive solutions and it is obtained by combining Theorem 3.1 with Lemma 3.2.

Theorem 3.2. *Suppose $f(x,t) \geq 0$ for a.a. $x \in [a,b]$ and for all $t \geq 0$. Assume that* (AR)*-condition holds and there exist two positive constants c, d, with $d < c$, such that* (h_1) *and* (h_2) *hold. Then, for each $\lambda \in \Lambda_{c,d}$, defined by* (3.6)*, problem* (D_λ) *admits at least two positive generalized solutions $u_1, u_2 \in W_0^{1,2}([a,b])$ with opposite energy sign.*

Remark 3.1. We underline that if the nonlinear term and the weight functions are continuous, i.e. $f \in C([a,b]\times\mathbb{R})$ and $\gamma, \delta \in C([a,b])$, then any generalized solution u is a classical solution, that is

- $u \in C^2([a,b])$;
- $u(a) = u(b) = 0$;
- $-u''(x) + \gamma(x)u'(x) + \delta(x)u(x) = \lambda f(x, u(x))$ for all $x \in]a,b[$.

4. Some consequences

In this section we point out some results for nonlinearities with separable variables, i.e. of type $f(x,t) = \alpha(x)g(t)$, and we also provide an example.

Let $\alpha : [a,b] \to \mathbb{R}$ be a function such that $\alpha \in L^1([a,b]), \alpha \not\equiv 0$ and $\alpha(x) \geq 0$ for a.a. $x \in [a,b]$ and let $g : \mathbb{R} \to \mathbb{R}$ be a nonnegative continuous function. Consider the following problem

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda\alpha(x)g(u) & \text{in }]a,b[, \\ u(a) = u(b) = 0, \end{cases} \tag{$\overline{D_\lambda}$}$$

where $\lambda > 0$ and $\gamma, \delta \in L^\infty([a,b])$ such that (2.1) holds, and put

$$\overline{K} = \frac{\displaystyle\int_{a+\frac{b-a}{4}}^{b-\frac{b-a}{4}} \alpha(x)\,\mathrm{d}x}{\|\alpha\|_1} K,$$

where K is given in (3.4). Also, we observe that in this case the Ambrosetti-Rabinowitz condition becomes the following

$$\text{there exist } s \in \mathbb{R}^+, \eta > 2 : \; 0 < \eta G(t) \leq t g(t) \qquad \forall t \geq s, \tag{$\overline{AR}$}$$

where $G(t) = \displaystyle\int_0^t g(\xi)\,\mathrm{d}\xi$ for all $t \in \mathbb{R}$. Taking Theorem 3.1 into account, we have the following result.

Theorem 4.1. *Assume that the ($\overline{AR}$)-condition holds and there exist two positive constants c, d with $d < c$, such that*

$$\frac{G(c)}{c^2} < \overline{K}\,\frac{G(d)}{d^2}. \tag{h$_2'$}$$

Then, for each $\lambda \in \overline{\Lambda}_{c,d}$, where

$$\overline{\Lambda}_{c,d} = \left]\frac{\tilde{C}}{\overline{K}\|\alpha\|_1}\frac{d^2}{G(d)}, \frac{\tilde{C}}{\|\alpha\|_1}\frac{c^2}{G(c)}\right[,$$

problem ($\overline{D_\lambda}$) admits at least two positive generalized solutions with opposite energy sign.

Proof. Our aim is to apply Theorem 3.2. Clearly, we need to verify only (h$_2$), since the other assumptions are satisfied. Hence, arguing as in the proof of Theorem 3.1 and taking into account that g is nonnegative, we derive

$$\frac{\sup_{\Phi(u)<r} \Psi(u)}{r} \leq \frac{\|\alpha\|_1}{\tilde{C}}\frac{G(c)}{c^2}, \tag{4.1}$$

where $\tilde{C}$ is defined in (3.5). Also, we get

$$\frac{\Psi(\tilde{u})}{\Phi(\tilde{u})} \geq \frac{\overline{K}\|\alpha\|_1}{\tilde{C}}\frac{G(d)}{d^2}. \tag{4.2}$$

Then, combining (h$_2'$), (4.1) and (4.2), it follows that (h$_2$) is verified and the proof is complete. □

A special case occurs for nonlinearities with sublinear behavior near zero.

Corollary 4.1. *Assume that*

$$\lim_{t\to 0^+} \frac{g(t)}{t} = +\infty, \tag{h_2''}$$

and $(\overline{AR})$*-condition holds. Then, for each* $\lambda \in]0, \lambda^*[$*, where*

$$\lambda^* = \frac{\tilde{C}}{\|\alpha\|_1} \sup_{c>0} \frac{c^2}{G(c)},$$

problem $(\overline{D_\lambda})$ *admits at least two positive generalized solutions with opposite energy sign.*

Proof. Fix $\lambda < \lambda^*$. So, there is $c > 0$ such that $\lambda < \frac{\tilde{C}}{\|\alpha\|_1} \frac{c^2}{G(c)}$. From (h_2''), one has

$$\lim_{s\to 0^+} \frac{\tilde{C}}{\overline{K}\|\alpha\|_1} \frac{s^2}{G(s)} = 0 < \lambda,$$

thus we can find $d < c$ such that

$$\frac{\tilde{C}}{\overline{K}\|\alpha\|_1} \frac{d^2}{G(d)} < \lambda.$$

Hence, the conclusion follows from Theorem 4.1. □

Remark 4.1. We note that in the particular case $\alpha \equiv 1$, problem $(\overline{D_\lambda})$ coincides with problem $(AD)_\lambda$ in [4], where existence of infinitely many solutions and three solutions are obtained. Hence, Theorem 4.1 and Corollary 4.1 are additional results on the existence of two positive generalized solutions for problem $(AD)_\lambda$. Moreover, we underline that the class of nonlinearities for which the problem admits two positive solutions is different because in our case we require an algebraic condition in an interval $[d, c]$ and the $(\overline{AR})$-condition at infinity. In [4] the authors obtain three classical solutions under different assumptions, namely the primitive of the nonlinearity has a more than quadratic growth in an interval $[c, d]$ and a behavior less than quadratic at infinity.

Next, we provide an example. In particular, we consider a problem with combined effects of concave and convex nonlinearities and we show that it admits two positive classical solutions with opposite energy sign.

Example 4.1. Consider the following problem

$$\begin{cases} -u'' + xu' - \log(x+2)u = \lambda(x^2+1)(\sqrt{|u|} + u^2) & \text{in }]0,1[, \\ u(0) = u(1) = 0. \end{cases} \tag{A}$$

Clearly, the functions

$$\gamma(x) = x, \quad \delta(x) = -\log(x+2), \quad \alpha(x) = x^2+1, \quad g(t) = \sqrt{|t|} + t^2,$$

are continuous and satisfy all the required assumptions for all $x \in [0,1]$, $t \in \mathbb{R}$ and

$$\operatorname*{ess\,inf}_{x\in[a,b]} \delta(x) = -\log 3 > -\pi^2.$$

The nonlinear term g is such that (h_2') holds, indeed

$$\lim_{t\to 0^+} \frac{g(t)}{t} = \lim_{t\to 0^+} \frac{\sqrt{t}+t^2}{t} = +\infty,$$

and fulfils ($\overline{AR}$)-condition. Furthermore, by standard computations we have

$$m = \frac{\sqrt{1-\frac{\log 3}{\pi^2}}}{e^{\frac{1}{4}}}, \quad M = 1, \quad \overline{K} = \frac{61(1-\frac{\log 3}{\pi^2})}{256\, e} \quad \text{and} \quad \lambda^* = \frac{3}{2\sqrt{e}}\left(1-\frac{\log 3}{\pi^2}\right).$$

Then, Corollary 4.1 and Remark 3.1 ensure that for any $\lambda \in]0, \lambda^*[$ problem (A) admits at least two positive classical solutions with opposite energy sign.

Remark 4.2. It is worth emphasizing that Corollary 4.1 is useful to solve general problems with concave-convex nonlinearities, first studied by Ambrosetti-Rabinowitz-Cerami in [2], as

$$g(t) = \begin{cases} t^q + t^r & \text{if } t \geq 0, \\ 0 & \text{if } t < 0, \end{cases}$$

with $0 < q < 1 < r < \infty$.

Indeed, it can be easily seen that assumption (h_2'') is satisfied and ($\overline{AR}$) holds for $\eta \in]2, r+1[$, since

$$\lim_{t\to+\infty} \frac{tg(t)}{\eta G(t)} = \frac{r+1}{\eta}.$$

Then, from Corollary 4.1 we have that for any $\lambda \in]0, \lambda^*[$ the following problem

$$\begin{cases} -u'' + \gamma(x)u' + \delta(x)u = \lambda\alpha(x)\,(u^q + u^r) & \text{in }]a, b[, \\ u(a) = u(b) = 0 \end{cases}$$

admits at least two positive generalized solutions with opposite energy sign. In addition, if $\gamma, \delta \in C([a, b])$ then the solutions are classical.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. W. O. Amrein, A. M. Hinz, D. B. Pearson, *Sturm-Liouville Theory: Past and Present*, Birkhäuser Verlag Basel/Switzerland, 2005. https://doi.org/10.1007/3-7643-7359-8
2. A. Ambrosetti, H. Brezis, G. Cerami, Combined effects of concave and convex non linearities in some elliptic problems, *J. Funct. Anal.*, **122** (1994), 519–543. https://doi.org/10.1006/jfan.1994.1078
3. E. Amoroso, G. Bonanno, G. D'Aguì, S. De Caro, S. Foti, D. O'Regan, et al., Second order differential equations for the power converters dynamical performance analysis, *Math. Methods Appl. Sci.*, **45** (2022), 5573–5591. https://doi.org/10.1002/mma.8127
4. E. Amoroso, G. Bonanno, G. D'Aguí, S. Foti, Multiple solutions for nonlinear Sturm-Liouville differential equations with possibly negative variable coefficients, *Nonlinear Anal. Real World Appl.*, **69** (2023), 103730. https://doi.org/10.1016/j.nonrwa.2022.103730
5. P. B. Bailey, J. Billingham, R. J. Cooper, W. N. Everitt, A. C. King, Q. Kong, et al., On some eigenvalue problems in fuel-cell dynamics, *Proc. R. Soc. Lond. (A)*, **459** (2003), 241–261. https://doi.org/10.1098/rspa.2002.1058
6. G. Bonanno, G. D'Aguì, A Neumann boundary value problem for the Sturm-Liouville equation, *Appl. Math. Comput.*, **208** (2009), 318–327. https://doi.org/10.1016/j.amc.2008.12.029
7. G. Bonanno, G. D'Aguì, Two non-zero solutions for elliptic Dirichlet problems, *Z. Anal. Anwend.*, **35** (2016), 449–464. https://doi.org/10.4171/zaa/1573
8. G. Bonanno, S. Heidarkhani, D. O'Regan, Nontrivial solutions for Sturm-Liouville systems via a local minimum theorem for functionals, *Bull. Aust. Math. Soc.*, **89** (2014), 8–18. https://doi.org/10.1017/S000497271300035X
9. J. P. Boyd, Sturm-Liouville eigenproblems with an interior pole, *J. Math. Phys.*, **22** (1981), 1575–1590. https://doi.org/10.1063/1.525100
10. H. Brézis, *Functional analysis, Sobolev spaces and partial differential equations*, Springer, New York, 2011. https://doi.org/10.1007/978-0-387-70914-7
11. J. R. Graef, S. Heidarkhani, L. Kong, Infinitely many solutions for systems of Sturm-Liouville boundary value problems, *Results Math.*, **66** (2014), 327–341. https://doi.org/10.1007/s00025-014-0379-1
12. J. R. Graef, S. Heidarkhani, L. Kong, Nontrivial solutions for systems of Sturm-Liouville boundary value problems, *Differ. Equ. Appl.*, **6** (2014), 255–265. https://doi.org/10.7153/dea-06-12
13. J. R. Graef, S. Heidarkhani, L. Kong, Multiple solutions for systems of Sturm- Liouville boundary value problems, *Mediterr. J. Math.*, **13** (2016), 1625–1640. https://doi.org/10.1007/s00009-015-0595-2

14. S. Heidarkhani, On a class of systems of n Neumann two-point boundary value Sturm-Liouville type equations, *Bull. Iran. Math. Soc.*, **39** (2013), 821–840.
15. M. S. Homer, Boundary value problems for the Laplace tidal wave equation, *Proc. R. Soc. Lond. (A)*, **428** (1990), 157–180. https://doi.org/10.1098/rspa.1990.0029
16. Z. Li, XB. Shu, T. Miao, The existence of solutions for Sturm–Liouville differential equation with random impulses and boundary value problems, *Bound. Value Probl.*, **2021** (2021), Article number 97. https://doi.org/10.1186/s13661-021-01574-x
17. J. Sun, H. Chen, Variational method to the impulsive equation with Neumann boundary conditions, *Bound. Value Probl.*, **2009** (2019), Article number 316812. https://doi.org/10.1155/2009/316812
18. Y. Tian, W. Ge, Multiple solutions of impulsive Sturm–Liouville boundary value problem via lower and upper solutions and variational methods, *J. Math. Anal. Appl.*, **387** (2012), 475–489. https://doi.org/10.1016/j.jmaa.2011.08.042