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Research article

Hankel determinants, Fekete-Szegö inequality, and estimates of initial coefficients for certain subclasses of analytic functions

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Abstract: In this paper, we define new subclasses of analytic functions related to a modified sigmoid function and analytic univalent function. Then, we attempt to investigate the upper bounds of the third and fourth Hankel determinant in the special case. Further, bound on third Hankel determinant of its inverse function is also investigated. In addition, we attempt to obtain the Fekete-Szegö inequality for the classes. Then, we estimate the bounds of initial coefficients for the function belongs to some kind of new subclasses when its inverse function also belongs to these new subclasses.

Keywords: analytic function; modified sigmoid function; subordination; Hankel determinant; upper bound; Fekete-Szegö inequality; initial coefficients
Mathematics Subject Classification: 30C45, 30C50

1. Introduction and definition

Let S denote the class of univalent functions which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{D}). \tag{1.1}$$

Let P represent a class of analytic functions within the unit disk $\mathbb{D}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}) \tag{1.2}$$

and satisfy the condition of $\Re(p(z)) > 0$. It is easy to know from the conclusion of [1], for $p(z) \in P$, there exists a Schwarz function $w(z)$, making

$$p(z) \in P \Leftrightarrow p(z) = \frac{1 + w(z)}{1 - w(z)}.$$

In 1976, Noonan and Thomas [2] defined the q^{th} Hankel determinant for a function $f \in S$ of form (1.1) as

$$H_{q,n}(f) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

where $a_1 = 1, n \geq 1, q \geq 1$. In particular, we have

$$H_{2,1}(f) = a_3 - a_2^2,$$

$$H_{2,2}(f) = a_2a_4 - a_3^2,$$

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)$$

and

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6\delta_1 + a_5\delta_2 - a_4\delta_3,$$

where

$$\delta_1 = a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2),$$

$$\delta_2 = a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3),$$

$$\delta_3 = a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_2a_4 - a_3^2).$$

Next, we recall the definition of subordination. We assume that f_1 and f_2 are two analytic functions in $\mathbb{D}$. Then, we say that the function f_1 is subordinate to the function f_2, as we write $f_1(z) \prec f_2(z)$, for all $z \in \mathbb{D}$. Then, there exists a Schwarz function $w(z)$ with $w(0) = 0$ and $|w(z)| < 1$ to satisfy

$$f_1(z) = f_2(w(z)).$$

Now, we consider the following class $S^*(g)$ as follows:

$$S^*(g) = \{f \in S : \frac{zf^{'}(z)}{f(z)} \prec g(z)\}, \tag{1.3}$$

where g is an analytic univalent function with positive real part in $\mathbb{D}$, and g maps $\mathbb{D}$ onto a region starlike with respect to $g(0) = 1$, $g^{'}(0) > 0$, and is symmetric about the real axis. The class $S^*(g)$ was introduced by Ma and Minda [3]. If we vary the function g on the right side of (1.3), we will obtain different results. In recent years, many researchers have also conducted a lot of research on this and obtained a series of conclusions. Some of them are as follows:

(1) For $g = \frac{2}{1+e^{-z}}$, which was defined in [4].
(2) For $g = \sqrt{1+z}$, it has been further studied in [5].
(3) For $g = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, it was introduced in [6] and further investigated in [7].
(4) For $g = e^z$, it was defined and studied in [8].
(5) For $g = z + \sqrt{1+z^2}$, the class is denoted by S_l^*, and it was further studied in [9].
(6) For $g = 1 + sinh^{-1}z$, the class $S_p^* = S^*(1 + sinh^{-1}z)$ was studied by Kumar and Arora [10].
(7) For $g = coshz$, the class $S_{cosh}^* = S^*(g(z))$ was introduced by Alotaibi et al. [11].

The Fekete-Szegö inequality is one of the inequalities for the coefficients of univalent analytic functions found by Fekete and Szegö. The Fekete-Szegö inequality of various analytic functions has been studied by many researchers in the last few decades, for example, Huo Tang defined certain class of analytic functions related to the sine function (see [12])

$$f^{'}(\zeta)^{\theta}(\frac{\zeta f^{'}(\zeta)}{f(\zeta)})^{1-\theta} \prec 1 + sin(\zeta); \quad (f \in S, 0 \leq \theta \leq 1)$$

and investigated the upper bound of the second Hankel determinant and the Fekete-Szegö inequality for functions in this class. Many papers have been devoted to researching the Fekete-Szegö inequality for various sub-class functions (see [13,14]). Therefore, the study of the Fekete-Szegö inequality for different analytic functions is valuable and of great significance.

In recent years, many papers have been devoted to finding the upper bounds of Hankel determinants for various sub-classes of analytic functions as well. For the basics and preliminaries, the readers are advised to see the academic achievements in [15–18]. Guangadharan studied a class of bounded turning functions related to the three leaf function in [19]. From this, it can be seen that the research on Hankel determinants of various analytic functions has become popular. Therefore, it is an interesting and hot topic to investigate the Hankel determinants for various classes of analytic functions. In addition, it is worth mentioning that a class of star like functions associated with the modified sigmoid function was defined by Goel and Kumar [20],

$$S_{S_G} = \{f \in S : \frac{zf^{'}(z)}{f(z)} \prec \frac{2}{1+e^{-z}}\}.$$

Apart from the above, the coefficient bounds for certain analytic functions have been studied by many researchers, see [21–25]. Further, many star like functions have been defined and studied as well, see [26–29]. Not long ago, another class of analytic functions associated with the modified sigmoid function was defined and studied by Muhammad Ghaffar Khan [4],

$$R_{S_G} = \{f \in S : f^{'}(z) \prec \frac{2}{1+e^{-z}}\}.$$

It is well known that for each univalent function $f \in S$, there is an inverse function $f^{-1}(w)$ which can be defined in ($|w| < r; r \geq \frac{1}{4}$), where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$

A function $f \in S$ is said to be bi-univalent in $\mathbb{D}$ if there exists a function $g \in S$ such that $g(z)$ is a univalent extension of f^{-1} to $\mathbb{D}$. Brannan [30] studied classes of bi-univalent functions and obtained estimates for their initial coefficients. Many classes of bi-univalent funtions were introduced and further studied in the past few years. Inspired by all the aforementioned works, in this paper we investigate another certain class of analytic functions $H(\lambda, \psi)$, which are related to the modified sigmoid function, and discuss the upper bound of the fourth-order Hankel determinant in special cases, here we use another method to obtain improved results compared to [20]. And we also obtain the upper bound of third-order Hankel determinant of its inverse function. Furthermore, we discuss the Fekete-Szegő inequality for functions in this class when $\lambda \in [0, 1]$ and $\psi = \frac{2}{1+e^{-z}}$. Finally, we estimate the upper bounds of the initial coefficients for functions in this class when $\lambda \in [0, 1]$, $\psi(0) = 1$, and $\psi'(0) > 0$, where its inverse function f^{-1} also belongs to this class.

Definition 1.1. *Assume that* $f \in S$, $0 \le \lambda \le 1$, $(f'(z))^{1-\lambda}$ *and* $(\frac{2zf'(z)}{f(z)-f(-z)})^{\lambda}$ *are analytic in* $\mathbb{D}$ *with* $f'(z) \neq 0$, *and* $f(z) \neq f(-z)$ *for all* $z \in \mathbb{D}\backslash\{0\}$. *Furthermore,* $(f'(z))^{\lambda}$=*1 at* $z = 0$, $\psi(z)$ *is a univalent and analytic function. Then,* $f(z)$ *is said to be in the class* $H(\lambda, \psi)$ *if the following condition is satisfied:*

$$(1-\lambda)(f'(z))^{1-\lambda} + \lambda(\frac{2zf'(z)}{f(z)-f(-z)})^{\lambda} \prec \psi(z).$$

For convenience, we denote

$$H(\lambda) = H(\lambda, \frac{2}{1+e^{-z}}).$$

Remark 1.1. For any $\lambda \in [0, 1]$, we have that $f(z) = z \in H(\lambda)$ always holds.

2. Third Hankel determinant

Below we will evaluate bounds of the first six initial coefficients and non-sharp bound of the third Hankel determinant for functions belonging to $H(1)$.

Theorem 2.1. *Let* $f \in H(1)$ *and be of form (1.1). Then,*

$$|a_2| \le \frac{1}{4}, \tag{2.1}$$

$$|a_3| \le \frac{1}{4}, \tag{2.2}$$

$$|a_4| \le \frac{1}{8}, \tag{2.3}$$

$$|a_5| \le \frac{1}{8}, \tag{2.4}$$

$$|a_6| \le \frac{731}{576}, \tag{2.5}$$

$$|a_7| \le \frac{388937}{241920}. \tag{2.6}$$

The first four inequalities are sharp.

We need the following lemmas to prove the above theorem:

Lemma 2.1. [4] *Let* $p \in P$, *then* $|c_n| \le 2$.

Lemma 2.2. [17] *Let* $p \in P$, *then for all* $n, m \in N$, *if* $0 \le \zeta \le 1$, *there is* $|c_{m+n} - \zeta c_m c_1| \le 2$. *If* $\zeta<0$ *or* $\zeta>1$, *there is* $|c_{m+n} - \zeta c_m c_1| \le 2|2\zeta - 1|$.

Lemma 2.3. [4] *Let* $p \in P$, *then*

$$|\alpha c_1^3 - \beta c_1 c_2 + \gamma c_3| \le 2|\alpha| + 2|\beta - 2\alpha| + 2|\alpha - \beta + \gamma|,$$

where α, β *and* γ *are real numbers.*

Lemma 2.4. [4] *Let* α,β,γ, *and* ζ *satisfy the inequalities* $0 < \gamma < 1, 0 < \beta < 1$, *and*

$$8\beta(1-\beta)[(\gamma\zeta - 2\alpha)^2 + (\gamma(\beta+\gamma) - \zeta)^2] + \gamma(1-\gamma)(\zeta - 2\beta\gamma)^2 \le 4\gamma^2(1-\gamma)^2\beta(1-\beta).$$

If $p \in P$, then

$$|\alpha c_1^4 + \beta c_2^2 + 2\gamma c_1 c_3 - \frac{3}{2}\zeta c_1^2 c_2 - c_4| \le 2.$$

Proof. If $f \in H(1)$, there exists a Schwarz function $w(z)$ to satisfy

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{2}{1 + e^{-w(z)}}.$$

Also, if $p \in P$, it can be written in terms of the Schwarz function $w(z)$ as

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots = \frac{1 + w(z)}{1 - w(z)},$$

or equivalently,

$$w(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2}c_1 z + (\frac{1}{2}c_2 - \frac{1}{4}c_1^2)z^2 + (\frac{1}{8}c_1^3 - \frac{1}{2}c_1 c_2 + \frac{1}{2}c_3)z^3 + \cdots. \tag{2.7}$$

Now, we set

$$\frac{2zf'(z)}{f(z) - f(-z)} = 1 + b_1 z + b_2 z^2 + b_3 z^3 + b_4 z^4 + b_5 z^5 + b_6 z^6 + \cdots = \frac{2}{1 + e^{-w(z)}}. \tag{2.8}$$

In addition,

$$\frac{2zf'(z)}{f(z) - f(-z)} = \frac{1 + 2a_2 z + 3a_3 z^2 + 5a_5 z^4 + 6a_6 z^5 + 7a_7 z^6 + \cdots}{1 + a_3 z^2 + a_5 z^4 + a_7 z^6 + \cdots}. \tag{2.9}$$

Using (2.8) and (2.9), we can get

$$b_1 = 2a_2, \tag{2.10}$$

$$b_2 = 2a_3, \tag{2.11}$$

$$b_3 = 4a_4 - 2a_2 a_3, \tag{2.12}$$

$$b_4 = 4a_5 - 2a_3^2, \tag{2.13}$$

$$b_5 = 6a_6 - 4a_3 a_4 + 2a_2 a_3^2 - 2a_2 a_5, \tag{2.14}$$

$$b_6 = 6a_7 - 4a_3 a_5 + 2a_3^2 - 2a_3 a_5. \tag{2.15}$$

Substituting (2.7) into the right side of (2.8), by simplifying, using (2.10)–(2.15), and comparing the coefficients on both sides of the equation, we can get

$$a_2 = \frac{1}{8}c_1, \tag{2.16}$$

$$a_3 = \frac{1}{2}(\frac{c_2}{4} - \frac{c_1^2}{8}), \tag{2.17}$$

$$a_4 = \frac{1}{4}(\frac{1}{24}c_1^3 - \frac{7}{32}c_1 c_2 + \frac{c_3}{4}), \tag{2.18}$$

$$a_5 = -\frac{1}{16}(\frac{1}{16}c_1^4 - \frac{9}{16}c_1^2 c_2 + c_1 c_3 + \frac{3}{8}c_2^2 - c_4), \tag{2.19}$$

$$a_6=\frac{1}{6}b_5+\frac{2}{3}a_3a_4-\frac{1}{3}a_2a_3^2+\frac{1}{3}a_2a_5, \tag{2.20}$$

$$a_7=\frac{1}{6}b_6+\frac{a_3(3a_5-a_3^2)}{3}+\frac{a_3a_5}{3}, \tag{2.21}$$

where

$$b_5=\frac{1}{3072}\left(\begin{array}{c}-5c_1^6+122c_1^4c_2-288c_1^3c_3-432c_1^2c_2^2+528c_1^2c_4+1056c_1c_2c_3\\ -768c_1c_5+176c_2^3-768c_2c_4-384c_3^2+768c_6\end{array}\right),$$

$$b_6=\frac{1}{5160960}\left(\begin{array}{c}-2537c_1^7-50400c_1^5c_2+204960c_1^4c_3+409920c_1^3c_2^2-483840c_4c_1^3\\ -1451520c_1^2c_2c_3+887040c_1^2c_5-483840c_1c_2^3+1774080c_1c_2c_4+\\ 887040c_1c_3^2-1290240c_1c_6+887040c_2^2c_3-1290240c_2c_5-1290240c_3c_4+1290240c_7\end{array}\right).$$

Applying Lemma 2.1, we have

$$|a_2|\le\frac{1}{4}.$$

The above inequality is sharp with extremal function $f(z)=\int_0^z\frac{2}{1+e^{-t}}dt$.

$$|a_3|=\frac{1}{8}|c_2-\frac{1}{2}c_1^2|\le\frac{1}{4}.$$

The above inequality is sharp for the function $p(z)=(1+z^2)/(1-z^2)$.
Applying Lemma 2.3, we have

$$|a_4|\le\frac{1}{4}[2|\frac{1}{24}|+2|\frac{7}{32}-\frac{1}{12}|+2|\frac{1}{24}-\frac{7}{32}+\frac{1}{4}|]=\frac{1}{8}.$$

The above inequality is sharp with extremal function $f(z)=\int_0^z\frac{2}{1+e^{-t^3}}dt$.
Applying Lemma 2.4, we have

$$|a_5|=|\frac{1}{16}(\frac{1}{16}c_1^4-\frac{9}{16}c_1^2c_3+c_1c_3+\frac{3}{8}c_2^2-c_4)|\le\frac{1}{8}.$$

The above inequality is sharp for the function $p(z)=(1+z^4)/(1-z^4)$.
Applying the triangle inequality, we have

$$|\frac{1}{6}b_5|\le\frac{1}{18432}\left(\begin{array}{c}122|c_1|^4|c_2-\frac{5}{122}c_1^2|+1056|c_1||c_3||c_2-\frac{3}{11}c_1^2|+528|c_1|^2|c_4-\frac{9}{11}c_2^2|\\ +768|c_6-c_1c_5|+768|c_2||c_4-\frac{88}{89}c_2^2|+384|c_3|^2\end{array}\right).$$

By applying Lemmas 2.1 and 2.2, we have

$$|\frac{1}{6}b_5|\le\frac{355}{288},$$

and then applying the triangle inequality and (2.1)–(2.4), we have

$$|\frac{2}{3}a_3a_4-\frac{1}{3}a_2a_3^2+\frac{1}{3}a_2a_5|\le\frac{2}{3}|a_3||a_4|+\frac{1}{3}|a_2||a_3|^2+\frac{1}{3}|a_2||a_5|\le\frac{7}{192},$$

and from (2.20) we can obtain

$$|a_6| \leq \frac{355}{288} + \frac{7}{192} = \frac{731}{576}.$$

By applying triangle inequality, we have

$$|\frac{1}{6}b_6| \leq \frac{1}{30965760}\begin{pmatrix} 20965760|c_1|^4|c_3 - \frac{105}{427}c_1c_2| + 483840|c_1|^3|c_4 - \frac{61}{72}c_1^2| + 1290240|c_1||c_6 - \frac{11}{16}c_1c_5| \\ +1774080|c_1||c_2||c_4 - \frac{9}{11}c_1c_3| + 887040|c_2|^2|c_3 - \frac{6}{11}c_1c_2| + 1290240|c_7 - c_2c_5| \\ +1290240|c_3||c_4 - \frac{11}{16}c_1c_3| + 2537|c_1|^7 \end{pmatrix}.$$

Then, from (2.2) and Lemmas 2.1, 2.2, and 2.4, we have

$$|\frac{1}{6}b_6| \leq \frac{381377}{241920},$$

$$|\frac{a_3(2a_5 - a_3^2)}{3}| = \frac{1}{24}|a_3||\frac{3}{32}c_1^4 + \frac{1}{2}c_2^2 + c_1c_3 - \frac{11}{16}c_1^2c_2 - c_4| \leq \frac{1}{48}.$$

From (2.2) and (2.4), we have

$$|\frac{a_3a_5}{3}| \leq \frac{1}{96}.$$

Then, applying the triangle inequality and (2.21), we can get

$$|a_7| \leq \frac{381377}{241920} + \frac{1}{48} + \frac{1}{96} = \frac{388937}{241920}.$$

This completes our proof.

Theorem 2.2. *If f of the form (1.1) belongs to $H(1)$, then*

$$|a_3 - a_2^2| \leq \frac{1}{4}.$$

The result is sharp for the function $p(z) = (1 + z^2)/(1 - z^2)$.

Proof. Using (2.16), (2.17), and Lemma 2.2, we have

$$|a_3 - a_2^2| = \frac{1}{8}|c_2 - \frac{5}{8}c_1^2| \leq \frac{1}{4}.$$

Theorem 2.3. *If f of the form (1.1) belongs to $H(1)$, then*

$$|a_2a_3 - a_4| \leq \frac{1}{8}.$$

The result is sharp with the extremal function $f(z) = \int_0^z \frac{2}{1+e^{-t^3}}dt$.

Proof. Using (2.16)–(2.18), we can get

$$|a_2a_3 - a_4| = |\frac{7}{384}c_1^3 - \frac{9}{128}c_1c_2 + \frac{1}{16}c_3|.$$

Applying Lemma 2.3,

$$|\frac{7}{384}c_1^3 - \frac{9}{128}c_1c_2 + \frac{1}{16}c_3| \leq 2|\frac{7}{384}| + 2|\frac{9}{128} - \frac{7}{192}| + 2|\frac{7}{384} - \frac{9}{128} + \frac{1}{16}| = \frac{1}{8}.$$

Theorem 2.4. *If f of the form (1.1) belongs to* $H(1)$*, then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{16}.$$

The result is sharp with the extremal function $f(z) = \int_0^z \frac{2}{1+e^{-t^3}}dt$.
Proof. Using (2.16)–(2.18), we can get

$$|a_2a_4 - a_3^2| = \frac{1}{64}|-\frac{1}{6}c_1^4 + \frac{9}{16}c_1^2c_2 + \frac{1}{2}c_1c_3 - c_2^2|.$$

Now, in order to get the desired bound, we shall prove that

$$|-\frac{1}{6}c_1^4 + \frac{9}{16}c_1^2c_2 + \frac{1}{2}c_1c_3 - c_2^2| \leq 4. \tag{2.22}$$

Next we will use the following Lemma:
Lemma 2.5. [17] *Let* $p \in P$*. Then, there exists some* x, y *with* $|x| \leq 1, |y| \leq 1$ *such that*

$$2c_2 = c_1^2 + x(4 - c_1^2),$$

$$4c_3 = c_1^3 + 2c_1x(4 - c_1^2) - (4 - c_1^2)c_1x^2 + 2(4 - c_1^2)(1 - |x|^2)y.$$

Using the invariant property under rotation, we can assume that $c = c_1 \in [0, 2]$, and then from Lemma 2.5, substituting the expression for c_2, c_3 and simplifying, we can obtain

$$-\frac{1}{6}c_1^4 + \frac{9}{16}c_1^2c_2 + \frac{1}{2}c_1c_3 - c_2^2 = -\frac{1}{96}c^4 + \frac{1}{32}c^2(4 - c^2)x - \frac{1}{4}(4 - c^2)(4 - \frac{1}{2}c^2)x^2 + \frac{1}{4}c(4 - c^2)(1 - |x|^2)y.$$

If $c = 0$, there is

$$|-\frac{1}{6}c_1^4 + \frac{9}{16}c_1^2c_2 + \frac{1}{2}c_1c_3 - c_2^2| = 4|x|^2 \leq 4.$$

If $c = 2$, there is

$$|-\frac{1}{6}c_1^4 + \frac{9}{16}c_1^2c_2 + \frac{1}{2}c_1c_3 - c_2^2| = \frac{1}{6}.$$

Next, we will discuss the case of $c \in (0, 2)$. At this time,

$$-\frac{1}{6}c_1^4 + \frac{9}{16}c_1^2c_2 + \frac{1}{2}c_1c_3 - c_2^2 = \frac{1}{4}c(4 - c^2)[px^2 + qx + t + (1 - |x|^2)y],$$

where

$$p = \frac{c^2 - 8}{2c}, \quad q = \frac{c}{8}, \quad t = -\frac{c^3}{24(4 - c^2)},$$

and then we denote

$$I = \frac{1}{4}c(4 - c^2)[px^2 + qx + t + (1 - |x|^2)y],$$

where $p < 0, q > 0$, and $t < 0$ always holds due to the fact that $c \in (0, 2)$. Then, by using the triangle inequality, we have

$$|I| \leq \frac{1}{4}c(4 - c^2)(1 - |x|^2 + |p||x|^2 + |q||x| + |t|) = \frac{1}{4}c(4 - c^2)[-(p + 1)|x|^2 + q|x| - t + 1].$$

Since $\frac{q}{2(p+1)} < 0$ always holds, we can obtain

$$|I| \leq \frac{1}{4}c(4-c^2)(-p+q-t) = \frac{5}{48}c^4 - \frac{11}{8}c^2 + 4 = f(c).$$

By computation, it can be revealed that

$$f(c) < max\{f(0), f(2)\} = 4.$$

In summary, $|I| \leq 4$, that is, (2.22) holds, which evidently yields

$$|-\frac{1}{6}c_1^4 + \frac{9}{16}c_1^2c_2 + \frac{1}{2}c_1c_3 - c_2^2| \leq 4.$$

This completes the proof.

Theorem 2.5. *If f of the form (1.1) belongs to $H(1)$, then*

$$|H_{3,1}(f)| \leq \frac{1}{16}.$$

Proof.

$$H_{3,1}(f) = a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2).$$

By using the triangle inequality, we have

$$|H_{3,1}(f)| \leq |a_3||a_2a_4 - a_3^2| + |a_4||a_4 - a_2a_3| + |a_5||a_3 - a_2^2|.$$

According to Theorem 2.1, we have

$$|a_3| \leq \frac{1}{4}, \quad |a_4| \leq \frac{1}{8}, \quad |a_5| \leq \frac{1}{8}.$$

According to Theorems 2.2–2.4, we have

$$|a_2a_4 - a_3^2| \leq \frac{1}{16}, \quad |a_4 - a_2a_3| \leq \frac{1}{8}, \quad |a_3 - a_2^2| \leq \frac{1}{4}.$$

Therefore,

$$|H_{3,1}(f)| \leq \frac{1}{16}.$$

3. Fourth Hankel determinant

Below we will evaluate the non-sharp bound of the fourth determinant for functions belonging to $H(1)$.

Theorem 3.1. *If f of the form (1.1) belongs to $H(1)$, then*

$$|a_2a_5 - a_3a_4| \leq \frac{1}{16}.$$

Proof. Using (2.16)–(2.19), we can get

$$|a_2a_5 - a_3a_4| = |\frac{1}{6144}c_1^5 - \frac{1}{3072}c_1^3c_2 - \frac{1}{256}c_1^2c_3 + \frac{1}{256}c_1c_2^2 + \frac{1}{128}c_1c_4 - \frac{1}{128}c_2c_3|.$$

Then, by applying the triangle inequality, we have

$$|a_2a_5 - a_3a_4| \leq \frac{1}{3072}|c_1^3(c_2 - \frac{c_1^2}{2})| + \frac{1}{128}|c_2(c_3 - \frac{c_1c_2}{2})| + \frac{1}{128}|c_1(c_4 - \frac{c_1c_3}{2})|.$$

We denote $|c_1| = c$, and from Lemmas 2.1 and 2.2 we can obtain

$$|c_1^3(c_2 - \frac{c_1^2}{2})| \leq c^3(2 - \frac{1}{2}c^2), \quad \frac{1}{128}|c_2(c_3 - \frac{c_1c_2}{2})| \leq \frac{1}{32}, \quad \frac{1}{128}|c_1(c_4 - \frac{c_1c_3}{2})| \leq \frac{c}{64}.$$

Thus,

$$|a_2a_5 - a_3a_4| \leq \frac{1}{3072}c^3(2 - \frac{c^2}{2}) + \frac{c}{64} + \frac{1}{32} = G(c),$$

$$G^{'}(c) = -\frac{5c^4}{6144} + \frac{c^2}{512} + \frac{1}{64} \geq 0 \quad c \in [0, 2].$$

Therefore,

$$G(c) \leq G(2) = \frac{1}{16}.$$

This completes the proof.

Theorem 3.2. *If f of the form (1.1) belongs to* $H(1)$*, then*

$$|a_5 - a_2a_4| \leq \frac{1}{8}.$$

Proof. By using (2.16), (2.18), and (2.19), we have

$$|a_5 - a_2a_4| = \frac{1}{16}|\frac{1}{12}c_1^4 - \frac{43}{64}c_1^2c_2 + \frac{9}{8}c_1c_3 + \frac{3}{8}c_2^2 - c_4|.$$

By applying Lemma 2.4, we can get the sharp result for the function $p(z) = (1 + z^4)/(1 - z^4)$.

Theorem 3.3. *If f of the form (1.1) belongs to* $H(1)$*, then*

$$|a_3a_5 - a_4^2| \leq \frac{689 + 144\sqrt{3}}{9216}.$$

Proof. Using (2.17)–(2.19), we can obtain

$$|a_3a_5 - a_4^2| = |\frac{5}{36864}c_1^6 - \frac{19}{12288}c_1^4c_2 + \frac{1}{384}c_1^3c_3 + \frac{47}{16384}c_1^2c_2^2 - \frac{1}{256}c_1^2c_4 - \frac{1}{1024}c_1c_2c_3 - \frac{3}{1024}c_2^3 + \frac{1}{128}c_2c_4 - \frac{1}{256}c_3^2|.$$

By applying the triangle inequality, we get

$$|a_3a_5 - a_4^2| \leq \frac{5}{36864}|c_1|^6 + \frac{1}{49152}|c_2||76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2| + \frac{1}{384}|c_3||c_1^3 - \frac{3}{2}c_3| + \frac{1}{128}|c_4||c_2 - \frac{1}{2}c_1^2|.$$

In order to get the desired bound, we shall prove that

$$|76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2| \leq 856.$$

Using the invariant property under rotation, we can assume that $c = c_1 \in [0,2]$, and then from Lemma 2.5, substituting the expression for c_2, c_3 and simplifying, we have

$$76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2 = \frac{107}{2}c^4 + \frac{51}{2}c^2(4-c^2)x + 48(4-c^2)(3-c^2)x^2 + 24c(4-c^2)(1-|x|^2)y.$$

If $c = 0$,

$$|76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2| = 576. \tag{3.1}$$

If $c = 2$,

$$|76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2| = 856. \tag{3.2}$$

If $c \neq 0$ and $c \neq 2$, we have

$$76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2 = 24c(4-c^2)[p + qx + tx^2 + (1-|x|^2)y],$$

where

$$p = \frac{107c^3}{48(4-c^2)}, \quad q = \frac{17c}{16}, \quad t = \frac{6-2c^2}{c}.$$

$p, q > 0$ always holds due to the fact that $c \in (0,2)$. Then, by using the triangle inequality, we have

$$|76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2| \leq 24c(4-c^2)(1-|x|^2 + p + q|x| + |t||x|^2).$$

We denote

$$I = 24c(4-c^2)(1-|x|^2 + p + q|x| + |t||x|^2).$$

For suitability, we divide the calculation in five cases:

Case (I). $t \leq 0$ if and only if $\sqrt{3} = c_1 \leq c < 2$. At this time,

$$I = 24c(4-c^2)[-(1+t)|x|^2 + q|x| + p + 1],$$

and when $c_1 \leq c < c_2$, there is $\frac{q}{2(1+t)} < 1$, where $c_2 = \frac{16+8\sqrt{247}}{81}$. Then, we have

$$I \leq 24c(4-c^2)\frac{-4(1+t)p - q^2}{-4(1+t)} + 24c(4-c^2) = \frac{107c^4}{2} + \frac{867}{128}c^4\frac{2+c}{3+2c} + 24c(4-c^2) = f_1(c).$$

By computation, it can be revealed that

$$f_1(c) < 856, \quad c \in [c_1, c_2). \tag{3.3}$$

Case (II). For $c \in [c_2, 2)$, there is $\frac{q}{2(1+t)} \geq 1$, and we then have

$$I \leq 24c(4-c^2)(p + q - t) = -20c^4 + 438c^2 - 576 = f_2(c).$$

By computation, we have

$$f_2(c) < 856, \quad c \in [c_2, 2). \tag{3.4}$$

Case (III). For $c_1 > c > c_3 = \frac{-16+8\sqrt{145}}{47}$, we have

$$I = 24c(4-c^2)[(t-1)|x|^2 + q|x| + p + 1],$$

where $\frac{q}{2(1-t)} < 1$, and we then obtain

$$I \le 24c(4-c^2)\frac{4(t-1)p-q^2}{4(t-1)} + 24c(4-c^2) = \frac{107c^4}{2} - \frac{867}{128}c^4\frac{2-c}{3-2c} + 24c(4-c^2) = f_3(c).$$

Now, computation reveals that

$$f_3(c) < 856, \quad c \in (c_3, c_1). \tag{3.5}$$

Case (IV). For $c_3 \ge c > c_4$, where $c_4 = \frac{3}{2}$, there is $\frac{q}{2(1-t)} > 1$. We can get

$$I \le 24c(4-c^2)(t+p+q) = 76c^4 - 234c^2 + 576 = f_4(c).$$

A computation shows that

$$f_4(c) < 856, c \in (c_4, c_3]. \tag{3.6}$$

Case (V). For the case of $c \in (0, c_4]$, we have

$$I = 24c(4-c^2)[(t-1)|x|^2 + q|x| + p + 1],$$

where $t-1 > 0$ holds for $c \in (0, \frac{3}{2})$. Thus, we have

$$I \le \max\{24c(4-c^2)(p+1), 24c(4-c^2)(p+q+t)\},$$

or, equivalently,

$$I \le \max\{\frac{107c^4}{2} + 24c(4-c^2), 76c^4 - 234c^2 + 576\}.$$

Now, we denote

$$g_1(c) = \frac{107c^4}{2} + 24c(4-c^2), \quad g_2(c) = 76c^4 - 234c^2 + 576$$

$$g_1^{'}(c) = 314c^3 - 72c^2 + 96,$$

$$g_1^{''}(c) = 144c(\frac{471c}{72} - 1).$$

$g_1^{'}(c)$ attains its minimum at $c_0 = \frac{72}{471}$, $g_1^{'}(c_0) > 0$, which evidently yields that $g_1^{'}(c) > 0$ holds for $c \in (0, \frac{3}{2})$. Therefore, $g_1(c) < g_1(\frac{3}{2}) = 333.84375$. On the other hand,

$$g_2(c) < \max\{g_2(0), g_2(\frac{3}{2})\} = 576.$$

Thus,

$$I < 576, \quad c \in (0, c_4]. \tag{3.7}$$

From (3.11)–(3.17), we conclude that $I \le 856$, which implies

$$|76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2| \le 856. \tag{3.8}$$

Next, we will use the following lemma:

Lemma 3.1. *Let $p(z) = 1 + c_1z + c_2z^2 + c_3z^3 + \cdots \in P$. Then, for any real number μ,*

$$|\mu c_3 - c_1^3| \le \begin{cases} 2|\mu - 4| & (\mu \le \frac{4}{3}), \\ 2\mu\sqrt{\frac{\mu}{\mu-1}} & (\frac{4}{3} < \mu). \end{cases}$$

By Lemmas 2.1, 2.2, and 3.1, we can obtain

$$\left|\frac{5}{36864}c_1^6\right| \le \frac{5}{576}, \quad \frac{1}{49152}|c_2||76c_1^4 - 141c_1^2c_2 + 48c_1c_3 + 144c_2^2| \le \frac{107}{3072}. \tag{3.9}$$

$$\frac{1}{384}|c_3||c_1^3 - \frac{3}{2}c_3| \le \frac{\sqrt{3}}{64}, \quad \frac{1}{128}|c_4||c_2 - \frac{1}{2}c_1^2| \le \frac{1}{32}. \tag{3.10}$$

From (3.9) and (3.10), we have

$$|a_3a_5 - a_4^2| \le \frac{689 + 144\sqrt{3}}{9216}.$$

This completes our proof.

Theorem 3.4. *If f of the form (1.1) belongs to $H(1)$, then*

$$|H_{4,1}(f)| \le \frac{215139562}{371589120} + \frac{3\sqrt{3}}{4096}.$$

Proof. We can write $H_{4,1}(f)$ as

$$H_{4,1}(f) = a_7H_{3,1}(f) - a_6\delta_1 + a_5\delta_2 - a_4\delta_3,$$

where

$$\delta_1 = a_3(a_2a_5 - a_3a_4) - a_4(a_5 - a_2a_4) + a_6(a_3 - a_2^2),$$

$$\delta_2 = a_3(a_3a_5 - a_4^2) - a_5(a_5 - a_2a_4) + a_6(a_4 - a_2a_3),$$

$$\delta_3 = a_4(a_3a_5 - a_4^2) - a_5(a_2a_5 - a_3a_4) + a_6(a_2a_4 - a_3^2).$$

By applying Theorems 2.1–2.5, 3.1–3.3, and the triangle inequality, we have

$$|H_{4,1}(f)| \le |a_7||H_{3,1}(f)| + |a_6||\delta_1| + |a_5||\delta_2| + |a_4||\delta_3|, \tag{3.11}$$

$$|a_7||H_{3,1}| \le \frac{388937}{241920} \times \frac{1}{16} = \frac{388937}{3870720}, \tag{3.12}$$

$$|\delta_1| \le |a_3||a_2a_5 - a_3a_4| + |a_4||a_5 - a_2a_4| + |a_6||a_3 - a_2^2| \le \frac{803}{2304}, \tag{3.13}$$

$$|\delta_2| \le |a_3||a_3a_5 - a_4^2| + |a_5||a_5 - a_2a_4| + |a_6||a_4 - a_2a_3| \le \frac{2371}{12288} + \frac{\sqrt{3}}{256}, \tag{3.14}$$

$$|\delta_3| \le |a_4||a_3a_5 - a_4^2| + |a_5||a_2a_5 - a_3a_4| + |a_6||a_2a_4 - a_3^2| \le \frac{2371 + 48\sqrt{3}}{24576}. \tag{3.15}$$

Thus, from (3.11)–(3.15), we obtain

$$|H_{4,1}(f)| \le \frac{215139562}{371589120} + \frac{3\sqrt{3}}{4096}.$$

4. The bound of initial coefficients and third Hankel determinant for f^{-1}

Theorem 4.1. *If the function $f \in H(1)$ given by (1.1) and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the analytic continuation to $\mathbb{D}$ of the inverse function of f with $|w| < r_0$, where $r_0 \geq \frac{1}{4}$ is the radius of the Koebe domain, then*

$$|d_2| \leq \frac{1}{4}, \tag{4.1}$$

$$|d_3| \leq \frac{1}{4}, \tag{4.2}$$

$$|d_4| \leq \frac{65}{384}, \tag{4.3}$$

$$|d_5| \leq \frac{167}{256}. \tag{4.4}$$

The first three inequalities are sharp.
Proof. If

$$f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$$

is the inverse function of f, it can be seen that

$$f^{-1}(f(z)) = f(f^{-1}(z)) = z.$$

Equivalently,

$$\sum_{n=1}^{\infty} d_n (z + \sum_{k=2}^{\infty} d_k w^k)^n = z \quad (d_1 = 1). \tag{4.5}$$

By comparing the coefficients on both sides of (4.5), we can obtain

$$d_2 = -a_2, \tag{4.6}$$

$$d_3 = 2a_2^2 - a_3, \tag{4.7}$$

$$d_4 = -(5a_2^3 - 5a_2 a_3 + a_4), \tag{4.8}$$

$$d_5 = 14a_2^4 - 21a_2^2 a_3 + 6a_2 a_4 + 3a_3^2 - a_5. \tag{4.9}$$

Applying Lemmas 2.1 and 2.2, (2.16), and (2.17), we have

$$|d_2| = |a_2| = |\frac{1}{8}c_1| \leq \frac{1}{4},$$

$$|d_3| = \frac{1}{8}|c_2 - \frac{3}{4}c_1^2| \leq \frac{1}{4}.$$

Applying Lemma 2.3 and (2.16)–(2.18), we have

$$|d_4| = |\frac{91}{1536}c_1^3 - \frac{17}{128}c_1 c_2 + \frac{1}{16}c_3| \leq 2 \cdot |\frac{91}{1536}| + 2 \cdot |\frac{17}{128} - \frac{91}{768}| + 2 \cdot |\frac{91}{1536} - \frac{17}{128} + \frac{1}{16}| = \frac{65}{384},$$

$$|d_5| = |\frac{97}{2048}c_1^4 - \frac{21}{128}c_1^2 c_2 + \frac{7}{64}c_1 c_3 + \frac{9}{128}c_2^2 - \frac{1}{16}c_4|.$$

By applying the triangle inequality, we can get

$$|d_5| \le |\frac{97}{2048}c_1^4 - \frac{21}{128}c_1^2c_2 + \frac{7}{64}c_1c_3| + |\frac{9}{128}c_2^2 - \frac{1}{16}c_4|.$$

Using Lemmas 2.1 and 2.3, we have

$$|\frac{97}{2048}c_1^4 - \frac{21}{128}c_1^2c_2 + \frac{7}{64}c_1c_3| = |c_1||\frac{97}{2048}c_1^3 - \frac{21}{128}c_1c_2 + \frac{7}{64}c_3| \le \frac{127}{256}.$$

Using Lemma 2.2, we obtain

$$|\frac{9}{128}c_2^2 - \frac{1}{16}c_4| = \frac{1}{16}|c_4 - \frac{9}{8}c_2^2| \le \frac{5}{32}.$$

Therefore,

$$|d_5| \le \frac{127}{256} + \frac{5}{32} = \frac{167}{256}.$$

This completes the proof.

Theorem 4.2. *If the function $f \in H(1)$ given by (1.1) and $f^{-1}(w) = w + \sum_{n=2}^{\infty} d_n w^n$ is the analytic continuation to $\mathbb{D}$ of the inverse function of f with $|w| < r_0$, where $r_0 \ge \frac{1}{4}$ is the radius of the Koebe domain, then*

$$|H_{3,1}(f^{-1})| \le \frac{34171}{147456}.$$

Proof. From Theorem 3.2, we have

$$|d_3 - d_2^2| = |a_2^2 - a_3| \le \frac{1}{4}. \tag{4.10}$$

Applying (4.6)–(4.8) and (2.16)–(2.18),

$$|d_2d_3 - d_4| = |3a_2^3 - 4a_2a_3 + a_4| = |\frac{73}{1536}c_1^3 - \frac{15}{128}c_1c_2 + \frac{1}{16}c_3|.$$

Using Lemma 2.3,

$$|d_2d_3 - d_4| \le 2|\frac{73}{1536}| + 2|\frac{15}{128} - \frac{73}{768}| + 2|\frac{73}{1536} - \frac{15}{128} + \frac{1}{16}| = \frac{59}{384}. \tag{4.11}$$

Applying (4.6)–(4.8) and (2.16)–(2.18),

$$|d_2d_4 - d_3^2| = |a_2^4 - a_2^2a_3 + a_2a_4 - a_3^2| = \frac{1}{64}|\frac{17}{192}c_1^4 + c_2^2 - \frac{7}{16}c_1^2c_2 - \frac{1}{2}c_1c_3|.$$

We denote $|c_1| = c \in [0, 2], |x| = t \in [0, 1]$, and referring to Lemma 2.5, we have

$$|c_2| \le \frac{c^2 + t(4 - c^2)}{2}, \quad |c_3| \le \frac{c^3}{4} + \frac{c(4 - c^2)t}{2} + \frac{(4 - c^2)ct^2}{4} + \frac{(4 - c^2)(1 - t^2)}{2}.$$

Using the triangle inequality, we have

$$|\frac{17}{192}c_1^4 + c_2^2 - \frac{7}{16}c_1^2c_2 - \frac{1}{2}c_1c_3| \le \frac{131}{192}c^4 + \frac{(4 - c^2)c}{4} + \frac{(c + 2)(c + 4)(c - 2)^2}{8}t^2 + \frac{31c^2(4 - c^2)}{32}t = F(c, t).$$

$$\frac{\partial F}{\partial t} = \frac{(c+2)(c+4)(c-2)^2}{4}t + \frac{31c^2(4-c^2)}{32} > 0.$$

Therefore,

$$F(c,t) \le F(c,1) = -\frac{31}{192}c^4 + \frac{19}{8}c^2 + 4 = G(c),$$

$$G^{'}(c) = \frac{19}{4}c(1 - \frac{31}{228}c^2) \ge 0.$$

This leads to

$$G(c) \le G(2) = \frac{131}{12},$$

$$|d_2d_4 - d_3^2| \le \frac{131}{12} \cdot \frac{1}{64} = \frac{131}{768}. \tag{4.12}$$

Applying (4.2)–(4.4), (4.10)–(4.12), and the triangle inequality, we have

$$|H_{3,1}(f)| = |d_3(d_2d_4-d_3^2)-d_4(d_4-d_2d_3)+d_5(d_3-d_2^2)| \le |d_3||d_2d_4-d_3^2|+|d_4||d_4-d_2d_3|+|d_5||d_3-d_2^2| \le \frac{34171}{147456},$$

which completes the proof.

5. The bound of coefficients and Fekete-Szegö inequality for $f \in H(\lambda)$

Theorem 5.1. *If $f \in H(\lambda)$ and is of the form (1.1), then*

$$|a_2| \le \frac{1}{8\lambda^2 - 8\lambda + 4},$$

$$|a_3| \le \frac{1}{10\lambda^2 - 12\lambda + 6},$$

$$|a_3 - \nu a_2^2| \le \begin{cases} \dfrac{\nu[3(1-\lambda)^2 + 2\lambda^2] - 2(1-\lambda)\lambda}{16[3(1-\lambda)^2+2\lambda^2][(1-\lambda)^2+\lambda^2]^2}, & \nu \ge t_1, \\ \dfrac{1}{2[3(1-\lambda)^2+2\lambda^2]}, & t_2 < \nu < t_1, \\ \dfrac{2(1-\lambda)\lambda - \nu[3(1-\lambda)^2+2\lambda^2]}{16[3(1-\lambda)^2+2\lambda^2][(1-\lambda)^2+\lambda^2]^2}, & \nu \le t_2, \end{cases} \tag{5.1}$$

where

$$t_1 = \frac{8[(1-\lambda)^2+\lambda^2]^2 + 2\lambda(1-\lambda)}{3(1-\lambda)^2+2\lambda^2}, \quad t_2 = \frac{2\lambda(1-\lambda) - 8[(1-\lambda)^2+\lambda^2]^2}{3(1-\lambda)^2+2\lambda^2}.$$

The result is sharp for the function $p(z) = (1+z^2)/(1-z^2)$.

Proof.

$$f^{'}(z) = 1 + 2a_2z + 3a_3z^2 + 4a_4z^3 + 5a_5z^4 + \cdots,$$

$$(1-\lambda)(f^{'}(z))^{1-\lambda} = (1-\lambda) + 2(1-\lambda)^2a_2z + (3(1-\lambda)^2a_3 - 2(1-\lambda)^2\lambda a_2^2)z^2 + \cdots, \tag{5.2}$$

$$\frac{2zf^{'}(z)}{f(z)-f(-z)} = 1 + 2a_2z + 2a_3z^2 + (4a_4 - 2a_2a_3)z^3 + \cdots,$$

$$\lambda(\frac{2zf'(z)}{f(z)-f(-z)})^{\lambda} = \lambda + 2\lambda^2 a_2 z + (2\lambda^2 a_3 - 2\lambda^2(1-\lambda)a_2^2)z^2 + \cdots. \tag{5.3}$$

In addition,

$$(1-\lambda)(f'(z))^{1-\lambda} + \lambda(\frac{2zf'(z)}{f(z)-f(-z)})^{\lambda} = \frac{2}{1+e^{-w(z)}} = 1 + \frac{c_1}{4}z + (\frac{c_2}{4} - \frac{c_1^2}{8})z^2 + \cdots, \tag{5.4}$$

where

$$w(z) = \frac{p(z)-1}{p(z)+1} = \frac{1}{2}c_1 z + (\frac{1}{2}c_2 - \frac{1}{4}c_1^2)z^2 + (\frac{1}{8}c_1^3 - \frac{1}{2}c_2 c_1 + \frac{1}{2}c_3)z^3 + \cdots.$$

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots \in P.$$

Substituting (5.2) and (5.3) into (5.4) and comparing the coefficients on both sides of (5.4), we can obtain

$$[2(1-\lambda)^2 + 2\lambda^2]a_2 = \frac{c_1}{4}, \tag{5.5}$$

$$[3(1-\lambda)^2 + 2\lambda^2]a_3 - [2(1-\lambda)^2\lambda + 2\lambda^2(1-\lambda)]a_2^2 = \frac{c_2}{4} - \frac{c_1^2}{8}. \tag{5.6}$$

From (5.5) and (5.6), we have

$$a_2 = \frac{c_1}{8[(1-\lambda)^2 + \lambda^2]}, \tag{5.7}$$

$$a_3 = \frac{1}{3(1-\lambda)^2 + 2\lambda^2}A, \tag{5.8}$$

where

$$A = \{\frac{(1-\lambda)\lambda}{32[(1-\lambda)^2 + \lambda^2]^2} - \frac{1}{8}\}c_1^2 + \frac{c_2}{4}.$$

Hence,

$$|a_3| = \frac{1}{4[3(1-\lambda)^2 + 2\lambda^2]}|c_2 - (\frac{1}{2} - \frac{(1-\lambda)\lambda}{8[(1-\lambda)^2 + \lambda^2]^2})c_1^2|,$$

and, since

$$|2(\frac{1}{2} - \frac{(1-\lambda)\lambda}{8[(1-\lambda)^2 + \lambda^2]^2}) - 1| = \frac{(1-\lambda)\lambda}{4[(1-\lambda)^2 + \lambda^2]^2} < 1 \quad \lambda \in [0,1],$$

we can apply Lemma 2.2 to get

$$|a_3| \le \frac{1}{10\lambda^2 - 12\lambda + 6}.$$

From (5.7) and (5.8), we can get

$$|a_3 - \nu a_2^2| = \frac{1}{4[3(1-\lambda)^2 + 2\lambda^2]}|c_2 - \{\frac{\nu[3(1-\lambda)^2 + 2\lambda^2]}{16[(1-\lambda)^2 + \lambda^2]^2} - \frac{(1-\lambda)\lambda}{8[(1-\lambda)^2 + \lambda^2]^2} + \frac{1}{2}\}c_1^2|.$$

Applying Lemma 2.2, we can obtain

$$|a_3 - \nu a_2^2| \le \frac{1}{2[3(1-\lambda)^2 + 2\lambda^2]}max\{1, |\frac{\nu[3(1-\lambda)^2 + 2\lambda^2] - 2(1-\lambda)\lambda}{8[(1-\lambda)^2 + \lambda^2]^2}|\}.$$

Then, we get (5.1), which completes the proof.

Corollary 5.1. *If $f \in H(\frac{1}{2})$ and is of the form (1.1), then*

$$|a_3 - \nu a_2^2| \leq \begin{cases} \frac{5\nu - 2}{20}, & \nu \geq 2, \\ \frac{2}{5}, & -\frac{6}{5} < \nu < 2, \\ \frac{2 - 5\nu}{20}, & \nu \leq -\frac{6}{5}. \end{cases} \tag{5.9}$$

6. Coefficient estimates

Now, we assume that $\psi(z)$ is an analytic and univalent function with positive real part in $\mathbb{D}$, and $\psi(z)$ satisfies the condition of $\psi(0) = 1$ and $\psi'(0) > 0$. It is easy to know that $\psi(z)$ has a series expansion of the form

$$\psi(z) = 1 + A_1 z + A_2 z^2 + A_3 z^3 + \cdots.$$

Next, we are going to estimate the upper bounds of the initial coefficients for f, where f and f^{-1} belong to $H(\lambda, \psi)$. Since $\psi'(0) > 0$, we have $A_1 > 0$.

Remark 6.1. *For $\psi(z) = \sqrt{1+z}$, $f(z) = z$, we have that $f(z)$ and $f^{-1}(z)$ belong to $H(\lambda, \psi)$ always holds.*

Theorem 6.1. *If f, g belong to $H(\lambda, \psi)$ and are of the form (1.1), where g is the inverse function of f, then we have*

$$|a_2| \leq min\{\frac{A_1}{2[\lambda^2 + (1-\lambda)^2]}, \sqrt{\frac{A_1 + |A_2 - A_1|}{7\lambda^2 - 8\lambda + 3}}\},$$

$$|a_3| \leq min\{\frac{A_1}{2\lambda^2 + 3(1-\lambda)^2} + \frac{A_1^2}{4[(1-\lambda)^2 + \lambda^2]^2}, \frac{A_1 + |A_2 - A_1|}{7\lambda^2 - 8\lambda + 3}\}.$$

Proof. Since $f, g \in H(\lambda, \psi)$, there exists two analytic functions $u, v : D \to D$, where $u(0) = v(0) = 0$, such that

$$(1-\lambda)(f'(z))^{1-\lambda} + \lambda(\frac{2zf'(z)}{f(z) - f(-z)})^{\lambda} = \psi(u(z)), \tag{6.1}$$

$$(1-\lambda)(g'(z))^{1-\lambda} + \lambda(\frac{2zg'(z)}{g(z) - g(-z)})^{\lambda} = \psi(v(z)). \tag{6.2}$$

Let us define the functions p and q by

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots = \frac{1 + u(z)}{1 - u(z)},$$

$$q(z) = 1 + q_1 z + q_2 z^2 + \cdots = \frac{1 + v(z)}{1 - v(z)}.$$

Or, equivalently,

$$u(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{1}{2} p_1 z + (\frac{p_2}{2} - \frac{p_1^2}{4}) z^2 + \cdots,$$

$$v(z)=\frac{q(z)-1}{q(z)+1}=\frac{1}{2}q_1z+(\frac{q_2}{2}-\frac{q_1^2}{4})z^2+\cdots.$$

In addition,

$$\psi(u(z))=1+\frac{1}{2}A_1p_1z+(A_1(\frac{1}{2}p_2-\frac{1}{4}p_1^2)+\frac{1}{4}A_2p_1^2)z^2+\cdots, \tag{6.3}$$

$$\psi(v(z))=1+\frac{1}{2}A_1q_1z+(A_1(\frac{1}{2}q_2-\frac{1}{4}q_1^2)+\frac{1}{4}A_2q_1^2)z^2+\cdots. \tag{6.4}$$

From (5.2), (5.3), (6.1), and (6.3), we have

$$2[\lambda^2+(1-\lambda)^2]a_2=\frac{1}{2}A_1p_1, \tag{6.5}$$

$$[3(1-\lambda)^2+2\lambda^2]a_3-2\lambda(1-\lambda)a_2^2=A_1(\frac{1}{2}p_2-\frac{1}{4}p_1^2)+\frac{1}{4}A_2p_1^2. \tag{6.6}$$

Since

$$g(z)=z-a_2z^2+(2a_2^2-a_3)z^3-(5a_2^3-5a_2a_3+a_4)z^4+\cdots,$$

we can obtain

$$(1-\lambda)(g^{'}(z))^{1-\lambda}+\lambda(\frac{2zg^{'}(z)}{g(z)-g(-z)})^{\lambda}=1-2[(1-\lambda)^2+\lambda^2]a_2z+[(3(1-\lambda)^2+2\lambda^2)(2a_2^2-a_3)-2\lambda(1-\lambda)a_2^2]z^2-\cdots. \tag{6.7}$$

By using (6.2), (6.4), and (6.7), we have

$$-2[(1-\lambda)^2+\lambda^2]a_2=\frac{1}{2}A_1q_1, \tag{6.8}$$

$$[3(1-\lambda)^2+2\lambda^2](2a_2^2-a_3)-2\lambda(1-\lambda)a_2^2=A_1(\frac{1}{2}q_2-\frac{1}{4}q_1^2)+\frac{1}{4}A_2q_1^2. \tag{6.9}$$

From (6.5) and (6.8), we can get

$$p_1=-q_1, \tag{6.10}$$

and

$$a_2^2=\frac{A_1^2(p_1^2+q_1^2)}{32[\lambda^2+(1-\lambda)^2]^2}. \tag{6.11}$$

Since $|p_i|\leq 2, |q_i|\leq 2 (i\in N^+)$, we obtain

$$|a_2|\leq\frac{A_1}{2[\lambda^2+(1-\lambda)^2]}. \tag{6.12}$$

By adding (6.6) to (6.9), we can get

$$a_2^2=\frac{2A_1(p_2+q_2)+(A_2-A_1)(p_1^2+q_1^2)}{8(7\lambda^2-8\lambda+3)}. \tag{6.13}$$

Since $|p_i|\leq 2$ *and* $|q_i|\leq 2$ $(i\in N^+)$, we can get

$$|a_2|\leq\sqrt{\frac{A_1+|A_2-A_1|}{7\lambda^2-8\lambda+3}}. \tag{6.14}$$

From (6.12) and (6.14), we can obtain the conclusion

$$|a_2| \leq min\{\frac{A_1}{2[\lambda^2+(1-\lambda)^2]}, \sqrt{\frac{A_1+|A_2-A_1|}{7\lambda^2-8\lambda+3}}\}.$$

By subtracting (6.6) from (6.9) and using (6.10), we have

$$a_3 = \frac{A_1(p_2-q_2)}{4[2\lambda^2+3(1-\lambda)^2]^2} + a_2^2. \tag{6.15}$$

Using (6.10) and (6.11) in (6.15), we can obtain

$$a_3 = \frac{A_1(p_2-q_2)}{4[3(1-\lambda)^2+2\lambda^2]} + \frac{A_1^2p_1^2}{16[(1-\lambda)^2+\lambda^2]^2}.$$

Therefore,

$$|a_3| \leq \frac{A_1}{2\lambda^2+3(1-\lambda)^2} + \frac{A_1^2}{4[(1-\lambda)^2+\lambda^2]^2}. \tag{6.16}$$

On the other hand, by using (6.10) and (6.13) in (6.15), we can obtain

$$a_3 = \frac{A_1(p_2-q_2)}{4[3(1-\lambda)^2+2\lambda^2]} + \frac{A_1(p_2+q_2)+(A_2-A_1)p_1^2}{4[3(1-\lambda)^2+2\lambda^2]-8\lambda(1-\lambda)},$$

or, equivalently,

$$a_3 = \left(\begin{array}{c} A_1p_2(\frac{1}{4(5\lambda^2-6\lambda+3)} + \frac{1}{4(7\lambda^2-8\lambda+3)}) + A_1q_2(\frac{1}{4(7\lambda^2-8\lambda+3)} - \frac{1}{4(5\lambda^2-6\lambda+3)}) \\ +\frac{(A_2-A_1)p_1^2}{4(7\lambda^2-8\lambda+3)} \end{array} \right).$$

Using the triangle inequality and Lemma 2.2, we can obtain

$$|a_3| \leq \frac{A_1}{7\lambda^2-8\lambda+3} + \frac{|A_2-A_1|}{7\lambda^2-8\lambda+3}. \tag{6.17}$$

From (6.16) and (6.17), we have

$$|a_3| \leq min\{\frac{A_1}{2\lambda^2+3(1-\lambda)^2} + \frac{A_1^2}{4[(1-\lambda)^2+\lambda^2]^2}, \frac{A_1+|A_2-A_1|}{7\lambda^2-8\lambda+3}\},$$

which completes the proof.

Corollary 6.1. *If f satisfies the condition of Theorem 6.1 and we let $\psi(z) = 1 + \frac{4}{3}z + \frac{2}{3}z^2$, then*

$$|a_2| \leq min\{\frac{2}{3[\lambda^2+(1-\lambda)^2]}, \sqrt{\frac{2}{7\lambda^2-8\lambda+3}}\},$$

$$|a_3| \leq min\{\frac{4}{3[2\lambda^2+3(1-\lambda)^2]} + \frac{4}{9[(1-\lambda)^2+\lambda^2]^2}, \frac{2}{7\lambda^2-8\lambda+3}\}.$$

Corollary 6.2. *If f satisfies the condition of Theorem 6.1 and we let $f \in H(0,\psi)$, then*

$$|a_2| \leq min\{\frac{A_1}{2}, \sqrt{\frac{A_1+|A_2-A_1|}{3}}\},$$

$$|a_3| \leq min\{\frac{A_1}{3} + \frac{A_1^2}{4}, \frac{A_1+|A_2-A_1|}{3}\}.$$

Let $f \in H(1, \psi)$*, then*

$$|a_2| \leq min\{\frac{A_1}{2}, \sqrt{\frac{A_1 + |A_2 - A_1|}{2}}\},$$

$$|a_3| \leq min\{\frac{A_1}{2} + \frac{A_1^2}{4}, \frac{A_1 + |A_2 - A_1|}{2}\}.$$

7. Conclusions

In the present work, we defined new subclasses of analytic functions associated with the modified sigmoid function. Then, we mainly get upper bounds of the third-order Hankel determinant and fourth-order Hankel determinant in certain conditions. We also get the upper bound of the third-order Hankel determinant of its inverse function in the specific conditions mentioned above. Next, we investigated the upper bound of the Fekete-Szegö inequality for the analytic functions in the class $H(\lambda)$. Finally, we estimated the upper bounds of the initial coefficients for the analytic functions in the class $H(\lambda, \psi)$, where $f^{-1}(z)$ also belongs to $H(\lambda, \psi)$. The purpose of our study is to stimulate the interest of scholars in the field and to further stimulate their research in this kind of subject. In fact, this kind of problem plays a very important role in many other problems of mathematical analysis.

We will further investigate the upper bounds of the third, fourth, and fifth-order Hankel determinants of functions belonging to $H(\lambda)$ or $H(\lambda, \psi)$ $(0 \leq \lambda \leq 1)$. We can also research the upper bounds of the third or fourth Hankel determinant of a class of functions defined in [12]. Recently, the problems of the quantum calculus happens to provide another popular and interesting direction for researchers in complex analysis, which is evidenced by the recently-published review article by Srivastava [31]. Hence, the quantum extension of the results shown in this paper is quite worthwhile to further research. Apart from the above, we are motivated to explore how to get the upper bound of the Hankel determinant of certain analytic functions by other methods, from which we may get more precise or sharp upper bounds.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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