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Research article

Some congruences for ℓ-regular partitions with certain restrictions

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Abstract: Let $\mathrm{pod}_\ell(n)$ and $\mathrm{ped}_\ell(n)$ denote the number of ℓ-regular partitions of a positive integer n into distinct odd parts and the number of ℓ-regular partitions of a positive integer n into distinct even parts, respectively. Our first goal in this note was to prove two congruence relations for $\mathrm{pod}_\ell(n)$. Furthermore, we found a formula for the action of the Hecke operator on a class of eta-quotients. As two applications of this result, we obtained two infinite families of congruence relations for $\mathrm{pod}_5(n)$. We also proved a congruence relation for $\mathrm{ped}_\ell(n)$. In particular, we established a congruence relation modulo 2 connecting $\mathrm{pod}_\ell(n)$ and $\mathrm{ped}_\ell(n)$.

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1. Introduction

For convenience, throughout this paper, we use the notation

$$f_k = \prod_{n=1}^{\infty}(1-q^{nk}), \quad k \geq 1.$$

A partition of a nonnegative integer n is a nonincreasing sequence of positive integers whose sum is n. We denote by $\mathrm{pod}(n)$ the number of partitions of n with odd parts distinct (and even parts are unrestricted). The generating function of $\mathrm{pod}(n)$ is given by

$$\sum_{n=0}^{\infty}\mathrm{pod}(n)q^n = \frac{1}{\psi(-q)} = \frac{f_2}{f_1 f_4}$$

where $\psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}$.

The congruence properties of pod(n) were first studied by Hirschhorn and Sellers [1] in 2010. They proved that for all $\alpha \geq 0$ and $n \geq 0$,

$$\text{pod}\left(3^{2\alpha+3} + \frac{23 \times 3^{2\alpha+2} + 1}{8}\right) \equiv 0 \pmod 3.$$

Using modular forms, Radu and Sellers [2] established several congruence relations modulo 5 and 7 for pod(n), such as

$$\text{pod}(135n + 8) \equiv \text{pod}(135n + 107) \equiv \text{pod}(135n + 116) \equiv 0 \pmod 5,$$
$$\text{pod}(567n + 260) \equiv \text{pod}(567n + 449) \equiv 0 \pmod 7.$$

For more details on pod(n), one can refer to [3–5].

Our goal in this paper is to find congruence properties of the function $\text{pod}_\ell(n)$, which enumerates the partitions of n into non-multiples of ℓ in which the odd parts are distinct (and even parts unrestricted). For example $\text{pod}_3(7) = 4$, where the relevant partitions are 7, 5+2, 4+2+1, 2+2+2+1. The generating function of $\text{pod}_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} \text{pod}_\ell(n) q^n = \frac{\psi(-q^\ell)}{\psi(-q)} = \frac{f_2 f_\ell f_{4\ell}}{f_1 f_4 f_{2\ell}}. \tag{1.1}$$

Recently, for each $\alpha \geq 0$, Gireesh, Hirschhorn, and Naika [6] have obtained the generating function for

$$\sum_{n=0}^{\infty} \text{pod}_3\left(3^\alpha n + \delta_\alpha\right) q^n$$

where $4\delta_\alpha \equiv -1 \pmod{3^\alpha}$ if α is even, and $4\delta_\alpha \equiv -1 \pmod{3^{\alpha+1}}$ if α is odd. Saika [7] also obtained the congruence properties for $\text{pod}_3(n)$ and found infnite families of congruences modulo 2 and 3. In addition, Veena and Fathima studied [8] the divisibility properties of $\text{pod}_3(n)$ by using the theory of modular forms, for example, for $k \geq 1$,

$$\lim_{X \to \infty} \frac{\#\left\{n \leq X : \text{pod}_3(n) \equiv 0 \pmod{3^k}\right\}}{X} = 1.$$

Similarly, by imposing restrictions on the even parts while the odd parts are unrestricted, we also obtain the ℓ-regular partitions with distinct odd parts. Let $\text{ped}_\ell(n)$ denote the number of ℓ-regular partitions of n with even parts distinct. The generating function of $\text{ped}_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} \text{ped}_\ell(n) q^n = \frac{f_4 f_\ell}{f_1 f_{4\ell}}. \tag{1.2}$$

Drema and Saikia [9] found some infinite families of congruences modulo 2 and 4 for $\text{ped}_\ell(n)$ when ℓ = 3, 5, 7 and 11. For example, for any prime $p \geq 5$, $\left(\frac{-6}{p}\right) = -1$ and $1 \leq r \leq p - 1$, then for any $\alpha \geq 0$, they proved the congruence relation

$$\text{ped}_3\left(6 \cdot p^{2\alpha+1}(pn + r) + \frac{11 \cdot p^{2\alpha} + 1}{4}\right) \equiv 0 \pmod 4.$$

In [10], Hemanthkumar, Bharadwaj, and Naika established several congruences modulo 16 and 24 for $\mathrm{pod}_9(n)$. The authors also proved an identity connecting $\mathrm{pod}_9(n)$ and $\mathrm{ped}_9(n)$

$$3\mathrm{pod}_9(2n+1) = \mathrm{ped}_9(2n+3).$$

In this work, we will continue to study the congruence properties of $\mathrm{pod}_\ell(n)$ and $\mathrm{ped}_\ell(n)$. The main purpose of this paper is to prove the following results.

Theorem 1. *For any $n \geq 0$, ℓ is even, and we have*

$$\sum_{i=0}^{2n-1} \mathrm{pod}_\ell(2n-i)\sigma_1(i) \equiv 0 \pmod 2$$

where $\sigma_k(n) = \sum_{d|n} d^k$ is the standard divisor function.

Theorem 2. *Let p_i be distinct odd primes. For any $m, n \geq 0$, $t > 0$, we have*

$$\mathrm{pod}_{2^m}\left(An - \frac{As - (2^m - 1)}{8}\right) \equiv 0 \pmod{2^t}$$

where $A = \prod_{i=1}^{t(2^m-1)} p_i$ and $s \in \mathbb{Z}$ satisfies $8 \,|As - (2^m - 1)$.

Theorem 3. *For any $n \geq 0$, $\alpha \geq 0$, we have*

$$\begin{aligned}\sum_{n=0}^{\infty} \mathrm{pod}_5\left(4\times 15^{2\alpha}n + \frac{15\times\left(1+15^{2\alpha-1}\right)}{2} - 8\right)q^n &\equiv \sum_{n=0}^{\infty} \mathrm{pod}_5\left(4\times 7^{2\alpha}n + \frac{7\times\left(1+7^{2\alpha-1}\right)}{2} - 4\right)q^n \\ &\equiv f_1^3 \pmod 2. \end{aligned} \tag{1.3}$$

Meanwhile, let $\ell \geq 1$, if $1 \leq m < 8\ell$, $m \equiv 1 \pmod 8$, and $\left(\frac{m}{\ell}\right) = -1$, then we have

$$\begin{aligned}\mathrm{pod}_5\left(4\times 15^{2\alpha}\ell n + \frac{15\times\left(1+15^{2\alpha-1}m\right)}{2} - 8\right) &\equiv \mathrm{pod}_5\left(4\times 7^{2\alpha}\ell n + \frac{7\times\left(1+7^{2\alpha-1}m\right)}{2} - 4\right) \\ &\equiv 0 \pmod 2. \end{aligned} \tag{1.4}$$

Theorem 4. *For any $n \geq 0$, the following statements hold:*

(i) *If $\ell > 0$ satisfying $\ell \equiv -1 \pmod 8$ and l is a prime with $l\,|\ell$, then*

$$\sum_{n=0}^{\infty} \mathrm{ped}_\ell\left(ln + \frac{l^2-1}{8}\right)q^n \equiv \frac{f_l^3}{f_{\ell/l}^3} \pmod 2. \tag{1.5}$$

(ii) *If l is a prime satisfying $l \equiv -1 \pmod 8$, then*

$$\mathrm{ped}_l\left(ln + \frac{l^2-1}{8}\right) \equiv \mathrm{pod}_l(n) \pmod 2. \tag{1.6}$$

2. Preliminaries

We begin with some background on modular forms to prove our main results.
Let $f(z) = \sum_{n=0}^{\infty} a(n)q^n$, if $f(z) \in M_k(\Gamma_0(N), \chi_N)$ is a modular form. Let p be a prime and the operators U_p and V_p are defined by

$$f(z)\big|U_p := \sum_{n=0}^{\infty} a(pn)q^n, \quad f(z)\big|V_p := \sum_{n=0}^{\infty} a(n)q^{pn}$$

which satisfies the following property:

$$\left(f(z) \cdot g(z)\big|V_p\right)\big|U_p = \left(f(z)\big|U_p\right) \cdot g(z).$$

The Hecke operator T_p is defined by

$$T_p := U_p + \chi_N(p)p^{k-1}V_p, \quad k \geq 1.$$

Lemma 1. *[12, P. 18], If $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $k = \sum_{\delta|N} r_\delta/2 \in \mathbb{Z}$, with the additional that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}, \qquad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}$$

then $f(z)$ satisfies

$$f\left(\frac{az+b}{cz+d}\right) = \chi_N(d)(cz+d)^k f(z), \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma_0(N).$$

Here, the character χ_N is defined by $\chi_N(d) := \left(\frac{(-1)^k s}{d}\right)$, where $s := \prod_{\delta|N} \delta^{r_\delta}$. Moreover, if $f(z)$ is holomorphic (resp. vanishes) at all of the cusps of $\Gamma_0(N)$, then $f(z) \in M_k(\Gamma_0(N), \chi_N)$ (resp. $S_k(\Gamma_0(N), \chi_N)$).

Lemma 2. *[12, P. 18], Let c,d, and N be positive integers with $d|N$ and $\gcd(c, d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Lemma 1 for N, then the order of vanishing of $f(z)$ at the cusp c/d is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d, \delta)^2 r_\delta}{\gcd\left(d, \frac{N}{\delta}\right) d\delta}.$$

3. Proofs of the results

Before proving Theorem 1, we state here the following identity:

$$\sum_{n=1}^{\infty} \frac{n^k q^n}{1-q^n} = \sum_{n=1}^{\infty} \sigma_k(n)q^n. \tag{3.1}$$

Proof of Theorem 1. Taking logarithms of relation (1.1), we find that

$$\log\left(\sum_{n=0}^{\infty} \mathrm{pod}_\ell(n)q^n\right) = \sum_{n=1}^{\infty} \log(1-q^{2n}) + \sum_{n=1}^{\infty} \log(1-q^{\ell n}) + \sum_{n=1}^{\infty} \log(1-q^{4\ell n})$$

$$-\sum_{n=1}^{\infty}\log(1-q^{n})-\sum_{n=1}^{\infty}\log(1-q^{4n})-\sum_{n=1}^{\infty}\log(1-q^{2\ell n}). \tag{3.2}$$

Using the differential operator $q\frac{d}{dq}$ to (3.2) yields

$$\left(\sum_{n=0}^{\infty} n\mathrm{pod}_{\ell}(n)q^{n-1}\right)\Big/\left(\sum_{n=0}^{\infty}\mathrm{pod}_{\ell}(n)q^{n}\right)=-2\sum_{n=1}^{\infty}\frac{nq^{4n-1}}{1-q^{4n}}-\ell\sum_{n=1}^{\infty}\frac{nq^{\ell n-1}}{1-q^{\ell n}}-4\ell\sum_{n=1}^{\infty}\frac{nq^{4\ell n-1}}{1-q^{4\ell n}}$$
$$+\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^{n}}+4\sum_{n=1}^{\infty}\frac{nq^{4n-1}}{1-q^{4n}}+2\ell\sum_{n=1}^{\infty}\frac{nq^{2\ell n-1}}{1-q^{2\ell n}} \tag{3.3}$$

namely,

$$\sum_{n=0}^{\infty} n\mathrm{pod}_{\ell}(n)q^{n}=\left(\sum_{n=0}^{\infty}\mathrm{pod}_{\ell}(n)q^{n}\right)\left(-2\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}}-\ell\sum_{n=1}^{\infty}\frac{nq^{\ell n}}{1-q^{\ell n}}-4\ell\sum_{n=1}^{\infty}\frac{nq^{4\ell n}}{1-q^{4\ell n}}\right.$$
$$\left.+\sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}}+4\sum_{n=1}^{\infty}\frac{nq^{4n}}{1-q^{4n}}+2\ell\sum_{n=1}^{\infty}\frac{nq^{2\ell n}}{1-q^{2\ell n}}\right).$$

Consequently, we can deduce the following congruence:

$$\sum_{n=0}^{\infty} n\mathrm{pod}_{\ell}(n)q^{n}\equiv\left(\sum_{n=0}^{\infty}\mathrm{pod}_{\ell}(n)q^{n}\right)\left(\sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}}-\ell\sum_{n=1}^{\infty}\frac{nq^{\ell n}}{1-q^{\ell n}}\right)$$
$$=\left(\sum_{n=0}^{\infty}\mathrm{pod}_{\ell}(n)q^{n}\right)\left(\sum_{n=1}^{\infty}\left(\sigma_1(n)-\ell\sigma_1\left(\frac{n}{\ell}\right)\right)q^{n}\right)\pmod{2}. \tag{3.4}$$

By equating the even terms coefficients, we find that

$$2n\mathrm{pod}_{\ell}(2n)=\sum_{i=0}^{2n-1}\mathrm{pod}_{\ell}(2n-i)\left(\sigma_1(i)-\ell\sigma_1\left(\frac{i}{\ell}\right)\right)\equiv 0\pmod{2}. \tag{3.5}$$

In particular, if ℓ is even, we have

$$\sum_{i=0}^{2n-1}\mathrm{pod}_{\ell}(2n-i)\sigma_1(i)\equiv 0\pmod{2}. \tag{3.6}$$

This completes the proof Theorem 1. □

Before we prove Theorem 2, we included here the following lemma, which was proved in a letter from Tate to Serre [11]. He proved that the action of the Hecke operators are locally nilpoten of modulo 2. For simplicity, we define $M_k := M_k(\mathrm{SL}_2\mathbb{Z})$, $S_k := S_k(\mathrm{SL}_2\mathbb{Z})$.

Lemma 3. *If $f(z)\in M_k\cap\mathbb{Z}[[q]]$ is a modular form, then there exists a positive integer $t\le\dim M_k\le\left[\frac{k}{12}\right]+1$ such that for any collection of distinct odd primes $p_1, p_2,\cdots,p_t$,*

$$f(z)\,\big|T_{p_1}\big|T_{p_2}\,|\cdots\,\big|T_{p_t}\equiv 0\pmod{2}.$$

That is, f has some degree of nilpotency that is bounded by $\left[\frac{k}{12}\right]+1$, which also bounds the degree of nilpotency of the image of f under any Hecke operator. Based on work of Tate, Mahlburg [13] proved some congruences modulo arbitrary powers of 2 for the coefficients of certain quotients of Eisenstein series. In particular, Boylan [14] gave a corollary of Tate's result, which asserts that for any $t \leq j$ if $f(z) \in S_{12j} \pmod 2$, that is,

$$f(z)\left|T_{p_1}\right|T_{p_2}\left|\cdots\right|T_{p_j} \equiv 0 \pmod 2.$$

By using the Boylan's result and similar techniques that are used by Mahlburg, we prove here Theorem 2.

Proof of Theorem 2. Setting $\ell = 2^m$ in (1.1), we have

$$\sum_{n=0}^{\infty} \mathrm{pod}_{2^m}(n) q^n = \frac{f_2 f_{2^m} f_{2^{m+2}}}{f_1 f_4 f_{2^{m+1}}} \equiv \frac{f_{2^m}^3}{f_1^3} \equiv f_1^{3(2^m-1)} \pmod 2. \tag{3.7}$$

Replacing q by q^8 in (3.7) and multiplying both sides by $q^{(2^m-1)}$, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_{2^m}(n) q^{8n+(2^m-1)} \equiv q^{(2^m-1)} \prod_{n=1}^{\infty} (1-q^n)^{24(2^m-1)} = \Delta(z)^{(2^m-1)} \pmod 2 \tag{3.8}$$

where $\Delta(z)$ is the Delta-function, which is the unique cusp form of weight 12 on $\mathrm{SL}_2(\mathbb{Z})$. $\Delta(z)^{(2^m-1)} \in S_{12(2^m-1)}(\mathrm{SL}_2(\mathbb{Z}))$ is a cusp form of weight $12(2^m-1)$. Hence, by using the Boylan's result, for any 2^m-1 distinct odd primes $p_1, p_2, \cdots, p_{2^m-1}$, we have

$$\Delta(z)^{(2^m-1)}\left|T_{p_1}\right|T_{p_2}\left|\cdots\right|T_{p_{2^m-1}} \equiv 0 \pmod 2.$$

Furthermore, it is well known that $\frac{1}{2}\Delta(z)^{(2^m-1)}\left|T_{p_1}\right|T_{p_2}\left|\cdots\right|T_{p_{2^m-1}} \in S_{12(2^m-1)}(\mathrm{SL}_2(\mathbb{Z}))$ is also a cusp form of weight $12(2^m-1)$. Applying Boylan's corollary once more, we choose any 2^m-1 distinct odd primes $p_{2^m}, p_{2^m+1}, \cdots, p_{2(2^m-1)}$ coprime to $p_1, p_2, \cdots, p_{2^m-1}$, and we can deduce that

$$\Delta(z)^{(2^m-1)}\left|T_{p_1}\right|T_{p_2}\left|\cdots\right|T_{p_{2(2^m-1)}} \equiv 0 \pmod{2^2}.$$

By iterating the above process, for any $t(2^m-1)$ distinct odd primes $p_1, p_2, \cdots, p_{t(2^m-1)}$, we can conclude that

$$\Delta(z)^{(2^m-1)}\left|T_{p_1}\right|T_{p_2}\left|\cdots\right|T_{p_{t(2^m-1)}} \equiv 0 \pmod{2^t}. \tag{3.9}$$

It follows from (3.8) that

$$\mathrm{pod}_{2^m}\left(\frac{p_1 p_2 \cdots p_{t(2^m-1)} n - (2^m-1)}{8}\right) \equiv 0 \pmod{2^t}. \tag{3.10}$$

Since $(8, p_i) = 1$, setting $A = \prod_{i=1}^{t(2^m-1)} p_i$, there exists a unique integer s such that $8 \,|As - (2^m-1)$. Replacing $8n+s$ by n in (3.10), we obtain

$$\mathrm{pod}_{2^m}\left(An - \frac{As-(2^m-1)}{8}\right) \equiv 0 \pmod{2^t}.$$

This completes the proof of Theorem 2. □

In order to prove the remaining theorems, we first establish an explicit formula for the action of the Hecke operator T_p on eta-quotient $\eta^3(z)\eta^3(Nz)$ for $N \equiv -1 \pmod 8$, where p is a prime satisfying $p\,|N$.

Lemma 4. *If $N > 0$ with $N \equiv -1 \pmod 8$, suppose p is a prime dividing N, then*

$$\eta^3(z)\eta^3(Nz)\,\big|T_p = (-1)^{(p-1)/2}p \cdot \eta^3(pz)\eta^3(Nz/p)\,. \tag{3.11}$$

Proof. We write

$$\eta^3(8z) = \sum_{k=1}^{\infty} k\left(\frac{-4}{k}\right)q^{k^2}.$$

Adding the U_p operator to $\eta^3(8z)$, we obtain

$$\eta^3(8z)\,\big|U_p = \sum_{n=1}^{\infty} k\left(\frac{-4}{k}\right)q^{k^2/p}. \tag{3.12}$$

Replacing k by pk in (3.12), we have

$$\eta^3(8z)\,\big|U_p = \sum_{k=1}^{\infty} pk\left(\frac{-4}{pk}\right)q^{pk^2} = \sum_{k=1}^{\infty} pk\left(\frac{-1}{p}\right)\left(\frac{-4}{k}\right)q^{pk^2} = (-1)^{(p-1)/2}p \cdot \eta^3(8pz). \tag{3.13}$$

Since $N \equiv -1 \pmod 8$, by Lemmas 1 and 2, we have that $\eta^3(8z)\eta^3(8Nz) \in S_3(\Gamma_0(8N), \chi_{2N})$ is a cusp form of weight 3. Adding the U_p operator to $\eta^3(8z)\eta^3(8Nz)$ and employing (3.13), we obtain

$$\begin{aligned}
\eta^3(8z)\eta^3(8Nz)\,\big|U_p &= \left(\sum_{k=1}^{\infty} k\left(\frac{-4}{k}\right)q^{k^2}\right)\left(\sum_{k=1}^{\infty} k\left(\frac{-4}{k}\right)q^{Nk^2}\right)\Big|U_p \\
&= \left(\sum_{k=1}^{\infty} pk\left(\frac{-4}{pk}\right)q^{pk^2}\right)\left(\sum_{k=1}^{\infty} k\left(\frac{-4}{k}\right)q^{Nk^2/p}\right) \\
&= (-1)^{(p-1)/2}p \cdot \eta^3(8pz)\eta^3\left(\frac{8Nz}{p}\right).
\end{aligned}$$

Since $p\,|N$, then $\chi_N(p) = 0$, and adding the T_p operator to $\eta^3(8z)\eta^3(8Nz)$, we deduce that

$$\eta^3(8z)\eta^3(8Nz)\,\big|T_p = \eta^3(8z)\eta^3(8Nz)\,\big|U_p = (-1)^{(p-1)/2}p \cdot \eta^3(8pz)\eta^3(8Nz/p). \tag{3.14}$$

Replacing q^8 by q in (3.14), we can deduce (3.11). □

To prove Theorem 3, we here need to verify some congruence relations. First, it is easy to check that if $f(z) \in M_k(\Gamma_0(N), \chi_N)$, then by definitions we have $f(z)\,\big|U_p = f(z)\,\big|T_p$, for p is a prime satisfying $p\,|N$. Second, by using Lemma 4, we can show the following congruences:

$$\eta^3(z)\eta^3(15z)\,|U_3 = \eta^3(z)\eta^3(15z)\,|T_3 = -3\eta^3(3z)\eta^3(5z) \equiv \eta^3(3z)\eta^3(5z) \pmod 2, \tag{3.15}$$

$$\eta^3(3z)\eta^3(5z)\,|U_5 = \eta^3(3z)\eta^3(5z)\,|T_5 = -5\eta^3(z)\eta^3(15z) \equiv \eta^3(z)\eta^3(15z) \pmod 2. \tag{3.16}$$

Based on the above results, now we are able to prove Theorem 3.

Proof of Theorem 3. Setting $\ell = 5$ in (1.1), we have

$$\sum_{n=0}^{\infty} \mathrm{pod}_5(n)q^n = \frac{f_2 f_5 f_{20}}{f_1 f_4 f_{10}} \equiv \frac{f_1 f_5 f_5^4}{f_1 f_1^4 f_5^2} \equiv \frac{f_5^3}{f_1^3} \pmod 2. \tag{3.17}$$

From [15, P. 60], Hirschhorn and Sellers proved the following 2-dissection of f_5/f_1:

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}. \tag{3.18}$$

Employing (3.18) in (3.17), we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_5(n)q^n \equiv \left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right)\frac{f_{10}}{f_2} \pmod 2. \tag{3.19}$$

Extracting those terms involving q^{2n} in (3.19) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_5(2n)q^n \equiv f_1 f_5 \equiv f_2\frac{f_5}{f_1} \pmod 2.$$

Applying (3.18) once more, it follows that

$$\sum_{n=0}^{\infty} \mathrm{pod}_5(2n)q^n \equiv f_2\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right) \pmod 2. \tag{3.20}$$

Extracting those terms involving q^{2n} in (3.20) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_5(4n)q^n \equiv f_1^3 \pmod 2. \tag{3.21}$$

Multiplying both sides by $q^2 f_{15}^3$ in (3.21), we have

$$\left(\sum_{n=0}^{\infty} \mathrm{pod}_5(4n)q^{n+2}\right) f_{15}^3 \equiv q^2 f_1^3 f_{15}^3 = \eta^3(z)\eta^3(15z) \pmod 2.$$

Applying the U_3, U_5 operators in sequence and employing (3.15) and (3.16), we find that

$$\begin{aligned}\left[\left(\sum_{n=0}^{\infty} \mathrm{pod}_5(4n)q^{n+2}\right) f_{15}^3\right] |U_3\ |U_5 &\equiv \eta^3(z)\eta^3(15z)\,|U_3\ |U_5 \equiv \eta^3(z)\eta^3(15z)\,|T_3\ |T_5 \\ &\equiv \eta^3(z)\eta^3(15z) \pmod 2\end{aligned}$$

which yields

$$\left(\sum_{n=0}^{\infty} \mathrm{pod}_5(60n-8)q^n\right) f_1^3 \equiv \eta^3(z)\eta^3(15z) \pmod 2. \tag{3.22}$$

Dividing both sides by $q^2 {f_1}^3$ in (3.22) and acting the U_{15} operator on both sides, we deduce that

$$\left(\sum_{n=0}^{\infty} \mathrm{pod}_5(900n+112)q^n\right) \equiv f_1^3 \pmod 2.$$

By induction, we can deduce that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} \mathrm{pod}_5\left(4\times 15^{2\alpha}n + \frac{15\times\left(1+15^{2\alpha-1}\right)}{2} - 8\right)q^n \equiv f_1^3 \pmod 2 \tag{3.23}$$

which is the first part of (1.3). Substituting q by q^8 and multiplying both sides by q, we obtain

$$\sum_{n=0}^{\infty} \mathrm{pod}_5\left(4\times 15^{2\alpha}n + \frac{15\times\left(1+15^{2\alpha-1}\right)}{2} - 8\right)q^{8n+1} \equiv qf_8^3 \equiv \Delta(z) \equiv \sum_{n=0}^{\infty} q^{(2n+1)^2} \pmod 2 \tag{3.24}$$

where $\Delta(z) = \prod_{n=1}^{\infty}(1-q^n)^{24}$. If $m \equiv 1 \pmod 8$ and $\left(\frac{m}{\ell}\right) = -1$, then $8\ell n + m$ cannot be a square. This implies that the coefficients of $q^{8\ell n+m}$ in the lefthand side of (3.24) must be even. Hence, we have

$$\mathrm{pod}_5\left(4\times 15^{2\alpha}\ell n + \frac{15\times\left(1+15^{2\alpha-1}m\right)}{2} - 8\right) \equiv 0 \pmod 2$$

which is the first part of (1.4). Similarly as in the preceding discussion, we observe that the eta quotient $\eta^3(z)\eta^3(7z) \in S_3(\Gamma_0(7), \chi_7)$ is a Hecke eigenform. Hnece, we can conclude that for $\alpha \geq 0$, the following congruence holds:

$$\sum_{n=0}^{\infty} \mathrm{pod}_5\left(4\times 7^{2\alpha}n + \frac{7\times\left(1+7^{2\alpha-1}\right)}{2} - 4\right)q^n \equiv f_1^3 \pmod 2$$

which is the second part of (1.3). Furthermore, for $m \equiv 1 \pmod 8$ and $\left(\frac{m}{\ell}\right) = -1$, we have

$$\mathrm{pod}_5\left(4\times 7^{2\alpha}\ell n + \frac{7\times\left(1+7^{2\alpha-1}m\right)}{2} - 4\right) \equiv 0 \pmod 2.$$

This completes the proof of Theorem 3. □

As another application of Lemma 4, we are now ready to prove a congruence relation modulo 2 for $\mathrm{ped}_\ell(n)$.

Proof of Theorem 4. By (1.2), we have

$$\sum_{n=0}^{\infty} \mathrm{ped}_\ell(n)q^n = \frac{f_4 f_\ell}{f_1 f_{4\ell}} \equiv \frac{f_1^4 f_\ell}{f_1 f_\ell^4} \equiv \frac{f_1^3}{f_\ell^3} \pmod 2.$$

Multiplying both sides by $q^{(\ell+1)/8} f_\ell^6$, we have

$$\left(\sum_{n=0}^{\infty} \mathrm{ped}_\ell(n)q^{n+(\ell+1)/8}\right) f_\ell^6 \equiv \eta^3(z)\eta^3(\ell z) \pmod 2. \tag{3.25}$$

Since $l\,|\,\ell$, acting the operator U_l to both sides of (3.25) and using Lemma 4, we have

$$\left(\sum_{n=0}^{\infty} \mathrm{ped}_\ell(n) q^{n+(\ell+1)/8}\right) f_\ell^6 \,|U_l \equiv \eta^3(z)\eta^3(\ell z)\,|U_l = \eta^3(z)\eta^3(\ell z)\,|T_l$$

$$= (-1)^{(l-1)/2} l \cdot \eta^3(lz)\eta^3(\ell z/l) \equiv \eta^3(lz)\eta^3(\ell z/l) \pmod 2.$$

Consequently,

$$\sum_{n=0}^{\infty} \mathrm{ped}_\ell\left(ln - \frac{\ell+1}{8}\right) q^{n-(l^2+\ell)/8l} \equiv \frac{f_l^3}{f_{\ell/l}^3} \pmod 2. \tag{3.26}$$

Replacing $n - (l^2 + \ell)/8l$ by n in (3.26), we obtain

$$\sum_{n=0}^{\infty} \mathrm{ped}_\ell\left(ln + \frac{l^2-1}{8}\right) q^n \equiv \frac{f_l^3}{f_{\ell/l}^3} \pmod 2.$$

This proves (1.5).

In particular, when $\ell = l$ is a prime, we obtain

$$\sum_{n=0}^{\infty} \mathrm{ped}_l\left(ln + \frac{l^2-1}{8}\right) q^n \equiv \frac{f_l^3}{f_1^3} \equiv \frac{f_2 f_l f_{4l}}{f_1 f_4 f_{2l}} \equiv \sum_{n=0}^{\infty} \mathrm{pod}_l(n) q^n \pmod 2. \tag{3.27}$$

Comparing the coefficients of q^n on both sides of (3.27), we obtain

$$\mathrm{ped}_l\left(ln + \frac{l^2-1}{8}\right) \equiv \mathrm{pod}_l(n) \pmod 2.$$

We deduce (1.6). □

4. Conclusions

In this paper, with the help of modular forms, we investigated on some congruence problems for ℓ-regular partitions with certain restrictions. In future studies, interested readers may examine whether these methods also can be extended to congruence problems for other types of partition functions.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares that he has no conflict of interest.

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