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Research article

Theoretical analysis of a class of φ-Caputo fractional differential equations in Banach space

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Abstract: A study of a class of nonlinear differential equations involving the φ-Caputo type derivative in a Banach space framework is presented. Weissinger's and Meir-Keeler's fixed-point theorems are used to achieve some quantitative results. Two illustrative examples are provided to justify the theoretical results.

Keywords: φ-Caputo derivative; fixed point theorem; Hausdorff measure of noncompactness
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1. Introduction

The present paper is devoted to analyzing the following problem with a constant coefficient $\varpi > 0$ of the form:

$$\begin{cases} \left({}^{c}\mathcal{D}_{a^+}^{\alpha;\varphi} + \varpi\, {}^{c}\mathcal{D}_{a^+}^{\alpha-1;\varphi}\right) y(t) = g(t, y(t)), \quad t \in \mathfrak{I} := [a, b], \\ y(a) = y'(a) = 0, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, ${}^{c}\mathcal{D}_{a^+}^{\theta;\varphi}$ is the Caputo fractional derivatives concerning φ of order $\theta \in \{\alpha, \alpha - 1\}$, $g : I \times \mathbb{X} \to \mathbb{X}$ is a function satisfying some hypotheses that will be precise later and $(\mathbb{X}, \|\cdot\|)$ is a real Banach space.

The study of differential equations involving integer or non-integer order derivatives has emerged as a pivotal tool for modeling complex phenomena across diverse scientific and engineering domains.

Extensive exploration by various authors, as reflected in references such as [1–4], underscores the multifaceted nature of this theoretical framework. In contemporary research, the incorporation of non-integer order derivatives, particularly through the φ-Caputo type introduced in [5], has gained prominence in concrete modeling. Applications range from anomalous diffusions, including ultra-slow processes [6], to financial models such as the Heston model [7], random walks [8], financial crises [9], and the Verhulst model [10]. The focus has intensified on both quantitative and qualitative properties of solutions for differential problems governed by φ-Caputo type derivatives, as evident in works like [11–13]. In this context, the present research seeks to advance the findings from [14] to a more general setting. Specifically, the investigation addresses scenarios where nonlinear forcing terms operate within infinite-dimensional Banach spaces. Employing a Weissinger type fixed point theorem, the study strategically sidesteps certain additional hypotheses identified in [14, Theorem 7]. Furthermore, through the integration of fixed point techniques and the measure of noncompactness (MNC) under specific growth and compactness assumptions, the research extends the existence result initially established in [14, Theorem 6]. This comprehensive approach aims to deepen the understanding of differential equations with non-integer order derivatives, offering insights into their behavior in complex and infinite-dimensional settings [15].

The current paper is divided into four sections: We collect in Section 2 the basic background needed in the remainder of the paper. In Section 3, Weissinger's and Meir-Keeler's fixed point theorems are used to obtain a new existence criterion. Finally, two illustrative examples are presented.

2. Preliminaries

Throughout this paper, we endow the space $C(\mathfrak{I}, \mathbb{X})$ of continuous functions $z : \mathfrak{I} \to \mathbb{X}$ by the norm

$$\|z\|_\infty = \sup_{t\in\mathfrak{I}} \|z(t)\|, \quad \text{for all } z \in C(\mathfrak{I}, \mathbb{X}).$$

$L^1(\mathfrak{I}, \mathbb{X})$ denotes the Banach space of Bochner integrable functions $z : \mathfrak{I} \to \mathbb{X}$ normed by

$$\|z\|_{L^1(\mathfrak{I},\mathbb{X})} = \int_a^b \|z(t)\|dt, \quad \text{for all } z \in L^1(\mathfrak{I}, \mathbb{X}).$$

Set

$$\mathbb{S}^{1,+}_{a^+}(\mathfrak{I}, \mathbb{R}) = \{\varphi : \varphi \in C^1(\mathfrak{I}, \mathbb{R}) \text{ and } \varphi'(t) > 0 \text{ for all } t \in \mathfrak{I}\}.$$

Letting $\varphi \in \mathbb{S}^{1,+}_{a^+}(\mathfrak{I}, \mathbb{R})$ for $s, t \in \mathfrak{I}$, $(s < t)$, we define

$$\varphi(t, s) = \varphi(t) - \varphi(s) \text{ and } \varphi(t, s)^\alpha = (\varphi(t) - \varphi(s))^\alpha .$$

Definition 2.1. [16] *The Mittag-Leffler function* $\mathbb{M}_\alpha(\cdot)$ *is given by*

$$\mathbb{M}_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha + 1)}, \quad (\alpha > 0,\ z \in \mathbb{R}),$$

where $\Gamma(\cdot)$ *is the gamma function.*

Definition 2.2. [5,17] *The φ-fractional integral of a function f of order $\alpha > 0$ is given by*

$$\mathcal{I}_{a^+}^{\alpha,\varphi} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \varphi(t,s)^{\alpha-1} \varphi'(s) f(s) ds, \quad t > a,$$

with $\varphi \in \mathbb{S}_{a^+}^{1,+}(\mathfrak{J}, \mathbb{R})$.

Lemma 2.1. [5,17] *Let $\alpha, \gamma > 0$, then*

$$\mathcal{I}_{a^+}^{\alpha;\varphi} \varphi(t,a)^{\gamma-1} = \frac{\Gamma(\gamma)}{\Gamma(\alpha+\gamma)} \varphi(t,a)^{\alpha+\gamma-1}.$$

Lemma 2.2. [18] *Let $\alpha > 1$ and $\varsigma > 0$. Then, for all $t \in \mathfrak{J}$ we have*

$$\mathcal{I}_{a^+}^{\alpha-1;\varphi} e^{\varsigma\varphi(t,a)} \leq \frac{1}{\varsigma^{\alpha-1}} e^{\varsigma\varphi(t,a)}.$$

Definition 2.3. [5] *Let $n-1 < \alpha \leq n$ with $n \in \mathbb{N}$, $\varphi \in \mathbb{S}_{a^+}^{1,+}(\mathfrak{J}, \mathbb{R})$. The left-sided φ-Caputo FDs of a function f of order α is defined as*

$$\left({}^c\mathcal{D}_{a^+}^{\alpha;\varphi} f\right)(t) = \mathcal{I}_{a^+}^{n-\alpha;\varphi} \left(\frac{1}{\varphi'(t)} \frac{d}{dt}\right)^n f(t).$$

Definition 2.4. [19] *Let $\mathbb{O} \subset \mathbb{X}$ be a bounded set. The Hausdorff MNC of $\mathbb{O}$ is defined by*

$$\Lambda(\mathbb{O}) = \inf\left\{\epsilon > 0 : \mathbb{O} \text{ has a finite } \epsilon-\text{net in } \mathbb{X}\right\}.$$

Lemma 2.3. [19] *Let $\mathbb{O}, \mathbb{V} \subset \mathbb{X}$ be bounded. Then, Λ satisfies:*

(1) $\Lambda(\mathbb{O}) = 0 \Longleftrightarrow \mathbb{O}$ *is relatively compact.*
(2) $\mathbb{O} \subset \mathbb{V} \Longrightarrow \Lambda(\mathbb{O}) \leq \Lambda(\mathbb{V})$.
(3) $\Lambda(\mathbb{O} \cup \mathbb{V}) = \max\{\Lambda(\mathbb{O}), \Lambda(\mathbb{V})\}$.
(4) $\Lambda(\mathbb{O}) = \Lambda(\overline{\mathbb{O}}) = \Lambda(\mathrm{conv}(\mathbb{O}))$, *where* $\mathrm{conv}\,\mathbb{O}$ *and* $\overline{\mathbb{O}}$ *denote the convex hull and the closure of* $\mathbb{O}$*, respectively.*
(5) $\Lambda(\mathbb{O} + \mathbb{V}) \leq \Lambda(\mathbb{O}) + \Lambda(\mathbb{V})$.
(6) $\Lambda(\lambda\mathbb{O}) \leq |\lambda|\Lambda(\mathbb{O})$, *for any* $\lambda \in \mathbb{R}$.

Lemma 2.4. [20] *Let $\mathbb{O} \subset \mathbb{X}$ be bounded. Then, for all ϵ, there is a sequence $\{x_n\}_{n=1}^{\infty} \subset \mathbb{O}$ such that*

$$\Lambda(\mathbb{O}) \leq 2\Lambda\left(\{x_n\}_{n=1}^{\infty}\right) + \epsilon.$$

The set $\mathbb{O} \subset L^1(\mathfrak{J}, \mathbb{X})$ is called uniformly integrable if for all $\zeta \in \mathbb{O}$ we have

$$\|\zeta(t)\| \leq \delta(t), \quad \text{a.e. } t \in \mathfrak{J},$$

with $\delta \in L^1(J, \mathbb{R}^+)$.

Lemma 2.5. [21] *Assume that $\{\zeta_n\}_{n=1}^{\infty} \subset L^1(\mathfrak{J}, \mathbb{X})$ is uniformly integrable, the map $t \longmapsto \Lambda\left(\{\zeta_n(t)\}_{n=1}^{\infty}\right)$ is measurable, and furthermore*

$$\Lambda\left(\left\{\int_a^t \zeta_n(s)\mathrm{d}s\right\}_{n=1}^{\infty}\right) \leq 2\int_a^t \Lambda\left(\{\zeta_n(s)\}_{n=1}^{\infty}\right)\mathrm{d}s.$$

We now recall respectively the theorems of Weissinger and Meir-Keeler that we will use in the following.

Theorem 2.1. [3] *Let* $(\mathbb{E}, \|\cdot\|)$ *be a Banach space and* $\Theta_n \geq 0$ *for every* $n \in \mathbb{N}$ *where* $\sum_{n=0}^{\infty} \Theta_n$ *converges. If the operator* $\mathcal{N} : \mathbb{E} \longrightarrow \mathbb{E}$ *satisfies*

$$d\left(\mathcal{N}^n u, \mathcal{N}^n v\right) \leq \Theta_n d(u, v), \quad u, v \in \mathbb{E},$$

for every $n \in \mathbb{N}$*, then* $\mathcal{N}$ *has a uniquely defined fixed point* u^**. Additionally, for any* $v_0 \in \mathbb{E}$*, the sequence* $\{\mathcal{N}^n v_0\}_{n=1}^{\infty}$ *converges to* u^**.*

Definition 2.5. [22] *Let* $\mathbb{B} \subset \mathbb{X}$ *be a nonempty set. We say that* $\mathcal{N} : \mathbb{B} \longrightarrow \mathbb{B}$ *is a Meir-Keeler condensing operator, if for any* $\epsilon > 0$*, there exists* $\eta > 0$ *such that*

$$\epsilon \leq \Lambda(\mathbb{A}) < \epsilon + \eta \implies \Lambda(\mathcal{N}\mathbb{A}) < \epsilon,$$

for any bounded subset $\mathbb{A}$ *of* $\mathbb{B}$*.*

Theorem 2.2. [22] *Let* $\mathbb{B}$ *be a nonempty, bounded, closed, and convex subset of a Banach space* $\mathbb{X}$*. If* $\mathcal{N} : \mathbb{B} \longrightarrow \mathbb{B}$ *is a continuous and Meir-Keeler condensing operator, then* $\mathcal{N}$ *has at least one fixed point and the set of all fixed points of* $\mathcal{N}$ *in* $\mathbb{B}$ *is compact.*

3. Main results

Theorem 3.1. *We impose the assumptions:*

(A1) $g : \mathfrak{J} \times \mathbb{X} \to \mathbb{X}$ *is a continuous function.*
(A2) There exists a constant $L_g > 0$ *such that*

$$\|g(t, u_1) - g(t, u_2)\| \leq L_g \|u_1 - u_2\|, \tag{3.1}$$

for any $u_1, u_2 \in \mathbb{X}$ *and* $t \in \mathfrak{J}$*.*

Then, problem (1.1) admits a unique solution on $\mathfrak{J}$*.*

Proof. According to [14, Theorem 1], let us introduce $\mathcal{L} : C(\mathfrak{J}, \mathbb{X}) \to C(\mathfrak{J}, \mathbb{X})$ given by

$$\mathcal{L}y(t) = (\alpha - 1) \int_a^t e^{-\varpi\varphi(t,s)} \left(\mathcal{I}_{a^+}^{\alpha-1;\varphi} g(\tau, y(\tau))\right)(s)\varphi'(s)ds, \quad t \in \mathfrak{J}. \tag{3.2}$$

Evidently, the solutions of problem (1.1) can be regarded as the fixed point of $\mathcal{L}$. We will show, with the aid of Theorem 2.1 and a suitably selected equivalent norm, that $\mathcal{L}$ admits a unique fixed point.

In this respect, let $y, z \in C(\mathfrak{J}, \mathbb{X})$. Then, for every $t \in \mathfrak{J}$ and $n \in \mathbb{N}$, since $0 < e^{-\varpi\varphi(t,s)} < 1$ for $a < s < t < b$, we have

$$\|\mathcal{L}y(t) - \mathcal{L}z(t)\| \leq (\alpha - 1) \int_a^t \left(\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)} \|g(\tau, y(\tau)) - g(\tau, z(\tau))\| d\tau\right) \varphi'(s)ds.$$

Using (A2) and Lemma 2.1, one gets

$$\begin{aligned}\|\mathcal{L}y(t)-\mathcal{L}z(t)\| &\le (\alpha-1)L_g\|y-z\|\int_a^t \varphi'(s)\left(\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau\right)ds\\ &\le (\alpha-1)L_g\|y-z\|\int_a^t \frac{\varphi'(s)\varphi(s,a)^{\alpha-1}}{\Gamma(\alpha)}ds\\ &\le \frac{L_g(\alpha-1)\varphi(t,a)^{\alpha}}{\Gamma(\alpha+1)}\|y-z\|.\end{aligned}$$

Again, by (A2), we obtain

$$\begin{aligned}\|\mathcal{L}^2y(t)-\mathcal{L}^2z(t)\| &\le \|\mathcal{L}(\mathcal{L}y(t))-\mathcal{L}(\mathcal{L}z(t))\|\\ &\le (\alpha-1)\int_a^t\left(\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\|g(\tau,\mathcal{L}y(\tau))-g(\tau,\mathcal{L}z(\tau))\|d\tau\right)\varphi'(s)ds\\ &\le (\alpha-1)L_g\int_a^t\left(\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\|\mathcal{L}y(\tau)-\mathcal{L}z(\tau)\|d\tau\right)\varphi'(s)ds\\ &\le \frac{(\alpha-1)^2L_g^2}{\Gamma(\alpha+1)}\|y-z\|\int_a^t\left(\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi(\tau,a)^{\alpha}d\tau\right)\varphi'(s)ds.\end{aligned}$$

Lemma 2.1 entails

$$\begin{aligned}\|\mathcal{L}^2y(t)-\mathcal{L}^2z(t)\| &\le (\alpha-1)^2L_g^2\|y-z\|\int_a^t \frac{\varphi(s,a)^{2\alpha-1}}{\Gamma(2\alpha)}\varphi'(s)ds\\ &\le \frac{L_g^2(\alpha-1)^2\varphi(t,a)^{2\alpha}}{\Gamma(2\alpha+1)}\|y-z\|.\end{aligned}$$

Repeating the process for $n=3,4,\cdots$, for each $t\in\mathfrak{J}$, it remains to show that

$$\|\mathcal{L}^ny(t)-\mathcal{L}^nz(t)\| \le \frac{\left(L_g(\alpha-1)\varphi(t,a)^{\alpha}\right)^n}{\Gamma(n\alpha+1)}\|y-z\|. \tag{3.3}$$

By induction, assume that (3.3) holds for some n and let us prove it for $n+1$.

One has

$$\begin{aligned}&\|\mathcal{L}^{n+1}y(t)-\mathcal{L}^{n+1}z(t)\|\\ &\le \|\mathcal{L}(\mathcal{L}^ny(t))-\mathcal{L}(\mathcal{L}^nz(t))\|\\ &\le (\alpha-1)\int_a^t\left(\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\|g(\tau,\mathcal{L}^ny(\tau))-g(\tau,\mathcal{L}^nz(\tau))\|d\tau\right)\varphi'(s)ds\\ &\le (\alpha-1)L_g\int_a^t\left(\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\|\mathcal{L}^ny(\tau)-\mathcal{L}^nz(\tau)\|d\tau\right)\varphi'(s)ds\\ &\le \frac{\left(L_g(\alpha-1)\right)^{n+1}}{\Gamma(n\alpha+1)}\|y-z\|\int_a^t\left(\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\varphi(\tau,a)^{n\alpha}d\tau\right)\varphi'(s)ds.\end{aligned}$$

Lemma 2.1 yields

$$\begin{aligned}\|\mathcal{L}^{n+1}y(t)-\mathcal{L}^{n+1}z(t)\| &\le \left(L_g(\alpha-1)\right)^{n+1}\|y-z\|\int_a^t \frac{\varphi(s,a)^{(n+1)\alpha-1}}{\Gamma(\alpha(n+1))}\varphi'(s)ds\\ &\le \frac{\left(L_g(\alpha-1)\varphi(t,a)^{\alpha}\right)^{n+1}}{\Gamma(\alpha(n+1)+1)}\|y-z\|.\end{aligned}$$

Hence, inequality (3.3) holds.

Therefore, we conclude that for all $n \in \mathbb{N}$, one has

$$\|\mathcal{L}^n y - \mathcal{L}^n z\| \le \frac{\left(L_g(\alpha-1)\varphi(b,a)^\alpha\right)^n}{\Gamma(n\alpha+1)}\|y-z\|, \quad y,z \in C(\mathfrak{I},\mathbb{X}).$$

By putting

$$\Theta_n = \frac{\left(L_g(\alpha-1)\varphi(b,a)^\alpha\right)^n}{\Gamma(n\alpha+1)}, \tag{3.4}$$

we observe that

$$\sum_{n=0}^{\infty}\Theta_n = \sum_{n=0}^{\infty}\frac{\left(L_g(\alpha-1)\varphi(b,a)^\alpha\right)^n}{\Gamma(n\alpha+1)} = \mathbb{M}_\alpha\left(L_g(\alpha-1)\varphi(b,a)^\alpha\right).$$

Finally, Theorem 2.1 entails that $\mathcal{L}$ admits a unique fixed point which is the unique global solution of problem (1.1). □

Now, we prove another existence result, which is based on Theorem 2.2.

Theorem 3.2. *Assume that assumption (A1) holds. Furthermore, we suppose:*

(A3) There exist continuous functions $\xi,\kappa : \mathfrak{I} \to \mathbb{R}_+$ such that

$$\|g(t,u)\| \le \xi(t) + \kappa(t)\|u\|, \quad u \in \mathbb{X},$$

for all $t \in \mathfrak{I}$.

(A4) There exists a continuous function $\sigma : \mathfrak{I} \to \mathbb{R}_+$ such that for each bounded set $\mathbb{U} \subset \mathbb{X}$, and each $t \in \mathfrak{I}$, we have

$$\Lambda(g(t,\mathbb{U})) \le \sigma(t)\Lambda(\mathbb{U}).$$

(A5) The following inequality holds:

$$(\alpha-1)(\xi^* + \kappa^* R)\frac{\varphi(b,a)^\alpha}{\Gamma(\alpha+1)} \le R,$$

with

$$R > 0, \quad \kappa^* = \sup_{t\in\mathfrak{I}}\kappa(t), \quad \textit{and} \quad \xi^* = \sup_{t\in\mathfrak{I}}\xi(t).$$

Then Eq (1.1) admits at least one solution.

Proof. Introduce again the operator $\mathcal{L}$ represented by (3.2) and define the ball

$$\mathbf{B}_R = \{y \in C(\mathfrak{I},\mathbb{X}) \ : \ \|y\|_\infty \le R\}.$$

Step 1. $\mathcal{L}$ is a self-mapping from $\mathbf{B}_R$ to $\mathbf{B}_R$. By (A3), we have

$$\begin{aligned}\|\mathcal{L}y(t)\| &\le (\alpha-1)\int_a^t\left(\int_a^s \tfrac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\|g(\tau,y(\tau))\|d\tau\right)\varphi'(s)ds \\ &\le (\alpha-1)\int_a^t\left(\int_a^s \tfrac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}(\xi(\tau)+\kappa(\tau)\|y(\tau)\|)d\tau\right)\varphi'(s)ds \\ &\le (\alpha-1)(\xi^*+\kappa^*\|y\|)\int_a^t\left(\int_a^s \tfrac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau\right)\varphi'(s)ds.\end{aligned}$$

Combining Lemma 2.1 and (A5), one gets

$$\begin{aligned}\|\mathcal{L}y(t)\| &\le (\alpha-1)(\xi^*+\kappa^*\|y\|)\tfrac{\varphi(b,a)^\alpha}{\Gamma(\alpha+1)}\\ &\le (\alpha-1)(\xi^*+\kappa^* R)\tfrac{\varphi(b,a)^\alpha}{\Gamma(\alpha+1)}\\ &\le R.\end{aligned}$$

Thus,

$$\|\mathcal{L}y\| \le R. \tag{3.5}$$

This shows that $\mathcal{L}$ is a self-mapping from $\mathbf{B}_R$ to $\mathbf{B}_R$.

Step 2. $\mathcal{L}$ is continuous. Let the sequence $\{y_n\}$ such that $y_n \to y$ in $\mathbf{B}_R$. For all $t \in \mathfrak{J}$, we obtain

$$\begin{aligned}&\|(\mathcal{L}y_n)(t)-(\mathcal{L}y)(t)\|\\ &\le (\alpha-1)\int_a^t \left(\int_a^s \tfrac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\|g(\tau,y_n(\tau))-g(\tau,y(\tau))\|d\tau\right)\varphi'(s)ds\\ &\le (\alpha-1)\|g(\cdot,y_n)-g(\cdot,y)\|\int_a^t\left(\int_a^s \tfrac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau\right)\varphi'(s)ds\\ &\le (\alpha-1)\tfrac{\varphi(b,a)^\alpha}{\Gamma(\alpha+1)}\|g(\cdot,y_n)-g(\cdot,y)\|.\end{aligned}$$

Since g is continuous, we have

$$\|\mathcal{L}y_n-\mathcal{L}y\| \to 0 \quad \text{as} \quad n\to\infty.$$

Step 3. $\mathcal{L}(\mathbf{B}_R)$ is equicontinuous. Letting $y \in \mathbf{B}_R$ and $a<t_1<t_2<b$, we get

$$\|(\mathcal{L}y)(t_2)-(\mathcal{L}y)(t_1)\| \le S_1+S_2,$$

where

$$S_1 = (\alpha-1)\int_{t_1}^{t_2}\varphi'(s)\left|e^{-\varpi\varphi(t_2,s)}\right|\int_a^s \frac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}\|g(\tau,y(\tau))\|d\tau ds,$$

and

$$S_2 = (\alpha-1)\int_a^{t_1}\varphi'(s)\left|e^{-\varpi\varphi(t_2,s)}-e^{-\varpi\varphi(t_1,s)}\right|\left\|\left(\mathcal{I}_{a^+}^{\alpha-1;\varphi}g(\tau,y(\tau))\right)(s)\right\|ds.$$

Since $e^{-\varpi\varphi(t_2,s)}<1$, making use of (A3), one obtains

$$\begin{aligned}S_1 &\le (\alpha-1)(\xi^*+\kappa^*\|y\|)\int_{t_1}^{t_2}\varphi'(s)\int_a^s \tfrac{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2}}{\Gamma(\alpha-1)}d\tau ds\\ &\le (\alpha-1)(\xi^*+\kappa^*\|y\|)\int_{t_1}^{t_2}\varphi'(s)\tfrac{\varphi(s,a)^{\alpha-1}}{\Gamma(\alpha)}ds\\ &\le \tfrac{(\alpha-1)(\xi^*+\kappa^*\|y\|)}{\Gamma(\alpha+1)}\left(\varphi(t_2,a)^\alpha-\varphi(t_1,a)^\alpha\right).\end{aligned}$$

Thus,

$$S_1 \longrightarrow 0 \quad \text{when} \quad t_2\longrightarrow t_1. \tag{3.6}$$

On the other side,

$$S_2 = (\alpha-1)\Big(e^{-\varpi\varphi(t_1)} - e^{-\varpi\varphi(t_2)}\Big)\int_a^{t_1} e^{\varpi\varphi(s)}\Big\|\Big(\mathcal{I}_{a^+}^{\alpha-1;\varphi} g(\tau, y(\tau))\Big)(s)\Big\|\varphi'(s)ds.$$

Thus,

$$S_2 \longrightarrow 0 \quad \text{when} \quad t_2 \longrightarrow t_1. \tag{3.7}$$

From (3.6) and (3.7), the equicontinuity of $\mathcal{L}(\mathbf{B}_R)$ results immediately.

Step 4. Now, we prove that $\mathcal{L} : \mathbf{B}_R \to \mathbf{B}_R$ satisfies Definition 2.5.

To do this, for every bounded subset $\mathbb{J} \subset C(\mathfrak{J}, \mathbb{X})$ we define the MNC as

$$\widehat{\Lambda}(\mathbb{J}) = \sup_{t\in\mathfrak{J}} e^{-\aleph t}\Lambda(\mathbb{J}(t)), \quad \aleph > 0. \tag{3.8}$$

Next, fixing $\epsilon > 0$, we show the existence of $\eta > 0$ such that

$$\epsilon \le \widehat{\Lambda}(\mathbb{U}) < \epsilon + \eta \;\Rightarrow\; \widehat{\Lambda}(\mathcal{L}\mathbb{U}) < \epsilon, \quad \text{for any } \mathbb{U} \subset \mathbf{B}_R. \tag{3.9}$$

Now, let $\mathbb{U} \subset \mathbf{B}_R$ and, using Lemma 2.4, it follows that for a given $\epsilon' > 0$. Then there exists a sequence $\{y_n\}_{n=1}^{\infty} \subset \mathbb{U}$ such that, for all $t \in \mathfrak{J}$,

$$\Lambda\Big(\mathcal{L}(\mathbb{U})(t)\Big) = \Lambda\Big(\{(\mathcal{L}(y))(t) \;:\; y \in \mathbb{U}\}\Big) \le 2\Lambda\Big(\{(\mathcal{L}(y_n))(t)\}_{n=1}^{+\infty}\Big) + \epsilon'. \tag{3.10}$$

Then, since $\varphi'(\cdot)\varphi(\cdot, a)^{\alpha-1} \in L^1(\mathfrak{J}, \mathbb{R})$, it is possible to choose $\aleph$ such that

$$q(\aleph) := \sup_{t\in\mathfrak{J}} \frac{4\sigma^*}{\Gamma(\alpha-1)} \int_a^t \varphi'(s)\varphi(s,a)^{\alpha-1} e^{-\aleph(t-s)} ds < \frac{1}{2}, \tag{3.11}$$

where $\sigma^* = \sup_{t\in\mathfrak{J}} \sigma(t)$. After that, from

$$\begin{aligned}(\mathcal{L}(y_n))(t) &= (\alpha-1)\int_a^t e^{-\varpi\varphi(t,s)}\mathcal{I}_{a^+}^{\alpha-1;\varphi} g(s, y_n(s))\varphi'(s)ds \\ &\le (\alpha-1)\int_a^t \varphi'(s)\mathcal{I}_{a^+}^{\alpha-1;\varphi} g(s, y_n(s))ds,\end{aligned} \tag{3.12}$$

we obtain

$$\Lambda\left(\{(\mathcal{L}(y_n))(t)\}_{n=1}^{+\infty}\right) \le \Lambda\left(\left\{(\alpha-1)\int_a^t \varphi'(s)\mathcal{I}_{a^+}^{\alpha-1;\varphi} g(s, y_n(s))ds\right\}_{n=1}^{+\infty}\right). \tag{3.13}$$

Next, using (A4), for all $\tau \in [a, s]$, we have

$$\begin{aligned}\Lambda\left(\left\{\varphi'(\tau)\varphi(s,\tau)^{\alpha-2} g(\tau, y_n(\tau))\right\}_{n=1}^{+\infty}\right) &\le \varphi'(\tau)\varphi(s,\tau)^{\alpha-2}\sigma(\tau)\Lambda(\{y_n(\tau)\}_{n=1}^{+\infty}) \\ &\le \sigma(\tau)\varphi'(\tau)\varphi(s,\tau)^{\alpha-2} e^{\aleph\tau} \sup_{a\le\tau\le s} e^{-\aleph\tau}\Lambda(\{y_n(\tau)\}_{n=1}^{+\infty}) \\ &\le \sigma(\tau)\varphi'(\tau)\varphi(s,\tau)^{\alpha-2} e^{\aleph\tau}\widehat{\Lambda}(\{y_n\}_{n=1}^{+\infty}).\end{aligned}$$

Thus, using Lemma 2.5, for all $t \in \mathfrak{J}$, $s \in [a, t]$, and $\tau \leq s$, one obtains

$$\begin{aligned}
&\Lambda\left(\left\{(\alpha-1)\int_a^t \varphi'(s)\mathcal{I}_{a^+}^{\alpha-1;\varphi} g(s, y_n(s))ds\right\}_{n=1}^{+\infty}\right) \\
&\leq \tfrac{4(\alpha-1)\sigma^*}{\Gamma(\alpha-1)}\widehat{\Lambda}(\{y_n\}_{n=1}^{+\infty})\int_a^t \varphi'(s)\int_a^s \varphi'(\tau)\varphi(s,\tau)^{\alpha-2}e^{\aleph\tau}d\tau ds \\
&\leq \tfrac{4(\alpha-1)\sigma^*}{\Gamma(\alpha-1)}\widehat{\Lambda}(\{y_n\}_{n=1}^{+\infty})\int_a^t \varphi'(s)e^{\aleph s}\int_a^s \varphi'(\tau)\varphi(s,\tau)^{\alpha-2}d\tau ds \\
&\leq \tfrac{4(\alpha-1)\sigma*}{\Gamma(\alpha)}\widehat{\Lambda}(\{y_n\}_{n=1}^{+\infty})\int_a^t \varphi'(s)\varphi(s,a)^{\alpha-1}e^{\aleph s}ds.
\end{aligned}$$

Multiplying both sides by $e^{-\aleph t}$, one obtains

$$\begin{aligned}
&\sup_{t\in\mathfrak{J}} e^{-\aleph t}\Lambda\left(\left\{(\alpha-1)\int_a^t \varphi'(s)\mathcal{I}_{a^+}^{\alpha-1;\varphi} g(s, y_n(s))ds\right\}_{n=1}^{+\infty}\right) \\
&\leq \tfrac{4\sigma^*}{\Gamma(\alpha-1)}\widehat{\Lambda}(\{y_n\}_{n=1}^{+\infty})\sup_{t\in\mathfrak{J}}\int_a^t \varphi'(s)\varphi(s,a)^{\alpha-1}e^{-\aleph(t-s)}ds.
\end{aligned} \tag{3.14}$$

So, by (3.11), (3.13), and (3.14), we have

$$\widehat{\Lambda}(\{(\mathcal{L}(y_n)\}_{n=1}^{+\infty}) \leq q(\aleph)\widehat{\Lambda}(\{y_n\}_{n=1}^{+\infty}) \leq q(\aleph)\widehat{\Lambda}(\mathbb{U}). \tag{3.15}$$

Next, by (3.10) and (3.15), fixing $\epsilon' > 0$, we have

$$\widehat{\Lambda}\Big(\mathcal{L}(\mathbb{U})\Big) \leq 2q(\aleph)\widehat{\Lambda}(\mathbb{U}) + \epsilon'.$$

Then,

$$\widehat{\Lambda}\Big(\mathcal{L}(\mathbb{U})\Big) \leq 2q(\aleph)\widehat{\Lambda}(\mathbb{U}). \tag{3.16}$$

Observe that, from the last estimates,

$$\widehat{\Lambda}(\mathcal{L}(\mathbb{U})) \leq 2q(\aleph)\widehat{\Lambda}(\mathbb{U}) < \epsilon \;\Rightarrow\; \widehat{\Lambda}(\mathbb{U}) < \frac{1}{2q(\aleph)}\epsilon.$$

Letting

$$\eta = \frac{1-2q(\aleph)}{2q(\aleph)}\epsilon, \tag{3.17}$$

one gets

$$\epsilon \leq \widehat{\Lambda}(\mathbb{U}) < \epsilon + \eta,$$

which means that $\mathcal{L} : \mathbf{B}_R \to \mathbf{B}_R$ satisfies Definition 2.5. Therefore, Theorem 2.2 entails that $\mathcal{L}$ admits at least one unique fixed point in $\mathbf{B}_R$ which is the solution of problem (1.1). □

4. Examples

Example 4.1. Let

$$\mathbb{X}_1 := \{u = (u_1, u_2, \cdots, u_n, \cdots) \;:\; u_n \to 0 \text{ as } n \to \infty\}$$

be the Banach space of real sequences converging to zero, equipped by

$$\|u\| = \sup_{n\geq 1} |u_n|.$$

Consider the following problem posed in $\mathbb{X}_1$:

$$\begin{cases} \left({}^c\mathcal{D}_{0^+}^{\alpha;\varphi} + \varpi {}^c\mathcal{D}_{0^+}^{\alpha-1;\varphi}\right) y(t) = g(t, y(t)),\ t \in \mathfrak{J} := [0,1], \\ y(0) = y'(0) = (0, 0, \cdots, 0, \cdots). \end{cases} \tag{4.1}$$

Note that, problem (4.1) is a particular case of (1.1), where:

$$\varphi(t) = e^t, \quad [a, b] = [0, 1]$$

and $g : [0,1] \times \mathbb{X}_1 \to \mathbb{X}_1$, given by

$$g(t, y) = \left\{ \frac{(1 + \sin(|y_n|))}{(e^t + 1)} + 13t^5 \cos(9t) \right\}_{n\geq 1}, \tag{4.2}$$

for $t \in [0, 1]$, $y = \{y_n\}_{n\geq 1} \in \mathbb{X}_1$.

Condition (A1) is satisfied. Moreover, for any $u_1, u_2 \in \mathbb{X}_1$ and $t \in [0, 1]$, we have

$$\begin{aligned} \|g(t, u_1) - g(t, u_2)\| &\leq \tfrac{1}{(e^t+1)} \|u_1 - u_2\| \\ &\leq \tfrac{1}{2} \|u_1 - u_2\|. \end{aligned}$$

So, condition (A2) is satisfied with

$$L_g = \frac{1}{2}.$$

Thus, with the assistance of Theorem 3.1, problem (4.1) has a unique solution $y \in C([0, 1], \mathbb{X}_1)$.

Example 4.2. Let

$$\mathbb{X}_2 := \left\{ u = (u_1, u_2, \cdots, u_n, \cdots) \ :\ \sum_{n=1}^{\infty} |u_n| < \infty \right\}$$

be the Banach space with the norm

$$\|u\| = \sum_{n=1}^{\infty} |u_n|.$$

We recall that the Hausdorff MNC in $(\mathbb{X}_2, \|\cdot\|)$ is defined as follows (see [19]):

$$\Lambda(\mathbb{B}) = \lim_{j\to\infty} \left[\sup_{u\in\mathbb{B}} \left(\sum_{n\geq j} |u_n| \right) \right].$$

Consider the following problem posed in $\mathbb{X}_2$:

$$\begin{cases} \left({}^c\mathcal{D}_{0^+}^{\alpha;\varphi} + \varpi {}^c\mathcal{D}_{0^+}^{\alpha-1;\varphi}\right) y(t) = g(t, y(t)),\ t \in \mathfrak{J} := [0, b], 0 < b < \left(\frac{\zeta}{0.2}\right)^{1/\alpha} \\ y(0) = y'(0) = (0, 0, \cdots, 0, \cdots), \end{cases} \tag{4.3}$$

where $\zeta = \min_{\alpha\in(1,2)} \alpha\Gamma(\alpha-1)$, and we take

$$[a,b] = [0,b], \ \varphi(t) = t$$

and $g : [0,b]\times\mathbb{X}_2 \to \mathbb{X}_2$, given by

$$g(t,y) = \left\{ \frac{1}{(e^{5t}+4)} \left(\frac{1}{2^n} + \ln(|y_n|+1) \right) \right\}_{n\geq 1}, \tag{4.4}$$

for $t \in [0,b]$, $y = \{y_n\}_{n\geq1} \in \mathbb{X}_2$.

Evidently, condition (A1) holds and

$$\|g(t,y)\| \leq \frac{1}{(e^{5t}+4)}(\|y_n\|+1), \quad y \in \mathbb{X}_2.$$

Thus, condition (A3) holds with $\xi(t) = \kappa(t) = \frac{1}{(e^{5t}+13)}$, and one gets $\xi^* = \kappa^* = 0.2$. On the other side, for any bounded set $\mathbb{U} \subset \mathbb{X}_2$, we obtain

$$\Lambda(g(t,\mathbb{U})) \leq \frac{1}{(e^{5t}+13)}\Lambda(\mathbb{U}), \quad \text{for any } \ t \in [0,b].$$

Hence, (A4) is verified. Now, we can choose R such that

$$\frac{(\alpha-1)\varphi(b,a)^\alpha \xi^*}{\Gamma(\alpha+1) - \kappa^*(\alpha-1)\varphi(b,a)^\alpha} \leq R.$$

This function satisfies condition (A5), and from $b < \left(\frac{\zeta}{0.2}\right)^{1/\alpha}$ we get

$$\Gamma(\alpha+1) > \kappa^*(\alpha-1)\varphi(b,a)^\alpha.$$

Finally, all the assumptions of Theorem 3.2 are verified, and thus problem (4.3) has at least one solution $y \in C([0,b],\mathbb{X}_2)$.

5. Conclusions

We concluded that the quantitative study for a class of nonlinear fractional differential equations involving φ-Caputo type of order $\alpha \in (1,2)$ in an infinite-dimensional Banach space framework is achieved. In this context, the results proved in [14, 23] can be regarded as a special case. Our proof combines results from MNC, Weissinger's, and Meir-Keeler's fixed point theorems. In the future, new work may explore some qualitative aspects of solutions to problem (1.1). Also, for more about fractional functions, we recommend [23–27].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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