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Research article

A novel iterative scheme for solving delay differential equations and third order boundary value problems via Green's functions

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Abstract: In this paper, we constructed a novel fixed point iterative scheme called the Modified-JK iterative scheme. This iteration process is a modification of the JK iterative scheme. Our scheme converged weakly to the fixed point of a nonexpansive mapping and strongly to the fixed point of a mapping satisfying condition (E). We provided some examples to show that the new scheme converges faster than some existing iterations. Stability and data dependence results were proved for this iteration process. To substantiate our results, we applied our results to solving delay differential equations. Furthermore, the newly introduced scheme was applied in approximating the solution of a class of third order boundary value problems (BVPs) by embedding Green's functions. Moreover, some numerical examples were presented to support the application of our results to BVPs via Green's function. Our results extended and generalized other existing results in literature.

Keywords: fixed point; rate of convergence; Garcia-Falset mapping; condition (E); boundary value problems; $\mathcal{J}$-stability; Modified-JK iterative scheme; delay differential equations
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1. Introduction

Let $\mathfrak{D}$ be a nonempty, closed, and convex subset of a Banach space $(\mathfrak{X}, \|\cdot\|)$. A self mapping $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$ is said to be nonexpansive if

$$\|\mathfrak{I}x - \mathfrak{I}y\| \leq \|x - y\|$$

for all $x, y \in \mathfrak{D}$. As a generalization to nonexpansive mapping, in 2008, Suzuki [1] introduced the generalized nonexpansive mapping. However, a more general class of mapping to the Suzuki's mapping was later introduced by García-Falset et al. [2] in 2011, and it is said to be a mapping satisfying condition (E).

A fixed point problem is a problem of finding a point x in the domain $\mathfrak{D}$ (i.e., $x \in \mathfrak{D}$) of an appropriate mapping $\mathfrak{I} : \mathfrak{D} \to \mathfrak{D}$ such that

$$\mathfrak{I}x = x \tag{1.1}$$

holds.

In this paper, $\mathscr{F}(\mathfrak{I}) \neq \emptyset$ represents the fixed point set that contains all fixed points of the mapping $\mathfrak{I}$. Fixed point theory has become quite useful in solving physical problems. The idea of fixed point theory usually involves the transformation of any problem to a fixed point equation of the form (1.1), where it is then approximated using a suitable fixed point iterative scheme in the framework of a suitable mapping. Moreover, fixed point theory has overtime penetrated a diverse area of application in applied sciences, engineering, and mathematics.

It is obvious that many physical problems are easily represented as differential equations and difference equations, including initial value problems and boundary value problems. Noticeably, some of these problems are difficult to solve by analytical method, such that transforming into a fixed point problem becomes a redeeming way out in an attempt to solve the problem. However, solving a fixed point equation analytically may pose some challenges of not admitting solution when the mapping fails to be a self map. Thus, using the suitable approximation method becomes appropriate and the use of fixed point iterative schemes turn out to be useful.

The use of an iterative scheme in solving fixed point problems has received remarkable attention with tremendous contributions from researchers and, therefore, has recorded many developments in literature; to mention but a few: Picard [3], Mann [4], AG [5], Picard-S [6], Agarwal et al. [7], and so on. Readers interested in this direction of research should consult [5, 8–11] and the references therein.

Our aim in this paper is to develop a novel fixed point iterative scheme called the Modified-JK iterative scheme and use it to approximate the solution of delay differential equations and a class of third order boundary value problems. Our scheme converges faster than some existing schemes, and our results extend and generalize many results in literature. We also provide some numerical examples to validate our results.

This paper is arranged as follows: Section 2 contains preliminaries, definitions, and lemmas. Section 3 is dedicated to main results, which comprises of weak and strong convergence results and stability and data dependence results. In Section 4, the main result is applied to the approximation of delay differential equations. Application of our main result to solving third order boundary value problems via Green's functions is contained in Section 5. The conclusion of this paper is presented in Section 6.

2. Preliminaries

The use of fixed point iterative scheme in solving some physical problems in mathematical sciences has become a very remarkable tool for the approximation of the solutions of several problems, including the ones that have appeared unsolvable by the use of available analytical methods. The following iterative scheme was introduced by Ahmad, et al. [12] in 2021, and they called it the JK iterative scheme:

$$\begin{cases} w_n = (1-\xi_n)u_n + \xi_n \Im u_n, \\ v_n = \Im w_n, \\ u_{n+1} = \Im[(1-\varpi_n)\Im w_n + \varpi_n \Im v_n], n \in \mathbb{N}. \end{cases} \tag{2.1}$$

In 2000, Noor [13] defined the iterative scheme called the Noor iterative scheme:

$$\begin{cases} w_n = (1-\gamma_n)u_n + \gamma_n \Im u_n, \\ v_n = (1-\xi_n)u_n + \xi_n \Im u_n + \xi_n \Im w_n, \\ u_{n+1} = (1-\varpi_n)u_n + \varpi_n \Im v_n, \;\; n \in \mathbb{N}. \end{cases} \tag{2.2}$$

Chugh et al. [14] in 2012 defined the CR iterative scheme:

$$\begin{cases} w_n = (1-\gamma_n)u_n + \gamma_n \Im u_n, \\ v_n = (1-\xi_n)\Im u_n + \xi_n \Im w_n, \\ u_{n+1} = (1-\varpi_n)v_n + \varpi_n \Im v_n, \;\; n \in \mathbb{N}. \end{cases} \tag{2.3}$$

Okeke [8] in 2019 introduced the Picard-Ishikawa hybrid iterative process:

$$\begin{cases} w_n = (1-\xi_n)u_n + \xi_n \Im u_n, \\ v_n = (1-\varpi_n)u_n + \varpi_n \Im w_n, \\ u_{n+1} = \Im v_n, \;\; n \in \mathbb{N}, \end{cases} \tag{2.4}$$

where it was shown to be faster than some existing iterative schemes in literature and the result was applied in approximating the solution of a class of delay differential equations.

Also, in 2022, Okeke, et al. [11] introduced the GA iterative scheme, defined as:

$$\begin{cases} v_n = (1-\xi_n)f_n + \xi_n \Im f_n, \\ u_n = (1-\varpi_n)f_n + \varpi_n \Im v_n, \\ t_n = \Im u_n, \\ h_n = \Im t_n, \\ f_{n+1} = \Im h_n, n \in \mathbb{N}. \end{cases} \tag{2.5}$$

Motivated by the results above, we introduce a new iterative scheme, which is a modification of the JK iterative scheme (2.1) by Ahmad et al. [12]. Our scheme is defined as follows:

$$\begin{cases} u_0 = u \in \mathfrak{D}, \\ w_n = \Im[(1-\xi_n)u_n + \xi_n \Im u_n], \\ v_n = \Im(\Im w_n), \\ u_{n+1} = \Im[(1-\varpi_n)\Im w_n + \varpi_n \Im v_n], \;\; n \in \mathbb{N}. \end{cases} \tag{2.6}$$

The following definitions and results will be useful in this paper.

Definition 2.1. *[1] Let $\mathfrak{D}$ be a nonempty, closed, convex subset of a Banach space $\mathfrak{X}$. Let $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$ be a mapping, then $\mathfrak{J}$ is said to satisfy condition (C) if the following condition holds*

$$\frac{1}{2}\|x - \mathfrak{J}x\| \leq \|x - y\| \Rightarrow \|\mathfrak{J}x - \mathfrak{J}y\| \leq \|x - y\|, \tag{C}$$

for all $x, y \in \mathfrak{D}$.

A mapping satisfying condition (C) is otherwise known as Suzuki type generalized nonexpansive mapping.

Definition 2.2. *[2] Let $\mathfrak{D}$ be a nonempty subset of a Banach space $\mathfrak{X}$. For $\mu \geq 1$, the mapping $\mathfrak{J} : \mathfrak{D} \to \mathfrak{X}$ is said to satisfy condition (E_μ) on $\mathfrak{D}$ if for all $x, y \in \mathfrak{D}$,*

$$\|x - \mathfrak{J}y\| \leq \mu\|x - \mathfrak{J}x\| + \|x - y\|. \tag{2.7}$$

Moreover, $\mathcal{J}$ satisfies condition (E) on $\mathfrak{D}$ whenever $\mathcal{J}$ satisfies (E_μ) for some $\mu \geq 1$.

The next definition gives the description of Opial property, which will be useful in proving one of our main results.

Definition 2.3. *[15] A Banach space $\mathfrak{X}$ is said to satisfy the Opial condition [16] if for each sequence $\{u_n\}$ in $\mathfrak{X}$, converging weakly to $u \in \mathfrak{X}$, we have*

$$\limsup_{n\to\infty} \|u_n - u\| < \limsup_{n\to\infty} \|u_n - w\|, \tag{2.8}$$

for all $w \in \mathfrak{X}$ such that $u \neq w$

Definition 2.4. *[17] Assume $\mathcal{J}, \mathfrak{S} : \mathfrak{D} \to \mathfrak{D}$ are two operators. We say that $\mathfrak{S}$ is an approximate operator of $\mathcal{J}$ for all $x \in \mathfrak{D}$ and a fixed ϵ if*

$$\|\mathcal{J}x - \mathfrak{S}x\| \leq \epsilon.$$

Lemma 2.1. *[18] Let $\mathfrak{X}$ be a uniformly convex Banach space and $\{\gamma_n\}_{n=0}^{\infty}$ be any sequence of numbers such that $0 < a \leq \gamma_n \leq b < 1$, $n \geq 1$, for $a, b \in \mathbb{R}$. Let $\{\varphi_n\}_{n=0}^{\infty}$ and $\{\omega_n\}_{n=0}^{\infty}$ be sequences in $\mathfrak{X}$ such that $\limsup_{n\to\infty} \|\varphi_n\| \leq r$, $\limsup_{n\to\infty} \|\omega_n\| \leq r$, and $\limsup_{n\to\infty} \|\gamma_n\varphi_n + (1 - \gamma_n)\omega_n\| = r$ for some $r \geq 0$, then $\lim_{n\to\infty} \|\varphi_n - \omega_n\| = 0$.*

Lemma 2.2. *[19] If $\rho \in [0, 1)$ is a real number and $\{\epsilon_n\}_{n=0}^{\infty}$ is a sequence of positive numbers such that $\lim_{n\to\infty} \epsilon_n = 0$, then for any sequence of positive numbers, $\{s_n\}_{n=0}^{\infty}$ satisfying $s_{n+1} \leq \rho s_n + \epsilon_n$, $(n = 0, 1, 2, ...)$, we have $\lim_{n\to\infty} s_n = 0$.*

Lemma 2.3. *[17] Let $\{\eta_n\}$ be a nonnegative sequence for which one assumes there exists $n_0 \in \mathbb{N}$ for all $n \geq n_0$. Suppose the following inequality is satisfied:*

$$\eta_{n+1} \leq (1 - \varphi_n)\eta_n + \varphi_n\varrho_n,$$

where $\varphi_n \in (0, 1)$, $\forall n \in \mathbb{N}$, $\sum_{n=0}^{\infty} \varphi_n = \infty$, and $\varrho_n \geq 0$ $\forall n \in \mathbb{N}$, then

$$0 \leq \limsup_{n\to\infty} \eta_n \leq \limsup_{n\to\infty} \varrho_n.$$

Lemma 2.4. *[1] Let $\mathcal{J}$ be a mapping on a subset $\mathfrak{D}$ of a Banach space $\mathfrak{X}$ with the Opial condition satisfying (2.8). Suppose that $\mathcal{J}$ is a Suzuki generalized nonexpansive mapping satisfying condition (C). If $\{u_n\}$ converges weakly to p^* and $\lim_{n\to\infty} \|\mathcal{J}u_n - u_n\| = 0$, then $\mathcal{J}p^* = p^*$. That is, $I - \mathcal{J}$ is demiclosed at zero.*

Lemma 2.5. *[20] Let $\{\varphi_n\}$ be a nonnegative sequence satisfying*

$$\eta_{n+1} \le (1 - \varphi_n)\eta_n,$$

where $\{\varphi_n\} \subset (0, 1)$, $\sum_{n=0}^{\infty} \varphi_n = \infty$, then $\lim_{n\to\infty} \eta_n = 0$.

The following is the characterization of the García-Falset mapping.

Proposition 2.1. *[2] Let $\mathfrak{D}$ be a nonempty subset of a uniformly convex Banach space and $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$.*

(a) If $\mathfrak{J}$ is a Suzuki mapping, then $\mathfrak{J}$ is a García-Falset mapping.
(b) If $\mathfrak{J}$ is a García-Falset mapping having $\mathscr{F}(\mathfrak{J}) \neq \emptyset$, then for every choice of $x \in \mathfrak{D}$ and $p \in \mathscr{F}(\mathfrak{J})$, $\|\mathfrak{J}x - p\| \le \|x - p\|$ holds.
(c) If $\mathfrak{J}$ is a García-Falset mapping, then the set $\mathscr{F}(\mathfrak{J})$ is closed.

Let $\mathfrak{D}$ be a nonempty subset of a Banach space $\mathfrak{X}$ and $\{u_n\}$ be a bounded sequence in $\mathfrak{X}$. For any $u \in \mathfrak{X}$, we define the asymptotic radius of $\{u_n\}$ at u by

$$\mathfrak{R}(u, \{u_n\}) = \limsup_{n\to\infty} \|u - u_n\|.$$

The asymptotic radius of $\{u_n\}$ relative to the subset $\mathfrak{D}$ is denoted by

$$\mathfrak{R}(\mathfrak{D}, \{u_n\}) = \inf\{\mathfrak{R}(u, \{u_n\}) : u \in \mathfrak{D}\}.$$

The asymptotic center of $\{u_n\}$ relative to $\mathfrak{D}$ is denoted by

$$\mathfrak{A}(\mathfrak{D}, \{u_n\}) = \{u \in \mathfrak{D} : \mathfrak{R}(u, \{u_n\}) = \mathfrak{R}(\mathfrak{D}, \{u_n\})\}$$

For a uniformly convex Banach space, the set $\mathfrak{A}(\mathfrak{D}, \{u_n\})$ is singleton [21]. Again, the set $\mathfrak{A}(\mathfrak{D}, \{u_n\})$ is convex and nonempty [22], provided $\mathfrak{D}$ is a weakly compact convex subset.

3. Main results

3.1. Weak and strong convergence theorems

Lemma 3.1. *Let $\mathfrak{D}$ be a nonempty, closed, and convex subset of a uniformly convex Banach space $\mathfrak{X}$. Assume $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$ is a mapping satisfying condition (E) with $\mathscr{F}(\mathfrak{J}) \neq \emptyset$. Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated by the Modified-JK iterative scheme (2.6) for $u_0 \in \mathfrak{D}$, then $\lim_{n\to\infty} \|u_n - p^*\|$ exists for all $p^* \in \mathscr{F}(\mathfrak{J})$.*

Proof. Let $p^* \in \mathscr{F}(\mathfrak{J})$, then using the Modified-JK (2.6) and Proposition 2.1(b), we have

$$\begin{aligned}\|w_n - p^*\| =& \|\mathfrak{J}[(1 - \xi_n)u_n + \xi_n\mathfrak{J}u_n] - p^*\| \\ \le& \|[(1 - \xi_n)u_n + \xi_n\mathfrak{J}u_n] - p^*\| \\ \le& (1 - \xi_n)\|u_n - p^*\| + \xi_n\|\mathfrak{J}u_n - p^*\| \\ \le& (1 - \xi_n)\|u_n - p^*\| + \xi_n\|u_n - p^*\| \\ \le& \|u_n - p^*\|. \end{aligned} \tag{3.1}$$

Using (3.1), we have

$$\begin{aligned}\|v_n - p^*\| =&\|\mathfrak{J}^2 w_n - p^*\|\\ \le&\|\mathfrak{J} w_n - p^*\|\\ \le&\|w_n - p^*\|\\ \le&\|u_n - p^*\|.\end{aligned} \tag{3.2}$$

Also, using (3.1) and (3.2), we have

$$\begin{aligned}\|u_{n+1} - p^*\| =&\|\mathfrak{J}[(1-\varpi_n)\mathfrak{J}w_n + \varpi_n\mathfrak{J}v_n] - p^*\|\\ \le&\|[(1-\varpi_n)\mathfrak{J}w_n + \varpi_n\mathfrak{J}v_n] - p^*\|\\ \le&(1-\varpi_n)\|\mathfrak{J}w_n - p^*\| + \varpi_n\|\mathfrak{J}v_n - p^*\|\\ \le&(1-\varpi_n)\|w_n - p^*\| + \varpi_n\|v_n - p^*\|\\ \le&(1-\varpi_n)\|w_n - p^*\| + \varpi_n\|w_n - p^*\|\\ =&\|w_n - p^*\|\\ \le&\|u_n - p^*\|.\end{aligned} \tag{3.3}$$

Noticeably, it follows that the sequence $\{\|u_n - p^*\|\}$ is decreasing and bounded. Hence, $\lim_{n\to\infty}\|u_n - p^*\|$ exists for $p^* \in \mathscr{F}(\mathfrak{J}) \neq \emptyset$. □

Lemma 3.2. *Let $\mathfrak{D}$ be a nonempty closed convex subset of a uniformly convex Banach space $\mathfrak{X}$. Assume that $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$ is a mapping satisfying condition (E). Let $\{u_n\}_{n=0}^{\infty}$ be a sequence generated by the Modified-JK iterative scheme (2.6), then $\mathscr{F}(\mathfrak{J}) \neq \emptyset$ if, and only if, $\{u_n\}$ is bounded and $\lim_{n\to\infty}\|\mathfrak{J}u_n - u_n\| = 0$.*

Proof. Suppose $\mathscr{F}(\mathfrak{J}) \neq \emptyset$ and $p^* \in \mathscr{F}(\mathfrak{J})$. By Lemma 3.1, we have that $\lim_{n\to\infty}\|u_n - p^*\|$ exists and $\{u_n\}_{n=0}^{\infty}$ is bounded. Let

$$\lim_{n\to\infty}\|u_n - p^*\| = r. \tag{3.4}$$

From (3.1) and (3.4),

$$\limsup_{n\to\infty}\|w_n - p^*\| \le \limsup_{n\to\infty}\|u_n - p^*\| = r \tag{3.5}$$

and

$$\limsup_{n\to\infty}\|v_n - p^*\| \le \limsup_{n\to\infty}\|u_n - p^*\| = r.$$

Since $\mathfrak{J}$ satisfies condition (E), and from Proposition 2.1(b), we have

$$\|\mathfrak{J}u_n - p^*\| = \|\mathfrak{J} - \mathfrak{J}p^*\| \le \|u_n - p^*\|$$

and

$$\limsup_{n\to\infty}\|\mathfrak{J}u_n - p^*\| \le \limsup_{n\to\infty}\|u_n - p^*\| = r.$$

Now, from (3.3) of Lemma 3.1, we have that

$$\|u_{n+1} - p^*\| \le \|w_n - p^*\|.$$

Taking lim inf of both sides, we have

$$r = \liminf_{n\to\infty} \|u_{n+1} - p^*\| \le \liminf_{n\to\infty} \|w_n - p^*\|,$$

which follows that

$$r \le \liminf_{n\to\infty} \|w_n - p^*\|. \tag{3.6}$$

Combining (3.5) and (3.6), we have

$$r \le \liminf_{n\to\infty} \|w_n - p^*\| \le \limsup_{n\to\infty} \|w_n - p^*\| \le r,$$

so that

$$\lim_{n\to\infty} \|w_n - p^*\| = r. \tag{3.7}$$

From (3.7), we have

$$\begin{aligned} r &= \lim_{n\to\infty} \|w_n - p^*\| \\ &= \lim_{n\to\infty} \|\mathfrak{J}[(1-\xi_n)u_n + \xi_n \mathfrak{J}u_n] - p^*\| \\ &\le \lim_{n\to\infty} \|(1-\xi_n)u_n + \xi_n \mathfrak{J}u_n - p^*\| \\ &\le \lim_{n\to\infty} \|(1-\xi_n)(u_n - p^*) + \xi_n(\mathfrak{J}u_n - p^*)\| \\ &\le \lim_{n\to\infty} \|(1-\xi_n)(u_n - p^*)\| + \lim_{n\to\infty} \|\xi_n(\mathfrak{J}u_n - p^*)\| \\ &\le \lim_{n\to\infty} \|(1-\xi_n)(u_n - p^*) + \xi_n(\mathfrak{J}u_n - p^*)\|. \end{aligned}$$

Set $\varphi = \mathfrak{J}u_n - p^*$, $\omega_n = u_n - p^*$, and $\gamma_n = \xi_n$. By Lemma 2.1, we have that $\lim_{n\to\infty} \|\mathfrak{J}u_n - u_n\| = 0$. Conversely, suppose $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|\mathfrak{J}u_n - u_n\| = 0$. We want to show that $\mathscr{F}(\mathfrak{J}) \neq \emptyset$.

Let $p^* \in \mathfrak{A}(\mathfrak{D}, \{u_n\})$, and we prove that $\mathfrak{J}p^* = p^*$. Since $\mathfrak{J}$ is a Garcìa-Falset mapping, we have that

$$\begin{aligned} \mathfrak{R}(\mathfrak{J}p^*, \{u_n\}) &= \limsup_{n\to\infty} \|u_n - \mathfrak{J}p^*\| \\ &\le \limsup_{n\to\infty} (\mu \|u_n - \mathfrak{J}u_n\| + \|u_n - p^*\|) \\ &= \limsup_{n\to\infty} \|u_n - p^*\| \\ &= \mathfrak{R}(p^*, \{u_n\}). \end{aligned}$$

It follows that $\mathfrak{J}p^* \in \mathfrak{A}(\mathfrak{D}, \{u_n\})$, and $\mathfrak{A}(\mathfrak{D}, \{u_n\})$ is a singleton set and contains exactly one element; therefore, $\mathfrak{J}p^* = p^*$ and $p^* \in \mathscr{F}(\mathfrak{J})$. Hence, completing the proof. □

The weak convergence result is as follows:

Theorem 3.1. *Let $\mathfrak{J}$ be a mapping satisfying condition (E), defined on a nonempty, closed, and convex subset $\mathfrak{D}$ of a Banach space $\mathfrak{X}$, which satisfies the Opial condition (2.8) with $\mathscr{F}(\mathfrak{J}) \neq \emptyset$. If for any arbitrary iterate $u_0 \in \mathfrak{D}$, $\{u_n\}$ is the iterative sequence defined by the iterative scheme (2.6) where ϖ_n and ξ_n are sequences of real numbers in* $[0, 1]$*, then $\{u_n\}$ converges weakly to a fixed point $p^* \in \mathscr{F}(\mathfrak{J})$.*

Proof. Let $p^* \in \mathscr{F}(\mathfrak{J})$. From Lemma 3.1, strong limit, $\lim_{n\to\infty} \|u_n - p^*\|$ exists. To prove weak convergence of the iterative scheme (2.6) to a fixed point of $\mathfrak{J}$, we need to show that $\{u_n\}$ has a unique weak limit, say, p_1 and p_2 of subsequence $\{u_{n_i}\}$ and $\{u_{n_j}\}$ of the sequence $\{u_n\}$, respectively.

By Lemma 3.2, it is clear that $\lim_{n\to\infty} \|u_n - \mathfrak{J}u_n\| = 0$, and by Lemma 2.4, $I - \mathfrak{J}$ is demiclosed at zero. Clearly, $p_1, p_2 \in \mathscr{F}(\mathfrak{J})$, and practically $(I - \mathfrak{J})p_1 = 0$, which follows that $\mathfrak{J}p_1 = p_1$. Similarly, $(I - \mathfrak{J})p_2 = 0$, which implies $\mathfrak{J}p_2 = p_2$.

Next, we want to show the uniqueness of the weak limits, that is, $p_1 \neq p_2$. To do this, we shall use the Opial condition to obtain the following:

$$\begin{aligned} \lim_{n\to\infty} \|u_n - p_1\| &= \lim_{i\to\infty} \|u_{n_i} - p_1\| \\ &< \lim_{i\to\infty} \|u_{n_i} - p_2\| \\ &= \lim_{n\to\infty} \|u_n - p_2\| \\ &= \lim_{j\to\infty} \|u_{n_j} - p_2\| \\ &< \lim_{j\to\infty} \|u_{n_j} - p_1\| \\ &= \lim_{n\to\infty} \|u_n - p_1\|. \end{aligned}$$

Clearly, this leads to a contradiction, so $p_1 = p_2$, which implies that there is only one limit point p^*. We conclude that $\{u_n\}$ converges weakly to a fixed point in $\mathscr{F}(\mathfrak{J})$. □

At this point, we consider the strong convergence results.

Theorem 3.2. *Let $\mathfrak{D}$ be a nonempty, closed, convex subset of a Banach space $\mathfrak{X}$. Suppose $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$ is a mapping satisfying condition (E) such that $\mathscr{F}(\mathfrak{J}) \neq \emptyset$. Let $\{u_n\}$ be a sequence generated by the Modified-JK iterative scheme (2.6). If $\mathfrak{D}$ is compact, then the sequence $\{u_n\}$ converges strongly to a fixed point $p^* \in \mathscr{F}(\mathfrak{J})$.*

Proof. Let $\mathfrak{D}$ be compact. Since $\mathfrak{D}$ is compact, we can find a subsequence $\{u_{n_i}\}$ of the sequence $\{u_n\}$, which has a limit ℓ such that $\lim_{i\to\infty} \|u_{n_i} - \ell\| = 0$.

Also, since $\mathfrak{J}$ is endowed with condition (E), we have

$$\|u_{n_i} - \mathfrak{J}\ell\| \leq \mu \|u_{n_i} - \mathfrak{J}u_{n_i}\| + \|u_{n_i} - \ell\|. \tag{3.8}$$

Taking $\lim_{i\to\infty}$ on both sides of (3.8) and the hypothesis of Lemma 3.2, we have

$$\lim_{i\to\infty} \|u_{n_i} - \mathfrak{J}\ell\| = 0,$$

that is, $u_{n_i} \to \mathfrak{J}\ell$.

Since $u_{n_i} \to \mathfrak{J}\ell$ and $u_{n_i} \to \ell$, we prove that $\mathfrak{J}\ell \to \ell$. Consequently,

$$\|\mathfrak{J}\ell - \ell\| \leq \|\mathfrak{J}\ell - u_{n_i}\| + \|u_{n_i} - \ell\| = \|u_{n_i} - \mathfrak{J}\ell\| + \|u_{n_i} - \ell\| = 0.$$

Obviously, $\|\mathfrak{J}\ell - \ell\| = 0$. It follows that $\mathfrak{J}\ell = \ell$. By Lemma 3.1, we can say that $\lim_{n\to\infty} \|u_n - \ell\|$ exists for all $\ell \in \mathscr{F}(\mathfrak{J})$.

Hence, we can say that $\mathfrak{J}$ converges strongly to a limit ℓ. Therefore, the proof is complete. □

The next result shows the strong convergence of $\{u_n\}$ to a fixed point of a contraction mapping in a uniformly convex Banach space $\mathfrak{X}$.

Theorem 3.3. *Let $\mathfrak{D}$ be a nonempty closed convex subset of a uniformly Banach space. Suppose $\mathfrak{I} : \mathfrak{D} \to \mathfrak{D}$ is a contraction mapping. Let $\{u_n\}$ be a sequence generated by the Modified-JK iterative scheme (2.6) with real sequences $\{\varpi_n\}, \{\xi_n\} \in (0, 1)$ satisfying $\sum_{k=0}^{\infty} \xi_k = \infty$, then $\{u_n\}$ converges strongly to the fixed point $p^* \in \mathscr{F}(\mathfrak{I}) \neq \emptyset$.*

Proof. From (2.6), let $\mathfrak{I}p^* = p^*$,

$$\begin{aligned}
\|w_n - p^*\| &= \|\mathfrak{I}[(1 - \xi_n)\mathfrak{I}u_n + \xi_n\mathfrak{I}u_n] - p^*\| \\
&\leq \delta\|(1 - \xi_n)\mathfrak{I}u_n + \xi_n\mathfrak{I}u_n - p^*\| \\
&\leq \delta\{(1 - \xi_n)\|u_n - p^*\| + \xi_n\|\mathfrak{I}u_n - p^*\|\} \\
&\leq \delta\{\|u_n - p^*\| + \delta\xi_n\|u_n - p^*\|\} \\
&\leq \delta(1 - \xi_n)\|u_n - p^*\| + \delta^2\xi_n\|u_n - p^*\| \\
&= [\delta(1 - \xi_n) + \delta^2\xi_n]\|u_n - p^*\| \\
&\leq \delta[1 - (1 - \delta)\xi_n]\|u_n - p^*\|.
\end{aligned}$$

$$\begin{aligned}
\|v_n - p^*\| &= \|\mathfrak{I}^2 w_n - p^*\| \\
&\leq \delta\|\mathfrak{I}w_n - p^*\| \\
&\leq \delta^2\|w_n - p^8\| \\
&\leq \delta^3[1 - (1 - \delta)\xi_n]\|u_n - p^*\|.
\end{aligned}$$

$$\begin{aligned}
\|u_{n+1} - p^*\| &= \|\mathfrak{I}[(1 - \varpi_n)\mathfrak{I}w_n + \varpi_n\mathfrak{I}v_n] - p^*\| \\
&\leq \delta\|[(1 - \varpi_n)\mathfrak{I}w_n + \varpi_n\mathfrak{I}v_n] - p^*\| \\
&\leq \delta(1 - \varpi_n)\|\mathfrak{I}w_n - p^*\| + \delta\varpi_n\|\mathfrak{I}v_n - p^*\| \\
&\leq \delta^2(1 - \varpi_n)\|w_n - p^*\| + \delta^2\varpi_n\|v_n - p^*\| \\
&\leq \delta^2(1 - \varpi_n)\|w_n - p^*\| + \delta^4\varpi_n\|w_n - p^*\| \\
&\leq \{\delta^2 - \delta^2\varpi_n + \delta^4\varpi_n\}\|w_n - p^*\| \\
&\leq \delta[\delta^2 - \delta^2\varpi_n + \delta^4\varpi_n][1 - (1 - \delta)\xi_n]\|u_n - p^*\|.
\end{aligned}$$

Since $\delta < 1$ and $\varpi_n \in [0, 1]$, then

$$\|u_{n+1} - p^*\| \leq \delta[1 - (1 - \delta)\xi_n]\|u_n - p^*\|.$$

By induction, we have that

$$\|u_1 - p^*\| \leq \delta[1 - (1 - \delta)\xi_0]\|u_0 - p^*\|,$$

so that

$$\|u_{n+1} - p^*\| \leq \delta^{(n+1)}\|u_0 - p^*\| \prod_{k=0}^{n}[1 - (1 - \delta)\xi_k].$$

Clearly, $\delta \in [0, 1)$ and $\xi_n \in (0, 1)$, which suffices that $[1 - (1 - \delta)\xi_n] < 1$ for all $n \in \mathbb{N}$.

From elementary analysis, we have that $1 - x \leq e^{-x}$ for $x \in (0, 1)$. Consequently, we have that

$$\begin{aligned}\|u_{n+1} - p^*\| &\leq \delta^{(n+1)}\|u_0 - p^*\| \prod_{k=0}^{n} e^{-(1-\delta)\xi_k} \\ &\leq \delta^{(n+1)}\|u_0 - p^*\| e^{-(1-\delta)\sum_{k=0}^{\infty}\xi_k}.\end{aligned}$$

Since $\delta \in [0, 1)$ and $\sum_{k=0}^{\infty} \xi_k = \infty$, then $e^{-(1-\delta)\sum_{k=0}^{n}\xi_k} \to 0$ as $n \to \infty$.

Hence, $\lim_{n\to\infty} u_n = p^*$. Therefore, $\{u_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of $\mathfrak{J}$. □

Example 1. *Let* $\mathfrak{X} = \mathbb{R}$ *and* $\mathfrak{D} = [5, 10]$. *Let the mapping* $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$ *be defined by*

$$\mathfrak{J}x = \begin{cases} \frac{x+5}{2}, & \text{if } x \in [5, 8], \\ 5, & \text{if } x \in (8, 10]. \end{cases}$$

We want to show that $\mathfrak{J}$ *is a mapping satisfying condition (E) for* $\mu \geq 1$. *To show that, we choose a specific value of* μ*, i.e.,* $\mu = 2$*, and consider the following cases:*

Case 1. *If* $x, y \in (8, 10]$*,*

$$\begin{aligned}\|x - \mathfrak{J}y\| &= |x - \mathfrak{J}y| = |x - 5| = |x - \mathfrak{J}x| \\ &\leq 2|x - \mathfrak{J}x| \leq 2|x - \mathfrak{J}x| + |x - y| \\ &= 2\|x - \mathfrak{J}x\| + \|x - y\|.\end{aligned}$$

Case 2. *If* $x, y \in [5, 8]$*,*

$$\begin{aligned}\|x - \mathfrak{J}y\| &= |x - \mathfrak{J}y| \\ &\leq |x - \mathfrak{J}x| + |\mathfrak{J}x - \mathfrak{J}y| \\ &= |x - \mathfrak{J}x| + \left|\frac{x+5}{2} - \frac{y+5}{2}\right| \\ &= |x - \mathfrak{J}x| + \frac{1}{2}|x - y| \\ &\leq |x - \mathfrak{J}x| + |x - y| \\ &\leq 2|x - \mathfrak{J}x| + |x - y| \\ &= 2\|x - \mathfrak{J}x\| + \|x - y\|.\end{aligned}$$

Case 3. *If* $x \in [5, 8]$*,* $y \in (8, 10]$*,*

$$\begin{aligned}\|x - \mathfrak{J}y\| &= |x - \mathfrak{J}y| \\ &= |x - 5| = 2\left|\frac{x-5}{2}\right| \\ &= 2\left|x - \left(\frac{x+5}{2}\right)\right| \\ &= 2|x - \mathfrak{J}x| \\ &= 2|x - \mathfrak{J}x| + |x - y| \\ &= 2\|x - \mathfrak{J}x\| + \|x - y\|.\end{aligned}$$

Case 4. *If* $y \in [5, 8]$ *and* $x \in (8, 10]$,

$$\begin{aligned}
\|x - \Im y\| &= |x - \Im y| \\
&= \left| x - \left(\frac{y+5}{2} \right) \right| \\
&= \left| \frac{2x - y - 5}{2} \right| = \left| \frac{x + x - y - 5}{2} \right| \\
&\leq \left| \frac{x-5}{2} \right| + \left| \frac{x-y}{2} \right| \\
&= \frac{1}{2} |x - 5| + \frac{1}{2} |x - y| \\
&\leq 2|x - 5| + |x - y| \\
&\leq 2|x - \Im x| + |x - y| \\
&= 2\|x - \Im x\| + \|x - y\|.
\end{aligned}$$

As illustrated in the different cases, we confirm that $\Im$ *is a mapping satisfying condition (E) on* $\mathfrak{D}$=[5, 10].

Next, for $n \in \mathbb{N}$, $u_0 = 6$ and choosing $\varpi_n = \frac{2}{3}$, $\xi_n = \frac{2}{3}$, we generate a table and figure such that from Tables 1 and 2 and Figures 1 and 2, it is clear that Modified-JK iterative scheme converges faster than Mann, JK, GA [11], Picard-Ishikawa, Noor, and CR iterative schemes.

Table 1. Comparison of speed of convergence of some iterative scheme for Example 1.

Step	Modified-JK	JK	Mann	Picard-Ishikawa
1	6.0000000000	6.0000000000	6.0000000000	6.0000000000
2	5.0700000000	5.1600000000	5.8000000000	5.3800000000
3	5.0049000000	5.0256000000	5.6400000000	5.1444000000
4	5.0003430000	5.0040960000	5.5120000000	5.0548720000
5	5.0000240100	5.0006553600	5.4096000000	5.0208513600
6	5.0000016807	5.0001048576	5.3276800000	5.0079235168
7	5.0000001176	5.0000167772	5.2621440000	5.0030109364
8	5.0000000082	5.0000026844	5.2097152000	5.0011441558
9	5.0000000006	5.0000004295	5.1677721600	5.0004347792
10	5.0000000000	5.0000000687	5.1342177280	5.0001652161
11	5.0000000000	5.0000000110	5.1073741824	5.0000627821
12	5.0000000000	5.0000000018	5.0858993459	5.0000238572
13	5.0000000000	5.0000000003	5.0687194767	5.0000090657
14	5.0000000000	5.0000000000	5.0549755814	5.0000034450
15	5.0000000000	5.0000000000	5.0439804651	5.0000013091
16	5.0000000000	5.0000000000	5.0351843721	5.0000004975
17	5.0000000000	5.0000000000	5.0281474977	5.0000001890
18	5.0000000000	5.0000000000	5.0225179981	5.0000000718
19	5.0000000000	5.0000000000	5.0180143985	5.0000000273
20	5.0000000000	5.0000000000	5.0144115188	5.0000000104

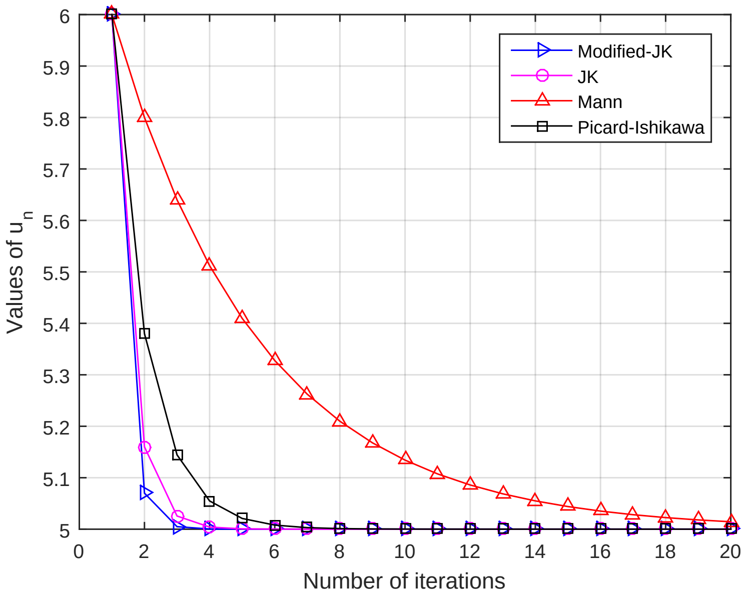

Figure 1. Graph corresponding to Table 1.

Table 2. Comparison of speed of convergence of some iterative scheme for Example 1.

Step	Modified-JK	GA	Noor	CR
1	6.0000000000	6.0000000000	6.0000000000	6.0000000000
2	5.0700000000	5.0950000000	5.7520000000	5.3680000000
3	5.0049000000	5.0090250000	5.5655040000	5.1354240000
4	5.0003430000	5.0008573750	5.4252590080	5.0498360320
5	5.0000240100	5.0000814506	5.3197947740	5.0183396598
6	5.0000016807	5.0000077378	5.2404856701	5.0067489948
7	5.0000001176	5.0000007351	5.1808452239	5.0024836301
8	5.0000000082	5.0000000698	5.1359956084	5.0009139759
9	5.0000000006	5.0000000066	5.1022686975	5.0003363431
10	5.0000000000	5.0000000006	5.0769060605	5.0001237743
11	5.0000000000	5.0000000001	5.0578333575	5.0000455489
12	5.0000000000	5.0000000000	5.0434906848	5.0000167620
13	5.0000000000	5.0000000000	5.0327049950	5.0000061684
14	5.0000000000	5.0000000000	5.0245941562	5.0000022700
15	5.0000000000	5.0000000000	5.0184948055	5.0000008354
16	5.0000000000	5.0000000000	5.0139080937	5.0000003074
17	5.0000000000	5.0000000000	5.0104588865	5.0000001131
18	5.0000000000	5.0000000000	5.0078650826	5.0000000416
19	5.0000000000	5.0000000000	5.0059145421	5.0000000153
20	5.0000000000	5.0000000000	5.0044477357	5.0000000056

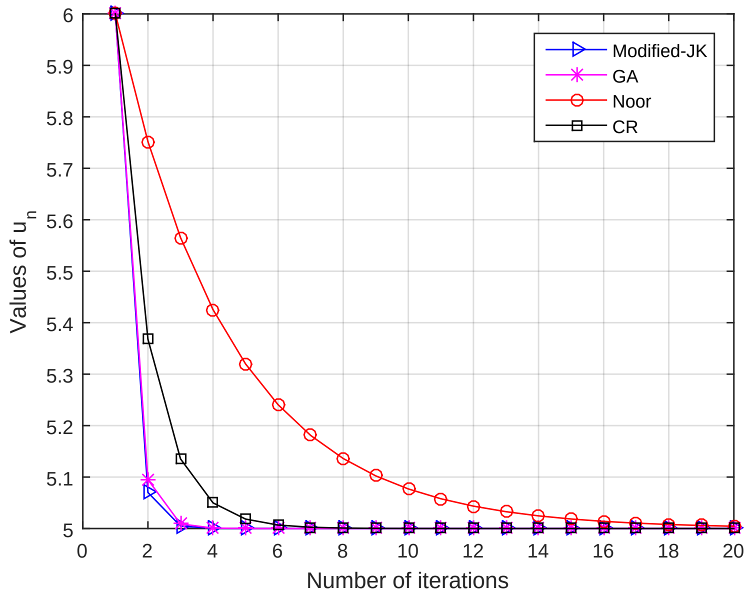

Figure 2. Graph corresponding to Table 2.

3.2. Stability and data dependence results

In this section, we discuss the stability and data dependence result of our new iterative scheme.

Theorem 3.4. *Let $\mathfrak{X}$ be a Banach space. Suppose $\mathfrak{J} : \mathfrak{D} \to \mathfrak{D}$ is a contraction mapping with $\delta \in [0, 1)$ and it has a fixed point $p^* \in \mathscr{F}(\mathfrak{J}) \neq \emptyset$. Suppose $\{u_n\}_{n=0}^{\infty}$ is a sequence generated by the Modified-JK fixed point iterative scheme (2.6) that converges to p^*, then (2.6) is stable with respect to $\mathfrak{J}$.*

Proof. Suppose that $\{m_n\}_{n=0}^{\infty} \subset \mathfrak{X}$ is an arbitrary sequence in $\mathfrak{D}$ and assume that the sequence generated by the Modified-JK is $u_{n+1} = f(\mathfrak{J}, u_n)$ converging to a unique fixed point p^*.

Let $\epsilon_n = \|m_{n+1} - f(\mathfrak{J}, m_n)\|$. Our aim is to show that $\lim_{n\to\infty} \epsilon_n = 0$ if, and only if, $\lim_{n\to\infty} \|m_n - p^*\| = 0$. Assume $\lim_{n\to\infty} \epsilon_n = 0$.

$$\begin{aligned}
\|m_{n+1} - p^*\| &= \|m_{n+1} - f(\mathfrak{J}, m_n) + f(\mathfrak{J}, m_n) - p^*\| \\
&\leq \|m_{n+1} - f(\mathfrak{J}, m_n)\| + \|f(\mathfrak{J}, m_n) - p^*\| \\
&\leq \epsilon_n + \|f(\mathfrak{J}, m_n) - p^*\| \\
&\leq \epsilon_n + \|\mathfrak{J}[(1 - \varpi_n)\mathfrak{J}w_n + \varpi_n\mathfrak{J}v_n] - p^*\| \\
&\leq \epsilon_n + \|(1 - \varpi_n)\mathfrak{J}w_n + \varpi_n\mathfrak{J}v_n - p^*\| \\
&\leq \epsilon_n + (1 - \varpi_n)\|\mathfrak{J}w_n - p^*\| + \varpi_n\|\mathfrak{J}v_n - p^*\| \\
&\leq \epsilon_n + (1 - \varpi_n)\|w_n - p^*\| + \varpi_n\|v_n - p^*\|. \qquad (3.9)
\end{aligned}$$

From (2.6),

$$\begin{aligned}
\|v_n - p^*\| &= \|\mathfrak{J}^2 w_n - p^*\| \\
&\leq \|\mathfrak{J}w_n - p^*\| \leq \|w_n - p^*\|
\end{aligned} \qquad (3.10)$$

and

$$\begin{aligned}\|w_n - p^*\| &= \|\mathfrak{J}[(1-\xi_n)\mathfrak{J}m_n + \xi_n\mathfrak{J}m_n] - p^*\| \\ &\le \|(1-\xi_n)\mathfrak{J}m_n + \xi_n\mathfrak{J}m_n - p^*\| \\ &\le (1-\xi_n)\|\mathfrak{J}m_n - p^*\| + \xi_n\|\mathfrak{J}m_n - p^*\| \\ &\le \|\mathfrak{J}m_n - p^*\| \\ &\le \|m_n - p^*\| \end{aligned} \tag{3.11}$$

Combining (3.9)–(3.11), we have

$$\|m_{n+1} - p^*\| \le \|m_n - p^*\|.$$

By Lemma 2.2, we have $\lim_{n\to\infty} \|m_n - p^*\| = 0$, that is, $\lim_{n\to\infty} m_n = p^*$. Conversely, assume $\lim_{n\to\infty} m_n = p^*$, then

$$\begin{aligned}\epsilon_n &= \|m_{n+1} - f(\mathfrak{J}, m_n)\| \\ &\le \|m_{n+1} - p^* + p^* - f(\mathfrak{J}, m_n)\| \\ &\le \|m_{n+1} - p^*\| + \|p^* - f(\mathfrak{J}, m_n)\| \\ &\le \|m_{n+1} - p^*\| + \|p^* - \mathfrak{J}[(1-\varpi_n)w_n + \varpi_n\mathfrak{J}v_n]\| \\ &\le \|m_{n+1} - p^*\| + \|[(1-\varpi_n)w_n + \varpi_n\mathfrak{J}v_n] - p^*\| \\ &\le \|m_{n+1} - p^*\| + (1-\varpi_n)\|w_n - p^*\| + \varpi_n\|\mathfrak{J}v_n - p^*\| \\ &\le \|m_{n+1} - p^*\| + (1-\varpi_n)\|w_n - p^*\| + \varpi_n\|v_n - p^*\| \\ &\le \|m_{n+1} - p^*\| + \|w_n - p^*\| \\ &\le \|m_{n+1} - p^*\| + \|m_n - p^*\|. \end{aligned}$$

Taking limit as $n \to \infty$ on both sides and having that $\lim_{n\to\infty} m_n = p^*$, we have that $\lim_{n\to\infty} \epsilon_n = 0$. Hence, the iterative scheme (2.6) is stable with respect to $\mathfrak{J}$. □

Theorem 3.5. *Let $\mathfrak{S}$ be an approximate operator to $\mathfrak{J}$ being a contraction mapping. Suppose $\{u_n\}_{n=0}^{\infty}$ is an iterative sequence generated by the Modified-JK iterative scheme (2.6) for $\mathfrak{J}$, and define the approximate scheme of the sequence $\{t_n\}_{n=0}^{\infty}$. Thus,*

$$\begin{cases} t_0 = t \in \mathfrak{D} \\ \mu_n = \mathfrak{S}[(1-\xi_n)t_n + \xi_n\mathfrak{S}t_n] \\ \theta_n = \mathfrak{S}^2\mu_n \\ t_{n+1} = \mathfrak{S}[(1-\varpi_n)\mathfrak{S}\mu_n + \varpi_n\mathfrak{S}\theta_n], \end{cases} \tag{3.12}$$

where $\{\varpi_n\}$ and $\{\xi_n\}$ are real sequences in $[0,1]$ satisfying the conditions (d_p) $\frac{1}{2} \le \xi_n$ and $\forall n \in \mathbb{N}$ and setting $\varpi_n = 1$. If $\mathfrak{J}p^* = p^*$ and $\mathfrak{S}\tau^* = \tau^*$ such that $\lim_{n\to\infty} \|t_n - \tau^*\| = 0$, then we have for $\delta \in (0,1)$, $\|p^* - \tau^*\| \le \frac{11\epsilon}{1-\delta}$, where $\epsilon > 0$ is a fixed constant.

Proof. Using (2.6) and (3.12),

$$\begin{aligned}
\|w_n-\mu_n\| =&\|\mathfrak{I}[(1-\xi_n)u_n+\xi_n\mathfrak{I}u_n]-\mathfrak{S}[(1-\xi_n)t_n+\xi_n\mathfrak{S}t_n]\| \\
\leq&\|\mathfrak{I}[(1-\xi_n)u_n+\xi_n\mathfrak{I}u_n]-\mathfrak{I}[(1-\xi_n)t_n+\xi_n\mathfrak{S}t_n] \\
&+\mathfrak{I}[(1-\xi_n)t_n+\xi_n\mathfrak{S}t_n]-\mathfrak{S}[(1-\xi_n)t_n+\xi_n\mathfrak{S}t_n]\| \\
\leq&\|\mathfrak{I}[(1-\xi_n)u_n+\xi_n\mathfrak{I}u_n]-\mathfrak{I}[(1-\xi_n)t_n+\xi_n\mathfrak{S}t_n]\| \\
&+\|\mathfrak{I}[(1-\xi_n)t_n+\xi_n\mathfrak{S}]-\mathfrak{S}[(1-\xi_n)t_n+\xi_n\mathfrak{S}t_n]\| \\
\leq&\delta\|(1-\xi_n)u_n+\xi_n\mathfrak{I}u_n-(1-\xi_n)t_n-\xi_n\mathfrak{S}t_n\|+\epsilon \\
\leq&\delta(1-\xi_n)\|u_n-t_n\|+\delta\xi_n\|\mathfrak{I}u_n-\mathfrak{S}t_n\|+\epsilon \\
\leq&\delta(1-\xi_n)\|u_n-t_n\|+\delta\xi_n\|\mathfrak{I}u_n-\mathfrak{I}t_n+\mathfrak{I}t_n-\mathfrak{S}t_n\|+\epsilon \\
\leq&\delta(1-\xi_n)\|u_n-t_n\|+\delta\xi_n\|\mathfrak{I}u_n-\mathfrak{I}t_n\|+\delta\xi_n\|\mathfrak{I}t_n-\mathfrak{S}t_n\|+\epsilon \\
\leq&\delta(1-\xi_n)\|u_n-t_n\|+\delta^2\xi_n\|u_n-t_n\|+\delta\xi_n\epsilon+\epsilon \\
=&[\delta(1-\xi_n)+\delta^2\xi_n]\|u_n-t_n\|+\delta\xi_n\epsilon+\epsilon \\
\leq&\delta[1-(1-\delta)\xi_n]\|u_n-t_n\|+\delta\xi_n\epsilon+\epsilon, \qquad (3.13)
\end{aligned}$$

$$\begin{aligned}
\|v_n-\theta_n\| =&\|\mathfrak{I}[\mathfrak{I}w_n]-\mathfrak{S}[\mathfrak{S}\mu_n]\| \\
\leq&\|\mathfrak{I}[\mathfrak{I}w_n]-\mathfrak{I}[\mathfrak{S}\mu_n]+\mathfrak{I}[\mathfrak{S}\mu_n]-\mathfrak{S}[\mathfrak{S}\mu_n]\| \\
\leq&\|\mathfrak{I}[\mathfrak{I}w_n]-\mathfrak{I}[\mathfrak{S}\mu_n]\|+\epsilon \\
\leq&\delta\|\mathfrak{I}w_n-\mathfrak{S}\mu_n\|+\epsilon \\
\leq&\delta\|\mathfrak{I}w_n-\mathfrak{I}\mu_n+\mathfrak{I}\mu_n-\mathfrak{S}\mu_n\|+\epsilon \\
\leq&\delta\|\mathfrak{I}w_n-\mathfrak{I}\mu_n\|+\delta\|\mathfrak{I}\mu_n-\mathfrak{S}\mu_n\|+\epsilon \\
\leq&\delta\|\mathfrak{I}w_n-\mathfrak{I}\mu_n\|+\delta\epsilon+\epsilon \\
\leq&\delta^2\|w_n-\mu_n\|+\delta\epsilon+\epsilon. \qquad (3.14)
\end{aligned}$$

Putting (3.13) in (3.14),

$$\begin{aligned}
\|v_n-\theta_n\| =& \delta^2\{\delta[1-(1-\delta)\xi_n]\|u_n-t_n\|+\delta\xi_n\epsilon+\epsilon\}+\delta\epsilon+\epsilon \\
\leq& \delta^3[1-(1-\delta)\xi_n]\|u_n-t_n\|+\delta^3\xi_n\epsilon+\delta^2\epsilon+\delta\epsilon+\epsilon. \qquad (3.15)
\end{aligned}$$

$$\begin{aligned}
\|u_{n+1}-t_{n+1}\| =&\|\mathfrak{I}[(1-\varpi_n)\mathfrak{I}w_n+\varpi_n\mathfrak{I}v_n]-\mathfrak{S}[(1-\varpi_n)\mathfrak{S}\mu_n+\varpi_n\mathfrak{S}\theta_n]\| \\
\leq&\|\mathfrak{I}[(1-\varpi_n)\mathfrak{I}w_n+\varpi_n\mathfrak{I}v_n]-\mathfrak{I}[(1-\varpi_n)\mathfrak{S}\mu_n+\varpi_n\mathfrak{S}\theta_n] \\
&+\mathfrak{I}[(1-\varpi_n)\mathfrak{S}\mu_n+\varpi_n\mathfrak{S}\theta_n]-\mathfrak{S}[(1-\varpi_n)\mathfrak{S}\mu_n+\varpi_n\mathfrak{S}\theta_n]\| \\
\leq&\delta\|[(1-\varpi_n)\mathfrak{I}w_n+\varpi_n\mathfrak{I}v_n]-[(1-\varpi_n)\mathfrak{S}\mu_n+\varpi_n\mathfrak{S}\theta_n]\|+\epsilon \\
\leq&\delta(1-\varpi_n)\|\mathfrak{I}w_n-\mathfrak{S}\mu_n\|+\delta\varpi_n\|\mathfrak{I}v_n-\mathfrak{S}\theta_n\|+\epsilon \\
\leq&\delta(1-\varpi_n)\|\mathfrak{I}w_n-\mathfrak{I}\mu_n+\mathfrak{I}\mu_n-\mathfrak{S}\mu_n\|+\delta\varpi_n\|\mathfrak{I}v_n-\mathfrak{I}\theta_n+\mathfrak{I}\theta_n-\mathfrak{S}\theta_n\|+\epsilon \\
\leq&\delta(1-\varpi_n)\|\mathfrak{I}w_n-\mathfrak{I}\mu_n\|+\delta(1-\varpi_n)\|\mathfrak{I}\mu_n-\mathfrak{S}\mu_n\| \\
&+\delta\varpi_n\|\mathfrak{I}v_n-\mathfrak{I}\theta_n\|+\delta\varpi_n\|\mathfrak{I}\theta_n-\mathfrak{S}\theta_n\|+\epsilon \\
\leq&\delta^2(1-\varpi_n)\|w_n-\mu_n\|+\delta(1-\varpi_n)\epsilon+\delta^2\varpi_n\|v_n-\theta_n\|+\delta\varpi_n\epsilon+\epsilon \\
=&\delta^2(1-\varpi_n)\|w_n-\mu_n\|+\delta^2\varpi_n\|v_n-\theta_n\|+\delta(1-\varpi_n)\epsilon+\delta\varpi_n\epsilon+\epsilon. \qquad (3.16)
\end{aligned}$$

Combining (3.13), (3.15), and (3.16),

$$\begin{aligned}
\|u_{n+1} - t_{n+1}\| =& \delta^2(1-\varpi_n)\{\delta[1-(1-\delta)\xi_n]\|u_n - t_n\| + \delta\xi_n\epsilon + \epsilon\} \\
&+ \delta^2\varpi_n\{\delta^3[1-(1-\delta)\xi_n]\|u_n - t_n\| + \delta^3\xi_n\epsilon + \delta^2\epsilon + \delta^2\epsilon + \delta\epsilon + \epsilon\} \\
&+ \delta(1-\varpi_n)\epsilon + \delta\varpi_n\epsilon + \epsilon \\
\le& \delta^3(1-\varpi_n)[1-(1-\delta)\xi_n]\|u_- t_n\| + \delta^3(1-\varpi_n)\xi_n\epsilon + \delta^2(1-\varpi_n)\epsilon \\
&+ \delta^5\varpi_n[1-(1-\delta)\xi_n]\|u_n - t_n\| + \delta^5\varpi_n\xi_n\epsilon + \delta^4\varpi_n\epsilon + \delta^3\varpi_n\epsilon \\
&+ \delta^2\varpi_n\epsilon + \delta(1-\varpi_n)\epsilon + \delta\varpi_n\epsilon + \epsilon \\
=& [\delta^3(1-\varpi_n) + \delta^5\varpi_n][1-(1-\delta)\xi_n]\|u_n - t_n\| + \delta^5\xi_n\epsilon + \delta^4\epsilon + \delta^3\epsilon \\
&+ \delta^2\epsilon + \delta\epsilon + \epsilon \\
\le& \delta^5[1-(1-\delta)\xi_n]\|u_n - t_n\| + \delta^5\xi_n\epsilon + \delta^3\epsilon + \delta^2\epsilon + \delta\epsilon + \epsilon.
\end{aligned} \tag{3.17}$$

Since $\delta < 1$, $n \in \mathbb{N}$, then $\delta^5 < 1$, $\delta^4 < 1$, $\delta^3 < 1$, $\delta^2 < 1$. We note that $1 - \xi_n \le \xi_n$, so

$$\begin{aligned}
\|u_{n+1} - t_{n+1}\| \le& [1-(1-\delta)\xi_n]\|u_n - t_n\| + \xi_n\epsilon + 5\epsilon \\
\le& [1-(1-\delta)\xi_n]\|u_n - t_n\| + \xi_n\epsilon + 5(1-\xi_n + \xi_n)\epsilon \\
\le& [1-(1-\delta)\xi_n]\|u_n - t_n\| + \xi_n(1-\delta)\frac{11\epsilon}{(1-\delta)}.
\end{aligned} \tag{3.18}$$

Let $\eta_n := \|u_n - t_n\|$, $\phi_n := \xi_n(1-\delta) \in (0,1)$ and $\varrho_n := \frac{11\epsilon}{1-\delta}$. From Lemma 2.3, it follows that $0 \le \limsup_{n\to\infty} \|u_n - t_n\| \le \limsup_{n\to\infty} \frac{11\epsilon}{1-\delta}$. From Theorem 3.3, it is obvious that $\lim_{n\to\infty} \|u_n - p^*\| = 0$. Consequently, with the assumption $\lim_{n\to\infty} \|t_n - \tau^*\| = 0$, we have

$$\|p^* - \tau^*\| \le \frac{11\epsilon}{1-\delta}.$$

Hence, the proof is complete. □

4. Applications to delay differential equations

Delay differential equations have been very useful in modeling some problems of applied sciences, like the biological science such as drug therapy, immune response, and epidemiology, and in other aspects of science, such as neural networks.

Let $C([a,b])$ denote the space of all continuous functions on the interval $[a,b]$, endowed with the max norm,

$$\|f - g\|_\infty = \max_{x\in[a,b]} |f(x) - g(x)|; \ \forall f, g \in C([a,b]),$$

such that the space $C([a,b], \|\cdot\|_\infty)$ is a Banach space.

In this section, our aim is to use our new fixed point iterative scheme, Modified-JK iterative scheme (2.6), to approximate the solution of the following delay differential equation,

$$y'(x) = f(x, y(x), y(x-\lambda)), \ \forall x \in [x_0, b] \tag{4.1}$$

with initial condition

$$y(x) = \aleph(x), \ x \in [x_0 - \lambda, x_0]. \tag{4.2}$$

To achieve our aim, the following axioms are considered:

Axiom 1. *Assume that the following conditions hold:*

($\mathscr{D}_1$) $x_0, b \in \mathbb{R}$, $\lambda > 0$;

($\mathscr{D}_2$) $f \in C([x_0, b] \times \mathbb{R}^2, \mathbb{R})$;

($\mathscr{D}_3$) $\aleph \in C([x_0 - \lambda, x_0], \mathbb{R})$;

($\mathscr{D}_4$) *there exists* $L_f > 0$, *such that*

$$|f(x, y_1, y_2) - f(x, g_1, g_2)| \leq L_f(|y_1 - g_1| + |y_2 - g_2|),$$

for all $y_j, g_j \in \mathbb{R}$, $j = 1, 2$, *and* $x \in [x_0, b]$;

($\mathscr{D}_5$) $2L_f(b - x_0) < 1$.

Next, the Eqs (4.1) and (4.2) are reformulated as the following integral equation:

$$y(x) = \begin{cases} \aleph(x), & x \in [x_0 - \lambda, x_0], \\ \aleph(x_0) + \int_{x_0}^{x} f(s, y(s), y(s - \lambda))ds, & x \in [x_0, b]. \end{cases} \tag{4.3}$$

The next theorem which is highlighted in [23] (and other references in literature on the subject matter) shows the existence of a prototype delay differential equation (4.1) and (4.2). The theorem will be useful in the proof of our main result for the convergence of the iterative scheme (2.6) to the solution of (4.1) and (4.2).

Theorem 4.1. *Suppose the conditions* ($\mathscr{D}_1$)–($\mathscr{D}_5$) *are satisfied, then the sequence* $\{u_n\}$ *generated by the Modified-JK iterative (2.6) has a unique solution* $y \in C([x_0 - \lambda, b], \mathbb{R}) \cap C'([x_0, b], \mathbb{R})$ *and* $y = \lim_{n\to\infty} T^n(x)$ *for any* $x \in C([x_0 - \lambda, b], \mathbb{R})$.

The next theorem represents our result on the aspect of this section.

Theorem 4.2. *Assume that conditions* ($\mathscr{D}_1$)–($\mathscr{D}_5$) *are satisfied, then the sequence* $\{u_n\}$ *generated by the Modified-JK iterative scheme (2.6) with* $\sum_{n=0}^{\infty} \varpi_n = \infty$, *converges to the solution* $p^* \in C([x_0 - \lambda, b], \mathbb{R}) \cap C'([x_0, b], \mathbb{R})$ *of the delay differential equation (4.1) and (4.2) for* $u_0 \in C([x_0 - \lambda, b], \mathbb{R})$.

Proof. Let

$$\Im y(x) = \begin{cases} \aleph(x), & x \in [x_0 - \lambda, x_0], \\ \aleph(x_0) + \int_{x_0}^{x} f(s, y(s), y(s - \lambda))ds, & x \in [x_0, b], \end{cases} \tag{4.4}$$

be an integral operator with respect to (4.3). Let $\{u_n\}$ be a sequence defined by the Modified-JK iterative scheme (2.6) for the operator (4.4). By Theorem 4.2, let p^* be the fixed point of $\Im$. We want to show that $\lim_{n\to\infty} \|u_n - p^*\| = 0$. Suppose $x \in [x_0 - \lambda, x_0]$, then it is easy to show that $\lim_{n\to\infty} \|u_n - p^*\| = 0$. Assume $x \in [x_0, b]$. We define $q_n = (1 - \xi_n)u_n + \xi_n \Im u_n$; $r_n = \Im q_n$ and $s_n = (1 - \varpi_n)\Im q_n + \varpi_n \Im r_n$, $n \in \mathbb{N}$.

Using (2.6), and ($\mathscr{D}_4$), we have

$$\begin{aligned} \|q_n - p^*\|_\infty =& \|(1 - \xi_n)u_n + \xi_n \Im u_n - p^*\|_\infty \\ \leq& (1 - \xi_n)\|u_n - p^*\|_\infty + \xi_n \|\Im u_n - \Im p^*\|_\infty \end{aligned}$$

$$
\begin{aligned}
&=(1-\xi_n)\|u_n-p^*\|_\infty+\xi_n \max_{x\in[x_0-\lambda,b]}|\Im u_n-\Im p^*|\\
&=(1-\xi_n)\|u_n-p^*\|_\infty+\xi_n \max_{x\in[x_0-\lambda,b]}\Big|\aleph(x_0)+\int_{x_0}^{x} f(s,u_n(s),u_n(s-\lambda))ds\\
&\quad-\aleph(x_0)+\int_{x_0}^{x} f(s,p^*(s),p^*(s-\lambda))ds\Big|\\
&=(1-\xi_n)\|u_n-p^*\|_\infty+\xi_n \max_{x\in[x_0-\lambda,b]}\Big|\int_{x_0}^{x} f(s,u_n(s),u_n(s-\lambda))ds\\
&\quad-\int_{x_0}^{x} f(s,p^*(s),p^*(s-\lambda))ds\Big|\\
&\le(1-\xi_n)\|u_n-p^*\|_\infty+\xi_n \max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x} L_f\Big(|u_n(s)-p^*(s)|\\
&\quad+|u_n(s-\lambda)-p^*(s-\lambda)|\Big)ds\\
&\le(1-\xi_n)\|u_n-p^*\|_\infty+\xi_n \int_{x_0}^{x} L_f\Big(\max_{x\in[x_0-\lambda,b]}|u_n(s)-p^*(s)|\\
&\quad+\max_{x\in[x_0-\lambda,b]}|u_n(s-\lambda)-p^*(s-\lambda)|\Big)ds\\
&\le(1-\xi_n)\|u_n-p^*\|_\infty+\xi_n \int_{x_0}^{x} L_f(\|u_n-p^*\|_\infty+\|u_n-p^*\|_\infty)ds\\
&\le(1-\xi_n)\|u_n-p^*\|_\infty+\xi_n \int_{x_0}^{x} 2L_f\|u_n-p^*\|_\infty ds\\
&\le(1-\xi_n)\|u_n-p^*\|_\infty+2\xi_n L_f(b-x_0)\|u_n-p^*\|_\infty\\
&\le[1-(1-2L_f(b-x_0))\xi_n]\|u_n-p^*\|_\infty.
\end{aligned}
$$

Using (2.6) and ($\mathscr{D}_4$),

$$
\begin{aligned}
\|r_n-p^*\|_\infty &=\|\Im q_n-p^*\|_\infty\\
&=\|\Im q_n-\Im p^*\|_\infty\\
&=\max_{x\in[x_0-\lambda,b]}|\Im q_n(s)-\Im p^*(s)|\\
&=\max_{x\in[x_0-\lambda,b]}\Big|\aleph(x_0)+\int_{x_0}^{x} f(s,q_n(s),q_n(s-\lambda))ds\\
&\quad-\aleph(x_0)+\int_{x_0}^{x} f(s,p^*(s),p^*(s-\lambda))ds\Big|\\
&=\max_{x\in[x_0-\lambda,b]}\Big|\int_{x_0}^{x} f(s,q_n(s),q_n(s-\lambda))ds-\int_{x_0}^{x} f(s,p^*(s),p^*(s-\lambda))ds\Big|\\
&\le\max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x}\Big|f(s,q_n(s),q_n(s-\lambda))-f(s,p^*(s),p^*(s-\lambda))\Big|ds\\
&\le\max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x} L_f\Big(|q_n(s)-p^*(s)|+|q_n(s-\lambda)-p^*(s-\lambda)|\Big)ds\\
&\le\int_{x_0}^{x} L_f\Big(\max_{x\in[x_0-\lambda,b]}|q_n(s)-p^*(s)|+\max_{x\in[x_0-\lambda,b]}|q_n(s-\lambda)-p^*(s-\lambda)|\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{x_0}^{x} L_f\Big(\|q_n(s) - p^*(s)\|_\infty + \|q_n(s-\lambda) - p^*(s-\lambda)\|_\infty\Big)ds \\
&= \int_{x_0}^{x} L_f\Big(\|q_n - p^*\|_\infty + \|q_n - p^*\|_\infty\Big)ds \\
&\leq \int_{x_0}^{x} 2L_f\|q_n - p^*\|_\infty ds \\
&\leq 2L_f(b - x_0)\|q_n - p^*\|_\infty \\
&\leq 2L_f(b - x_0)\Big(1 - (1 - 2L_f(b - x_0))\xi_n\Big)\|u_n - p^*\|_\infty.
\end{aligned}
$$

Using (2.6), ($\mathscr{D}_4$), and ($\mathscr{D}_5$), we have

$$
\begin{aligned}
\|u_{n+1} - p^*\|_\infty =& \|s_n - p^*\|_\infty \\
=& \|(1-\varpi_n)\Im q_n + \varpi_n \Im r_n - p^*\|_\infty \\
\leq& (1-\varpi_n)\|\Im q_n - \Im p^*\|_\infty + \varpi_n\|\Im r_n - p^*\|_\infty \\
=& (1-\varpi_n)\max_{x\in[x_0-\lambda,b]}|\Im q_n - \Im p^*| + \varpi_n \max_{x\in[x_0-\lambda,b]}|\Im r_n - \Im p^*| \\
=& (1-\varpi_n)\max_{x\in[x_0-\lambda,b]}\Big|\aleph(x_0) + \int_{x_0}^{x} f(s, q_n(s), q_n(s-\lambda))ds \\
& - \aleph(x_0) - \int_{x_0}^{x} f(s, p^*(s), p^*(s-\lambda))ds\Big| \\
& + \varpi_n \max_{x\in[x_0-\lambda,b]}\Big|\aleph(x_0) + \int_{x_0}^{x} f(s, r_n(s) r_n(s-\lambda))ds \\
& - \aleph(x_0) - \int_{x_0}^{x} f(s, p^*(s), p^*(s-\lambda))ds\Big| \\
=& (1-\varpi_n)\max_{x\in[x_0-\lambda,b]}\Big|\int_{x_0}^{x} f(s, q_n(s), q_n(s-\lambda))ds - \int_{x_0}^{x} f(s, p^*(s), p^*(s-\lambda))ds\Big| \\
& + \varpi_n \max_{x\in[x_0-\lambda,b]}\Big|\int_{x_0}^{x} f(s, r_n(s), r_n(s-\lambda))ds - \int_{x_0}^{x} f(s, p^*(s), p^*(s-\lambda))ds\Big| \\
\leq& (1-\varpi_n)\max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x}\Big|f(s, q_n(s), q_n(s-\lambda)) - f(s, p^*(s), p^*(s-\lambda))\Big|ds \\
& + \varpi_n \max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x}\Big|f(s, r_n(s), r_n(s-\lambda)) - f(s, p^*(s), p^*(s-\lambda))\Big|ds \\
\leq& (1-\varpi_n)\max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x} L_f\Big(|q_n(s) - p^*(s)| + |q_n(s-\lambda) - p^*(s-\lambda)|\Big)ds \\
& + \varpi_n \max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x} L_f\Big(|r_n(s) - p^*(s)| + |r_n(s-\lambda) - p^*(s-\lambda)|\Big)ds \\
\leq& (1-\varpi_n)\max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x} L_f\Big(|q_n - p^*| + |q_n - p^*|\Big)ds \\
& + \varpi_n \max_{x\in[x_0-\lambda,b]}\int_{x_0}^{x} L_f\Big(|r_n - p*| + |r_n - p^*|\Big)ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq (1-\varpi_n)\int_{x_0}^{x} L_f\Big(\max_{x\in[x_0-\lambda,b]}|q_n-p^*| + \max_{x\in[x_0-\lambda,b]}|q_n-p^*|\Big)ds \\
&\quad + \varpi_n\int_{x_0}^{x} L_f\Big(\max_{x\in[x_0-\lambda,b]}|r_n-p^*| + \max_{x\in[x_0-\lambda,b]}|r_n-p^*|\Big)ds \\
&= (1-\varpi_n)\int_{x_0}^{x} L_f\Big(\|q_n-p^*\|_\infty + \|q_n-p^*\|_\infty\Big)ds \\
&\quad + \varpi_n\int_{x_0}^{x} L_f\Big(\|r_n-p^*\|_\infty + \|r_n-p^*\|_\infty\Big)ds \\
&\leq (1-\varpi_n)\int_{x_0}^{x} 2L_f\|q_n-p^*\|_\infty ds + \varpi_n\int_{x_0}^{x} 2L_f\|r_n-p^*\|_\infty ds \\
&\leq 2(1-\varpi_n)L_f(b-x_0)\|q_n-p^*\|_\infty + 2\varpi_n L_f(b-x_0)\|r_n-p^*\|_\infty \\
&\leq 2L_f(b-x_0)\Big\{(1-\varpi_n)\Big([1-(1-2L_f(b-x_0))\xi_n]\|u_n-p^*\|_\infty\Big) \\
&\quad + \varpi_n\Big(2L_f(b-x_0)[1-(1-2L_f(b-x_0))\xi_n]\|u_n-p^*\|_\infty\Big)\Big\} \\
&\leq 2L_f(b-x_0)[1-(1-2L_f(b-x_0))\xi_n]\|u_n-p^*\|_\infty\{(1-\varpi_n)+\varpi_n 2L_f(b-x_0)\} \\
&= 2L_f(b-x_0)[1-(1-2L_f(b-x_0)\varpi_n][1-(1-2L_f(b-x_0))\xi_n]\|u_n-p^*\|_\infty.
\end{aligned}
$$

It is convenient to state that $[1-(1-2L_f(b-x_0))\xi_n] < 1$, so that by $(\mathscr{D}_5)$, we have that

$$
\|u_{n+1}-p^*\| \leq [1-(1-2L_f(b-x_0))\varpi_n]\|u_n-p^*\|_\infty. \tag{4.5}
$$

Let $\varphi_n := (1-2L_f(b-x_0))\varpi_n < 1$, for $\varpi_n \in (0,1)$ such that $\sum_{n=0}^{\infty}\varpi_n = \infty$ and $\eta_n := \|u_n-p^*\|_\infty$.

Hence, (4.5) becomes

$$
\eta_{n+1} \leq (1-\varphi_n)\eta_n.
$$

Clearly, from Lemma 2.5, we have that $\lim_{n\to\infty}\|u_n-p^*\| = 0$, thereby completing the proof. □

Obviously, our result extends other existing results (see, for example, [10, 20, 24] and references therein).

5. Applications to third order boundary value problems

5.1. Construction of Green's functions

Here, we consider the construction of Green's functions for the following third order boundary value problems (BVP):

$$
L[h] \equiv g_1(t)h'''(t) + g_2(t)h''(t) + g_3(t)h'(t) + g_4(t)h(t) = \Phi(t), \tag{5.1}
$$

where $t \in [a,b]$, accompanied by the following boundary conditions (BCs):

$$
\begin{aligned}
B_\alpha[h] &= \varphi_1 h(\alpha) + \varphi_2 h'(\alpha) + \varphi_3 h''(\alpha) = \varphi, \\
B_\beta[h] &= \lambda_1 h(\beta) + \lambda_2 h'(\beta) + \lambda_3 h''(\beta) = \lambda, \\
B_\gamma[h] &= \psi_1 h(\gamma) + \psi_2 h'(\gamma) + \psi_3 h''(\gamma) = \psi,
\end{aligned} \tag{5.2}
$$

for $\gamma = \alpha$ or $\gamma = \beta$.

$L[h]$ is linear, the righthand side of (5.1) can be written compactly as $\Phi(t, h(t), h'(t), h''(t))$, which may be linear or nonlinear, and φ, λ, ψ are constants. The homogeneous part $L[h] = 0$ of (5.1) can be solved to obtain corresponding linearly independent complementary solutions: h_1, h_2, and h_3.

The Green function is a piecewise function expressed as a linear combination of the complementary solutions h_1, h_2, and h_3. Thus,

$$G(t,s) = \begin{cases} a_1h_1 + a_2h_2 + a_3h_3, & a < t < s, \\ b_1h_1 + b_2h_2 + b_3h_3, & s < t < b, \end{cases} \tag{5.3}$$

where a_j, b_j $(j = 1, 2, 3)$ are constants that can determined through the following axioms:

(A_1) G satisfies the associated boundary conditions:

$$B_\alpha[G(t,s)] = B_\beta[G(t,s)] = B_\gamma[G(t,s)] = 0.$$

(A_2) G is continuous at $t = s$:

$$a_1h_1(s) + a_2h_2(s) + a_3h_3(s) = b_1h_1(s) + b_2h_2(s) + b_3h_3(s).$$

(A_3) G' is continuous at $t = s$:

$$a_1h_1'(s) + a_2h_2'(s) + a_3h_3'(s) = b_1h_1'(s) + b_2h_2'(s) + b_3h_3'(s).$$

(A_4) G'' has jump discontinuity at $t = s$:

$$a_1h_1''(s) + a_2h_2''(s) + a_3h_3''(s) + \frac{1}{k(s)} = b_1h_1''(s) + b_2h_2''(s) + b_3h_3''(s).$$

The Green function represents the solution of the boundary value problem that takes the form

$$-L[G(t,s)] = \delta(t-s), \tag{5.4}$$

where δ is the Kronecker Delta that is subject to the homogeneous boundary conditions

$$B_\alpha[G(t,s)] = B_\beta[G(t,s)] = B_\gamma[G(t,s)] = 0.$$

Notably, for operators that are self-adjoint, the righthand side of (5.4) will be $-\delta(t-s)$. The Green function expressed in (5.3) can be obtained from the homogeneous part in the form $L[G(t,s)] = 0$ for $t \neq s$.

5.2. Modified-JK-Green iterative scheme

In this section, we will embed the Green function in the Modified-JK fixed point iterative scheme (2.6). To achieve this, we consider the nonlinear boundary value problem:

$$L[h] + N[h] = \Phi(t, h), \tag{5.5}$$

where $L[h]$ and $N[h]$ are, respectively, linear and nonlinear functions in h, and $\Phi(t,h)$ is a function in h that may be linear or nonlinear.

Suppose that h_p is a particular solution of the nonhomogeneous part of (5.5). We can define an integral operator in terms of the Green function, and the particular solution h_p,

$$\mathscr{M}[h_p]=\int_a^b G(t,s)L[h_p]ds, \tag{5.6}$$

where G is the Green function corresponding to the linear differential operator $L[h]$. Setting $h_p=\Omega$, (5.6) becomes

$$\mathscr{M}(\Omega)=\int_a^b G(t,s)L[\Omega]ds. \tag{5.7}$$

Clearly, Ω is a fixed point if, and only if, Ω is a solution of (5.5). Rewriting (5.7), we have

$$\begin{aligned}\mathscr{M}(\Omega)&=\int_a^b G(t,s)[L[\Omega]+N[\Omega]-\Phi(t,\Omega)-N[\Omega]+\Phi(t,\Omega)]ds\\&\leq\int_a^b G(t,s)[L[\Omega]+N[\Omega]-\Phi(t,\Omega)]ds\\&\quad+\int_a^b G(t,s)[\Phi(t,\Omega)-N[\Omega]]ds\\&=\Omega+\int_a^b G(t,s)[L[\Omega]+N[\Omega]-\Phi(t,\Omega)]ds,\end{aligned} \tag{5.8}$$

where $h_p=\Omega=\int_a^b G(t,s)[\Phi(t,\Omega)-N[\Omega]]ds$. Applying the Modified-JK iterative scheme (2.6), we have

$$\begin{cases}w_n=\mathscr{M}[(1-\xi_n)u_n+\xi_n\mathscr{M}[u_n]],\\v_n=\mathscr{M}^2[w_n],\\u_{n+1}=\mathscr{M}[(1-\varpi_n)\mathscr{M}[w_n]+\varpi_n\mathscr{M}[v_n]],\end{cases}$$

where $\{\xi_n\}$ and $\{\varpi_n\}$ are real sequences in $[0,1]$ for all $n\in\mathbb{N}$.

We have

$$\begin{aligned}w_n=&\Big[(1-\xi_n)u_n+\xi_n\{u_n+\int_a^b G(t,s)(L[u_n]+N[u_n]-\Phi(t,u_n))ds\}\Big]\\&+\int_a^b G(t,s)\Big(L[(1-\xi_n)u_n+\xi_n\{u_n+\int_a^b G(t,s)(L[u_n]+N[u_n]-\Phi(t,u_n))ds\}]\\&+N[(1-\xi_n)u_n+\xi_n\{u_n+\int_a^b G(t,s)(L[u_n]+N[u_n]-\Phi(s,u_n))ds\}]\\&-\Phi(s,[(1-\xi_n)u_n+\xi_n\{u_n+\int_a^b G(t,s)(L[u_n]+N[u_n]-\Phi(s,u_n))ds\}])\Big)ds,\\v_n=&\mathscr{M}[w_n]+\int_a^b G(t,s)\Big(L[\mathscr{M}(w_n)]+N[\mathscr{M}[w_n]]-\Phi(s,\mathscr{M}[w_n])\Big)ds,\\u_{n+1}=&[(1-\varpi_n)\mathscr{M}[w_n]+\varpi_n\mathscr{M}[v_n]]+\int_a^b G(t,s)\Big(L[(1-\varpi_n)\mathscr{M}[w_n]+\varpi_n\mathscr{M}[v_n]]\end{aligned}$$

$$+ N[(1-\varpi_n)\mathscr{M}[w_n] + \varpi_n \mathscr{M}[v_n]] - \Phi(s, [(1-\varpi_n)\mathscr{M}[w_n] + \xi_n \mathscr{M}[v_n]])\Big)ds.$$

Furthermore,

$$\begin{aligned}
w_n =& \Big[u_n + \xi_n \int_a^b G(t,s)\Big(L[u_n] + N[u_n] - \Phi(t,u_n)\Big)ds\Big] \\
&+ \int_a^b G(t,s)\Big(L[u_n + \xi_n \int_a^b G(t,s)(L[u_n] + N[u_n] - \Phi(t,u_n))ds] \\
&+ N[u_n + \xi_n \int_a^b G(t,s)(L[u_n] + N[u_n] - \Phi(s,u_n))ds] \\
&- \Phi(s, [u_n + \xi_n \int_a^b G(t,s)(L[u_n] + N[u_n] - \Phi(s,u_n))ds])\Big)ds, \\
v_n =& \Big[w_n + \int_a^b G(t,s)(lw_n + N[w_n] - \Phi(s,w_n))ds\Big] \\
&+ \int_a^b G(t,s)\Big(L[w_n + \int_a^b G(t,s)(L[w_n] + N[w_n] - \Phi(s,w_n))ds] \\
&+ N[w_n + \int_a^b G(t,s)(L[w_n] + N[w_n] - \Phi(s,w_n))ds] \\
&- \Phi(s, [w_n + \int_a^b G(t,s)(L[w_n] + N[w_n] - \Phi(s,w_n))ds])\Big)ds, \\
u_{n+1} =& \Big[(1-\varpi_n)[w_n + \int_a^b G(t,s)(L[w_n] + N[w_n] - \Phi(s,w_n))ds] \\
&+ \varpi_n[v_n + \int_a^b G(t,s)(L[v_n] + N[v_n] - \Phi(s,v_n))ds]\Big] \\
&+ \int_a^b G(t,s)\Big(L\Big[(1-\varpi_n)[w_n + \int_a^b G(t,s)(L[w_n] + N[w_n] - \Phi(s,w_n))ds] \\
&+ \varpi_n[v_n + \int_a^b G(t,s)(L[v_n] + N[v_n] - \Phi(s,v_n))ds]\Big] \\
&+ N\Big[(1-\varpi_n)[w_n + \int_a^b G(t,s)(L[w_n] + N[w_n] - \Psi(t,w_n))ds] \\
&+ \varpi_n[v_n + \int_a^b G(t,s)(L[v_n] + N[v_n] - \Phi(s,w_n))ds]\Big] \\
&- \Phi\Big(s, [(1-\xi_n)[w_n + \int_a^b G(t,s)(L[w_n] + N[w_n] - \Phi(s,w_n))ds]] \\
&+ \varpi_n[v_n + \int_a^b G(t,s)(L[v_n] + N[v_n] - \Psi(s,v_n))ds]\Big)ds\Big)ds. \qquad (5.9)
\end{aligned}$$

5.3. Convergence analysis result

In this section, our aim is to show the convergence analysis of our iterative scheme (2.6) for a BVP using the Green function. Without loss of generality, we consider the BVP:

$$-h'''(t) = \Phi(t, h(t), h'(t), h''(t)) \qquad (5.10)$$

with accompanying BCs

$$h(1) = R, h''(1) = S, h(2) = T.$$

By solving the homogeneous part $h'''(t) = 0$, the following Green function is imminent,

$$G(t,s) = \begin{cases} a_1t^2 + a_2t + a_3, 1 \le t \le s \le 2, \\ b_1t^2 + b_2t + b_3, 1 \le s \le t \le 2. \end{cases}$$

By Axioms (A_1)–(A_4), the constants a_1, a_2, a_3, b_1, b_2, and b_3 can be obtained. After obtaining the values of the constants, the Green function becomes:

$$G(t,s) = \begin{cases} \frac{-1}{2}s^2 + 2s - 2 + (\frac{1}{2}s^2 - 2s + 2)t, & 1 \le t \le s \le 2, \\ -s^2 + 2s - 2 + (\frac{1}{2}s^2 - 2s + 2)t - \frac{1}{2}t^2, & 1 \le s \le t \le 2. \end{cases}$$

Hence, the Modified-JK-Green iterative scheme (5.9) is given as:

$$\begin{cases} w_n = \Im_G[(1-\xi_n)u_n + \xi_n \Im_G u_n], \\ v_n = \Im_G^2 w_n, \\ u_{n+1} = \Im_G[(1-\varpi_n)\Im_G w_n + \varpi_n \Im_G v_n], \; n \in \mathbb{N}, \end{cases} \tag{5.11}$$

where $\Im_G : C^2([1,2]) \to C^2([1,2])$ is an operator defined by

$$\Im_G(u) = u + \int_1^2 G(t,s)(u''' - \Phi(s,u,u',u''))ds \tag{5.12}$$

and the initial iterate u_0 satisfies the homogeneous equation $u_0''' = 0$ and the boundary conditions $u_0(1) = R$, $u_0''(1) = S$, and $u_0(2) = T$.

Again, if we use integration by part for $\int_1^2 G(t,s)u'''ds$, as expressed in (5.12), with the condition that $\int_1^2 \frac{\partial^3 G(t,s)}{\partial s^3} h(s)ds = \int_1^2 \delta(t-s)h(s)ds$, then we have that

$$\Im_G(u) = (2-t)R + \frac{1}{2}(t^2 - 3t + 2)S + (t-1)T - \int_1^2 G(t,s)\Phi(s,u,u',u'')ds. \tag{5.13}$$

The next result is to show that the operator $\Im_G$ is a contraction on the Banach space $C^2([1,2])$ with regards to the norm

$$\|u\|_{C^2} = \sum_{j=0}^{2} \sup_{[1,2]} |u^{(j)}(s)|$$

under some weakened conditions on Φ.

Furthermore, we shall show that under certain hypotheses on Φ, $\Im_G$ is a Zamfirescu operator.

Theorem 5.1. *Suppose Φ, which appears in the expression of $\Im_G$, satisfying a Lipschitz condition of the form:*

$$|\Phi(s,u,u',u'') - \Phi(s,g,g',g'')| \le \wp_1|u(s) - g(s)| + \wp_2|u'(s) - g'(s)| + \wp_3|u''(s) - g''(s)|, \tag{5.14}$$

where $\wp_1, \wp_2$, and $\wp_3$ are positive constants such that

$$\frac{1}{8}\max\{\wp_1, \wp_2, \wp_3\} \leq 1.$$

The operator $\Im_G$ is a contraction on the Banach space $C^2([1,2], \|\cdot\|_{C^2})$, and the sequence $\{u_n\}$ defined by the Modified-JK-Green iterative scheme (2.6) converges to the fixed point of $\Im_G$.

Proof. Suppose $u_1, u_2 \in C^2([1,2])$ so that by the Lipschitz condition (5.13), we have

$$\begin{aligned}
|\Im_G(u_1) - \Im_G(u_2)| =& \Big|\int_1^2 G(t,s)\Phi(s,u_1,u_1',u_1'')ds - \int_1^2 G(t,s)\Phi(s,u_2,u_2',u_2'')ds\Big| \\
=& \Big|\int_1^2 G(t,s)\Big(\Phi(s,u,u_1',u_1'') - \Phi(s,u_2,u_2',u_2'')\Big)ds\Big| \\
\leq& \int_1^2 |G(t,s)||\Phi(s,u_1,u_1',u_1'') - \Phi(s,u_2,u_2',u_2'')|ds \\
\leq& \Big(\sup_{[1,2]\times[1,2]} |G(t,s)|\Big)\int_1^2 |\Phi(s,u_1,u_1',u_1'') - \Phi(s,u_2,u_2',u_2'')|ds \\
=& G(\frac{3}{4},1)\int_1^2 |\Phi(s,u_1,u_1',u_1'') - \Phi(s,u_2,u_2',u_2'')|ds \\
=& \frac{1}{8}\int_1^2 |\Phi(s,u_1,u_1',u_1'') - \Phi(s,u_2,u_2',u_2'')|ds \\
\leq& \frac{1}{8}\int_1^2 \Big\{\wp_1|u_1(s) - u_2(s)| + \wp_2|u_1'(s) - u_2'(s)| + \wp_3|u_1''(s) - u_2''(s)|\Big\}ds \\
\leq& \frac{1}{8}\max\{\wp_1,\wp_2,\wp_3\}\int_1^2 \Big(\sum_{j=0}^{2} |u_1^{(j)}(s) - u_2^{(j)}(s)|\Big)ds \\
\leq& \frac{1}{8}\max\{\wp_1,\wp_2,\wp_3\}\|u_1 - u_2\|_{C^2} \\
<& \|u_1 - u_2\|_{C^2}.
\end{aligned} \tag{5.15}$$

This follows that $\Im_G$ is a contraction.

On the other hand, we want to show that the sequence $\{u_n\}$ defined by the Modified-JK-Green iterative scheme (5.9) converges strongly to the fixed point of the operator $\Im_G$. Since $\Im_G$ is a contraction, it is obvious from the Banach contraction principle that the existence of a unique fixed point, p^* of $\Im_G$ in the Banach space $C^2([1,2], \|\cdot\|_{C^2})$ is guaranteed. That is, we shall show that $\lim_{n\to\infty} \|u_n - p^*\| = 0$. Using iterative scheme (5.11), we have

$$\begin{aligned}
\|w_n - p^*\| =& \|\Im_G[(1-\xi_n)u_n + \xi_n\Im_G u_n] - p^*\| \\
\leq& \delta\|(1-\xi_n)u_n + \xi_n\Im_G u_n - p^*\| \\
\leq& \delta(1-\xi_n)\|u_n - p^*\| + \delta^2\xi_n\|u_n - p^*\| \\
=& \delta[1 - (1-\delta)\xi_n]\|u_n - p^*\|.
\end{aligned} \tag{5.16}$$

Again, using (5.11) and (5.16),

$$\begin{aligned}\|v_n - p^*\| =&\|\Im_G^2 w_n - p^*\| \\ \leq&\delta\|\Im_G w_n - p^*\| \\ \leq&\delta^2\|w_n - p^*\| \\ =&\delta^3[1-(1-\delta)\varpi_n]\|u_n - p^*\|.\end{aligned} \tag{5.17}$$

Next, using (5.11) and (5.17),

$$\begin{aligned}\|u_{n+1} - p^*\| =&\|\Im_G[(1-\varpi_n)\Im_G w_n + \varpi_n \Im_G v_n] - p^*\| \\ \leq&\delta\|(1-\varpi_n)\Im_G w_n + \varpi_n \Im_G v_n - p^*\| \\ \leq&\delta(1-\xi_n)\|\Im_G w_n - p^*\| + \delta\xi_n\|\Im_G v_n - p^*\| \\ \leq&\delta^2(1-\varpi_n)\|w_n - p^*\| + \delta^2\varpi_n\|v_n - p^*\| \\ \leq&\delta^2(1-\varpi_n)\|w_n - p^*\| + \delta^4\varpi_n\|w_n - p^*\| \\ =&[\delta^2(1-\varpi_n) + \delta^4\varpi_n]\|w_n - p^*\| \\ =&\delta^2[(1-\varpi_n) + \delta^2\varpi_n]\|w_n - p^*\| \\ =&\delta^2[1-(1-\delta^2)\varpi_n]\|w_n - p^*\|.\end{aligned} \tag{5.18}$$

Combining (5.16) and (5.18), we have

$$\|u_{n+1} - p^*\| \leq \delta^3[1-(1-\delta)\xi_n][1-(1-\delta^2)\varpi_n]\|u_n - p^*\|.$$

Since $\delta \in [0,1)$ and $\varpi_n, \xi_n \in [o,1]$, then $[1-(1-\delta^2)\varpi_n] < 1$. This implies that

$$\|u_{n+1} - p^*\| \leq \delta^3[1-(1-\delta)\xi_n]\|u_n - p^*\|.$$

By induction, we have

$$\|u_{n+1} - p^*\| \leq \delta^{3(n+1)}[1-(1-\delta)\xi_n]\|u_0 - p^*\|$$

$$\|u_{n+1} - p^*\| \leq \delta^{3(n+1)}\|u_0 - p\|\prod_{k=0}^{n}[1-(1-\delta)\xi_k].$$

Recalling that $\delta \in [0,1)$, $\varpi_n \in [0,1]$ for all $n \in \mathbb{N}$, and from elementary analysis $1 - x \leq e^{-x}$ for $x \in (0,1)$, we have that

$$\begin{aligned}\|u_{n+1} - p^*\| \leq&\delta^{3(n+1)}\|u_0 - p^*\|^{n+1}\prod_{k=0}^{n} e^{-(1-\delta)\xi_k} \\ \leq&\delta^{3(n+1)}\|u_0 - p^*\|^{n+1} e^{-(1-\delta)\sum_{k=0}^{\infty}\xi_k}.\end{aligned}$$

Clearly, if $\sum_{k=0}^{\infty}\xi_k = \infty$ such that $e^{-(1-\delta)\sum_{k=0}^{\infty}\xi_k} \to 0$ as $n \to \infty$, then $\lim_{n\to\infty}\|u_n - p^*\| = 0$.

Hence, the proof is complete. □

5.4. Numerical example

Example 2. *Consider the BVP*

$$\begin{cases} y'''(t) + y(t)y''(t) - (y'(t))^2 + 1 = 0, \\ y(0) = y'(0) = y(1) = 0. \end{cases} \tag{5.19}$$

The corresponding Green function to Eq (5.19) is given:

$$G(t,s) = \begin{cases} (\frac{-1}{2}s^2 + s)t^2 - st + \frac{s^2}{2}, & 0 < s < t, \\ (\frac{-s^2}{2} + s - \frac{1}{2})t^2, & t < s < 1. \end{cases} \tag{5.20}$$

Embedding (5.19) and (5.20) in the iterative scheme (5.9), we have

$$\begin{aligned} w_n =& \Big[u_n + \xi_n \int_0^t \Big((\frac{-1}{2}s^2 + s)t^2 - st + \frac{s^2}{2} \Big) \Big(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1 \Big) ds \\ &+ \xi_n \int_t^1 \Big((\frac{-s^2}{2} + s - \frac{1}{2})t^2 \Big) \Big(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1 \Big) ds \Big] \\ &+ \int_0^t \Big((\frac{-1}{2}s^2 + s)t^2 - st + \frac{s^2}{2} \Big) \\ &\times \Bigg(\Big(u_n + \xi_n \int_0^1 G(t,s)(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1) ds \Big)''' \\ &+ \Big((u_n + \xi_n \int_0^1 G(t,s)(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1) ds \Big) \\ &\times \Big((u_n + \xi_n \int_0^1 G(t,s)(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1) ds \Big)'' \\ &- \Big(\Big((u_n + \xi_n \int_0^1 G(t,s)(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1) ds \Big)' \Big)^2 + 1 \Bigg) ds \\ &+ \int_t^1 (\frac{-s^2}{2} + s - \frac{1}{2})t^2 \Bigg(\Big(u_n + \xi_n \int_0^1 G(t,s)(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1) ds \Big)''' \\ &+ \Big((u_n + \xi_n \int_0^1 G(t,s)(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1) ds \Big) \\ &\times \Big((u_n + \xi_n \int_0^1 G(t,s)(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1) ds \Big)'' \\ &- \Big(\Big((u_n + \xi_n \int_0^1 G(t,s)(u_n'''(s) + u_n(s)u_n''(s) - (u_n'(s))^2 + 1) ds \Big)' \Big)^2 + 1 \Bigg) ds, \\ v_n =& \Big[w_n + \int_0^t \Big((\frac{-1}{2}s^2 + s)t^2 - st + \frac{s^2}{2} \Big) \Big(w_n'''(s) + w_n(s)w_n''(s) - (w_n'(s))^2 + 1 \Big) ds \\ &+ \int_t^1 \Big((\frac{-s^2}{2} + s - \frac{1}{2})t^2 \Big) \Big(w_n'''(s) + w_n(s)w_n''(s) - (w_n'(s))^2 + 1 \Big) ds \Big] \\ &+ \int_0^t \Big((\frac{-1}{2}s^2 + s)t^2 - st + \frac{s^2}{2} \Big) \end{aligned}$$

$$
\begin{aligned}
&\times\Big(\Big(w_n+\int_0^1 G(t,s)(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1)\Big)'''\\
&+\Big(w_n+\int_0^1 G(t,s)(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1)\Big)\\
&\times\Big(w_n+\int_0^1 G(t,s)(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1)\Big)''\\
&-\Big(\Big(w_n+\int_0^1 G(t,s)(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1)\Big)'\Big)^2+1\Big)ds\\
&+\int_t^1\Big(\Big(\frac{-s^2}{2}+s-\frac{1}{2}\Big)t^2\Big)\Big(\Big(w_n+\int_0^1 G(t,s)(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1)\Big)'''\\
&+\Big(w_n+\int_0^1 G(t,s)(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1)\Big)\\
&\times\Big(w_n+\int_0^1 G(t,s)(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1)\Big)''\\
&-\Big(\Big(w_n+\int_0^1 G(t,s)(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1)\Big)'\Big)^2+1\Big)ds,\\
u_{n+1}=&\Big[(1-\varpi_n)\Big[w_n+\int_0^t\Big(\Big(\frac{-1}{2}s^2+s\Big)t^2-st+\frac{s^2}{2}\Big)\\
&\times\Big(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1\Big)ds\\
&+\int_t^1\Big(\Big(\frac{-s^2}{2}+s-\frac{1}{2}\Big)t^2\Big)\Big(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1\Big)ds\Big]\\
&+\varpi_n\Big[v_n+\int_0^t\Big(\Big(\frac{-1}{2}s^2+s\Big)t^2-st+\frac{s^2}{2}\Big)\Big(v_n'''(s)+v_n(s)v_n''(s)-(v_n'(s))^2+1\Big)ds\\
&+\int_t^1\Big(\Big(\frac{-s^2}{2}+s-\frac{1}{2}\Big)t^2\Big)\Big(v_n'''(s)+v_n(s)v_n''(s)-(v_n'(s))^2+1\Big)ds\Big]\Big]\\
&+\int_0^t\Big(\Big(\frac{-1}{2}s^2+s\Big)t^2-st+\frac{s^2}{2}\Big)\\
&\times\Big(\Big((1-\varpi_n)\Big[w_n+\int_0^1 G(t,s)\Big(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1\Big)ds\Big]\\
&+\varpi_n\Big[v_n+\int_0^1 G(t,s)\Big(v_n'''(s)+v_n(s)v_n''(s)-(v_n'(s))^2+1\Big)ds\Big]\Big)'''\\
&+\Big((1-\varpi_n)\Big[w_n+\int_0^1 G(t,s)\Big(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1\Big)ds\Big]\\
&+\varpi_n\Big[v_n+\int_0^1 G(t,s)\Big(v_n'''(s)+v_n(s)v_n''(s)-(v_n'(s))^2+1\Big)ds\Big]\Big)\\
&\times\Big((1-\xi_n)\Big[w_n+\int_0^1 G(t,s)\Big(w_n'''(s)+w_n(s)w_n''(s)-(w_n'(s))^2+1\Big)ds\Big]
\end{aligned}
$$

$$+ \varpi_n\Big[v_n + \int_0^1 G(t,s)\Big(v_n'''(s) + v_n(s)v_n''(s) - (v_n'(s))^2 + 1\Big)ds\Big]\Big)''$$
$$- \Big(\Big((1-\varpi_n)\Big[w_n + \int_0^1 G(t,s)\Big(w_n'''(s) + w_n(s)w_n''(s) - (w_n'(s))^2 + 1\Big)ds\Big]$$
$$+ \varpi_n\Big[v_n + \int_0^1 G(t,s)\Big(v_n'''(s) + v_n(s)v_n''(s) - (v_n'(s))^2 + 1\Big)ds\Big]\Big)'\Big)^2 + 1\Big)ds.$$

With the right choice of $\varpi_n, \xi_n \in [0,1]$ and by considering the minimization of the L^2-norm of the residual error, a better computation is achieved. The initial iterate $u_0 = 0$ satisfies the homogeneous equation $u''' = L[u] = 0$ and the accompanying boundary conditions.

Our iterative scheme, Modified-JK-Green (5.9) converges faster than Picard-Green [25, 26], Mann-Green [27], Khan-Green [28], and Ishikawa-Green [29] iterative schemes.

6. Conclusions

We have been able to show that our new iterative scheme (2.6) converges faster than all of JK, Mann, Picard-Ishikawa, GA, Noor, and CR iterative schemes for Example 1, as shown in Tables 1 and 2 and Figures 1 and 2, which indicates that the mapping converges to a fixed point $5 \in \mathscr{F}(\mathfrak{J})$. A result on using our scheme to find the solution of a delay differential equation was proved in a uniformly convex Banach space. Also, our scheme formulated in terms of the Green function and defined as the Modified-JK-Green iterative scheme was used to approximate the solution of a third order BVP with examples shown in Example 2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interests.

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