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Research article

Soft closure spaces via soft ideals

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Abstract: This paper was devoted to defining new soft closure operators via soft relations and soft ideals, and consequently new soft topologies. The resulting space is a soft ideal approximation. Many of the well known topological concepts were given in the soft set-topology. Particularly, it introduced the notations of soft accumulation points, soft continuous functions, soft separation axioms, and soft connectedness. Counterexamples were introduced to interpret the right implications. Also, a practical application of the new soft approximations was explained by an example of a real-life problem.

Keywords: soft binary relations; soft approximation space; soft connectedness; soft separation axioms; soft ideal
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1. Introduction

The paper [1] by Pawlak was the first article focused on the rough area between the interior set A° and the closure set $\overline{A}$ of a subset A in a universal set X. This idea led to many applications in decision theory. The theory of rough sets is constructed using the equivalence classes as its building blocks.

The most efficacious tools to study the generalization of rough set theory are the neighborhood systems. The main idea in this theory is the upper and lower approximations that have been defined using different types of neighborhoods instead of equivalence classes such as left and right neighborhoods [2–5], minimal left neighborhoods [6] and minimal right neighborhoods [7], and the intersection of minimal left and right neighborhoods [8]. Afterwards, the approximations by minimal right neighborhoods which are determined by reflexive relations that form the base of the topological space defined in [9]. In 2018, Dai et al. [10] presented new kind of neighborhoods, namely the maximal right neighborhoods which were determined by similarity relations and have been used to propose three new kinds of approximations. Dai et al.'s approximations [10] differed from Abo-Tabl's approximations [9] in that the corresponding upper and lower approximations, boundary regions, accuracy measures, and roughness measures in two types of Dai et al.'s approximations [10] had a monotonicity. Later on, Al-shami [11] embraced a new type of neighborhood systems namely, the intersection of maximal right and left neighborhoods, and then used this type to present new approximations. These approximations improved the accuracy measures more than Dai et al.'s approximations [10]. Al-shami's [11] accuracy measures preserved the monotonic property under any arbitrary relation. The paper [12], by Molodtsov, was the first article that defined the notion "soft set", and it has many applications in uncertainty area or ambiguity decision. A theoretical research on soft set theory was given in [13] by Maji et al. The paper [14] by Ali et al. proposed many soft set-theoretical notions such as union, intersection, difference and complement. [15–20] objected to developing the theory and the applications of soft sets. In [21], the authors introduced the soft ideal notion. It is a completely new approach for modeling vagueness and uncertainty by reducing the boundary region and increasing the accuracy of a rough set which helped scholars to solve many real-life problems [4, 22–25]. Recently, many extensions of the classical rough set approximations have been applied to provide new rough paradigms using certain topological structures and concepts like subset neighborhoods, containment neighborhoods, and maximal and minimal neighborhoods to deal with rough set notions and address some real-life problems [2, 4, 26–28]. Numerous researchers have recently examined some topological concepts, including continuity, separation axioms, closure spaces, and connectedness in ideal approximation spaces [29–31]. Ordinary rough sets were defined using an equivalence relation R on X, and produced two approximations, one is lower and second is upper. The space (X, R) is named approximation space. In the soft case, soft roughness used soft relations [32]. Some researchers transferred the common definitions in set-topology to soft set-topology, depending on that soft topology is an extension to the usual topology as explained in [15]. Many researchers objected to the basics of set-topology and subsequently the well-known embedding theorems but in point of view of soft set-topology with some real-life applications (see [33–38]). This paper used the notion of soft binary relations to ensure that the soft interior and soft closure in approximation spaces utilizing soft ideal to generate soft ideal approximation topological spaces based on soft minimal neighborhoods. We illustrated that soft rough approximations [17] are special cases of the current soft ideal approximations. Soft accumulation points, soft exterior sets, soft dense sets, and soft nowhere dense sets with respect to these spaces were defined and studied, and we gave some examples. We introduce and study soft ideal accumulation points in such spaces under a soft ideal defined on the given soft ideal. Soft separation axioms with respect to these soft ideal approximation spaces are reformulated via soft relational concepts and compared with examples to show their implications. In addition, we reformulate and study soft connectedness in these soft ideal approximation spaces. Finally, we defined soft boundary region and soft accuracy measure with respect to our soft ideal approximation spaces. We added two real life examples to illustrate the importance of the results obtained in this paper.

This paper is divided into 6 sections beyond the introduction and the preliminaries. Section 3 defined the soft approximation spaces using a soft ideal. Section 4 is the main section of the manuscript and displays the properties of soft sets in the soft ideal approximation spaces. It has been generated using the concepts of $R < x > R$, soft neighborhoods and soft ideals. We study the main properties in soft ideal approximation spaces which are generalizations of the same properties of ideal approximation spaces given by Abbas et al. [31] and provide various illustrative examples. Section 5 introduced soft lower separation axioms via soft binary relations and soft ideal as a generalization of lower separation axioms given in [31]. We scrutinized its essential characterizations of some of its relationships associated with the soft ideal closure operators. Some illustrative examples are given. Section 6 reformulated and studied soft connectedness in [31] with respect to these soft ideal approximation spaces. Some examples are submitted to explain the definitions. Section 7 is devoted to comparing between the current purposed methods in Definitions 3.4–3.6 and to demonstrate that the method given in Definition 3.6 is the best in terms of developing the soft approximation operators and the values of soft accuracy. That is, the third approach in Definition 3.6 produces soft accuracy measures of soft subsets higher than their counterparts displayed in previous method 2.4 in [17]. Moreover, we applied these approaches to handle real-life problems. Section 8 is the conclusion.

2. Preliminaries

Through this paper, X stands for the universal set of objects, E denotes the set of parameters, $\mathcal{L}_E$ denotes for a soft ideal, R_E as a soft binary relation, $P(X)$ represents all subsets of X, and $SS(X)$ refers to the set of all soft subsets of X. All basic notions and notations of soft sets are found in [12, 13, 15, 39, 40].

If (F, E) is a soft set of X and $x \in X$, then $x \check{\in} (F, E)$ whenever $x \in F(e)$ for each $e \in E$. A soft set (F, E) of X with $F(e) = \{x\}$ for each $e \in E$ is called a singleton soft set or a soft point and it is represented by x_E or (x, E). Let (F_1, E), $(F_2, E) \check{\in} SS(X)_E$. Then, (F_1, E) is a soft subset of (F_2, E), represented by $(F_1, E) \sqsubseteq (F_2, E)$, if $F_1(e) \subseteq F_2(e), \forall e \in E$. In that case, (F_1, E) is called a soft subset of (F_2, E) and (F_2, E) is said to be a soft supset of (F_1, E), $(F_2, E) \sqsupseteq (F_1, E)$. Two soft subset (F_1, E) and (F_2, E) over X are called equal if (F_1, E) is soft subset of (F_2, E) and (F_1, E) is soft supset of (F_2, E). A soft set (F, E) over X is called a NULL soft set written as Φ if for each $e \in E, F(e) = \phi$. Let A be a non-empty subset of X, then $\tilde{A}_E$ or $\tilde{A}$ represents the absolute soft set (A, E) of X in which $A(e) = A$, for each $e \in E$. The soft intersection (resp. soft union) of (F_1, E) and (F_2, E) over X denoted by $(F_1 \sqcap F_2, E)$ (resp. $(F_1 \sqcup F_2, E)$) and defined as $(F_1 \sqcap F_2)(e) = F_1(e) \cap F_2(e)$ (resp. $(F_1 \sqcup F_2)(e) = F_1(e) \cup F_2(e)$) for each $e \in E$. Complementing a soft set (F, E) is represented by $(F, E)^c$ and it is defined as $(F, E)^c = (F^c, E)$ where $F^c : E \longrightarrow P(X)$ is a mapping defined by $F^c(e) = X - F(e)$ for all $e \in E$, and F^c is then a soft complement function of F.

Definition 2.1. [32] Let $(R, E) = R_E$ be a soft set of $X \times X$, that is $R : E \longrightarrow P(X \times X)$. Then, R_E is said to be a soft binary relation of X. R_E is a collection of parameterized binary relations of X, from that $R(e)$ is a binary relation on X for all parameters $e \in E$. The set of all soft binary relations of X is denoted by $SBr(X)$.

Definition 2.2. [15] Let $\tilde{\tau}$ be a collection of soft sets over a universe X with a fixed set of parameters E. Then, $\tilde{\tau} \sqsubseteq SS(X)_E$ is called a soft topology on X if

(1) $\tilde{X}, \Phi_E \check{\in} \tilde{\tau}$,
(2) the intersection of any two soft sets in $\tilde{\tau}$ belongs to $\tilde{\tau}$,
(2) the union of any number of soft sets in in $\tilde{\tau}$ belongs to $\tilde{\tau}$.

The triplet $(X, \tilde{\tau}, E)$ is called a soft topological space over X.

Definition 2.3. [28] A mapping $Cl : SS(X)_E \longrightarrow SS(X)_E$ is called a soft closure operator on X if it satisfies these properties for every $(F,E),\ (G,E) \check{\in} SS(X)_E$:

(1) $Cl(\Phi) = \Phi$,
(2) $(F,E) \subseteq Cl(F,E)$,
(3) $Cl[(F,E) \sqcup (G,E)] = Cl(F,E) \sqcup Cl(G,E)$,
(4) $Cl(Cl(F,E)) = Cl(F,E)$.

Definition 2.4. [17] Let $R : E \longrightarrow P(X_1 \times X_2)$ and $A \subseteq X_2$. Then, the sets $\underline{R}^A(e), \overline{R}^A(e)$ could be defined by

$$\underline{R}^A(e) = \{x \in X_1 : \phi \neq xR(e) \subseteq A\},$$
$$\overline{R}^A(e) = \{x \in X_1 : xR(e) \cap A \neq \phi\}$$

where $xR(e) = \{y \in X_2 : (x,y) \in R(e)\}$. Moreover, $\underline{R} : E \longrightarrow P(X_1)$ and $\overline{R} : E \longrightarrow P(X_1)$ and we say (X_1, X_2, R) a generalized soft approximation space.

Definition 2.5. [21] Let $\mathcal{L}_E$ be a non-empty family of soft sets of X. Then, $\mathcal{L}_E \sqsubseteq SS(X)_E$ is said to be a soft ideal on X if the following properties are fulfilled:

(1) $\Phi \check{\in} \mathcal{L}_E$,
(2) $(F,E) \check{\in} \mathcal{L}_E$ and $(G,E) \sqsubseteq (F,E)$ imply $(G,E) \check{\in} \mathcal{L}_E$,
(3) $(F,E),\ (G,E) \check{\in} \mathcal{L}_E$ imply $(F,E) \sqcup (G,E) \check{\in} \mathcal{L}_E$.

3. Soft approximation spaces via soft ideals

In this section, we define the soft approximation spaces using soft ideals.

Definition 3.1. Let R_E be a soft binary relation of X and $(x,y) \in X \times X$. Then, $(x,y) \check{\in} R$ whenever $(x,y) \in R(e)$ for each $e \in E$.

Definition 3.2. Let R_E be a soft binary relation of X. Then, the soft afterset of $x \check{\in} \tilde{X}$ is $xR = \{y \check{\in} \tilde{X} : (x,y) \check{\in} R\}$. Also, the soft foreset of $x \check{\in} \tilde{X}$ is $Rx = \{y \check{\in} \tilde{X} : (y,x) \check{\in} R\}$.

Definition 3.3. Let R_E be a soft binary relation over X. Then, a soft set $< x > R : E \longrightarrow P(X)$ is defined by

$$< x > R = \begin{cases} \sqcap_{x \in yR}(yR) & \text{if } \exists y : x \check{\in} yR, \\ \Phi & o.w. \end{cases}$$

Also, $R < x >: E \longrightarrow P(X)$ is the intersection of all foresets containing x, that is,

$$R < x >= \begin{cases} \sqcap_{x \in yR}(Ry) & \text{if } \exists y : x \check{\in} Ry, \\ \Phi & o.w. \end{cases}$$

Also, $R < x > R = R < x > \sqcap < x > R$.

Lemma 3.1. *Let R_E be a soft binary relation over X. Then,*

(1) If $x \check{\in} <y>R$, then $<x>R \sqsubseteq <y>R$.
(2) If $x \check{\in} R<y>R$, then $R<x>R \sqsubseteq R<y>R$.

Proof. (1) Let $z \check{\in} <x>R = \sqcap_{x \check{\in} wR}(wR)$. Then, z is contained in any wR which contain x, and since x is contained in any uR which contains y, we have $z \check{\in} <y>R$. Hence, $<x>R \sqsubseteq <y>R$.
(2) Straightforward from part (1).

□

Definition 3.4. Let R_E be a soft binary relation of X. For a soft set $(F,E) \check{\in} SS(X)_E$, the *soft lower approximation* $\underline{Apr}^1_S(F,E)$ and the *soft upper approximation* $\overline{Apr}^1_S(F,E)$ are defined by:

$$\underline{Apr}^1_S(F,E) = \{x \check{\in} (F,E) : <x>R \sqsubseteq (F,E)\}, \tag{3.1}$$

$$\overline{Apr}^1_S(F,E) = (F,E) \sqcup \{x \check{\in} \tilde{X} : <x>R \sqcap (F,E) \neq \Phi\}. \tag{3.2}$$

Theorem 3.1. *Let (F,E), $(G,E) \check{\in} SS(X)_E$. The soft upper approximation defined by Eq (3.2) has the following properties:*

(1) $\overline{Apr}^1_S(\Phi) = \Phi$ and $\overline{Apr}^1_S(\tilde{X}) = \tilde{X}$,
(2) $(F,E) \subseteq \overline{Apr}^1_S(F,E)$,
(3) $(F,E) \sqsubseteq (G,E) \Rightarrow \overline{Apr}^1_S(F,E) \sqsubseteq \overline{Apr}^1_S(G,E)$,
(4) $\overline{Apr}^1_S[(F,E) \sqcap (G,E)] \sqsubseteq \overline{Apr}^1_S(F,E) \sqcap \overline{Apr}^1_S(G,E)$,
(5) $\overline{Apr}^1_S[(F,E) \sqcup (G,E)] = \overline{Apr}^1_S(F,E) \sqcup \overline{Apr}^1_S(G,E)$,
(6) $\overline{Apr}^1_S(\overline{Apr}^1_S(F,E)) = \overline{Apr}^1_S(F,E)$,
(7) $\overline{Apr}^1_S(F,E) = [\underline{Apr}^1_S(F,E)^c]^c$.

Proof. (1), (2) It is clear from Definition 3.4.
(3) Let $x \check{\in} \overline{Apr}^1_S[(F,E)$. Then, $<x>R \sqcap (F,E) \neq \Phi$. Since $(F,E) \sqsubseteq (G,E)$, $<x>R \sqcap (G,E) \neq \Phi$. Therefore, $x \check{\in} \overline{Apr}^1_S(G,E)$. Hence, $\overline{Apr}^1_S(F,E) \sqsubseteq \overline{Apr}^1_S(G,E)$.
(4) Immediately by part (3).
(5) $\overline{Apr}^1_S[(F,E) \sqcup (G,E)] = [(F,E) \sqcup (G,E)] \sqcup \{x \check{\in} \tilde{X} : <x>R \sqcap [(F,E) \sqcup (G,E)] \neq \Phi\}$. Then, $\overline{Apr}^1_S[(F,E) \sqcup (G,E)] = [(F,E) \sqcup \{x \check{\in} \tilde{X} : <x>R \sqcap (F,E) \neq \Phi\}] \sqcup [(G,E) \sqcup \{x \check{\in} \tilde{X} : <x>R \sqcap (G,E) \neq \Phi\}]$. Hence, $\overline{Apr}^1_S[(F,E) \sqcup (G,E)] = \overline{Apr}^1_S((F,E)) \sqcup \overline{Apr}^1_S((G,E))$.
(6) From part (2), we have $\overline{Apr}^1_S(F,E) \sqsubseteq \overline{Apr}^1_S(\overline{Apr}^1_S(F,E))$.
Conversely, let $x \check{\in} \overline{Apr}^1_S(\overline{Apr}^1_S(F,E))$. Then, $<x>R \sqcap \overline{Apr}^1_S(F,E) \neq \Phi$. Thus, there exists $y \check{\in} <x>R \sqcap \overline{Apr}^1_S(F,E)$. That means $<y>R \sqsubseteq <x>R$ (by Lemma 3.1 part (1)) and $<x>R \sqcap (F,E) \neq \Phi$. Hence, $x \check{\in} \overline{Apr}^1_S(F,E)$. This completes the proof.

(7)

$$\begin{aligned}[\underline{Apr}^1_S(F,E)^c]^c &= [(F,E)^c \sqcap \{x \,\check{\in}\, \tilde{X} \ :< x > R \sqsubseteq (F,E)^c\}]^c \\ &= (F,E) \sqcup \{x \,\check{\in}\, \tilde{X} \ :< x > R \sqcap (F,E) \neq \Phi\} \\ &= \overline{Apr}^1_S(F,E).\end{aligned}$$

□

Example 3.1. *Let $X = \{a,b,c,d\}$, $E = \{e_1, e_2\}$ and $R_E = \{(e_1, \{(a,a),(a,b),(b,d),(c,d),(d,c),(d,d)),$ $(e_2, \{(a,a),(a,b),(a,c),(b,d),(b,c),(c,d),(d,c),(d,d),(d,b))\}\}$. Then, we have $< a > R =< b > R = \{(e_1, \{a,b\}),(e_2, \{a,b\})\}, < c > R = \{(e_1, \{c,d\}),(e_2, \{c,d\})\}$, $< d > R = \{(e_1, \{d\}),(e_2, \{d\})\}$. Suppose $(F_1, E) = \{(e_1, \{a,c\}),(e_2, \{a,c\})\}$ and $(F_2, E) = \{(e_1, \{a,d\}),(e_2, \{a,d\})\}$. Therefore, $\overline{Apr}^1_S(F_1,E) = (F,E) \sqcup \{x \,\check{\in}\, \tilde{X} \ :< x > R \sqcap (F,E) \neq \Phi\} = \{(e_1, \{a,b,c\}),(e_2, \{a,b,c\})\}$, $\overline{Apr}^1_S(F_2,E) = \tilde{X}$ and $\overline{Apr}^1_S[(F_1,E) \sqcap (F_2,E)] = \{(e_1, \{a,b\}),(e_2, \{a,b\})\}$. Hence, $\overline{Apr}^1_S[(F_1,E) \sqcap (F_2,E)] \neq \overline{Apr}^1_S(F_1,E) \sqcap \overline{Apr}^1_S(F_2,E)$.*

Corollary 3.1. *Let R_E be a soft binary relation of X. Then, the soft operator $\overline{Apr}^1_S : SS(X)_E \longrightarrow SS(X)_E$ is said to be a soft closure operator and $(X, \overline{Apr}^1_S)$ is standing for a soft closure space. Moreover, it induces a soft topology on X written as $\tilde{\tau}^1_S$ and defined by $\tilde{\tau}^1_S = \{(F,E) \,\check{\in}\, SS(X)_E : \overline{Apr}^1_S(F,E)^c = (F,E)^c\}$.*

Theorem 3.2. *Let (F,E), $(G,E) \,\check{\in}\, SS(X)_E$. The soft lower approximation defined by Eq (3.1) has the following properties:*

(1) $\underline{Apr}^1_S(\Phi) = \Phi$ and $\underline{Apr}^1_S(\tilde{X}) = \tilde{X}$,
(2) $\underline{Apr}^1_S(F,E) \sqsubseteq (F,E)$,
(3) $(F,E) \sqsubseteq (G,E) \Rightarrow \underline{Apr}^1_S(F,E) \sqsubseteq \underline{Apr}^1_S(G,E)$,
(4) $\underline{Apr}^1_S[(F,E) \sqcap (G,E)] = \underline{Apr}^1_S(F,E) \sqcap \underline{Apr}^1_S(G,E)$,
(5) $\underline{Apr}^1_S[(F,E) \sqcup (G,E)] \sqsupseteq \underline{Apr}^1_S(F,E) \sqcup \underline{Apr}^1_S(G,E)$,
(6) $\underline{Apr}^1_S(\underline{Apr}^1_S(F,E)) = \underline{Apr}^1_S(F,E)$,
(7) $\underline{Apr}^1_S(F,E) = [\overline{Apr}^1_S(F,E)^c]^c$.

Proof. It is the same as given in Theorem 3.1. □

Note that the equality in Theorem 3.2 part (5) did not hold in general (see Example 3.1). Take $(F_1, E) = \{(e_1, \{b,c\}),(e_2, \{b,c\})\}$ and $(F_2, E) = \{(e_1, \{b,d\}),(e_2, \{b,d\})\}$. Then, $\underline{Apr}^1_S(F_1,E) = \{x \,\check{\in}\, (F_1,E) \ :< x > R \sqsubseteq (F_1,E)\} = \Phi$, $\underline{Apr}^1_S(F_2,E) = \{(e_1, \{d\}),(e_2, \{d\})\}$ and $\underline{Apr}^1_S[(F_1,E) \sqcup (F_2,E)] = \{(e_1, \{c,d\}),(e_2, \{c,d\})\}$, which means that $\underline{Apr}^1_S[(F,E) \sqcup (G,E)] \neq \underline{Apr}^1_S(F,E) \sqcup \underline{Apr}^1_S(G,E)$.

Definition 3.5. Let R_E be a soft binary relation over X and $\mathcal{L}_E$ a soft ideal on X. For any soft set $(F,E)\,\check{\in}\,SS(X)_E$, the *soft lower approximation* and the *soft upper approximation* of (F,E) by $\mathcal{L}_E$, denoted by $\underline{Apr}^2_S(F,E)$ and $\overline{Apr}^2_S(F,E)$ are defined by:

$$\underline{Apr}^2_S(F,E) = \{x\,\check{\in}\,(F,E)\ :<x>R\sqcap(F,E)^c\,\check{\in}\,\mathcal{L}_E\}, \tag{3.3}$$

$$\overline{Apr}^2_S(F,E) = (F,E)\sqcup\{x\,\check{\in}\,\tilde{X}\ :<x>R\sqcap(F,E)\,\check{\notin}\,\mathcal{L}_E\}. \tag{3.4}$$

Theorem 3.3. *Let $(F,E),\ (G,E)\,\check{\in}\,SS(X)_E$. The soft upper approximation defined by Eq (3.4) has the following properties:*

(1) $\overline{Apr}^2_S(\Phi)=\Phi$ *and* $\overline{Apr}^2_S(\tilde{X})=\tilde{X}$,
(2) $(F,E)\subseteq\overline{Apr}^2_S(F,E)$,
(3) $(F,E)\sqsubseteq(G,E)\Rightarrow\overline{Apr}^2_S(F,E)\sqsubseteq\overline{Apr}^2_S(G,E)$,
(4) $\overline{Apr}^2_S[(F,E)\sqcap(G,E)]\sqsubseteq\overline{Apr}^2_S(F,E)\sqcap\overline{Apr}^2_S(G,E)$,
(5) $\overline{Apr}^2_S[(F,E)\sqcup(G,E)]=\overline{Apr}^2_S(F,E)\sqcup\overline{Apr}^2_S(G,E)$,
(6) $\overline{Apr}^2_S(\overline{Apr}^2_S(F,E))=\overline{Apr}^2_S(F,E)$,
(7) $\overline{Apr}^2_S(F,E)=[\underline{Apr}^2_S(F,E)^c]^c$.

Proof. (1), (2) Direct from Definition 3.5.

(3) Let $x\,\check{\in}\,\overline{Apr}^2_S[(F,E)$. Thus, $<x>R\sqcap(F,E)\,\check{\notin}\,\mathcal{L}_E$. Since $(F,E)\sqsubseteq(G,E)$ and $\mathcal{L}_E$ is a soft ideal, $<x>R\sqcap(G,E)\,\check{\notin}\,\mathcal{L}_E$. Therefore, $x\,\check{\in}\,\overline{Apr}^2_S(G,E)$. Hence, $\overline{Apr}^2_S(F,E)\sqsubseteq\overline{Apr}^2_S(G,E)$.

(4) Straightforward by part (3).

(5) $\overline{Apr}^2_S[(F,E)\sqcup(G,E)]=[(F,E)\sqcup(G,E)]\sqcup\{x\,\check{\in}\,\tilde{X}\ :<x>R\sqcap[(F,E)\sqcup(G,E)]\,\check{\notin}\,\mathcal{L}_E\}$. Then, $\overline{Apr}^2_S[(F,E)\sqcup(G,E)]=[(F,E)\sqcup\{x\,\check{\in}\,\tilde{X}\ :<x>R\sqcap(F,E)\,\check{\notin}\,\mathcal{L}_E\}]\sqcup[(G,E)\sqcup\{x\,\check{\in}\,\tilde{X}\ :<x>R\sqcap(G,E)\,\check{\notin}\,\mathcal{L}_E\}]$. Hence, $\overline{Apr}^2_S[(F,E)\sqcup(G,E)]=\overline{Apr}^2_S((F,E))\sqcup\overline{Apr}^2_S((G,E))$.

(6) From part (2), we have $\overline{Apr}^2_S(F,E)\sqsubseteq\overline{Apr}^2_S(\overline{Apr}^2_S(F,E))$.
Conversely, let $x\,\check{\in}\,\overline{Apr}^2_S(\overline{Apr}^2_S(F,E))$. Then, $<x>R\sqcap\overline{Apr}^2_S(F,E)\,\check{\notin}\,\mathcal{L}_E$. Therefore, $<x>R\sqcap\overline{Apr}^1_S(F,E)\neq\Phi$. Thus, there exists $y\,\check{\in}\ <x>R\sqcap\overline{Apr}^2_S(F,E)$. That means $<y>R\sqsubseteq<x>R$ (by Lemma 3.1 part (1)) and $<y>R\sqcap(F,E)\,\check{\notin}\,\mathcal{L}_E$. Then, $<x>R\sqcap(F,E)\,\check{\notin}\,\mathcal{L}_E$. Hence, $x\,\check{\in}\,\overline{Apr}^2_S(F,E)$. This completes the proof.

(7)

$$\begin{aligned}[\underline{Apr}^2_S(F,E)^c]^c &= [(F,E)^c\sqcap\{x\,\check{\in}\,\tilde{X}\ :<x>R\sqcap(F,E)\,\check{\in}\,\mathcal{L}_E\}]^c\\ &= (F,E)\sqcup\{x\,\check{\in}\,\tilde{X}\ :<x>R\sqcap(F,E)\,\check{\notin}\,\mathcal{L}_E\}\\ &= \overline{Apr}^2_S(F,E).\end{aligned}$$

□

Corollary 3.2. *Let R_E be a soft binary relation over X and $\mathcal{L}_E$ be a soft ideal on X. Then, the soft operator $\overline{Apr}^2_S : SS(X)_E\longrightarrow SS(X)_E$ is said to be a soft closure operator and $(X,\overline{Apr}^2_S)$ is standing for a soft closure space. Moreover, it induces a soft topology on X written as $\tilde{\tau}^2_S$ and defined by $\tilde{\tau}^2_S=\{(F,E)\,\check{\in}\,SS(X)_E : \overline{Apr}^2_S(F,E)^c=(F,E)^c\}$.*

Theorem 3.4. *Let $(F,E),\ (G,E) \check{\in} SS(X)_E$. The soft lower approximation defined by Eq (3.3) has the following properties:*

(1) $\underline{Apr}^2_S(\Phi) = \Phi$ *and* $\underline{Apr}^2_S(\tilde{X}) = \tilde{X}$,
(2) $\underline{Apr}^2_S(F,E) \sqsubseteq (F,E)$,
(3) $(F,E) \sqsubseteq (G,E) \Rightarrow \underline{Apr}^2_S(F,E) \sqsubseteq \underline{Apr}^2_S(G,E)$,
(4) $\underline{Apr}^2_S[(F,E) \sqcap (G,E)] = \underline{Apr}^2_S(F,E) \sqcap \underline{Apr}^2_S(G,E)$,
(5) $\underline{Apr}^2_S[(F,E) \sqcup (G,E)] \sqsupseteq \underline{Apr}^2_S(F,E) \sqcup \underline{Apr}^2_S(G,E)$,
(6) $\underline{Apr}^2_S(\underline{Apr}^2_S(F,E)) = \underline{Apr}^2_S(F,E)$,
(7) $\underline{Apr}^2_S(F,E) = [\overline{Apr}^2_S(F,E)^c]^c$.

Proof. It is similar to that was given in Theorem 3.3. □

Definition 3.6. Let R_E be a soft binary relation over X and $\mathcal{L}_E$ be a soft ideal on X. For any soft set $(F,E) \check{\in} SS(X)_E$, the *soft lower approximation* and *soft upper approximation* of (F,E) by $\mathcal{L}_E$, denoted by $\underline{Apr}^3_S(F,E)$ and $\overline{Apr}^3_S(F,E)$ are defined by:

$$\underline{Apr}^3_S(F,E) = \{x \check{\in} (F,E)\ : R < x > R \sqcap (F,E)^c \check{\in} \mathcal{L}_E\}, \tag{3.5}$$

$$\overline{Apr}^3_S(F,E) = (F,E) \sqcup \{x \check{\in} \tilde{X}\ : R < x > R \sqcap (F,E) \check{\notin} \mathcal{L}_E\}. \tag{3.6}$$

Theorem 3.5. *Let $(F,E),\ (G,E) \check{\in} SS(X)_E$. The soft upper approximation defined by Eq (3.6) has the following properties:*

(1) $\overline{Apr}^3_S(\Phi) = \Phi$ *and* $\overline{Apr}^3_S(\tilde{X}) = \tilde{X}$,
(2) $(F,E) \subseteq \overline{Apr}^3_S(F,E)$,
(3) $(F,E) \sqsubseteq (G,E) \Rightarrow \overline{Apr}^3_S(F,E) \sqsubseteq \overline{Apr}^3_S(G,E)$,
(4) $\overline{Apr}^3_S[(F,E) \sqcap (G,E)] \sqsubseteq \overline{Apr}^3_S(F,E) \sqcap \overline{Apr}^3_S(G,E)$,
(5) $\overline{Apr}^3_S[(F,E) \sqcup (G,E)] = \overline{Apr}^3_S(F,E) \sqcup \overline{Apr}^3_S(G,E)$,
(6) $\overline{Apr}^3_S(\overline{Apr}^3_S(F,E)) = \overline{Apr}^3_S(F,E)$,
(7) $\overline{Apr}^3_S(F,E) = [\underline{Apr}^3_S(F,E)^c]^c$.

Proof. It is clear from Theorem 3.3. □

Corollary 3.3. *Let R_E be a soft binary relation over X and $\mathcal{L}_E$ be a soft ideal on X. Then, the soft operator $\overline{Apr}^3_S : SS(X)_E \longrightarrow SS(X)_E$ is said to be a soft closure operator and $(X, \overline{Apr}^3_S)$ is standing for a soft closure space. In addition, $(X, R_E, \mathcal{L}_E)$ is said to be a soft ideal approximation space. Moreover, it induces a soft topology on X written as $\tilde{\tau}^3_S$ and defined by $\tilde{\tau}^3_S = \{(F,E) \check{\in} SS(X)_E : \overline{Apr}^3_S(F,E)^c = (F,E)^c\}$. It is clear that $\tilde{\tau}^1_S \sqsubseteq \tilde{\tau}^2_S \sqsubseteq \tilde{\tau}^3_S$.*

Theorem 3.6. *Let $(F,E),\ (G,E) \check{\in} SS(X)_E$. The soft lower approximation defined by Eq (3.5) has the following properties:*

(1) $\underline{Apr}^3_S(\Phi) = \Phi$ *and* $\underline{Apr}^3_S(\tilde{X}) = \tilde{X}$,
(2) $\underline{Apr}^3_S(F, E) \sqsubseteq (F, E)$,
(3) $(F, E) \sqsubseteq (G, E) \Rightarrow \underline{Apr}^3_S(F, E) \sqsubseteq \underline{Apr}^3_S(G, E)$,
(4) $\underline{Apr}^3_S[(F, E) \sqcap (G, E)] = \underline{Apr}^3_S(F, E) \sqcap \underline{Apr}^3_S(G, E)$,
(5) $\underline{Apr}^3_S[(F, E) \sqcup (G, E)] \sqsupseteq \underline{Apr}^3_S(F, E) \sqcup \underline{Apr}^3_S(G, E)$,
(6) $\underline{Apr}^3_S(\underline{Apr}^3_S(F, E)) = \underline{Apr}^3_S(F, E)$,
(7) $\underline{Apr}^3_S(F, E) = [\overline{Apr}^3_S(F, E)^c]^c$.

Corollary 3.4. *Let R_E be a soft binary relation over X, $(F, E) \check{\in} SS(X)_E$ and $\mathcal{L}_E$ be a soft ideal on X. Then,*

$$\underline{Apr}^1_S(F, E) \sqsubseteq \underline{Apr}^2_S(F, E) \sqsubseteq \underline{Apr}^3_S(F, E) \sqsubseteq (F, E) \subseteq \overline{Apr}^3_S(F, E) \subseteq \overline{Apr}^2_S(F, E) \subseteq \overline{Apr}^1_S(F, E).$$

Proof. Direct from Definitions 3.4–3.6, using Lemma 3.1. □

4. Properties of soft sets in the soft ideal approximation spaces

We dedicate this is the main section of the manuscript to display the properties of soft sets in the soft ideal approximation spaces. It has been generated using the concepts of $R < x > R$, soft neighborhoods and soft ideals. We study the main properties in soft ideal approximation spaces which are generalizations of the same properties of ideal approximation spaces given by Abbas et al. in [31] and provide various illustrative examples.

Lemma 4.1. *Let $(X, R_E, \mathcal{L}_E)$ is be a soft ideal approximation space. Then,*

(1) $\underline{Apr}^1_S(< x > R) =< x > R$,
(2) $\underline{Apr}^2_S(< x > R) =< x > R$,
(3) $\underline{Apr}^3_S(R < x > R) = R < x > R$.

Proof. We will ensure that item (1) and the other items will be similar. From Theorem 3.3 part (3), it is clear that $\underline{Apr}^2_S(< x > R) \sqsubseteq < x > R$.
Conversely, we will ensure that $< x > R \sqsubseteq \underline{Apr}^2_S(< x > R)$. Let $y \check{\in} < x > R$. Then, by Lemma 3.1 part(1), $< y > R \sqsubseteq < x > R$. Thus, $< y > R \sqcap (< x > R)^c = \Phi$. So, $< y > R \sqcap (< x > R)^c \check{\in} \mathcal{L}_E$. Hence, $y \check{\in} \underline{Apr}^2_S(< x > R)$. Thus, $< x > R \sqsubseteq \underline{Apr}^2_S(< x > R)$.
□

Proposition 4.1. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space. For $x \neq y \check{\in} \tilde{X}$,*

(1) $x \check{\in} \overline{Apr}^1_S(y_E)$ *iff* $< x > R \sqcap y_E \neq \Phi$ *and* $x \check{\notin} \overline{Apr}^1_S(y_E)$ *iff* $< x > R \sqcap y_E = \Phi$,
(2) $x \check{\in} \overline{Apr}^2_S(y_E)$ *iff* $< x > R \sqcap y_E \check{\notin} \mathcal{L}_E$ *and* $x \check{\notin} \overline{Apr}^2_S(y_E)$ *iff* $< x > R \sqcap y_E \check{\in} \mathcal{L}_E$,
(3) $x \check{\in} \overline{Apr}^3_S(y_E)$ *iff* $R < x > R \sqcap y_E \check{\notin} \mathcal{L}_E$ *and* $x \check{\notin} \overline{Apr}^3_S(y_E)$ *iff* $R < x > R \sqcap y_E \check{\in} \mathcal{L}_E$.

Proof. We will prove the second statement and the others will be similar. Let $x \check{\in} \overline{Apr}^2_S(y_E)$. Then, $x \check{\in} [y_E \sqcup \{z \check{\in} \tilde{X} : < z > R \sqcap y_E \check{\notin} \mathcal{L}_E\}]$. Thus, $< x > R \sqcap y_E \check{\notin} \mathcal{L}_E$. Conversely, let $< x > R \sqcap y_E \check{\notin} \mathcal{L}_E$. Then, by Definition 3.6, $x \check{\in} \overline{Apr}^2_S(y_E)$. □

Proposition 4.2. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space and $< x > R \tilde{\in} \mathcal{L}_E$. Then,*

(1) $\underline{Apr}^1_S(x_E) = x_E = \overline{Apr}^1_S(x_E)$,
(2) $\underline{Apr}^2_S(x_E) = x_E = \overline{Apr}^2_S(x_E)$,
(3) $\underline{Apr}^3_S(x_E) = x_E = \overline{Apr}^3_S(x_E)$.

Proof. We will prove that the second statement and the others will be similar. Let $< x > R \tilde{\in} \mathcal{L}_E$. Then, $< x > R \sqcap [x_E]^c \tilde{\in} \mathcal{L}_E$. Thus, $x \tilde{\in} \underline{Apr}^2_S(x_E)$. So, $\underline{Apr}^2_S(x_E) = x_E$. Also, $< x > R \tilde{\in} \mathcal{L}_E$ induces that $< x > R \sqcap y_E \tilde{\in} \mathcal{L}_E$ for all $y \tilde{\in} X$. Hence, $\overline{Apr}^2_S(x_E) = x_E$. □

Theorem 4.1. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space and $x \tilde{\in} \tilde{X}, (F, E) \tilde{\in} SS(X)_E$. If $< x > R \sqcap (F, E) \tilde{\in} \mathcal{L}_E$, then*

(1) $< x > R \sqcap \overline{Apr}^1_S(F, E) = \Phi$,
(2) $< x > R \sqcap \overline{Apr}^2_S(F, E) \tilde{\in} \mathcal{L}_E$,
(3) $R < x > R \sqcap \overline{Apr}^3_S(F, E) \tilde{\in} \mathcal{L}_E$.

Proof. We will prove the second part and the others will be similar. Suppose $< x > R \sqcap (F, E) \tilde{\in} \mathcal{L}_E$. It is clear that $[< x > R - x_E] \sqcap (F, E) \tilde{\in} \mathcal{L}_E$. Then, $x \tilde{\notin} D^*_S(F, E)$. Thus, $< x > R \sqcap D^*_S(F, E) = \Phi$. So, $< x > R \sqcap D^*_S(F, E) \tilde{\in} \mathcal{L}_E$. Hence, $[< x > R \sqcap (F, E) \sqcup D^*_S(F, E)] \tilde{\in} \mathcal{L}_E$. Therefore, $< x > R \sqcap \overline{Apr}^2_S(F, E) \tilde{\in} \mathcal{L}_E$. □

Definition 4.1. Let $(X, R_E, \mathcal{L}_E)$ be a *soft ideal approximation space* and $(F, E) \tilde{\in} SS(X)_E$. The soft exterior of (F, E) is $Ext^i_S(F, E) = \underline{Apr}^i_S(F, E)^c$, $i \in \{1, 2, 3\}$.

Lemma 4.2. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space and $(F, E), (G, E) \tilde{\in} SS(X)_E$. For $i \in \{1, 2, 3\}$, we have*

(1) $Ext^i_S(\Phi) = \tilde{X}$ *and* $Ext^i_S(\tilde{X}) = \Phi$,
(2) $Ext^i_S(F, E) \sqsubseteq (F, E)^c$,
(3) $(F, E) \sqsubseteq (G, E) \Rightarrow Ext^i_S(F, E) \sqsubseteq Ext^i_S(G, E)$,
(4) $Ext^i_S[(F, E) \sqcup (G, E)] = Ext^i_S(G, E) \sqcap Ext^i_S(F, E)$,
(5) $\underline{Apr}^i_S(F, E) = Ext^i_S[Ext^i_S(F, E)]$,
(6) $Ext^i_S(F, E) = Ext^i_S([Ext^i_S(F, E)]^c)$.

Proof. Straightforward from Theorems 3.2, 3.4, and 3.6. □

Definition 4.2. Let $(X, R_E, \mathcal{L}_E)$ be a *soft ideal approximation space* and $(F, E) \tilde{\in} SS(X)_E$. Then, a soft point $x_E \tilde{\in} SS(X)_E$ is called:

(i) A *soft accumulation point* of (F, E) if $(< x > R - x_E) \sqcap (F, E) \neq \Phi$.
The set of all soft ideal accumulation points of (F, E) is written as $D_S(F, E)$, that is,

$$D_S(F, E) = \{x_E \tilde{\in} SS(X)_E \;:\; (< x > R - x_E) \sqcap (F, E) \neq \Phi\}.$$

(ii) A $*$-*soft ideal accumulation point* of (F, E) if $(< x > R - x_E) \sqcap (F, E) \tilde{\notin} \mathcal{L}_E$.
The set of all $*$-soft ideal accumulation points of (F, E) is written as $D^*_S(F, E)$, that is,

$$D^*_S(F, E) = \{x_E \tilde{\in} SS(X)_E \;:\; (< x > R - x_E) \sqcap (F, E) \tilde{\notin} \mathcal{L}_E\}.$$

(iii) A $**$-*soft ideal accumulation point* of (F,E) if $(R < x > R - x_E) \sqcap (F,E) \check{\notin} \mathcal{L}_E$.
The set of all $**$-soft ideal accumulation points of (F,E) is written as $D_S^{**}(F,E)$, that is,

$$D_S^{**}(F,E) = \{x_E \check{\in} SS(X)_E \ : \ (R < x > R - x_E) \sqcap (F,E) \check{\notin} \mathcal{L}_E\}.$$

Lemma 4.3. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space and $(F,E) \check{\in} SS(X)_E$. Then,*

(1) $\overline{Apr}_S^1(F,E) = (F,E) \sqcup D_S(F,E)$,
(2) $\overline{Apr}_S^1(F,E) = (F,E)$ *iff* $D_S(F,E) \sqsubseteq (F,E)$,
(3) $\overline{Apr}_S^2(F,E) = (F,E) \sqcup D_S^*(F,E)$,
(4) $\overline{Apr}_S^2(F,E) = (F,E)$ *iff* $D_S^*(F,E) \sqsubseteq (F,E)$,
(5) $\overline{Apr}_S^3(F,E) = (F,E) \sqcup D_S^{**}(F,E)$,
(6) $\overline{Apr}_S^3(F,E) = (F,E)$ *iff* $D_S^{**}(F,E) \sqsubseteq (F,E)$.

Proof. We will prove that the third and forth statements and the others will be similar.

(3) Let $x \check{\in} \overline{Apr}_S^2(F,E)$. Then, $x \check{\in} [(F,E) \sqcup \{y_E \check{\in} SS(X)_E \ : < y > R \sqcap (F,E) \check{\notin} \mathcal{L}_E\}]$. Then, we have either $x \check{\in} (F,E)$, that is,

$$x \check{\in} (F,E) \sqcup D_S^*(F,E) \tag{4.1}$$

or $x \check{\notin} (F,E)$. So, $x \check{\in} \{y_E \check{\in} SS(X)_E :< y > R \sqcap (F,E) \check{\notin} \mathcal{L}_E\}$. In the latter case, we have $(< x > R - x_E) \sqcap (F,E) \check{\notin} \mathcal{L}_E$. Hence, $x \check{\in} D_S^*(F,E)$, that is,

$$x \check{\in} (F,E) \sqcup D_S^*(F,E). \tag{4.2}$$

From Eqs (4.1) and (4.2), $\overline{Apr}_S^2(F,E) \sqsubseteq (F,E) \sqcup D_S^*(F,E)$. Conversely, let $x \check{\in} (F,E) \sqcup D_S^*(F,E)$. Then, we have either $x \check{\in} (F,E)$, that is,

$$x \check{\in} \overline{Apr}_S^2(F,E) \tag{4.3}$$

or $x \check{\notin} (F,E)$. Thus, $x \check{\in} D_S^*(F,E)$. So $(< x > R - x_E) \sqcap (F,E) \check{\notin} \mathcal{L}_E$. Hence, $x \check{\in} \overline{Apr}_S^2(F,E)$, that is,

$$x \check{\in} \overline{Apr}_S^2(F,E). \tag{4.4}$$

From Eqs (4.3) and (4.4), $(F,E) \sqcup D_S^*(F,E) \sqsubseteq \overline{Apr}_S^2(F,E)$.
Therefore, $\overline{Apr}_S^2(F,E) = (F,E) \sqcup D_S^*(F,E)$.

(4) Let $x \check{\notin} (F,E)$, that is, $x \check{\notin} \overline{Apr}_S^2(F,E)$. Then, $< x > R \sqcap (F,E) \check{\in} \mathcal{L}_E$. Thus, $(< x > R - x_E) \sqcap (F,E) \check{\in} \mathcal{L}_E$ and $x \check{\notin} D_S^*(F,E)$. Conversely, let $D_S^*(F,E) \sqsubseteq (F,E)$. Then, by part (1), $D_S^*(F,E) \sqcup (F,E) = \overline{Apr}_S^2(F,E) = (F,E)$.

□

Lemma 4.4. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space and $(F,E), (G,E) \check{\in} SS(X)_E$. Then,*

(1) *if* $(F,E) \sqsubseteq (G,E)$, *then* $D_S^*(F,E) \sqsubseteq D_S^*(G,E)$ *and* $D_S^{**}(F,E) \sqsubseteq D_S^{**}(G,E)$,
(2) $D_S^*[(F,E) \sqcup (G,E)] = D_S^*(F,E) \sqcup D_S^*(F,E)$ *and* $D_S^{**}[(F,E) \sqcup (G,E)] = D_S^{**}(F,E) \sqcup D_S^{**}(F,E)$,

(3) $D_S^*[(F,E)\sqcap(G,E)] \sqsubseteq D_S^*(F,E)\sqcap D_S^*(F,E)$ *and* $D_S^{**}[(F,E)\sqcap(G,E)] \sqsubseteq D_S^{**}(F,E)\sqcap D_S^{**}(F,E)$,
(4) $D_S^*[(F,E)\sqcup D_S^*(F,E)] \sqsubseteq (F,E)\sqcup D_S^*(F,E)$ *and* $D_S^{**}[(F,E)\sqcup D_S^{**}(F,E)] \sqsubseteq (F,E)\sqcup D_S^{**}(F,E)$.

Proof. (1) Suppose $(F,E)\sqsubseteq(G,E)$ and let $x \tilde{\in} D_S^*(F,E)$. Then, $[<x>R-x_E]\sqcap(F,E)\check{\notin}\mathcal{L}_E$. Thus, $[<x>R-x_E]\sqcap(G,E)\check{\notin}\mathcal{L}_E$. So, $x\tilde{\in} D_S^*(G,E)$. The second part is easily proved.

(2) Since $(F,E)\sqsubseteq(F,E)\sqcup(G,E)$ and $(G,E)\sqsubseteq(F,E)\sqcup(G,E)$, by part (1), we have $D_S^*(F,E)\sqcup D_S^*(G,E)\sqsubseteq D_S^*(F,E)\sqcup(G,E))$.
Conversely, let $x\check{\notin}(D_S^*(F,E)\sqcup D_S^*(G,E)$. Then, $x\check{\notin}D_S^*(F,E)$ and $x\check{\notin}D_S^*(G,E)$. Thus, $(<x>R-x_E)\sqcap(F,E)\tilde{\in}\mathcal{L}_E$ and $(<x>R-x_E)\sqcap(G,E)\tilde{\in}\mathcal{L}_E$. So, $(<x>R-x_E)\sqcap(F,E)\sqcup(G,E))\tilde{\in}\mathcal{L}_E$. Hence, $x\tilde{\in}D_S^*[(F,E)\sqcup(G,E)]$. The proof of the second part is similar.

(3) Similar to part (2).

(4) Let $x\check{\notin}(F,E)\sqcup D_S^*(F,E)$. It is obvious that $x\check{\notin}(F,E)$ and $(<x>R-x_E)\sqcap(F,E)\tilde{\in}\mathcal{L}_E$. Then, $<x>R\sqcap(F,E)\tilde{\in}\mathcal{L}_E$. Thus, $x\check{\notin}\overline{Apr}_S^2(F,E)$. So, $x\check{\notin}\overline{Apr}_S^2(\overline{Apr}_S^2(F,E))$. Hence, $x\check{\notin}D_S^*(\overline{Apr}_S^2(F,E)) = D_S^*(F,E)\sqcup D_S^*(F,E))$. Therefore, $D_S^*(F,E)\sqcup D_S^*(F,E)\sqsubseteq(F,E)\sqcup D_S^*(F,E)$. The proof of the second part is similar. □

Corollary 4.1. *Let* $(X,R_E,\mathcal{L}_E)$ *be any soft ideal approximation space and* $(F,E)\tilde{\in}SS(X)_E$. *Then,*

$$D_S^{**}(F,E)\sqsubseteq D_S^*(F,E)\sqsubseteq D_S(F,E).$$

Proof. Let $x\check{\notin}D_S(F,E)$. Then, $(<x>R-x_E)\sqcap(F,E)=\Phi$. Thus, $(<x>R-x_E)\sqcap(F,E)\tilde{\in}\mathcal{L}_E$. So, $x\check{\notin}D_S^*(F,E)$ and $(R<x>R-x_E)\sqcap(F,E)\tilde{\in}\mathcal{L}_E$, where $R<x>R\sqsubseteq<x>R$. Hence, $x\check{\notin}D_S^{**}(F,E)$. Therefore, $D_S^{**}(F,E)\sqsubseteq D_S^*(F,E)\sqsubseteq D_S(F,E)$. □

Remark 4.1. The converse of the previous result is not true.

Example 4.1. *Let* $X=\{a,b,c\}$ *associated with a set of parameters* $E=\{e_1,e_2\}$. *Let* R_E *be a soft relation of X and* $\mathcal{L}_E$ *be a soft ideal on X, defined respectively by:*

$$R=\{(e_1,\{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c)\}),(e_2,\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,b),$$
$$(c,c)\})\}$$

$\mathcal{L}_E=\{\Phi,(F_1,E),(F_2,E),(F_3,E)\}$ *where,*

$$(F_1,E)=\{(e_1,\{c\}),(e_2,\phi)\},(F_2,E)=\{(e_1,\phi),(e_2,\{c\})\},(F_3,E)=\{(e_1,\{c\}),(e_2,\{c\})\}.$$

Then, $<a>R=\{(e_1,\{a,b,c\}),(e_2,\{a,b,c\})\}$, $<b>R=\{(e_1,\{b,c\}),(e_2,\{b,c\})\}$, $<c>R=c_E$. *Also,* $R<a>=a_E$, $R<b>=\{(e_1,\{a,b\}),(e_2,\{a,b\})\}$, $R<c>=<a>R$. *Thus,* $R<a>R=a_E$, $R<b>R=b_E$, $R<c>R=c_E$. *Suppose* $(F,E)=\{(e_1,\{b,c\}),(e_2,\{b,c\})\}$. *Then, we have:*

$$(<a>R-a_E)\sqcap(F,E)=(F,E)\neq\Phi,$$
$$(<b>R-b_E)\sqcap(F,E)=c_E\neq\Phi,$$
$$(<c>R-c_E)\sqcap(F,E)=\Phi.$$

Thus, $a\tilde{\in}D_S(F,E)$, $b\tilde{\in}D_S(F,E)$, $c\check{\notin}D_S(F,E)$. *So,* $D_S(F,E)=\{(e_1,\{a,b\}),(e_2,\{a,b\})\}$. *On the other hand, we get:*

$$(< a > R - a_E) \sqcap (F,E) = (F,E) \check{\notin} \mathcal{L}_E,$$

$$(< b > R - b_E) \sqcap (F,E) = (F_3,E) \check{\in} \mathcal{L}_E,$$

$$(< c > R - c_E) \sqcap (F,E) = \Phi \check{\in} \mathcal{L}_E.$$

Thus, $a \check{\in} D^*_S(F,E)$, $b \check{\notin} D^*_S(F,E)$, $c \check{\notin} D^*_S(F,E)$. *Hence,* $D^*_S(F,E) = a_E$. *Also, we have:*

$$(R < a > R - a_E) \sqcap (F,E) = \Phi \check{\in} \mathcal{L}_E,$$

$$(R < b > R - b_E) \sqcap (F,E) = \Phi \check{\in} \mathcal{L}_E,$$

$$(R < c > R - c_E) \sqcap (F,E) = \Phi \check{\in} \mathcal{L}_E.$$

Then, $a \check{\notin} D^{**}_S(F,E)$, $b \check{\notin} D^{**}_S(F,E)$, $c \check{\notin} D^{**}_S(F,E)$. *Thus,* $D^{**}_S(F,E) = \Phi$. *So,* $D_S(F,E) \not\sqsubseteq D^*_S(F,E) \not\sqsubseteq D^{**}_S(F,E)$.

Definition 4.3. Let $(X, R_E, \mathcal{L}_E)$ be any soft ideal approximation space and $(F,E) \check{\in} SS(X)_E$. Then, (F,E) is said to be:

(i) *soft dense* if $\overline{Apr}^1_S(F,E) = \tilde{X}$,
(ii) $*$-*soft ideal dense* if $\overline{Apr}^2_S(F,E) = \tilde{X}$,
(iii) $**$-*soft ideal dense* if $\overline{Apr}^3_S(F,E) = \tilde{X}$,
(iv) *soft nowhere dense* if $\underline{Apr}^1_S(\overline{Apr}^1_S(F,E)) = \Phi$,
(v) $*$-*soft ideal nowhere dense* if $\underline{Apr}^1_S(\overline{Apr}^2_S(F,E)) = \Phi$,
(vi) $**$-*soft ideal nowhere dense* if $\underline{Apr}^1_S(\overline{Apr}^3_S(F,E)) = \Phi$.

Corollary 4.2. *Let* $(X, R_E, \mathcal{L}_E)$ *be any soft ideal approximation space and* $(F,E) \check{\in} SS(X)_E$. *Then,*

(1) $**$*-soft ideal dense* $\Longrightarrow$ $*$*-soft ideal dense* $\Longrightarrow$ *soft dense,*
(2) soft nowhere dense $\Longrightarrow$ $*$*-soft ideal nowhere dense* $\Longrightarrow$ $**$*-soft ideal nowhere dense.*

Proof. Immediately from Definition 4.3 and part (3) of Theorem 3.5. □

Example 4.2. *Let* $X = \{a,b,c\}$, $E = \{e_1,e_2\}$,
$R_E = \{(e_1,\{(a,a),(a,b),(b,b),(b,c),(c,c),(d,d),(d,b)),(e_2,\{(a,a),(a,b),(a,c),(b,b),(b,c),(c,c),(d,d),(d,b))\}\}$ *and* $\mathcal{L}_E = \{\Phi,(F_1,E),(F_2,E),(F_3,E)\}$, *where*
$(F_1,E) = \{(e_1,\{a\}),(e_2,\phi)\},(F_2,E) = \{(e_1,\phi),(e_2,\{a\})\},(F_3,E) = \{(e_1,\{a\}),(e_2,\{a\})\}$.
Therefore, we have $< a > R = \{(e_1,\{a,b\}),(e_2,\{a,b\})\}$, $< b > R = \{(e_1,\{b\}),(e_2,\{b\})\}$, $< c > R = \{(e_1,\{c\}),(e_2,\{c\})\}$, $< d > R = \{(e_1,\{b,d\}),(e_2,\{b,d\})\}$. *Also,* $R < a >= \{(e_1,\{a\}),(e_2,\{a\})\}$, $R < b >= \{(e_1,\{b\}),(e_2,\{b\})\}$, $R < c >= \{(e_1,\{b,c\}),(e_2,\{b,c\})\}$ $R < d >= \{(e_1,\{d\}),(e_2,\{d\})\}$. *Thus,* $R < a > R = \{(e_1,\{a\}),(e_2,\{a\})\}$, $R < b > R = \{(e_1,\{b\}),(e_2,\{b\})\}$, $R < c > R = \{(e_1,\{c\}),(e_2,\{c\})\}$ $R < d > R = \{(e_1,\{d\}),(e_2,\{d\})\}$. *Suppose* $(F,E) = \{(e_1,\{b,c\}),(e_2,\{b,c\})\}$. *Then,* $\overline{Apr}^2_S(F,E) = (F,E) \sqcup \{x \check{\in} \tilde{X} :< x > R \sqcap (F,E) \check{\notin} \mathcal{L}_E\} = \tilde{X}$. *Also,* $\overline{Apr}^3_S(F,E) = (F,E) \sqcup \{x \check{\in} \tilde{X} : R < x > R \sqcap (F,E) \check{\notin} \mathcal{L}_E\} = (F,E) \neq \tilde{X}$. *Hence,* (F,E) *is* $*$*-soft ideal dense but not* $**$*-soft ideal dense.*

Corollary 4.3. *Let $(X, R_E, \mathcal{L}_E)$ be any soft ideal approximation space and $(F, E) \check{\in} SS(X)_E$. Then,*

(1) If (F, E) is soft dense, then $[\overline{Apr}^1_S(F, E)]^c$ is soft nowhere dense.
(2) If (F, E) is $*$-soft ideal dense, then $[\overline{Apr}^2_S(F, E)]^c$ is $*$-soft ideal nowhere dense.
(3) If (F, E) is $**$-soft ideal dense, then $[\overline{Apr}^3_S(F, E)]^c$ is $**$-nowhere dense .

Proof. (1) Suppose (F, E) is soft dense. Then, $\overline{Apr}^1_S(F, E) = \tilde{X}$. Thus, $[\overline{Apr}^1_S(F, E)]^c = \Phi$ and $\overline{Apr}^1_S[(\overline{Apr}^1_S(F, E))^c] = \Phi$. So, $\underline{Apr}^1_S[\overline{Apr}^1_S(\overline{Apr}^1_S(F, E))^c)] = \Phi$. Hence, $[\overline{Apr}^1_S(F, E)]^c$ is nowhere soft dense.

(2) Suppose (F, E) is $*$-soft ideal dense. Then, $\overline{Apr}^2_S(F, E) = \tilde{X}$. Thus, $[\overline{Apr}^2_S(F, E)]^c = \Phi$. So, $\overline{Apr}^2_S[(\overline{Apr}^2_S(F, E))^c] = \Phi$ and $\underline{Apr}^1_S[\overline{Apr}^2_S((\overline{Apr}^2_S(F, E))^c)] = \Phi$. Hence, $[\overline{Apr}^2_S(F, E)]^c$ is $*$-soft ideal nowhere dense.

(3) Similar to part (2).

□

5. Lower soft separation axioms in soft ideal approximation spaces

In this section, we introduce soft lower separation axioms via soft binary relations and soft ideal as a generalization of lower separation axioms given in [31]. We scrutinize its essential characterizations and infer some of its relationships associated with the soft ideal closure operators. Some illustrative examples are given. In an approximation space (X, R) where R is an equivalence relation on X, a general topology is generated by the lower approximations $L(A)$ or the upper approximations $U(A)$ of any subset as follows. $\tau_R = \{A \subseteq X : A = L(A)\}$ or $\tau_R = \{A \subseteq X : A^c = U(A^c)\}$. In the soft case, it is an extension of the same definitions.

Definition 5.1. (1) A *soft approximation space* (X, R_E) is said to be a soft-T_0 *space* if $\forall x \neq y \check{\in} \tilde{X}$, there exists $(F, E) \check{\in} SS(X)_E$ such that

$$x \check{\in} \underline{Apr}^1_S(F, E),\ y \check{\notin} (F, E) \text{ or } y \check{\in} \underline{Apr}^1_S(F, E),\ x \check{\notin} (F, E).$$

(2) A *soft ideal approximation space* $(X, R_E, \mathcal{L}_E)$ is said to be a soft-T^*_0 *space* if $\forall x \neq y \check{\in} \tilde{X}$, there exists $(F, E) \check{\in} SS(X)_E$ such that

$$x \check{\in} \underline{Apr}^2_S(F, E),\ y \check{\notin} (F, E) \text{ or } y \check{\in} \underline{Apr}^2_S(F, E),\ x \check{\notin} (F, E).$$

(3) A *soft ideal approximation space* $(X, R_E, \mathcal{L}_E)$ is said to be a soft-T^{**}_0 *space* if $\forall x \neq y \check{\in} \tilde{X}$, there exists $(F, E) \check{\in} SS(X)_E$ such that

$$x \check{\in} \underline{Apr}^3_S(F, E),\ y \check{\notin} (F, E) \text{ or } y \check{\in} \underline{Apr}^3_S(F, E),\ x \check{\notin} (F, E).$$

Proposition 5.1. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T^*_0 space.
(2) $\overline{Apr}^2_S(x_E) \neq \overline{Apr}^2_S(y_E)$ for all $x \neq y \check{\in} \tilde{X}$.

Proof.

(1) $\Rightarrow$ (2): For each $x \neq y \check{\in} \tilde{X}$, by part (1), there exists $(F,E) \check{\in} SS(X)_E$ such that $x \check{\in} \underline{Apr}^2_S(F,E)$, $y \check{\notin} (F,E)$. Thus, $< x > R \sqcap (F,E)^c \check{\in} \mathcal{L}_E$, $y \check{\in} (F,E)^c$. So, $< x > R \sqcap y_E \check{\in} \mathcal{L}_E$ and by Proposition 4.1 part (1), $x \check{\notin} \overline{Apr}^2_S(y_E)$. Similarly, we can prove that $y \check{\notin} \overline{Apr}^2_S(x_E)$. Therefore, $\overline{Apr}^2_S(x_E) \neq \overline{Apr}^2_S(y_E)$.

(2) $\Rightarrow$ (1): Suppose part (2) holds and let $x \neq y \check{\in} \tilde{X}$. Then, $x \check{\notin} \overline{Apr}^2_S(y_E)$ or $y \check{\notin} \overline{Apr}^2_S(x_E)$. By Proposition 4.1 part (2), $< x > R \sqcap y_E \check{\in} \mathcal{L}_E$ or $< y > R \sqcap x_E \check{\in} \mathcal{L}_E$. Thus, $[x \check{\in} \underline{Apr}^2_S(y_E)^c, y \check{\notin} (y_E)^c]$ or $[y \check{\in} \underline{Apr}^2_S(x_E)^c, x \check{\notin} (x_E)^c]$. Therefore, $\tilde{X}$ is soft-T_0^* *space*.

□

Corollary 5.1. *For a soft approximation space (X, R_E), these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T_0 space.
(2) $\overline{Apr}^1_S(x_E) \neq \overline{Apr}^1_S(y_E)$ for each $x \neq y \check{\in} \tilde{X}$.

Corollary 5.2. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T_0^{**} space.
(2) $\overline{Apr}^3_S(x_E) \neq \overline{Apr}^3_S(y_E)$ for all $x \neq y \check{\in} \tilde{X}$.

Definition 5.2. (1) A *soft approximation space* (X, R_E) is said to be a soft-T_1 *space* if $\forall x \neq y \check{\in} \tilde{X}$, there exist $(F,E), (G,E) \check{\in} SS(X)_E$ such that

$$x \check{\in} \underline{Apr}^1_S(F,E),\ y \check{\notin} (F,E) \text{ and } y \check{\in} \underline{Apr}^1_S(G,E),\ x \check{\notin} (G,E).$$

(2) A *soft ideal approximation space* $(X, R_E, \mathcal{L}_E)$ is said to be a soft-T_1^* *space* if $\forall x \neq y \check{\in} \tilde{X}$, there exist $(F,E), (G,E) \check{\in} SS(X)_E$ such that

$$x \check{\in} \underline{Apr}^2_S(F,E),\ y \check{\notin} (F,E) \text{ and } y \check{\in} \underline{Apr}^2_S(G,E),\ x \check{\notin} (G,E).$$

(3) A *soft ideal approximation space* $(X, R_E, \mathcal{L}_E)$ is said to be a soft-T_1^{**} *space* if $\forall x \neq y \check{\in} \tilde{X}$, there exist $(F,E), (G,E) \check{\in} SS(X)_E$ such that

$$x \check{\in} \underline{Apr}^3_S(F,E),\ y \check{\notin} (F,E) \text{ and } y \check{\in} \underline{Apr}^3_S(G,E),\ x \check{\notin} (G,E).$$

Proposition 5.2. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T_1^* space.
(2) $\overline{Apr}^2_S(x_E) = x_E$ for all $x \check{\in} \tilde{X}$.
(3) $D_S^*(x_E) = \Phi$ for each $x \check{\in} \tilde{X}$.

Proof. (1) $\Rightarrow$ (2): Suppose $(X, R_E, \mathcal{L}_E)$ is a soft-T_1^* *space* and let $x \check{\in} \tilde{X}$. Thus, for $y \check{\in} \tilde{X} - x_E$, $x \neq y$ and $\exists (F,E) \check{\in} SS(X)_E$ such that $y \check{\in} \underline{Apr}^2_S(F,E)$, $x \check{\notin} (F,E)$. Thus, $< y > R \sqcap (F,E)^c \check{\in} \mathcal{L}_E$, $x \check{\in} (F,E)^c$. So, $< y > R \sqcap x_E \check{\in} \mathcal{L}_E$, that is, $y \check{\notin} \overline{Apr}^2_S(x_E)$. Hence, $\overline{Apr}^2_S(x_E) = x_E$.

(2) $\Rightarrow$ (3): Suppose part (2) holds and let $x \check{\in} \tilde{X}$. Then, $\overline{Apr}^2_S(x_E) = x_E \sqcup D^*_S x_E$ but $x \check{\notin} D^*_S x_E$. Thus, $D^*_S x_E = \Phi$.

(3) $\Rightarrow$ (1): Suppose part (3) holds and $x \neq y \check{\in} \tilde{X}$. By part (3), $D^*_S x_E = D^*_S y_E = \Phi$. Thus, $\overline{Apr}^2_S(x_E) = x_E$ and $\overline{Apr}^2_S(y_E) = y_E$, that is, $\underline{Apr}^2_S(x_E)^c = (x_E)^c$ and $\underline{Apr}^2_S(y_E)^c = (y_E)^c$. So, there exist $(x_E)^c$ and $(y_E)^c \check{\in} SS(X)_E$ such that $y \check{\in} \underline{Apr}^2_S(x_E)^c$, $x \check{\notin} (x_E)^c$ and $x \check{\in} \underline{Apr}^2_S(y_E)^c$, $y \check{\notin} (y_E)^c$. Therefore, $\tilde{X}$ is a soft-T^*_1 *space*.

□

Corollary 5.3. *For a soft approximation space (X, R_E), these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T_1 space.
(2) $\overline{Apr}^1_S(x_E) = x_E$ for all $x \check{\in} \tilde{X}$.
(3) $D_S(x_E) = \Phi$ for each $x \check{\in} \tilde{X}$.

Corollary 5.4. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T^{**}_1 space.
(2) $\overline{Apr}^3_S(x_E) = x_E$ for all $x \check{\in} \tilde{X}$.
(3) $D^{**}_S(x_E) = \Phi$ for each $x \check{\in} \tilde{X}$.

Definition 5.3. (1) A *soft approximation space* (X, R_E) is said to be a soft-R_0 *space* if, for all $x \neq y \check{\in} \tilde{X}$,

$$\overline{Apr}^1_S(x_E) = \overline{Apr}^1_S(y_E) \text{ or } \overline{Apr}^1_S(x_E) \sqcap \overline{Apr}^1_S(y_E) = \Phi.$$

(2) A *soft ideal approximation space* $(X, R_E, \mathcal{L}_E)$ is said to be a soft-R^*_0 *space* if, for all $x \neq y \check{\in} \tilde{X}$,

$$\overline{Apr}^2_S(x_E) = \overline{Apr}^2_S(y_E) \text{ or } \overline{Apr}^2_S(x_E) \sqcap \overline{Apr}^2_S(y_E) = \Phi.$$

(3) A *soft ideal approximation space* $(X, R_E, \mathcal{L}_E)$ is said to be a soft-R^{**}_0 *space* if, for all $x \neq y \check{\in} \tilde{X}$,

$$\overline{Apr}^3_S(x_E) = \overline{Apr}^3_S(y_E) \text{ or } \overline{Apr}^3_S(x_E) \sqcap \overline{Apr}^3_S(y_E) = \Phi.$$

Proposition 5.3. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these properties are equivalent:*

(1) $\tilde{X}$ is a soft-R^*_0 space,
(2) if $x \check{\in} \overline{Apr}^2_S(y_E)$, then $y \check{\in} \overline{Apr}^2_S(x_E)$ for all $x \neq y \check{\in} \tilde{X}$.

Proof.

(1) $\Rightarrow$ (2): Suppose statement (1) holds, and let $x \neq y$ be two soft points in $(X, R_E, \mathcal{L}_E)$. Then, $\overline{Apr}^2_S(x_E) = \overline{Apr}^2_S(y_E)$ or $\overline{Apr}^2_S(x_E) \sqcap \overline{Apr}^2_S(y_E) = \Phi$.

If $\overline{Apr}^2_S(x_E) = \overline{Apr}^2_S(y_E)$, then $y \check{\in} \overline{Apr}^2_S(x_E)$ and $x \check{\in} \overline{Apr}^2_S(y_E)$.

If $\overline{Apr}^2_S(x_E) \sqcap \overline{Apr}^2_S(y_E) = \Phi$, then $x_E \sqcap \overline{Apr}^2_S(y_E) = \Phi$ and $y_E \sqcap \overline{Apr}^2_S(x_E) = \Phi$. Thus, $x \check{\notin} \overline{Apr}^2_S(y_E)$ and $y \check{\notin} \overline{Apr}^2_S(x_E)$). So, $x \check{\notin} \overline{Apr}^2_S(y_E)$ and $y \check{\notin} \overline{Apr}^2_S(x_E)$. Hence, in either case, statement (2) holds.

(2) $\Rightarrow$ (1): Suppose that statement (2) holds and let $x \neq y \,\breve{\in}\, \tilde{X}$. Then, we have

$$\text{either } [x \,\breve{\in}\, \overline{Apr}_S^2(y_E) \text{ and } y \,\breve{\in}\, \overline{Apr}_S^2(x_E)] \text{ or } [x \,\breve{\notin}\, \overline{Apr}_S^2(y_E) \text{ and } y \,\breve{\notin}\, \overline{Apr}_S^2(x_E)].$$

If $x \,\breve{\in}\, \overline{Apr}_S^2(y_E)$ and $y \,\breve{\in}\, \overline{Apr}_S^2(x_E)$, then

$$\overline{Apr}_S^2(x_E) = \overline{Apr}_S^2(y_E). \tag{5.1}$$

If $x \,\breve{\notin}\, \overline{Apr}_S^2(y_E)$ and $y \,\breve{\notin}\, \overline{Apr}_S^2(x_E)$, then

$$\overline{Apr}_S^2(x_E) \sqcap \overline{Apr}_S^2(y_E) = \Phi. \tag{5.2}$$

From (5.1) and (5.2), the proof is complete. □

Corollary 5.5. *For a soft approximation space (X, R_E), these properties are equivalent:*

(1) $\tilde{X}$ is a soft-R_0 space,
(2) if $x \,\breve{\in}\, <y> R$, then $y \,\breve{\in}\, <x> R$ for any $x \neq y \,\breve{\in}\, \tilde{X}$.

Corollary 5.6. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these properties are equivalent:*

(1) $\tilde{X}$ is a soft-R_0^{**} space,
(2) if $x \,\breve{\in}\, \overline{Apr}_S^3(y_E)$, then $y \,\breve{\in}\, \overline{Apr}_S^3(x_E)$ for all $x \neq y \,\breve{\in}\, \tilde{X}$.

Definition 5.4. (1) A *soft approximation space* (X, R_E) is said to be a soft-T_2 *space* if $\forall x \neq y \,\breve{\in}\, \tilde{X}$, there exist $(F, E), (G, E) \,\breve{\in}\, SS(X)_E$ such that

$$x \,\breve{\in}\, \underline{Apr}_S^1(F, E),\ y \,\breve{\in}\, \underline{Apr}_S^1(G, E) \text{ and } (F, E) \sqcap (G, E) = \Phi.$$

(2) A *soft ideal approximation space* $(X, R_E, \mathcal{L}_E)$ is said to be a soft-T_2^* *space* if $\forall x \neq y \,\breve{\in}\, \tilde{X}$, there exist $(F, E), (G, E) \,\breve{\in}\, SS(X)_E$ such that

$$x \,\breve{\in}\, \underline{Apr}_S^2(F, E),\ y \,\breve{\in}\, \underline{Apr}_S^2(G, E) \text{ and } (F, E) \sqcap (G, E) = \Phi.$$

(3) A *soft ideal approximation space* $(X, R_E, \mathcal{L}_E)$ is said to be a soft-T_2^{**} *space* if $\forall x \neq y \,\breve{\in}\, \tilde{X}$, there exist $(F, E), (G, E) \,\breve{\in}\, SS(X)_E$ such that

$$x \,\breve{\in}\, \underline{Apr}_S^3(F, E),\ y \,\breve{\in}\, \underline{Apr}_S^3(G, E), \text{ and } (F, E) \sqcap (G, E) = \Phi.$$

Theorem 5.1. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T_2^* space,
(2) $\exists (F, E) \,\breve{\in}\, SS(X)_E : x \,\breve{\in}\, \underline{Apr}_S^2(F, E), y \,\breve{\in}\, [\overline{Apr}_S^2(F, E)]^c$ for any $x \neq y \,\breve{\in}\, \tilde{X}$.

Proof. (1) $\Rightarrow$ (2): Suppose $\tilde{X}$ is a soft-T_2^* *space* and let $x \neq y \,\breve{\in}\, \tilde{X}$. Then, there exist $(F, E),\ (G, E) \,\breve{\in}\, SS(X)_E$ such that $x \,\breve{\in}\, \underline{Apr}_S^2(F, E),\ y \,\breve{\in}\, \underline{Apr}_S^2(G, E)$ and $(F, E) \sqcap (G, E) = \Phi$. Thus, $<y> R \sqcap (G, E)^c \,\breve{\in}\, \mathcal{L}_E$ and $(F, E) \sqsubseteq (G, E)^c$. So $[<y> R - x_E] \sqcap (F, E) \,\breve{\in}\, \mathcal{L}_E$, that is, $y \,\breve{\notin}\, D_S^*(F, E)$. Hence, $\underline{Apr}_S^2(G, E) \sqcap D_S^*(F, E) = \Phi$ and $\underline{Apr}_S^2(G, E) \sqcap (F, E) = \Phi$, that is, $\underline{Apr}_S^2(G, E) \sqcap \overline{Apr}_S^2(F, E) = \Phi$. Therefore, $x \,\breve{\in}\, \underline{Apr}_S^2(F, E), y \,\breve{\in}\, \underline{Apr}_S^2(G, E) \sqsubseteq [\overline{Apr}_S^2(F, E)]^c$.

(2) $\Rightarrow$ (1): Suppose part (2) holds and let $x \neq y \check{\in} \tilde{X}$. Then, there exists $(F,E) \check{\in} SS(X)_E$ such that $x \check{\in} \underline{Apr}^2_S(F,E)$, $y \check{\in} [\overline{Apr}^2_S(F,E)]^c$. Let $(G,E) = [\overline{Apr}^2_S(F,E)]^c$. Then, $(G,E) = \underline{Apr}^2_S(F,E)^c$ (from Theorem 3.3 part (7)) and so $\underline{Apr}^2_S(G,E) = \underline{Apr}^2_S[\underline{Apr}^2_S(F,E)^c] = \underline{Apr}^2_S(F,E)^c = (G,E)$. Also, $(F,E) \sqcap (G,E) = (F,E) \sqcap \underline{Apr}^2_S(F,E)^c \sqsubseteq (F,E) \sqcap (F,E)^c = \Phi$. Hence, $\tilde{X}$ is a soft-T_2^* *space*.

□

Corollary 5.7. *For a soft approximation space (X, R_E), these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T_2 space,
(2) $\exists (F,E) \check{\in} SS(X)_E : x \check{\in} \underline{Apr}^1_S(F,E), y \check{\in} [\overline{Apr}^1_S(F,E)]^c$ for all $x \neq y \check{\in} \tilde{X}$.

Corollary 5.8. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these properties are equivalent:*

(1) $\tilde{X}$ is a soft-T_2^{**} space,
(2) $\exists (F,E) \check{\in} SS(X)_E : x \check{\in} \underline{Apr}^3_S(F,E), y \check{\in} [\overline{Apr}^3_S(F,E)]^c$ for all $x \neq y \check{\in} \tilde{X}$.

Corollary 5.9. *For a soft ideal approximation space $(X, R_E, \mathcal{L}_E)$, these conditions hold:*

(1) soft-T_1= soft-R_0 + soft-T_0,
(2) soft-T_1^*= soft-R_0^* + soft-T_0^*,
(3) soft-T_1^{**}= soft-R_0^{**} + soft-T_0^{**}.

Proof. Straightforward from Definition 5.3, Propositions 5.1 and 5.2, and Corollaries 5.1–5.4. □

Remark 5.1. From Definitions 5.1, 5.2, and 5.4 we have the following implication.

$$\begin{array}{ccccc}
\text{soft-}T_2 & \Longrightarrow & \text{soft-}T_1 & \Longrightarrow & \text{soft-}T_0 \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{soft-}T_2^* & \Longrightarrow & \text{soft-ı}T_1^* & \Longrightarrow & \text{soft-}T_0^* \\
\Downarrow & & \Downarrow & & \Downarrow \\
\text{soft-}T_2^{**} & \Longrightarrow & \text{soft-}T_1^{**} & \Longrightarrow & \text{soft-}T_0^{**}
\end{array}$$

Example 5.1. *(1) Let $X = \mathbb{Z}$ with $E = \{e_1, e_2\}$ and $R : E \longrightarrow P(\mathbb{Z} \times \mathbb{Z})$ be a soft relation over $\mathbb{Z} \times \mathbb{Z}$ defined by $R(e_1) = \mathbb{Z} \times \mathbb{Z}$, $R(e_2) = \mathbb{N} \times \mathbb{N}$ and $\mathcal{L}_E = \{(F,E) \check{\in} SS(X)_E : (F,E)$ is a finite soft set$\}$. Thus,*

$$\underline{Apr}^1_S(F,E) = \begin{cases} (F,E) & \text{if } (F,E)^c \check{\in} \mathcal{L}_E, \\ \Phi & \text{otherwise.} \end{cases}$$

Thus, $\forall x \neq y \check{\in} \tilde{\mathbb{Z}}$, we have:

$$x \check{\in} \underline{Apr}^1_S(y_E)^c = (y_E)^c,\ y \check{\notin} (y_E)^c \text{ and } y \check{\in} \underline{Apr}^1_S(x_E)^c = (x_E)^c,\ x \check{\notin} (x_E)^c.$$

So, $\tilde{\mathbb{Z}}$ is a soft-T_1 space. But $\tilde{\mathbb{Z}}$ is not a soft-T_2 space, since if $x \check{\in} \underline{Apr}^1_S(F,E)$, $y \check{\in} \underline{Apr}^1_S(G,E)$ and $(F,E) \sqcap (G,E) = \Phi$, then $\underline{Apr}^1_S(F,E) \sqcap \underline{Apr}^1_S(G,E) = \Phi$ and $\underline{Apr}^1_S(F,E) \sqsubseteq [\underline{Apr}^1_S(G,E)]^c$ which is impossible because $\underline{Apr}^1_S(F,E)$ is infinite soft set and $[\underline{Apr}^1_S(G,E)]^c$ is finite soft set.

(2) From part (1), we have

$$\underline{Apr}^2_S(F,E) = \underline{Apr}^3_S(F,E) = \begin{cases} (F,E) & \text{if } (F,E)^c \check{\in} \mathcal{L}_E, \\ \Phi & \text{otherwise.} \end{cases}$$

Then, $\forall x \neq y \check{\in} \tilde{\mathbb{Z}}$, *we have:*

$$x \check{\in} \underline{Apr}^2_S(y_E)^c = (y_E)^c,\ y \check{\notin} (y_E)^c \text{ and } y \check{\in} \underline{Apr}^2_S(x_E)^c = (x_E)^c,\ x \check{\notin} (x_E)^c.$$

$$x \check{\in} \underline{Apr}^3_S(y_E)^c = (y_E)^c,\ y \check{\notin} (y_E)^c \text{ and } y \check{\in} \underline{Apr}^3_S(x_E)^c = (x_E)^c,\ x \check{\notin} (x_E)^c.$$

Hence, $\tilde{\mathbb{Z}}$ *is soft-*T_1^* *and soft-*T_1^{**}*. However,* $\tilde{\mathbb{Z}}$ *is neither soft-*T_2^* *space nor soft-*T_2^{**}*. By the same way, any one can add examples to show that the above implication is not reversible.*

Definition 5.5. Let (X, R_E) and $(Y, (R_2)_H)$ be two soft approximation spaces and let $\mathcal{L}_E$ a soft ideal on X. Then,

(1) a function $f_{\rho\varrho} : SS(X)_E \longrightarrow SS(Y)_H$ is said to be *soft continuous* if $(\underline{Apr}^1_S)_E[f^{-1}_{\rho\varrho}(G,H)] \sqsupseteq f^{-1}_{\rho\varrho}[(\underline{Apr}^1_S)_H(G,H)]$, that is, $(\overline{Apr}^1_S)_E[f^{-1}_{\rho\varrho}(G,H)] \sqsubseteq f^{-1}_{\rho\varrho}[(\overline{Apr}^1_S)_H(G,H)]$ for all $(G,H) \check{\in} SS(Y)_H$.

(2) A function $f_{\rho\varrho} : SS(X)_E \longrightarrow SS(Y)_H$ is said to be $*$*-soft continuous* if $(\underline{Apr}^2_S)_E[f^{-1}_{\rho\varrho}(G,H)] \sqsupseteq f^{-1}_{\rho\varrho}[(\underline{Apr}^1_S)_H(G,H)]$, that is, $(\overline{Apr}^2_S)_E[f^{-1}_{\rho\varrho}(G,H)] \sqsubseteq f^{-1}_{\rho\varrho}[(\overline{Apr}^1_S)_H(G,H)]$ for all $(G,H) \check{\in} SS(Y)_H$.

(1) A function $f_{\rho\varrho} : SS(X)_E \longrightarrow SS(Y)_H$ is said to be $**$*-soft continuous* if $(\underline{Apr}^3_S)_E[f^{-1}_{\rho\varrho}(G,H)] \sqsupseteq f^{-1}_{\rho\varrho}[(\underline{Apr}^1_S)_H(G,H)]$, that is, $(\overline{Apr}^3_S)_E[f^{-1}_{\rho\varrho}(G,H)] \sqsubseteq f^{-1}_{\rho\varrho}[(\overline{Apr}^1_S)_H(G,H)]$ for all $(G,H) \check{\in} SS(Y)_H$.

Remark 5.2. From Corollary 3.4, we have the following implications:

Soft continuous $\Longrightarrow$ $*$-soft continuous $\Longrightarrow$ $**$-soft continuous.

Example 5.2. *Let* $X = \{a, b, c\}$ *associated with the parameters* $E = \{e_1, e_2\}$*. Let* $(R_1)_E$ *be a soft relation of X, and* $\mathcal{L}_E$ *be a soft ideal on X, defined respectively by:*

$(R_1)_E = \{(e_1, \{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,c)\}), (e_2, \{(a,a),(a,b),(a,c),(b,b),(b,c)\})\}$,

$\mathcal{L}_E = \{\Phi, (F_1,E), (F_2,E), (F_3,E), (F_4,E), (F_5,E), (F_6,E), (F_7,E), (F_8,E), (F_9,E), (F_{10},E), (F_{11},E), (F_{12},E), (F_{13},E), (F_{14},E)(F_{15},E)\}$

where

$$(F_1,E) = \{(e_1,\{b\}),(e_2,\phi)\}, (F_2,E) = \{(e_1,\{c\}),(e_2,\phi)\}, (F_3,E) = \{(e_1,\{b,c\}),(e_2,\phi)\},$$

$$(F_4,E) = \{(e_1,\phi),(e_2,\{b\})\}, (F_5,E) = \{(e_1,\phi),(e_2,\{c\})\}, (F_6,E) = \{(e_1,\phi),(e_2,\{b,c\})\},$$

$$(F_7,E) = \{(e_1,\{b\}),(e_2,\{b\})\}, (F_8,E) = \{(e_1,\{b\}),(e_2,\{c\})\}, (F_9,E) = \{(e_1,\{b\}),(e_2,\{b,c\})\},$$

$$(F_{10},E) = \{(e_1,\{c\}),(e_2,\{b\})\}, (F_{11},E) = \{(e_1,\{c\}),(e_2,\{c\})\}, (F_{12},E) = \{(e_1,\{c\}),(e_2,\{b,c\})\},$$

$$(F_{13},E)=\{(e_1,\{b,c\}),(e_2,\{b\})\},(F_{14},E)=\{(e_1,\{b,c\}),(e_2,\{c\})\},$$
$$(F_{15},E)=\{(e_1,\{b,c\}),(e_2,\{b,c\})\}.$$

Then, $<a>R_1=\{(e_1,\{a,b,c\}),(e_2,\{a,b,c\})\}$, $<b>R_1=\{(e_1,\{b,c\}),(e_2,\{b,c\})\}=<c>R_1$. *Also,* $R_1<a>=a_E$, $R_1<b>=\{(e_1,\{a,b\}),(e_2,\{a,b\})\}$, $R_1<c>=\Phi$. *Thus,* $R_1<a>R_1=a_E$, $R_1<b>R_1=b_E$, $R_1<c>R_1=\Phi$. *On the other hand, let* $Y=\{u,v,w\}$ *associated with the parameters* $H=\{h_1,h_2\}$. *Let* $(R_2)_H$ *be a soft relation over Y defined by:*
$(R_2)_H=\{(h_1,\{(u,u),(u,v),(v,u),(v,v),(v,w),(w,u),(w,w)\}),(h_2,\{(u,u),(u,v),(v,u),(v,v),(w,w)\})\}$. *Then,* $<u>R_2=\{(h_1,\{u,v\}),(h_2,\{u,v\})\}=<v>R_2$, $<w>R_2=w_H$. *Also,* $R_2<u>=\{(h_1,\{u,v\}),(h_2,\{u,v\})\}=R_2<v>$, $R_2<w>=w_H$. *Thus,* $R_2<u>R_2=\{(h_1,\{u,v\}),(h_2,\{u,v\})\}=R_2<v>R_2$, $R_2<w>R_2=w_H$. *Now, define the function* $f_{\rho\varrho}:SS(X)_E\longrightarrow SS(Y)_H$, *where* $\rho:E\longrightarrow H$ *is a function defined by* $\rho(e_1)=h_1,\rho(e_2)=h_2$ *and* $\varrho:X\longrightarrow Y$ *is a function defined by* $\varrho(a)=\varrho(b)=u$, $\varrho(c)=w$.

By calculating $(\underline{Apr}^2_S)_E[f^{-1}_{\rho\varrho}(G,H)]$ *and* $f^{-1}_{\rho\varrho}[(\underline{Apr}^1_S)_H(G,H)]$ *of a soft set* $(G,H)\,\breve{\in}\,SS(Y)_H$, *it is clear that* $f_{\rho\varrho}$ *is* $*$-soft continuous. *However,* $f_{\rho\varrho}$ *is not soft continuous, where*
$(\underline{Apr}^1_S)_E[f^{-1}_{\rho\varrho}(w_H)]=\Phi\not\sqsupseteq f^{-1}_{\rho\varrho}[(\underline{Apr}^1_S)_H(w_H)]=c_E$.

Theorem 5.2. *Let* $f_{\rho\varrho}:SS(X)_E\longrightarrow SS(Y)_H$ *be an injective soft continuous function. Then,* $(X,(R_1)_E,\mathcal{L}_E)$ *is a soft* T^*_i*-space if* $(Y,(R_2)_H)$ *is a soft-*T_i *space for* $i\in\{0,1,2\}$.

Proof. Suppose $(Y,(R_2)_H)$ is a soft-T_i *space* for $i\in\{0,1,2\}$ and let $x_1\neq x_2$ in $\tilde{X}$. For $i=2$, since $f_{\rho\varrho}$ is injective, $f_{\rho\varrho}(x_1,E)\neq f_{\rho\varrho}(x_2,E)\ \breve{\in}\ SS(Y)_H$. Then, by the hypothesis, there exist (G_1,H), $(G_2,H)\,\breve{\in}\,SS(Y)_H$ such that $f_{\rho\varrho}(x_1,E)\sqsubseteq(\underline{Apr}^1_S)_H(G_1,H)$, $f_{\rho\varrho}(x_2,E)\sqsubseteq(\underline{Apr}^1_S)_H(G_2,H))$ and $(G_1,H)\sqcap(G_2,H)=\Phi_H$, that is, $x_1\,\breve{\in}\,f^{-1}_{\rho\varrho}[(\underline{Apr}^1_S)_H(G_1,H)]$, $x_2\,\breve{\in}\,f^{-1}_{\rho\varrho}[(\underline{Apr}^1_S)_H(G_2,H)]$ and
$f^{-1}_{\rho\varrho}(G_1,H)\sqcap f^{-1}_{\rho\varrho}(G_2,H)=\Phi_H$.
Since $f_{\rho\varrho}$ is soft continuous, $x_1\,\breve{\in}\,(\underline{Apr}^1_S)_E[f^{-1}_{\rho\varrho}(G_1,H)]$, $x_2\,\breve{\in}\,(\underline{Apr}^1_S)_E[f^{-1}_{\rho\varrho}(G_2,H)]$. Thus,
$x_1\,\breve{\in}\,(\underline{Apr}^2_S)_E[f^{-1}_{\rho\varrho}(G_1,H)],x_2\,\breve{\in}\,(\underline{Apr}^2_S)_E[f^{-1}_{\rho\varrho}(G_2,H)]$ that is there exist
$(F_1,E)=f^{-1}_{\rho\varrho}(G_1,H)$, $(F_2,E)=f^{-1}_{\rho\varrho}(G_2,H)\ \breve{\in}\ SS(X)_E$ such that $x_1\,\breve{\in}\,(\underline{Apr}^2_S)_E(F_1,E)$, $x_2\,\breve{\in}\,(\underline{Apr}^2_S)_E(F_2,E)$ and $(F_1,E)\sqcap(F_2,E)=\Phi_E$. So, $(X,R_E,\mathcal{L}_E)$ is a soft $-T^*_2$ space. For $i\in\{0,1\}$ the proofs are similar. □

Corollary 5.10. *Let* $f_{\rho\varrho}:SS(X)_E\longrightarrow SS(Y)_H$ *be an injective soft continuous function. Thus,* (X,R_E) *is a soft* T_i*-space if* $(Y,(R_2)_H)$ *is a soft-*T_i *space for* $i\in\{0,1,2\}$.

Corollary 5.11. *Let* $f_{\rho\varrho}:SS(X)_E\longrightarrow SS(Y)_H$ *be an injective soft continuous function. Then,* $(X,R_E,\mathcal{L}_E)$ *is a soft* T^{**}_i*-space if* $(Y,(R_2)_H)$ *is a soft-*T_i *space for* $i\in\{0,1,2\}$.

6. Connectedness in soft ideal approximation spaces

In this section, We reformulate and study soft connectedness in [31] with respect to these soft ideal approximation spaces. Some examples are submitted to explain the definitions.

Definition 6.1. Let (X,R_E) be a soft approximation space. Then,

(1) (F,E), $(G,E)\,\breve{\in}\,SS(X)_E$ are called *soft separated sets* if $\overline{Apr}^1_S(F,E)\sqcap(G,E)=(F,E)\sqcap\overline{Apr}^1_S(G,E)=\Phi$.

(2) $\tilde{A} \check{\in} SS(X)_E$ is said to be a *soft disconnected set* if there exist soft separated sets $(F, E),\ (G, E) \check{\in} SS(X)_E$ such that $\tilde{A} \sqsubseteq (F, E) \sqcup (G, E)$. $\tilde{A}$ is said to be *soft connected* if it is not soft disconnected.

(3) (X, R_E) is said to be a *soft disconnected space* if there exist soft separated sets $(F, E), (G, E) \check{\in} SS(X)_E$ such that $(F, E) \sqcup (G, E) = \tilde{X}$. (X, R_E) is said to be a *soft connected space* if it is not soft disconnected space.

Definition 6.2. Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space. Then,

(1) $(F, E),\ (G, E) \check{\in} SS(X)_E$ are called $*$- *soft separated* (resp. $**$- *soft separated*) *sets* if $\overline{Apr}^2_S(F, E) \sqcap (G, E) = (F, E) \sqcap \overline{Apr}^2_S(G, E) = \Phi$ (resp. $\overline{Apr}^3_S(F, E) \sqcap (G, E) = (F, E) \sqcap \overline{Apr}^3_S(G, E) = \Phi$).

(2) $\tilde{A} \check{\in} SS(X)_E$ is called a $*$-*soft disconnected* (resp. $**$-*soft disconnected*) *set* if there exist $*$-soft separated (resp. $**$-soft separated) sets $(F, E),\ (G, E) \check{\in} SS(X)_E$ such that $\tilde{A} \sqsubseteq (F, E) \sqcup (G, E)$. $\tilde{A}$ is said to be $*$-*soft connected* (resp. $**$-*soft connected*) if it is not $*$-soft disconnected (resp. $**$-soft disconnected).

(3) $(X, R_E, \mathcal{L}_E)$ is called a $*$-*soft disconnected* (resp. $**$-*soft disconnected*) *space* if there exist $*$-soft separated (resp. $**$-soft separated) sets $(F, E),\ (G, E) \check{\in} SS(X)_E$ such that $(F, E) \sqcup (G, E) = \tilde{X}$. $(X, R_E, \mathcal{L}_E)$ is called a $*$-*soft connected* (resp. $**$-*soft connected*) *space* if it is not a $*$-soft disconnected (resp. $**$-soft disconnected) space.

Remark 6.1. The following implications are correct:

$$\text{soft separated} \Longrightarrow *\text{-soft separated} \Longrightarrow **\text{-soft separated},$$

and so

$$**\text{-soft connected} \Longrightarrow *\text{-soft connected} \Longrightarrow \text{soft connected}.$$

Example 6.1. *Let $X = \{a, b, c\}$ associated with a set of parameters $E = \{e_1, e_2\}$. Let R_E be a soft relation over X defined by:*

$$R_E = \{(e_1, \{(a, a), (a, b), (a, c), , (b, a), (b, b), (b, c), (c, c)\}), (e_2, \{(a, a), (a, b), (a, c), (b, b), (b, c)\})\}$$

Then, $< a > R = \{(e_1, \{a, b, c\}), (e_2, \{a, b, c\})\},\ \ < b > R = \{(e_1, \{b, c\}), (e_2, \{b, c\})\}$, $< c > R = \{(e_1, \{b, c\}), (e_2, \{b, c\})\}$. *Also,* $R < a >= a_E,\ R < b >= \{(e_1, \{a, b\}), (e_2, \{a, b\})\},\ R < c >= \Phi$. *Thus,* $R < a > R = a_E,\ R < b > R = b_E,\ R < c > R = \Phi$.

(1) Let $\mathcal{L}_E$ be a soft ideal on X defined by:

$$\mathcal{L}_E = \{\Phi, (F_1, E), (F_2, E), (F_3, E), (F_4, E), (F_5, E), (F_6, E)\}$$

where

$$(F_1, E) = \{(e_1, \{b\}), (e_2, \phi)\}, (F_2, E) = \{(e_1, \phi), (e_2, \{b\})\}, (F_3, E) = \{(e_1, \{c\}), (e_2, \phi)\},$$

$$(F_4, E) = \{(e_1, \phi), (e_2, \{c\})\}, (F_5, E) = \{(e_1, \{b\}), (e_2, \{c\})\}, (F_6, E) = \{(e_1, \{b, c\}), (e_2, \{b, c\})\}.$$

Then, we have

$$\overline{Apr}^1_S b_E = \overline{Apr}^1_S c_E = \overline{Apr}^1_S(\widetilde{\{b, c\}_E}) = \overline{Apr}^1_S(\widetilde{\{a, b\}_E}) = \overline{Apr}^1_S(\widetilde{\{a, c\}_E}) = \tilde{X},\ \overline{Apr}^1_S a_E = a_E.$$

Thus, (X, R_E) is a soft connected space. However, we get

$$\tilde{X} = a_E \sqcup \widetilde{\{b,c\}}_E,\ \overline{Apr}^2_S a_E \sqcap \widetilde{\{b,c\}}_E = a_E \sqcap \overline{Apr}^2_S(\widetilde{\{b,c\}}_E) = \Phi.$$

So, $(X, R_E, \mathcal{L}_E)$ is not a $*$-soft connected space.

(2) *Consider $\mathcal{L}_E = \{\Phi, (F_1, E), (F_2, E), (F_3, E)\}$ where*

$$(F_1, E) = \{(e_1, \{a\}), (e_2, \phi)\}, (F_2, E) = \{(e_1, \phi), (e_2, \{a\})\}, (F_3, E) = \{(e_1, \{a\}), (e_2, \{a\})\}.$$

Then, we get

$$\overline{Apr}^2_S b_E = \overline{Apr}^2_S c_E = \overline{Apr}^2_S(\widetilde{\{b,c\}}_E) = \overline{Apr}^2_S(\widetilde{\{a,b\}}_E) = \overline{Apr}^2_S(\widetilde{\{a,c\}}_E) = \tilde{X},\ \overline{Apr}^2_S a_E = a_E.$$

Thus, $(X, R_E, \mathcal{L}_E)$ is a $*$-soft connected space. However, we have

$$\tilde{X} = a_E \sqcup \widetilde{\{b,c\}}_E,\ \overline{Apr}^3_S a_E \sqcap \widetilde{\{b,c\}}_E = a_E \sqcap \overline{Apr}^3_S(\widetilde{\{b,c\}}_E) = \Phi.$$

So, $(X, R_E, \mathcal{L}_E)$ is not a $**$-soft connected space.

Proposition 6.1. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space. Then, these properties are equivalent:*

(1) *$(X, R_E, \mathcal{L}_E)$ is $*$-soft connected,*
(2) *for each $(F, E),\ (G, E) \check{\in} SS(X)_E$ with $(F, E) \sqcap (G, E) = \Phi$, $\underline{Apr}^2_S(F, E) = (F, E)$, $\underline{Apr}^2_S(G, E) = (G, E)$ and $(F, E) \sqcup (G, E) = \tilde{X}$, $(F, E) = \Phi$ or $(G, E) = \Phi$,*
(3) *for each $(F, E),\ (G, E) \check{\in} SS(X)_E$ with $(F, E) \sqcap (G, E) = \Phi$, $\overline{Apr}^2_S(F, E) = (F, E)$, $\overline{Apr}^2_S(G, E) = (G, E)$ and $(F, E) \sqcup (G, E) = \tilde{X}$, $(F, E) = \Phi$ or $(G, E) = \Phi$.*

Proof. (1) $\Rightarrow$ (2): Suppose part (1) holds and let $(F, E),\ (G, E) \check{\in} SS(X)_E$ with $\underline{Apr}^2_S(F, E) = (F, E)$, $\underline{Apr}^2_S(G, E) = (G, E)$ such that $(F, E) \sqcap (G, E) = \Phi$ and $(F, E) \sqcup (G, E) = \tilde{X}$. Then,

$$\overline{Apr}^2_S(F, E) \sqsubseteq \overline{Apr}^2_S(G, E)^c = [\underline{Apr}^2_S(G, E)]^c = (G, E)^c,$$

$$\overline{Apr}^2_S(G, E) \sqsubseteq \overline{Apr}^2_S(F, E)^c = [\underline{Apr}^2_S(F, E)]^c = (F, E)^c.$$

Thus, $\overline{Apr}^2_S(F, E) \sqcap (G, E) = (F, E) \sqcap \overline{Apr}^2_S(G, E) = \Phi$. So, $(F, E),\ (G, E)$ are $*$-soft separated sets. Since $(F, E) \sqcup (G, E) = \tilde{X}$, $(F, E) = \Phi$ or $(G, E) = \Phi$ by part (1).

(2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) Clear. □

Corollary 6.1. *Let (X, R_E) be a soft approximation space. Then, these properties are equivalent:*

(1) *(X, R_E) is soft connected,*
(2) *for each $(F, E),\ (G, E) \check{\in} SS(X)_E$ with $(F, E) \sqcap (G, E) = \Phi$, $\underline{Apr}^1_S(F, E) = (F, E)$, $\underline{Apr}^1_S(G, E) = (G, E)$ and $(F, E) \sqcup (G, E) = \tilde{X}$, $(F, E) = \Phi$ or $(G, E) = \Phi$,*
(3) *for each $(F, E),\ (G, E) \check{\in} SS(X)_E$ with $(F, E) \sqcap (G, E) = \Phi$, $\overline{Apr}^1_S(F, E) = (F, E)$, $\overline{Apr}^1_S(G, E) = (G, E)$ and $(F, E) \sqcup (G, E) = \tilde{X}$, $(F, E) = \Phi$ or $(G, E) = \Phi$.*

Corollary 6.2. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space. Then, these properties are equivalent:*

(1) $(X, R_E, \mathcal{L}_E)$ *is* $**$*-soft connected.*
(2) *For each* (F,E), $(G,E) \check{\in} SS(X)_E$ *with* $(F,E) \sqcap (G,E) = \Phi$, $\underline{Apr}_S^3(F,E) = (F,E)$, $\underline{Apr}_S^3(G,E) = (G,E)$ *and* $(F,E) \sqcup (G,E) = \tilde{X}$, $(F,E) = \Phi$ *or* $(G,E) = \Phi$.
(3) *For each* (F,E), $(G,E) \check{\in} SS(X)_E$ *with* $(F,E) \sqcap (G,E) = \Phi$, $\overline{Apr}_S^3(F,E) = (F,E)$, $\overline{Apr}_S^3(G,E) = (G,E)$ *and* $(F,E) \sqcup (G,E) = \tilde{X}$, $(F,E) = \Phi$ *or* $(G,E) = \Phi$.

Theorem 6.1. *Let* $(X, R_E, \mathcal{L}_E)$ *be a soft ideal approximation space and* $(F,E) \check{\in} SS(X)_E$ *be* $*$*-soft connected. If* (F_1,E), $(F_2,E) \check{\in} SS(X)_E$ *are* $*$*-soft separated sets with* $(F,E) \sqsubseteq (F_1,E) \sqcup (F_2,E)$, *then either* $(F,E) \sqsubseteq (F_1,E)$ *or* $(F,E) \sqsubseteq (F_2,E)$.

Proof. Suppose (F_1,E), (F_2,E) are $*$-soft separated sets with $(F,E) \sqsubseteq (F_1,E) \sqcup (F_2,E)$. Then, we have

$$\overline{Apr}_S^2(F_1,E) \sqcap (F_2,E) = (F_1,E) \sqcap \overline{Apr}_S^2(F_2,E) = \Phi,\ (F,E) = [(F,E) \sqcap (F_1,E)] \sqcup [(F,E) \sqcap (F_2,E)].$$

On the other hand, we get
$\overline{Apr}_S^2[(F,E) \sqcap (F_1,E)] \sqcap [(F,E) \sqcap (F_2,E)] \sqsubseteq \overline{Apr}_S^2(F,E) \sqcap \overline{Apr}_S^2(F_1,E) \sqcap [(F,E) \sqcap (F_2,E)] = \overline{Apr}_S^2(F,E) \sqcap (F,E) \sqcap \overline{Apr}_S^2(F_1,E) \sqcap (F_2,E) = (F,E) \sqcap \Phi = \Phi$. Also,
$\overline{Apr}_S^2[(F,E) \sqcap (F_2,E)] \sqcap [(F,E) \sqcap (F_1,E)] \sqsubseteq \overline{Apr}_S^2(F,E) \sqcap \overline{Apr}_S^2(F_2,E) \sqcap [(F,E) \sqcap (F_1,E)] = \overline{Apr}_S^2(F,E) \sqcap (F,E) \sqcap \overline{Apr}_S^2(F_2,E) \sqcap (F_1,E) = (F,E) \sqcap \Phi = \Phi$. Thus, $[(F,E) \sqcap (F_1,E)]$ and $[(F,E) \sqcap (F_2,E)]$ are $*$-soft separated sets with $(F,E) = [(F,E) \sqcap (F_1,E)] \sqcup [(F,E) \sqcap (F_2,E)]$. However, (F,E) is $*$-soft connected, which implies that $(F,E) \sqsubseteq (F_1,E)$ or $(F,E) \sqsubseteq (F_2,E)$. □

Corollary 6.3. *Let* (X, R_E) *be a soft approximation space and* $(F,E) \check{\in} SS(X)_E$ *be soft connected. If* (F_1,E), $(F_2,E) \check{\in} SS(X)_E$ *are soft separated sets with* $(F,E) \sqsubseteq (F_1,E) \sqcup (F_2,E)$, *then either* $(F,E) \sqsubseteq (F_1,E)$ *or* $(F,E) \sqsubseteq (F_2,E)$.

Corollary 6.4. *Let* $(X, R_E, \mathcal{L}_E)$ *be a soft ideal approximation space and* $(F,E) \check{\in} SS(X)_E$ *be* $**$*-soft connected. If* (F_1,E), $(F_2,E) \check{\in} SS(X)_E$ *are* $**$*-soft separated sets with* $(F,E) \sqsubseteq (F_1,E) \sqcup (F_2,E)$, *then either* $(F,E) \sqsubseteq (F_1,E)$ *or* $(F,E) \sqsubseteq (F_2,E)$.

Theorem 6.2. *Let* $f_{\rho\varrho} : (X, R_E, \mathcal{L}_E) \longrightarrow (Y, (R_2)_H)$ *be a* $*$*-soft continuous function. Then,*
$f_{\rho\varrho}(F,E) \check{\in} SS(Y)_H$ *is a soft connected set if* $(F,E) \check{\in} SS(X)_E$ *is* $*$*-soft connected.*

Proof. Assume that (F,E) is $*$-soft connected in $(X, R_E, \mathcal{L}_E)$. Suppose that $f_{\rho\varrho}(F,E)$ is soft disconnected. Thus, there exist two soft separated sets (G_1,H), $(G_2,H) \check{\in} SS(Y)_H$ with $f_{\rho\varrho}(F,E) \sqsubseteq (G_1,H) \sqcup (G_2,H)$, that is, $(\overline{Apr}_S^1)_H(G_1,H) \sqcap (G_2,H) = (G_1,H) \sqcap (\overline{Apr}_S^1)_H(G_2,H) = \Phi$. Since $f_{\rho\varrho}$ is $*$-soft continuous, $(F,E) \sqsubseteq f_{\rho\varrho}^{-1}(G_1,H) \sqcup f_{\rho\varrho}^{-1}(G_2,H)$. Thus, we have
$(\overline{Apr}_S^2)_E[f_{\rho\varrho}^{-1}(G_1,H)] \sqcap f_{\rho\varrho}^{-1}(G_2,H) \sqsubseteq f_{\rho\varrho}^{-1}[(\overline{Apr}_S^1)_H(G_1,H)] \sqcap f_{\rho\varrho}^{-1}(G_2,H)$
$= f_{\rho\varrho}^{-1}[(\overline{Apr}_S^1)_H(G_1,H) \sqcap (G_2,H)] = f_{\rho\varrho}^{-1}(\Phi) = \Phi$. Also, we have
$(\overline{Apr}_S^2)_E[f_{\rho\varrho}^{-1}(G_2,H)] \sqcap f_{\rho\varrho}^{-1}(G_1,H) \sqsubseteq f_{\rho\varrho}^{-1}[(\overline{Apr}_S^1)_H(G_2,H)] \sqcap f_{\rho\varrho}^{-1}(G_1,H)$
$= f_{\rho\varrho}^{-1}[(\overline{Apr}_S^1)_H(G_2,H) \sqcap (G_1,H)] = f_{\rho\varrho}^{-1}(\Phi) = \Phi$.
So, $f_{\rho\varrho}^{-1}(G_1,H)$ and $f_{\rho\varrho}^{-1}(G_2,H)$ are $*$-soft separated sets in $(X, R_E, \mathcal{L}_E)$, that is,
$(F,E) \sqsubseteq f_{\rho\varrho}^{-1}(G_1,H) \sqcup f_{\rho\varrho}^{-1}(G_2,H)$. Hence, (F,E) is $*$-soft disconnected, which contradicts that (F,E) is $*$-soft connected. Therefore, $f_{\rho\varrho}(F,E)$ is a soft connected set in $(Y, (R_2)_H)$. □

Corollary 6.5. *Let $f_{\rho\varrho} : (X, R_E) \to (Y, (R_2)_H)$ be a soft continuous function. Then, $f_{\rho\varrho}(F, E) \check{\in} SS(Y)_H$ is soft connected set, if $(F, E) \check{\in} SS(X)_E$ is soft connected .*

Corollary 6.6. *Let $f_{\rho\varrho} : (X, R_E, \mathcal{L}_E) \longrightarrow (Y, (R_2)_H)$ be a $**$-soft continuous function. Then, $f_{\rho\varrho}(F, E) \check{\in} SS(Y)_H$ is soft connected set if $(F, E) \check{\in} SS(X)_E$ is $**$-soft connected .*

7. Soft boundary region and soft accuracy measure

Herein, we first compare the current purposed methods in Definitions 3.4–3.6 and demonstrate that the method given in Definition 3.6 is the best in terms of developing the soft approximation operators and the values of soft accuracy. Then, we clarify that the third approach in Definition 3.6 produces soft accuracy measures of soft subsets higher than their counterparts displayed in the previous method 2.4 in [17]. Moreover, we applied these approaches to handle real-life problems.

Definition 7.1. Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space. Then, the soft boundary region $Bnd^i_S(F, E)$ of a soft set $(F, E) \check{\in} SS(X)_E$ and the soft accuracy measure $Acc^i_S(F, E)$ of an absolute soft set $(F, E) \check{\in} SS(X)_E$, $i \in \{1, 2, 3\}$ with respect to the soft binary relation R_E are defined respectively by:

$$Bnd^i_S(F, E) = \overline{Apr}^i_S(F, E) - \underline{Apr}^i_S(F, E),\ Acc^i_S(F, E) = \frac{|\underline{Apr}^i_S(F, E)|}{|\overline{Apr}^i_S(F, E)|}, i \in \{1, 2, 3\}$$

where $(F, E) \neq \Phi$. Note that $|\tilde{A}_E| = |A|$ denotes the cardinality of set $A \subseteq X$.

Proposition 7.1. *Let $(X, R_E, \mathcal{L}_E)$ be a soft ideal approximation space and $(F, E) \check{\in} SS(X)_E$. Then,*

(1) $Bnd^3_S(F, E) \sqsubseteq Bnd^2_S(F, E) \sqsubseteq Bnd^1_S(F, E)$.

(2) $Acc^1_S(F, E) \leqslant Acc^2_S(F, E) \leqslant Acc^3_S(F, E)$.

Proof. (1) Let $x \check{\in} Bnd^3_S(F, E) = \overline{Apr}^3_S(F, E) - \underline{Apr}^3_S(F, E)$. Then, from Corollary 3.4, we have $x \check{\in} \overline{Apr}^2_S(F, E) - \underline{Apr}^2_S(F, E) = Bnd^2_S(F, E)$. Again, by Corollary 3.4, if $x \check{\in} Bnd^2_S(F, E) = \overline{Apr}^2_S(F, E) - \underline{Apr}^2_S(F, E)$, then $x \check{\in} \overline{Apr}^1_S(F, E) - \underline{Apr}^1_S(F, E) = Bnd^1_S(F, E)$. Hence, $Bnd^3_S(F, E) \sqsubseteq Bnd^2_S(F, E) \sqsubseteq Bnd^1_S(F, E)$.

(2) From Corollary 3.4, we have

$$\begin{aligned} Acc^1_S(F, E) &= \frac{|\underline{Apr}^1_S(F, E)|}{|\overline{Apr}^1_S(F, E)|} \leqslant \frac{|\underline{Apr}^2_S(F, E)|}{|\overline{Apr}^2_S(F, E)|} = Acc^2_S(F, E) \\ &\leqslant \frac{|\underline{Apr}^1_S(F, E)|}{|\overline{Apr}^1_S(F, E)|} = Acc^1_S(F, E). \end{aligned}$$

□

Proposition 7.2. *Let $(X, R_E, (\mathcal{L}_1)_E)$ and $(X, R_E, (\mathcal{L}_2)_E)$ be soft ideal approximation spaces such that $(\mathcal{L}_1)_E \sqsubseteq (\mathcal{L}_2)_E$. Thus, for each $(F, E) \check{\in} SS(X)_E$ we have*

(1) $(\underline{Apr}^2_S)_{(\mathcal{L}_1)_E}(F,E) \sqsubseteq (\underline{Apr}^2_S)_{(\mathcal{L}_2)_E}(F,E)$.

(2) $(\overline{Apr}^2_S)_{(\mathcal{L}_2)_E}(F,E) \sqsubseteq (\overline{Apr}^2_S)_{(\mathcal{L}_1)_E}(F,E)$.

(3) $(Bnd^2_S)_{(\mathcal{L}_2)_E}(F,E) \sqsubseteq (Bnd^2_S)_{(\mathcal{L}_1)_E}(F,E)$.

(4) $(Acc^2_S)_{(\mathcal{L}_1)_E}(F,E) \leqslant (Acc^2_S)_{(\mathcal{L}_2)_E}(F,E)$.

Proof.

(1) Let $x \check{\in} (\underline{Apr}^2_S)_{(\mathcal{L}_1)_E}(F,E)$. Then, $< x > R \sqcap (F,E)^c \check{\in} (\mathcal{L}_1)_E$. Since $(\mathcal{L}_1)_E \sqsubseteq (\mathcal{L}_2)_E$. Thus, $< x > R \sqcap (F,E)^c \check{\in} (\mathcal{L}_2)_E$. Therefore, $x \check{\in} (\underline{Apr}^2_S)_{(\mathcal{L}_2)_E}(F,E)$. Hence, $(\underline{Apr}^2_S)_{(\mathcal{L}_1)_E}(F,E) \sqsubseteq (\underline{Apr}^2_S)_{(\mathcal{L}_2)_E}(F,E)$.

(2) Let $x \check{\in} (\overline{Apr}^2_S)_{(\mathcal{L}_2)_E}(F,E)$. Then, $< x > R \sqcap (F,E)^c \check{\notin} (\mathcal{L}_2)_E$. Since $(\mathcal{L}_1)_E \sqsubseteq (\mathcal{L}_2)_E$. Thus, $< x > R \sqcap (F,E)^c \check{\notin} (\mathcal{L}_1)_E$. Therefore, $x \check{\in} (\overline{Apr}^2_S)_{(\mathcal{L}_1)_E}(F,E)$. Hence, $(\overline{Apr}^2_S)_{(\mathcal{L}_2)_E}(F,E) \subseteq (\overline{Apr}^2_S)_{(\mathcal{L}_1)_E}(F,E)$.

(3), (4): It is immediately obtained by parts (1) and (2). □

Corollary 7.1. *Let $(X, R_E, (\mathcal{L}_1)_E)$, and $(X, R_E, (\mathcal{L}_2)_E)$ be soft ideal approximation spaces such that $(\mathcal{L}_1)_E \sqsubseteq (\mathcal{L}_2)_E$. Thus, for each $(F,E) \check{\in} SS(X)_E$ we have*

(1) $(\underline{Apr}^3_S)_{(\mathcal{L}_1)_E}(F,E) \sqsubseteq (\underline{Apr}^3_S)_{(\mathcal{L}_2)_E}(F,E)$.

(2) $(\overline{Apr}^3_S)_{(\mathcal{L}_2)_E}(F,E) \subseteq (\overline{Apr}^3_S)_{(\mathcal{L}_1)_E}(F,E)$.

(3) $(Bnd^3_S)_{(\mathcal{L}_2)_E}(F,E) \sqsubseteq (Bnd^3_S)_{(\mathcal{L}_1)_E}(F,E)$.

(4) $(Acc^3_S)_{(\mathcal{L}_1)_E}(F,E) \leqslant (Acc^3_S)_{(\mathcal{L}_2)_E}(F,E)$.

Remark 7.1. Proposition 7.2 shows that the soft boundary region of a soft set $(F,E) \check{\in} SS(X)_E$ decreases as the soft ideal increases as illustrated in the next example.

Example 7.1. *Let $X = \{a,b,c\}$ associated with a set of parameters $E = \{e_1, e_2\}$. Let R_E be a soft relation over X. Let $(\mathcal{L}_1)_E$, $(\mathcal{L}_2)_E$ be soft ideals on X, defined respectively by:*

$$R_E = \{(e_1, \{(a,a),(a,b),(a,c),(b,b),(b,c)\}), (e_2, \{(a,a),(a,c),(b,a),(b,b),(b,c)\})\}$$

$$(\mathcal{L}_1)_E = \{\Phi, \{(e_1,\{a\}),(e_2,\phi)\}\}$$

$$(\mathcal{L}_2)_E = SS(\{a,c\})_E = \{(F,E) : (F,E) \text{ is a soft set over } \{a,c\}\}.$$

Therefore, $< a > R = \{(e_1,\{a\}),(e_2,\{a\})\}$, $< b > R = \{(e_1,\{b,c\}),(e_2,\{b,c\})\} =< c > R$.
Let $(F,E) = \{(e_1,\{c\}),(e_2,\phi)\}$. *Then,*

$$(Bnd^2_S)_{(\mathcal{L}_1)_E}(F,E) = (\overline{Apr}^2_S)_{(\mathcal{L}_1)_E}(F,E) - (\underline{Apr}^2_S)_{(\mathcal{L}_1)_E}(F,E) = (\widetilde{\{b,c\}})_E - \Phi = (\widetilde{\{b,c\}})_E.$$

Also

$$(Bnd^2_S)_{(\mathcal{L}_2)_E}(F,E) = (\overline{Apr}^2_S)_{(\mathcal{L}_2)_E}(F,E) - (\underline{Apr}^2_S)_{(\mathcal{L}_2)_E}(F,E) = \{(e_1,\{c\}),(e_2,\phi)\} - \Phi$$

$$= \{(e_1,\{c\}),(e_2,\Phi)\}.$$

It is clear that $(Bnd^2_S)_{(\mathcal{L}_2)_E}(F,E) \sqsubseteq (Bnd^2_S)_{(\mathcal{L}_1)_E}(F,E)$.

Remark 7.2. From Proposition 7.1, one can deduce that Definition 3.6 improves the soft boundary region which means decreasing for a soft set $(F, E) \check{\in} SS(X)_E$, and improves the soft accuracy measure which means increasing for that soft set $(F, E) \check{\in} SS(X)_E$ by increasing the soft lower approximation and decreasing the soft upper approximation in comparison to the methods in Definitions 3.4, 3.5, and Definition 2.4 in [17]. So, the suggested method in Definition 3.6 is more accurate in decision-making. As a special case:

(1) If R_E is soft symmetric relation, then the soft approximations in Definition 3.6 coincide with the soft approximations in Definition 3.5.
(2) If $\mathcal{L}_E = \Phi$ and R_E is soft symmetric relation, then the soft approximations in Definition 3.5 coincide with the soft approximations in Definition 3.5.
(3) If $\mathcal{L}_E = \Phi$, $E = \{e\}$ and R_E is soft reflexive and soft transitive relation, then the soft approximations in Definition 3.6 coincide with the previous soft approximations in [17].

Example 7.2. ***Selection of a house:***
Considering $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ is a collection of six houses where H ={expensive, beautiful, cheap, in green surroundings, wooden modern, in good repair, in bad repair} be a set of parameters. Suppose ***Mr.Z*** *wants to purchase a house on the following parametric set E = {beautiful, cheap, in green surroundings, wooden, in good repair}. Consider $E = \{e_1, e_2, e_3, e_4, e_5\}$.*
Define a soft equivalence relation $R : E \longrightarrow P(X \times X)$. The soft equivalence classes for each $e \in E$ are obtained as follows:

$$\text{For } R(e_1):\ \text{are } \{x_1, x_3\}, \{x_2, x_4, x_5, x_6\}. \text{ For } R(e_2):\ \text{are } \{x_1, x_2, x_4, x_5\}, \{x_3\}, \{x_6\}.$$

$$\text{For } R(e_3):\ \text{are } \{x_1, x_2, x_4, x_5, x_6\}, \{x_3\}. \text{ For } R(e_4):\ \text{are } \{x_1, x_3, x_6\}, \{x_2, x_4, x_5\}.$$

$$\text{For } R(e_5):\ \text{is } \{x_1, x_2, x_3, x_4, x_5, x_6\}.$$

Therefore, $R < x_1 > R = (\widetilde{\{x_1\}})_E$, $R < x_2 > R = R < x_4 > R = R < x_5 > R = (\{x_2, \widetilde{x_4}, x_5\})_E$, $R < x_3 > R = (\widetilde{\{x_3\}})_E$, $R < x_6 > R = (\widetilde{\{x_6\}})_E$. Consider $\mathcal{L}_E = SS(\{x_1, x_3, x_5\})_E = \{(F, E) : (F, E)$ is a soft set over $\{x_1, x_3, x_5\}\}$ be a soft ideal over X. The soft representation of the equivalence relation R_E is explained in Table 1. In Table 2, the soft approximations, soft boundary region, and soft accuracy measure of a soft set $(F, E) \check{\in} SS(X)_E$ by using our suggested method in Definition 3.6. This method is the best tool to help ***Mr.Z*** *in his decision-making about selecting the house that is most suitable to his choice of parameters. For example, take $(\{x_2, \widetilde{x_3}, x_4\})_E$, then from Table 2, the soft lower and soft upper approximations, soft boundary region, and soft accuracy measure are $(\widetilde{\{x_3\}})_E$, $(\{x_2, x_3, \widetilde{x_4}, x_5\})_E$, $(\{x_2, \widetilde{x_4}, x_5\})_E$, and $1/4$, respectively. One can see that* ***Mr.Z*** *will decide to buy the house x_3 according to his choice parameters in E.*

Table 1. Soft equivalence relation representation of houses under consideration.

	e_1	e_2	e_3	e_4	e_5
(x_1, x_1)	1	1	1	1	1
(x_1, x_2)	0	1	1	0	1
(x_1, x_3)	1	0	0	1	1
(x_1, x_4)	0	1	1	0	1
(x_1, x_5)	0	1	1	0	1
(x_1, x_6)	0	0	1	1	1
(x_2, x_1)	0	1	1	0	1
(x_2, x_2)	1	1	1	1	1
(x_2, x_3)	0	0	0	0	1
(x_2, x_4)	1	1	1	1	1
(x_2, x_5)	1	1	1	1	1
(x_2, x_6)	1	0	1	0	1
(x_3, x_1)	1	0	0	1	1
(x_3, x_2)	0	0	0	0	1
(x_3, x_3)	1	1	1	1	1
(x_3, x_4)	0	0	0	0	1
(x_3, x_5)	0	0	0	0	1
(x_3, x_6)	0	0	0	1	1
(x_4, x_1)	0	1	1	0	1
(x_4, x_2)	1	1	1	1	1
(x_4, x_3)	0	0	0	0	1
(x_4, x_4)	1	1	1	1	1
(x_4, x_5)	1	1	1	1	1
(x_4, x_6)	1	0	1	0	1
(x_5, x_1)	0	1	1	0	1
(x_5, x_2)	1	1	1	1	1
(x_5, x_3)	0	0	0	0	1
(x_5, x_4)	1	1	1	1	1
(x_5, x_5)	1	1	1	1	1
(x_5, x_6)	1	0	1	0	1
(x_6, x_1)	0	0	1	1	1
(x_6, x_2)	1	0	1	0	1
(x_6, x_3)	0	0	0	1	1
(x_6, x_4)	1	0	1	0	1
(x_6, x_5)	1	0	1	0	1
(x_6, x_6)	1	1	1	1	1

Table 2. Soft approximations, soft boundary region and soft accuracy measure of a soft set $(F, E) \check{\in} SS(X)_E$ of Definition 3.6.

$(F, E) \check{\in} SS(X)_E$	$\underline{Apr}^3_S(F, E)$	$\overline{Apr}^3_S(F, E)$	$Bnd^3_S(F, E)$	$Acc^3_S(F, E)$
$(\widetilde{\{x_1, x_2, x_3\}})_E$	$(\widetilde{\{x_1, x_3\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	2/5
$(\widetilde{\{x_1, x_3, x_4\}})_E$	$(\widetilde{\{x_1, x_3\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	2/5
$(\widetilde{\{x_1, x_2, x_5\}})_E$	$(\widetilde{\{x_1\}})_E$	$(\widetilde{\{x_1, x_2, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	1/4
$(\widetilde{\{x_1, x_3, x_6\}})_E$	$(\widetilde{\{x_1, x_3, x_6\}})_E$	$(\widetilde{\{x_1, x_3, x_6\}})_E$	Φ	1
$(\widetilde{\{x_1, x_4, x_6\}})_E$	$(\widetilde{\{x_1, x_6\}})_E$	$(\widetilde{\{x_1, x_2, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	2/5
$(\widetilde{\{x_2, x_3, x_4\}})_E$	$(\widetilde{\{x_3\}})_E$	$(\widetilde{\{x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	1/4
$(\widetilde{\{x_2, x_3, x_5\}})_E$	$(\widetilde{\{x_3\}})_E$	$(\widetilde{\{x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	1/4
$(\widetilde{\{x_2, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	Φ	1
$(\widetilde{\{x_2, x_4, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_6\}})_E$	Φ	1
$(\widetilde{\{x_2, x_5, x_6\}})_E$	$(\widetilde{\{x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	1/4
$(\widetilde{\{x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_3\}})_E$	$(\widetilde{\{x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	1/4
$(\widetilde{\{x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	1/4
$(\widetilde{\{x_1, x_2, x_3, x_4\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_5\}})_E$	4/5
$(\widetilde{\{x_1, x_2, x_5, x_6\}})_E$	$(\widetilde{\{x_1, x_6\}})_E$	$(\widetilde{\{x_1, x_2, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	2/5
$(\widetilde{\{x_1, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_1, x_3\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	2/5
$(\widetilde{\{x_1, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_1, x_6\}})_E$	$(\widetilde{\{x_1, x_2, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5\}})_E$	2/5
$(\widetilde{\{x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_2, x_3, x_4, x_5\}})_E$	Φ	1
$(\widetilde{\{x_2, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_4, x_5, x_6\}})_E$	Φ	1
$(\widetilde{\{x_1, x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4, x_5\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4, x_5\}})_E$	Φ	1
$(\widetilde{\{x_1, x_2, x_3, x_4, x_6\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4, x_6\}})_E$	$(\widetilde{\{x_1, x_2, x_3, x_4, x_6\}})_E$	Φ	1
$(\widetilde{\{x_1, x_3, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_1, x_3, x_6\}})_E$	$\widetilde{X}$	$(\widetilde{\{x_4, x_5\}})_E$	1/2
$(\widetilde{\{x_2, x_3, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_3, x_4, x_5, x_6\}})_E$	$(\widetilde{\{x_2, x_3, x_4, x_5, x_6\}})_E$	Φ	1

Example 7.3. ***Selection of a car:***
*Suppose a person **Mr.Z** wants to buy a car from the alternatives $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}$. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ be the universe of ten different cars and let $E = \{e_1, e_2, e_3\}$ be the set of attributes, where e_1 refers to price, e_2 refers to color, and e_3 refers to car brands.*
The parameters are characterized as follows:

- *The price of a car includes under 30 lacs, between 31 and 35 lacs, and between 36 and 40 lacs.*
- *The car brand includes Honda Accord, Audi, Mercedes Benz, and BMW.*
- *The color of a car includes black, white, and silver.*

*Define a soft equivalence relation $R : E \longrightarrow P(X \times X)$ for each $e \in E$ which describes the advantages of the car for which the person **Mr.Z** will buy. The soft equivalence classes for each $e \in E$ are obtained* *as follows:*

$$\text{For } R(e_1): \text{ are } \{x_1, x_{10}\}, \{x_2, x_4, x_6, x_7\}, \{x_3, x_5, x_8, x_9\},$$

which means that the price of cars x_1 and x_{10} is under 30 lacs; the price of cars x_2, x_4, x_6, and x_7 is between 31 and 35 lacs; and the price of cars x_3, x_5, x_8, and x_9 is between 36 and 40 lacs.

$$\text{For } R(e_2): \text{ are } \{x_1\}, \{x_2\}, \{x_3, x_4, x_5, x_7, x_8, x_9, x_{10}\}, \{x_6\},$$

which represents that the brand of car x_1 is Honda Accord; the brand of car x_2 is Audi; the brand of cars $x_3, x_4, x_5, x_7, x_8, x_9$, and x_{10} is Mercedes Benz; and the brand of car x_6 is BMW. For $R(e_3)$: are $\{x_{10}\}, \{x_6\}, \{x_1, x_2, x_3, x_4, x_5, x_7, x_8, x_9\}$, which represents that the color of cars $x_1, x_2, x_3, x_4, x_5, x_7, x_8$, and x_9 is black; the color of car x_{10} is white; and the color of car x_6 is silver.
Therefore, $R < x_1 > R = (\widetilde{\{x_1\}})_E$, $R < x_2 > R = (\widetilde{\{x_2\}})_E$, $R < x_6 > R = (\widetilde{\{x_6\}})_E$, $R < x_{10} > R = (\widetilde{\{x_{10}\}})_E$, $R < x_4 > R = R < x_7 > R = (\widetilde{\{x_4, x_7\}})_E$, $R < x_3 > R = R < x_5 > R = R < x_8 > R = R < x_9 > R = (\widetilde{\{x_3, x_5, x_8, x_9\}})_E$.
*Consequently, anyone can offer a soft ideal to extend an example similar to the one in Table 2 to help **Mr.Z** in his decision-making about selecting the car that is most suitable according to the given parameters.*
*For example, let $\mathcal{L}_E = SS(\widetilde{\{x_2, x_6, x_{10}\}})_E = \{(F, E) : (F, E) \text{ is a soft set over } \{x_2, x_6, x_{10}\}\}$ be a soft ideal over X and $(F, E) = (\widetilde{\{x_1, x_4, x_8\}})_E \tilde{\in} SS(X)_E$ consisting of these cars which are most acceptable for **Mr.Z**. Thus, $\underline{Apr}^3_S(F, E) = (\widetilde{\{x_1\}})_E$, $\overline{Apr}^3_S(F, E) = (\widetilde{\{x_1, x_3, x_4, x_5, x_7, x_8, x_9\}})_E$,*
*$Bnd^3_S(F, E) = (\widetilde{\{x_3, x_4, x_5, x_7, x_8, x_9\}})_E$ and $Acc^3_S(F, E) = 1/7$. **Mr.Z** will buy the car x_1 which is under 30 lacs, a Honda Accord, and is white.*

8. Conclusions

This paper introduced new soft closure operators based on soft ideals, defining soft topological spaces. To that end, soft accumulation points, soft subspaces, and soft lower separation axioms of such spaces are defined and studied. Moreover, soft connectedness in these spaces is defined, which enables us to make more generalizations and studies. The obtained results are newly presented and could enrich soft topology theory. Finally, applications in multi criteria group decision making by using our methods to present the importance of our soft ideals approximations have been presented.

As it is well-known that the soft interior and soft closure topological operators behave similarly to the lower and upper soft approximations. So, in forthcoming works, we plan to study the counterparts of these models via topological structures. In addition, we will benefit from the hybridization of rough set theory with some approaches, such as fuzzy sets and soft fuzzy sets, to introduce these approximation spaces via these hybridized frames and show their role in efficiently dealing with uncertain knowledge.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflicts of interest

The authors declare that they have no conflicts of interest.

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