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Research article

Contraction of variational principle and optical soliton solutions for two models of nonlinear Schrödinger equation with polynomial law nonlinearity

Aly R. Seadawy* **and Bayan Alsaedi**

Department of Mathematics, Faculty of Science, Taibah University, Al-Madinah Al-Munawarah, Saudi Arabia

* **Correspondence:** Email: aabdelalim@taibahu.edu.sa.

Abstract: Our study analyzes the two models of the nonlinear Schrödinger equation (NLSE) with polynomial law nonlinearity by powerful and comprehensible techniques, such as the variational principle method and the amplitude ansatz method. We will derive the functional integral and the Lagrangian of these equations, which illustrate the system's dynamic. The solutions of these models will be extracted by selecting the trial ansatz functions based on the Jost linear functions, which are continuous at all intervals. We start with the Jost function that has been approximated by a piecewise linear function with a single nontrivial variational parameter in three cases from a region of a rectangular box, then use this trial function to obtain the functional integral and the Lagrangian of the system without any loss. After that, we approximate this trial function by piecewise linear ansatz function in two cases of the two-box potential, then approximate it by quadratic polynomials with two free parameters rather than a piecewise linear ansatz function, and finally, will be approximated by the tanh function. Also, we utilize the amplitude ansatz method to extract the new solitary wave solutions of the proposed equations that contain bright soliton, dark soliton, bright-dark solitary wave solutions, rational dark-bright solutions, and periodic solitary wave solutions. Furthermore, conditions for the stability of the solutions will be submitted. These answers are crucial in applied science and engineering and will be introduced through various graphs such as 2D, 3D, and contour plots.

Keywords: nonlinear Schrödinger equation with polynomial law nonlinearity; variational principle method; amplitude ansatz method; soliton solutions
Mathematics Subject Classification: 35B10, 35C07, 35C08, 35Q35

1. Introduction

The nonlinear Schrödinger equations (NLSEs) are extensively used to describe numerous crucial phenomena and dynamic processes in various fields such as fluid dynamics, plasma, chemistry, biology, optical fibers [1–4], nuclear physics, stochastic mechanics, biomolecule dynamics, dynamics of accelerators, and Bose-Einstein condensates [5–10]. The last few decades have seen significant advancements in the field of nonlinear optics [11–13]. These equations also appear in other forms of nonlinearities, such as cubic-quintic (CQ), cubic-quintic-septic (CQS), power-law, logarithmic nonlinearities, and various other forms. There is a growing interest in studying soliton pulses that can propagate without changing their shape in optical fibers [14, 15].

Therefore, obtaining the exact soliton solutions of these NLSEs can help us to understand these phenomena better. In recent years, several effective approaches have been developed to construct accurate solutions of these equations. For example, the simple equation method [16], Kudryashov's method [17], the Jacobi elliptic function method [18], the inverse scattering method [19], the extended trial function method [20], the tanh method [21], the F-expansion method [22], generalized extended tanh-function method, the sine–cosine method [23], the generalized Riccati equation method [24], the new ϕ^6-model expansion method [25], Hirota bilinear method [26], the exp-function method [27], the Darboux transformation [28], the auxiliary equation method [29], the Binary Bell polynomials [30], the extended hyperbolic function method [31], the homogeneous balance Method [32], the $(\frac{G'}{G})$-expansion method [33, 34], the exponential rational function method [35], the Bäcklund transformation [36], the homotopy perturbation method [37], the modified Kudryashov method, the sine-Gordon expansion approach [38], the Riccati-Bernoulli sub-ODE method [39], the modified extended direct algebraic method [40], the truncated Painlevé expansion method [41], the $(G'/G, 1/G)$-expansion method [42], the $\tan(\phi/2)$-expansion method [43, 44], the soliton ansatz method [45, 46], and so on.

Researches have been conducted on the study of NLSEs with the polynomial law of nonlinearity. For instance, Seadawy et al. [47] established soliton solutions using the extended simplest equation method, and they also described this model [48], which includes the conservation principles for optical soliton (OS) with polynomial and triple power law nonlinearities, while Aziz et al. [49] discovered chirped soliton solutions by utilizing Jacobi elliptic functions. Furthermore, this model was discussed by Dieu-donne et al. [50], who examined the optical soliton (OS) solutions with two types of nonlinearities, which are triple power and polynomial laws. Sugati et al. [51] also examined this model, obtaining the traveling pulse solutions to the NLSE when it is treated with both spatio-temporal dispersion (STD) and group velocity dispersion (GVD).

In this paper, our aim is to investigate soliton solutions for two models of the NLSE that carry the polynomial law of nonlinearity (cubic-quintic-septic). We will use the variational principle based on finding Lagrangian, which we will then apply it with different trial functions that have one or two nontrivial variational parameters. Furthermore, we will employ another technique called the amplitude ansatz method to extract new solitary wave solutions.

The paper is categorized as follows. In Section 2, we discuss model-I of NLSE with polynomial law nonlinearity in terms of formulation of the variational principle and finding solitary wave solutions. In Section 3, we apply the same steps on model-II of NLSE with polynomial law nonlinearity. Finally, the work concludes in Section 4.

2. Schrödinger's nonlinear equation with the polynomial nonlinear law model I

2.1. Formulation of the variational principle for the Schrödinger's nonlinear equation with the polynomial nonlinear law

Nonlinear Schrödinger equation (NLSE) with the polynomial nonlinear law is [47]:

$$i\Gamma_t + a\Gamma_{xx} + b_1\Gamma|\Gamma|^2 + b_2\Gamma|\Gamma|^4 + b_3\Gamma|\Gamma|^6 = 0. \tag{2.1}$$

Such that Γ is a complex function on the form: $\Gamma(x,t) = \Theta(x,t) + i\Psi(x,t)$. Since Θ and Ψ are real functions of x and t, furthermore $|\Gamma|^2 = (\Theta + i\Psi)(\Theta - i\Psi)$. Using the variational approach, we will search for solutions to NLSE with the polynomial nonlinear law. As a result, we derive Γ with respect to x and t to investigate the existence of a Lagrangian and the invariant variational principle for this equation, which are expressed in the following way:

Let M and N are functionals in Θ and Ψ:

$$\begin{aligned} M(\Theta,\Psi) =& \frac{\partial\Theta}{\partial t} + a\frac{\partial^2\Psi}{\partial x^2} + b_1\Psi\Theta^2 + b_1\Psi^3 + b_2\Psi\Theta^4 + 2b_2\Theta^2\Psi^3 + b_2\Psi^5 + b_3\Psi\Theta^6 \\ &+ 3b_3\Theta^4\Psi^3 + 3b_3\Theta^2\Psi^5 + b_3\Psi^7, \end{aligned} \tag{2.2}$$

$$\begin{aligned} N(\Theta,\Psi) =& -\frac{\partial\Psi}{\partial t} + a\frac{\partial^2\Theta}{\partial x^2} + b_1\Theta^3 + b_1\Theta\Psi^2 + b_2\Theta^5 + 2b_2\Psi^2\Theta^3 + b_2\Theta\Psi^4 + b_3\Theta^7 \\ &+ 3b_3\Psi^2\Theta^5 + 3b_3\Theta^3\Psi^4 + b_3\Theta\Psi^6. \end{aligned} \tag{2.3}$$

Put $\Theta = \lambda\Theta$ and $\Psi = \lambda\Psi$,

$$\begin{aligned} \int_0^1 M(\lambda\Theta,\lambda\Psi)\,d\lambda =& \frac{1}{2}\frac{\partial\Theta}{\partial t} + \frac{1}{2}a\frac{\partial^2\Psi}{\partial x^2} + \frac{1}{4}b_1\Psi\Theta^2 + \frac{1}{4}b_1\Psi^3 + \frac{1}{6}b_2\Psi\Theta^4 + \frac{1}{3}b_2\Theta^2\Psi^3 \\ &+ \frac{1}{6}b_2\Psi^5 + \frac{1}{8}b_3\Psi\Theta^6 + \frac{3}{8}b_3\Theta^4\Psi^3 + \frac{3}{8}b_3\Theta^2\Psi^5 + \frac{1}{8}b_3\Psi^7, \end{aligned}$$

$$\begin{aligned} \int_0^1 N(\lambda\Theta,\lambda\Psi)\,d\lambda =& -\frac{1}{2}\frac{\partial\Psi}{\partial t} + \frac{1}{2}a\frac{\partial^2\Theta}{\partial x^2} + \frac{1}{4}b_1\Theta^3 + \frac{1}{4}b_1\Theta\Psi^2 + \frac{1}{6}b_2\Theta^5 + \frac{1}{3}b_2\Psi^2\Theta^3 \\ &+ \frac{1}{6}b_2\Theta\Psi^4 + \frac{1}{8}b_3\Theta^7 + \frac{3}{8}b_3\Psi^2\Theta^5 + \frac{3}{8}b_3\Theta^3\Psi^4 + \frac{1}{8}b_3\Theta\Psi^6. \end{aligned}$$

The consistency conditions are expressed as follows [52, 53], where Θ_x, Ψ_x, Θ_t and Ψ_t stand for the partial derivatives of Θ and Ψ with respect to variables x and t. If the system of Eqs (2.2) and (2.3) satisfies the the previously mentioned conditions, then a functional integral $J(\Theta,\Psi)$ can be written down using the formula given by Tonti [53]:

$$\begin{aligned} J(\Theta,\Psi) = \int_\omega \Big[& \frac{1}{4}b_1\Theta^4 + \frac{1}{6}b_2\Theta^6 + \frac{1}{8}b_3\Theta^8 + \frac{1}{2}b_1\Theta^2\Psi^2 + \frac{1}{2}b_2\Theta^4\Psi^2 + \frac{1}{2}b_3\Theta^6\Psi^2 \\ &+ \frac{1}{4}b_1\Psi^4 + \frac{1}{2}b_2\Theta^2\Psi^4 + \frac{3}{4}b_3\Theta^4\Psi^4 + \frac{1}{6}b_2\Psi^6 + \frac{1}{2}b_3\Theta^2\Psi^6 + \frac{1}{8}b_3\Psi^8 \\ &+ \frac{1}{2}\Psi\Theta_t - \frac{1}{2}\Theta\Psi_t + \frac{1}{2}a\Theta\Theta_{xx} + \frac{1}{2}a\Psi\Psi_{xx} \Big]\, d\omega, \end{aligned}$$

where $d\omega = dxdt$.

The terms $\Theta\Theta_{xx}$ and $\Psi\Psi_{xx}$ solve integration by parts and choosing the boundary on Θ_x and Ψ_x to be such that the boundary terms vanish, we get

$$
\begin{aligned}
J(\Theta,\Psi) = \int_{\omega}\Big[&\frac{1}{4}b_1\Theta^4 + \frac{1}{6}b_2\Theta^6 + \frac{1}{8}b_3\Theta^8 + \frac{1}{2}b_1\Theta^2\Psi^2 + \frac{1}{2}b_2\Theta^4\Psi^2 + \frac{1}{2}b_3\Theta^6\Psi^2 \\
&+ \frac{1}{4}b_1\Psi^4 + \frac{1}{2}b_2\Theta^2\Psi^4 + \frac{3}{4}b_3\Theta^4\Psi^4 + \frac{1}{6}b_2\Psi^6 + \frac{1}{2}b_3\Theta^2\Psi^6 + \frac{1}{8}b_3\Psi^8 \\
&+ \frac{1}{2}\Psi\Theta_t - \frac{1}{2}\Theta\Psi_t - \frac{1}{2}a\Theta_x^2 - \frac{1}{2}a\Psi_x^2\Big]\, d\omega,
\end{aligned} \tag{2.4}
$$

where $d\omega = dxdt$.

We obtain the Lagrangian L as

$$
\begin{aligned}
L(\Theta,\Psi) =&\frac{1}{4}b_1\Theta^4 + \frac{1}{6}b_2\Theta^6 + \frac{1}{8}b_3\Theta^8 + \frac{1}{2}b_1\Theta^2\Psi^2 + \frac{1}{2}b_2\Theta^4\Psi^2 + \frac{1}{2}b_3\Theta^6\Psi^2 \\
&+ \frac{1}{4}b_1\Psi^4 + \frac{1}{2}b_2\Theta^2\Psi^4 + \frac{3}{4}b_3\Theta^4\Psi^4 + \frac{1}{6}b_2\Psi^6 + \frac{1}{2}b_3\Theta^2\Psi^6 + \frac{1}{8}b_3\Psi^8 \\
&+ \frac{1}{2}\Psi\Theta_t - \frac{1}{2}\Theta\Psi_t - \frac{1}{2}a\Theta_x^2 - \frac{1}{2}a\Psi_x^2.
\end{aligned} \tag{2.5}
$$

We find the value of L in the Euler-Lagrange equations to validate our interpretations:

$$
\frac{\partial L}{\partial \Theta} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Theta_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Theta_x}\right) = 0,
$$

$$
\frac{\partial L}{\partial \Psi} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \Psi_t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \Psi_x}\right) = 0.
$$

The resulting derivatives give us the system of Eqs (2.2) and (2.3).

2.1.1. The rectangular box

We demonstrate the simplest example of the application of this technique, by taking the box-shaped initial pulse and an ansatz based on linear Jost functions in a single nontrivial variational parameter in three cases.

Case 1: We use the following ansatz for $\Theta(x,t)$ and $\Psi(x,t)$ functions:

$$
\Theta(x,t) = \begin{cases} 100\exp(-\mu(t-5)(x-5)), & \text{at} \quad x>5,\ t>5, \\ (t+5)(x+5), & \text{at} \quad |x|<5,\ |t|<5, \\ 0, & \text{at} \quad x<-5,\ t<-5, \end{cases} \tag{2.6}
$$

$$
\Psi(x,t) = \begin{cases} 0, & \text{at} \quad x>5,\ t>5, \\ (5-t)(5-x), & \text{at} \quad |x|<5,\ |t|<5, \\ 100\exp(\mu(t+5)(x+5)), & \text{at} \quad x<-5,\ t<-5. \end{cases} \tag{2.7}
$$

We found the values of the Lagrangian L from substituting Eqs (2.6) and (2.7) into Eq (2.5) (see Figure 1):

$$J(\Theta,\Psi)=\int_{-\infty}^{-5}\int_{-\infty}^{-5}L\,dxdt+\int_{-5}^{5}\int_{-5}^{5}L\,dxdt+\int_{5}^{\infty}\int_{5}^{\infty}L\,dxdt,$$

where

$$\int_{-\infty}^{-5}\int_{-\infty}^{-5}L\,dxdt=\int_{5}^{\infty}\int_{5}^{\infty}L\,dxdt=0.$$

By using Mathematica program we get the value of $J(\Theta,\Psi)$ at interval $-5<x<5,\ -5<t<5$ as

$$\begin{aligned}J(\Theta,\Psi)=&-\frac{10000a}{3}+\frac{1850000000b_1}{9}+\frac{304000000000000b_2}{441}\\&+\frac{17600000000000000000b_3}{567}+\frac{25000}{3}.\end{aligned}\tag{2.8}$$

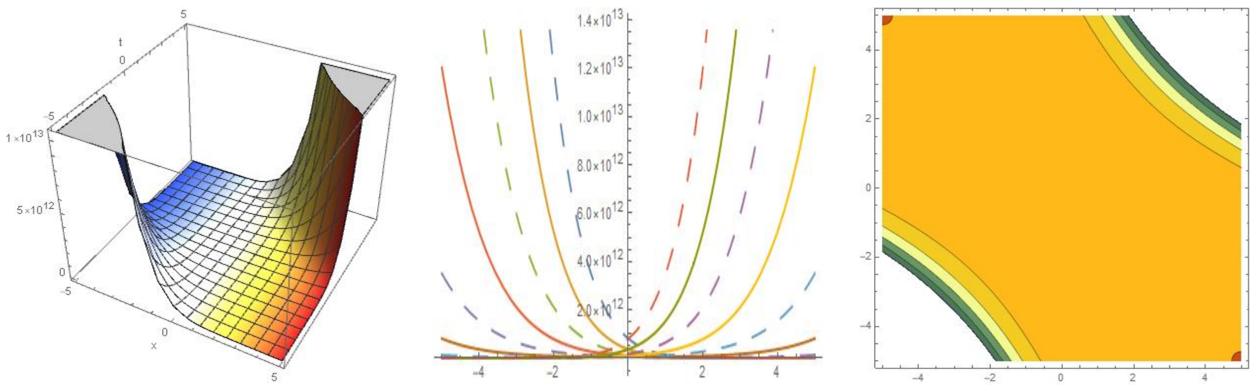


Figure 1. Show an example of the Lagrangian $L(x,t)$ with Eq (2.5), in the interval $-5<x<5$ and $-5<t<5$, by choosing the trial functions (2.6) and (2.7) at $a=3,\ b_1=2,\ b_2=\frac{1}{2},\ b_3=\frac{1}{6}$.

Case 2: Assume $\Theta(x,t)$ and $\Psi(x,t)$ for the Jost function as the following:

$$\Theta(x,t)=\begin{cases}20\exp(-\mu(t-5)(x-5)), & \text{at}\quad x>5,\ t>5,\\ t+x+10, & \text{at}\quad |x|<5,\ |t|<5,\\ 0, & \text{at}\quad x<-5,\ t<-5,\end{cases}\tag{2.9}$$

$$\Psi(x,t)=\begin{cases}0, & \text{at}\quad x>5,\ t>5,\\ 10-t-x, & \text{at}\quad |x|<5,\ |t|<5,\\ 20\exp(\mu(t+5)(x+5)), & \text{at}\quad x<-5,\ t<-5.\end{cases}\tag{2.10}$$

We found the values of the Lagrangian L from substituting by Eqs (2.9) and (2.10) into Eq (2.5), then we get

$$J(\Theta,\Psi)=\frac{100}{63}\left(-63a+882000b_1+145800000b_2+28120000000b_3+630\right).\tag{2.11}$$

Case 3:

$$\Theta(x,t)=\begin{cases}\frac{1}{2}(\exp(6\pi-t-x)-\exp(-2\pi-t-x)), & \text{at} \quad x>\pi,\ t>\pi,\\ \sinh(t+x+2\pi), & \text{at} \quad |x|<\pi,\ |t|<\pi,\\ 0, & \text{at} \quad x<-\pi,\ t<-\pi,\end{cases} \tag{2.12}$$

$$\psi(x,t)=\begin{cases}0, & \text{at} \quad x>\pi,\ t>\pi,\\ \sinh(2\pi-t-x), & \text{at} \quad |x|<\pi,\ |t|<\pi,\\ \frac{1}{2}(\exp(6\pi+t+x)-\exp(-2\pi+t+x)), & \text{at} \quad x<-\pi,\ t<-\pi.\end{cases} \tag{2.13}$$

We found the values of the Lagrangian L from substituting by Eqs (2.12) and (2.13) into Eq (2.5), then we have

$$\begin{aligned} J(\Theta,\Psi)=&-1.02783\times10^{10}a+2.64108\times10^{19}b_1+1.60864\times10^{29}b_2\\ &+1.39506\times10^{39}b_3+2.83012\times10^{6}.\end{aligned} \tag{2.14}$$

2.1.2. The two-box potential

Case 1: We try the following Jost functions:

$$\Theta(x,t)=\begin{cases}\frac{1}{2}\Big(\exp(\pi^2(3+2\alpha+\alpha^2)-\pi t-\pi x) & \\ -\exp(\pi^2(1-2\alpha-\alpha^2)-\pi t-\pi x)\Big), & \text{at} \quad x>\pi,\ t>\pi,\\ \sinh((\pi+\alpha t)(\pi+\alpha x)), & \text{at} \quad 0<x<\pi,\ 0<t<\pi,\\ \sinh((t+\pi)(x+\pi)), & \text{at} \quad -\pi<x<0,\ -\pi<t<0,\\ 0, & \text{at} \quad x<-\pi,\ t<-\pi,\end{cases} \tag{2.15}$$

$$\Psi(x,t)=\begin{cases}0, & \text{at} \quad x>\pi,\ t>\pi,\\ \sinh((\pi-t)(\pi-x)), & \text{at} \quad 0<x<\pi,\ 0<t<\pi,\\ \sinh((\pi-\alpha t)(\pi-\alpha x)), & \text{at} \quad -\pi<x<0,\ -\pi<t<0,\\ \frac{1}{2}\Big(\exp(\pi^2(3+2\alpha+\alpha^2)+\pi t+\pi x) & \\ -\exp(\pi^2(1-2\alpha-\alpha^2)+\pi t+\pi x)\Big), & \text{at} \quad x<-\pi,\ t<-\pi.\end{cases} \tag{2.16}$$

This ansatz now contains nontrivial variational parameter α. Substituting Eqs (2.15) and (2.16) into Eq (2.5) (see Figure 2), then we obtain the functional integral at $\alpha=-1$ as follows:

$$J(\Theta,\Psi)=-4.43569\times10^{7}a+1.13554\times10^{14}b_1+6.23167\times10^{21}b_2+4.89178\times10^{29}b_3. \tag{2.17}$$

Also, we can use $\alpha=2$, then we get

$$\begin{aligned}\int_{-\pi}^{0}\int_{-\pi}^{0}L\,dxdt=\int_{0}^{\pi}\int_{0}^{\pi}L\,dxdt=&-4.42653\times10^{75}a+5.59314\times10^{148}b_1+5.89626\times10^{224}b_2\\ &+8.85443\times10^{300}b_3-2.66975\times10^{36}.\end{aligned}$$

The functional integral at $\alpha=2$ becomes

$$\begin{aligned} J(\Theta,\Psi)=&-1.77562\times10^{76}a+4.12756\times10^{150}b_1+4.35525\times10^{226}b_2\\ &+6.5433\times10^{302}b_3-5.33949\times10^{36}.\end{aligned} \tag{2.18}$$

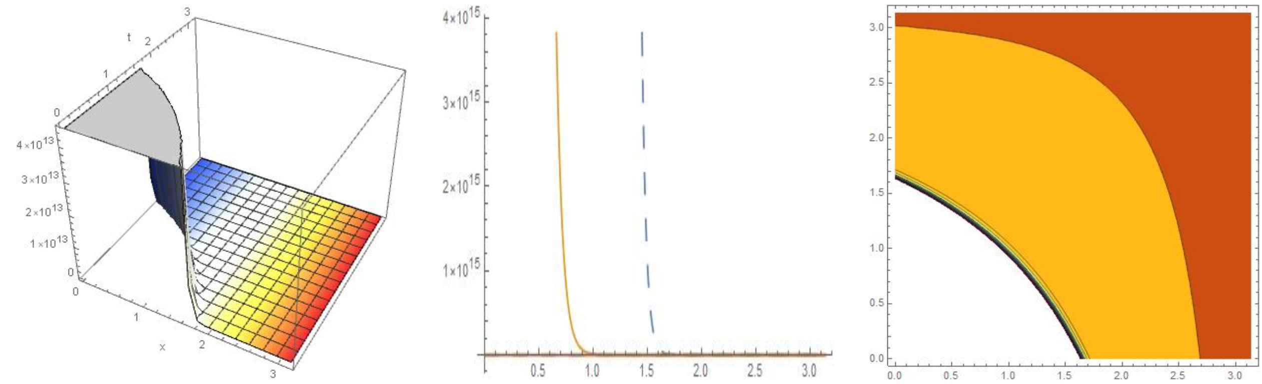


Figure 2. Show an example of the Lagrangian $L(x,t)$ with Eq (2.5), in the interval $0 < x < \pi$ and $0 < t < \pi$, by choosing the trial functions (2.15) and (2.16) at the parameter $\alpha = -1$ and $a = 3$, $b_1 = 2$, $b_2 = \frac{1}{2}$, $b_3 = \frac{1}{6}$.

Case 2: We now try the following Jost function:

$$\Theta(x,t) = \begin{cases} \frac{1}{2}\Big(\exp(4\pi - t - x + 2\pi\alpha) \\ -\exp(-t - x - 2\pi\alpha)\Big), & \text{at} \quad x > \pi,\ t > \pi, \\ \sinh(2\pi + \alpha t + \alpha x), & \text{at} \quad 0 < x < \pi,\ 0 < t < \pi, \\ \sinh(2\pi + t + x), & \text{at} \quad -\pi < t < 0,\ -\pi < x < 0, \\ 0, & \text{at} \quad x < -\pi,\ t < -\pi, \end{cases} \tag{2.19}$$

$$\Psi(x,t) = \begin{cases} 0, & \text{at} \quad x > \pi,\ t > \pi, \\ \sinh(2\pi - t - x), & \text{at} \quad 0 < x < \pi,\ 0 < t < \pi, \\ \sinh(2\pi - \alpha t - \alpha x), & \text{at} \quad -\pi < x < 0,\ -\pi < t < 0, \\ \frac{1}{2}\Big(\exp(4\pi + t + x + 2\pi\alpha) \\ -\exp(t + x - 2\pi\alpha)\Big), & \text{at} \quad x < -\pi,\ t < -\pi. \end{cases} \tag{2.20}$$

This ansatz contains nontrivial variational parameter α. Substituting Eqs (2.19) and (2.20) into Eq (2.5), then we get the functional integral at $\alpha = 0.5$ as follows:

$$\begin{aligned} J(\Theta,\Psi) = &- 1.84004 \times 10^7 a + 2.29576 \times 10^{14} b_1 + 2.6187 \times 10^{21} b_2 \\ &+ 4.24151 \times 10^{28} b_3 + 270561. \end{aligned} \tag{2.21}$$

Also, we can take $\alpha = -0.2$, then we obtain

$$\begin{aligned} \int_{-\pi}^{0}\int_{-\pi}^{0} L\,dxdt = \int_{0}^{\pi}\int_{0}^{\pi} L\,dxdt = &- 13516.2a + 2.22124 \times 10^9 b_1 + 7.54018 \times 10^{13} b_2 \\ &+ 3.79568 \times 10^{18} b_3 + 19035.7. \end{aligned}$$

The functional integral at $\alpha = -0.2$ becomes

$$\begin{aligned} J(\Theta,\Psi) = &- 28484a + 4.44353 \times 10^9 b_1 + 1.50805 \times 10^{14} b_2 \\ &+ 7.59137 \times 10^{18} b_3 + 38071.3. \end{aligned} \tag{2.22}$$

2.1.3. The quadratic polynomials

We assume the Jost function by quadratic polynomials at interval $|x| < 5,\ |t| < 5$:

Case 1: We use the following Jost functions:

$$\Theta(x,t) = \begin{cases} (100 + 10000\alpha)\exp(-\mu(t-5)(x-5)), & \text{at} \quad x > 5,\ t > 5, \\ (t+5)(x+5) + \alpha(5+t)^2(5+x)^2, & \text{at} \quad |x| < 5,\ |t| < 5, \\ 0, & \text{at} \quad x < -5,\ t < -5, \end{cases} \tag{2.23}$$

$$\Psi(x,t) = \begin{cases} 0, & \text{at} \quad x > 5,\ t > 5, \\ (5-t)(5-x) + \alpha(5-t)^2(5-x)^2, & \text{at} \quad |x| < 5,\ |t| < 5, \\ (100 + 10000\alpha)\exp(\mu(t+5)(x+5)), & \text{at} \quad x < -5,\ t < -5. \end{cases} \tag{2.24}$$

We found the values of Lagrangian calculation from Eqs (2.23) and (2.24) into Eq (2.5), and we get

$$\begin{aligned} J(\Theta,\Psi) =& 5.55556\times 10^{6}\alpha^2 - 2.66667\times 10^{7}a\alpha^2 - 500000a\alpha + 555556\alpha \\ & - 3333.33a + 8.65053\times 10^{30}\alpha^8 b_3 + 7.81255\times 10^{29}\alpha^7 b_3 \\ & + 1.97241\times 10^{23}\alpha^6 b_2 + 3.11121\times 10^{28}\alpha^6 b_3 + 1.38897\times 10^{22}\alpha^5 b_2 \\ & + 7.14358\times 10^{26}\alpha^5 b_3 + 6.1741\times 10^{15}\alpha^4 b_1 + 4.13321\times 10^{20}\alpha^4 b_2 \\ & + 1.03576\times 10^{25}\alpha^4 b_3 + 3.12755\times 10^{14}\alpha^3 b_1 + 6.67171\times 10^{18}\alpha^3 b_2 \\ & + 9.72777\times 10^{22}\alpha^3 b_3 + 6.14172\times 10^{12}\alpha^2 b_1 + 6.18569\times 10^{16}\alpha^2 b_2 \\ & + 5.79228\times 10^{20}\alpha^2 b_3 + 5.61111\times 10^{10}\alpha b_1 + 3.14172\times 10^{14}\alpha b_2 \\ & + 2.00529\times 10^{18}\alpha b_3 + 2.05556\times 10^{8} b_1 + 6.89342\times 10^{11} b_2 \\ & + 3.10406\times 10^{15} b_3 + 8333.33. \end{aligned} \tag{2.25}$$

We choose $a = 3$, $b_1 = 2$, $b_2 = \frac{1}{2}$, and $b_3 = \frac{1}{6}$, then the functional integral $J(\Theta,\Psi)$ becomes

$$\begin{aligned} J(\Theta,\Psi) =& 1.44175\times 10^{30}\alpha^8 + 1.30209\times 10^{29}\alpha^7 + 5.18545\times 10^{27}\alpha^6 \\ & + 1.19067\times 10^{26}\alpha^5 + 1.72648\times 10^{24}\alpha^4 + 1.62163\times 10^{22}\alpha^3 \\ & + 9.65689\times 10^{19}\alpha^2 + 3.34372\times 10^{17}\alpha + 5.17688\times 10^{14}. \end{aligned} \tag{2.26}$$

Derive Eq (2.26) with respect to α, then values of α are:

$$\begin{aligned} &\alpha = -0.0136245, \quad \alpha = -0.012952 - 0.00218808i, \quad \alpha = -0.012952 + 0.00218808i, \\ &\alpha = -0.0110959 - 0.00364364i, \quad \alpha = -0.0110959 + 0.00364364i, \\ &\alpha = -0.0086518 - 0.00394467i, \quad \alpha = -0.0086518 + 0.00394467i. \end{aligned}$$

We substitute the exact roots of α into Eq (2.26) give the following analytical equations:

$$\begin{aligned} J(\Theta,\Psi) &= 4.83907\times 10^{10}, \\ J(\Theta,\Psi) &= 3.84513\times 10^{10} - 4.55557\times 10^{10}i, \\ J(\Theta,\Psi) &= 3.84513\times 10^{10} + 4.55557\times 10^{10}i, \\ J(\Theta,\Psi) &= -3.14336\times 10^{9} - 1.06115\times 10^{11}i, \end{aligned} \tag{2.27}$$

$$\begin{aligned} J(\Theta,\Psi) &= -3.14336\times 10^{9}+1.06115\times 10^{11}i,\\ J(\Theta,\Psi) &= -1.8701\times 10^{11}-2.04551\times 10^{11}i,\\ J(\Theta,\Psi) &= -1.8701\times 10^{11}+2.04551\times 10^{11}i. \end{aligned}$$

Case 2: We try the following Jost functions:

$$\Theta(x,t)=\begin{cases}(20+200\alpha)\exp(-\mu(t-5)(x-5)), & \text{at} \quad x>5,\ t>5,\\ (10+t+x)+\alpha(t+5)^2+\alpha(x+5)^2, & \text{at} \quad |x|<5,\ |t|<5,\\ 0, & \text{at} \quad x<-5,\ t<-5,\end{cases} \tag{2.28}$$

$$\Psi(x,t)=\begin{cases}0, & \text{at} \quad x>5,\ t>5,\\ (10-t-x)+\alpha(5-t)^2+\alpha(5-x)^2, & \text{at} \quad |x|<5,\ |t|<5,\\ (20+200\alpha)\exp(\mu(t+5)(x+5)), & \text{at} \quad x<-5,\ t<-5.\end{cases} \tag{2.29}$$

We found the values of Lagrangian calculation from Eqs (2.28) and (2.29) into Eq (2.5) (see Figure 3), and we have

$$\begin{aligned} J(\Theta,\Psi) =&50000\alpha^2-13333.3a\alpha^2-2000a\alpha+15000\alpha-100a+9.90698\times 10^{17}\alpha^8 b_3\\ &+9.04639\times 10^{17}\alpha^7 b_3+5.97273\times 10^{13}\alpha^6 b_2+3.65565\times 10^{17}\alpha^6 b_3\\ &+4.25597\times 10^{13}\alpha^5 b_2+8.55415\times 10^{16}\alpha^5 b_3+4.78095\times 10^{9}\alpha^4 b_1\\ &+1.28615\times 10^{13}\alpha^4 b_2+1.27048\times 10^{16}\alpha^4 b_3+2.39683\times 10^{9}\alpha^3 b_1\\ &+2.11556\times 10^{12}\alpha^3 b_2+1.22953\times 10^{15}\alpha^3 b_3+4.62698\times 10^{8}\alpha^2 b_1\\ &+2.00368\times 10^{11}\alpha^2 b_2+7.59375\times 10^{13}\alpha^2 b_3+4.08889\times 10^{7}\alpha b_1\\ &+1.03937\times 10^{10}\alpha b_2+2.7454\times 10^{12}\alpha b_3+1.4\times 10^{6} b_1\\ &+2.31429\times 10^{8} b_2+4.46349\times 10^{10} b_3+1000. \end{aligned} \tag{2.30}$$

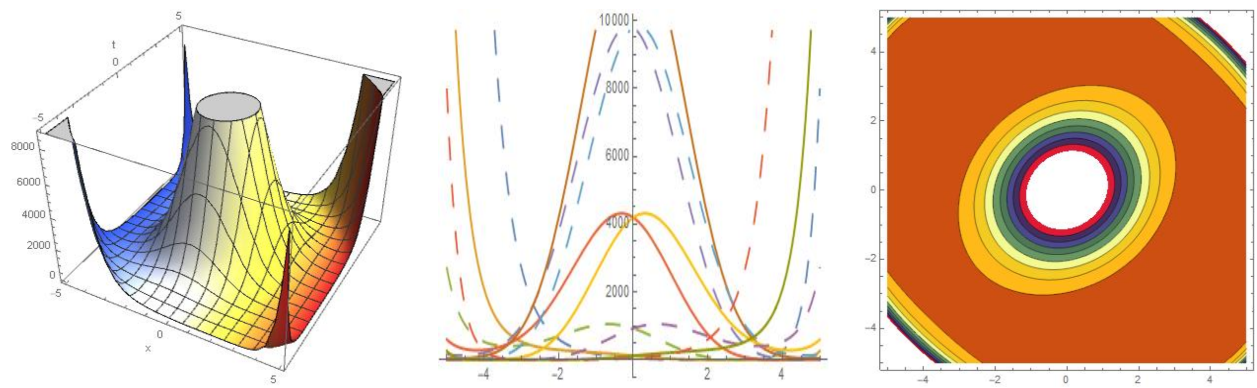


Figure 3. Show an example of the Lagrangian $L(x,t)$ with Eq (2.5), in the interval $-5<x<5$ and $-5<t<5$, by choosing the trial functions (2.28) and (2.29) at the parameter $\alpha=-0.131984$ and $a=6$, $b_1=3$, $b_2=\frac{1}{4}$, $b_3=\frac{1}{3}$.

We put $a = 6$, $b_1 = 3$, $b_2 = \frac{1}{4}$, and $b_3 = \frac{1}{3}$, then the functional integral $J(\Theta, \Psi)$ becomes

$$\begin{aligned} J(\Theta, \Psi) =& 3.30233 \times 10^{17}\alpha^8 + 3.01546 \times 10^{17}\alpha^7 + 1.2187 \times 10^{17}\alpha^6 \\ &+ 2.85245 \times 10^{16}\alpha^5 + 4.23816 \times 10^{15}\alpha^4 + 4.10379 \times 10^{14}\alpha^3 \\ &+ 2.5364 \times 10^{13}\alpha^2 + 9.17853 \times 10^{11}\alpha + 1.49404 \times 10^{10}. \end{aligned} \tag{2.31}$$

Derive Eq (2.31) with respect to α, then values of α are:

$$\begin{aligned} &\alpha = -0.131984, \quad \alpha = -0.129184 - 0.0176052i, \quad \alpha = -0.129184 + 0.0176052i, \\ &\alpha = -0.117682 - 0.0366915i, \quad \alpha = -0.117682 + 0.0366915i, \\ &\alpha = -0.0866387 - 0.051817i, \quad \alpha = -0.0866387 + 0.051817i. \end{aligned}$$

We substitute the exact roots of α into Eq (2.31) give the following analytical equations:

$$\begin{aligned} J(\Theta, \Psi) &= 207360, \\ J(\Theta, \Psi) &= -102453 - 293854i, \\ J(\Theta, \Psi) &= -102453 + 293854i, \\ J(\Theta, \Psi) &= -1.38256 \times 10^6 + 542198i, \\ J(\Theta, \Psi) &= -1.38256 \times 10^6 - 542198i, \\ J(\Theta, \Psi) &= -1.31177 \times 10^7 + 1.75734 \times 10^7 i, \\ J(\Theta, \Psi) &= -1.31177 \times 10^7 - 1.75734 \times 10^7 i. \end{aligned} \tag{2.32}$$

2.2. *Exact solitary wave solutions of the Schrödinger's nonlinear equation with the polynomial nonlinear law model I*

We inspect here the exact solutions of the nonlinear Schrödinger equation with the polynomial nonlinear law. This equation is defined as Eq (2.1).

Case 1: We suppose the ansatz function of the NLSE with the polynomial nonlinear law is in the form of a bright solitary wave solution

$$h_1(x,t) = A\,\mathrm{sech}\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right),\ \Gamma(x,t) = A\,\mathrm{sech}\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right)e^{i(kx-\omega t)}, \tag{2.33}$$

where A, w and v are the amplitude, the pulse width, and velocity of soliton in normalized unites. Substituting from Eq (2.33) in Eq (2.1), and separating the real and imaginary parts, we obtain

$$\begin{aligned} &-ak^2v + av\mathrm{w}^2 + v\omega + \left(A^2 b_1 v - 2av\mathrm{w}^2\right)\mathrm{sech}^2\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right) \\ &+A^4 b_2 v\,\mathrm{sech}^4\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right) + A^6 b_3 v\,\mathrm{sech}^6\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right) = 0, \end{aligned} \tag{2.34}$$

$$\left(\mathrm{w} - 2akv\mathrm{w}\right)\tanh\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right) = 0. \tag{2.35}$$

Equating the coefficients of the linearly independent terms to zero, we obtain the dynamical system in $A, \mathrm{w}, v, k, \omega, a, b_1, b_2, b_3$ by solving this system we get:

Family I:

$$A=\pm\frac{\mathrm{w}}{\sqrt{b_1kv}},\quad a=\frac{1}{2kv},\quad \omega=\frac{k^2-\mathrm{w}^2}{2kv}. \tag{2.36}$$

The sufficient conditions for solitary wave solution stability are

$$b_1kv>0,\quad kv\neq 0. \tag{2.37}$$

Family II:

$$A=\pm\sqrt{\frac{k-2v\omega}{b_1v}},\quad a=\frac{1}{2kv},\quad \mathrm{w}=\pm\sqrt{k\left(k-2v\omega\right)}, \tag{2.38}$$

provided that

$$\frac{k-2v\omega}{b_1v}>0,\quad kv\neq 0,\quad k\left(k-2v\omega\right)>0. \tag{2.39}$$

Family III:

$$A=\pm\mathrm{w}\sqrt{\frac{2\omega}{b_1k^2-b_1\mathrm{w}^2}},\quad a=\frac{\omega}{k^2-\mathrm{w}^2},\quad v=\frac{k^2-\mathrm{w}^2}{2k\omega}, \tag{2.40}$$

whenever

$$\frac{\omega}{b_1\left(k^2-\mathrm{w}^2\right)}>0,\quad k\omega\neq 0,\quad k^2-\mathrm{w}^2\neq 0. \tag{2.41}$$

Family IV:

$$\begin{aligned}A&=\pm\sqrt{\frac{1}{b_1}\left(\frac{\sqrt{v^4\omega^2+v^2\mathrm{w}^2}}{v^2}-\omega\right)},\quad a=\frac{\sqrt{v^4\omega^2+v^2\mathrm{w}^2}-v^2\omega}{2v^2\mathrm{w}^2},\\ k&=\frac{\sqrt{v^4\omega^2+v^2\mathrm{w}^2}}{v}+v\omega,\end{aligned} \tag{2.42}$$

provided that

$$v^4\omega^2+v^2\mathrm{w}^2>0,\quad \frac{1}{b_1}\left(\frac{\sqrt{v^4\omega^2+v^2\mathrm{w}^2}}{v^2}-\omega\right)>0,\quad v^2\mathrm{w}^2\neq 0. \tag{2.43}$$

Then, the solutions of the NLSE with the polynomial nonlinear law as bright solitary wave solutions are (see Figures 4 and 5):

$$\Gamma_{11}(x,t)=\pm\frac{\mathrm{w}}{\sqrt{b_1kv}}\operatorname{sech}\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(kx-\omega t)}, \tag{2.44}$$

$$\Gamma_{12}(x,t)=\pm\sqrt{\frac{k-2v\omega}{b_1v}}\operatorname{sech}\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(kx-\omega t)}, \tag{2.45}$$

$$\Gamma_{13}(x,t)=\pm\mathrm{w}\sqrt{\frac{2\omega}{b_1k^2-b_1\mathrm{w}^2}}\operatorname{sech}\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(kx-\omega t)}, \tag{2.46}$$

$$\Gamma_{14}(x,t)=\pm\sqrt{\frac{1}{b_1}\left(\frac{\sqrt{v^4\omega^2+v^2\mathrm{w}^2}}{v^2}-\omega\right)}\operatorname{sech}\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(kx-\omega t)}. \tag{2.47}$$

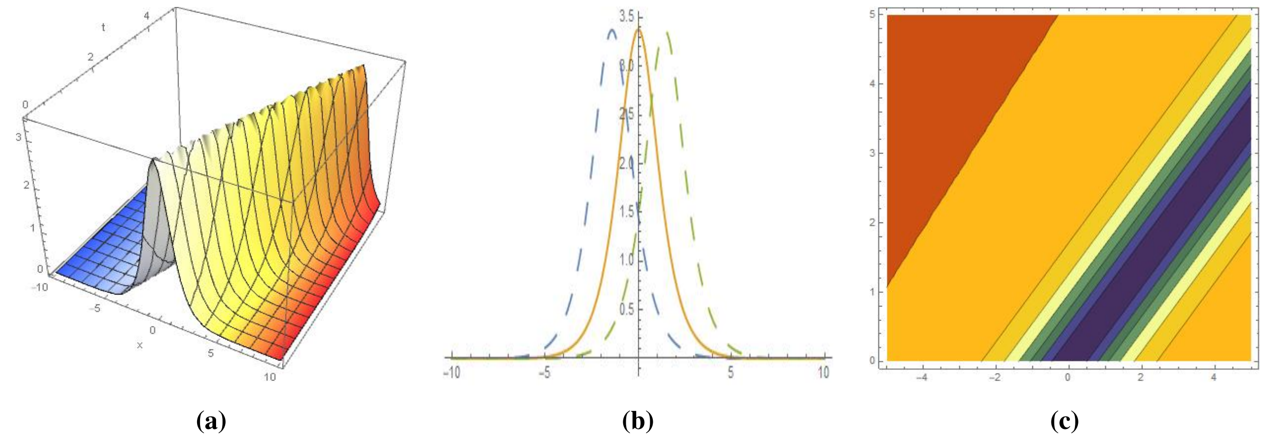


Figure 4. Representation of solitary wave solution Γ_{11}. These figures are obtained by $b_1 = 0.5$, $k = 0.25$, $v = 0.7$, $\mathrm{w} = 1$, $\omega = 0.3$, (4a) is plotted in 3D while (4b) is plotted in 2D at different positions, and (4c) plotted as contour.

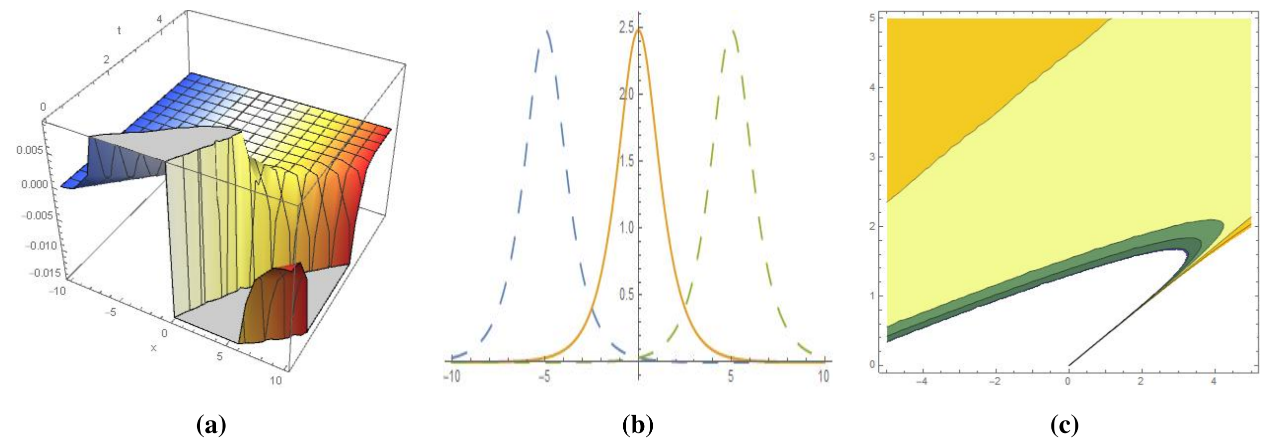


Figure 5. Representation of solitary wave solution Γ_{13}. These figures are obtained by $b_1 = 0.25$, $k = 0.3$, $v = 0.2$, $\mathrm{w} = 1$, $\omega = 0.7$, (5a) is plotted in 3D while (5b) is plotted in 2D at different positions, and (5c) plotted as contour.

Case 2: Another choice of the dark solitary wave solution of the NLSE with the polynomial nonlinear law is

$$\begin{aligned} h_2(x,t) &= A + B\tanh\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right), \\ \Gamma(x,t) &= \left(A + B\tanh\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right)\right)e^{i(kx-\omega t)}. \end{aligned} \tag{2.48}$$

By replacement from Eq (2.48) in Eq (2.1) and separating the real and imaginary parts

$$\begin{aligned} &-aAk^2 + A^7b_3 + 21A^5b_3B^2 + A^5b_2 + 35A^3b_3B^4 + 10A^3b_2B^2 + A^3b_1 \\ &+7Ab_3B^6 + 5Ab_2B^4 + 3Ab_1B^2 + A\omega + \Big(-21A^5b_3B^2 - 70A^3b_3B^4 \\ &-10A^3b_2B^2 - 21Ab_3B^6 - 10Ab_2B^4 - 3Ab_1B^2\Big) \times \operatorname{sech}^2\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right) \\ &+\Big(35A^3b_3B^4 + 21Ab_3B^6 + 5Ab_2B^4\Big) \times \operatorname{sech}^4\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right) \end{aligned}$$

$$
\begin{aligned}
&-7Ab_3B^6\,\mathrm{sech}^6\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)+\Big(-aBk^2+7A^6b_3B+35A^4b_3B^3\\
&+5A^4b_2B+21A^2b_3B^5+10A^2b_2B^3+3A^2b_1B+b_3B^7+b_2B^5+b_1B^3\\
&+B\omega\Big)\times\tanh\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)+\Big(-2aB\mathrm{w}^2-35A^4b_3B^3-42A^2b_3B^5\\
&-10A^2b_2B^3-3b_3B^7-2b_2B^5-b_1B^3\Big)\times\tanh\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)\mathrm{sech}^2\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)\\
&+\Big(21A^2b_3B^5+3b_3B^7+b_2B^5\Big)\tanh\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)\mathrm{sech}^4\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)\\
&-B^7b_3\,\tanh\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)\mathrm{sech}^6\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)=0,
\end{aligned}
\tag{2.49}
$$

$$
\Big(2aBk\mathrm{w}-\frac{B\mathrm{w}}{v}\Big)\mathrm{sech}^2\Big(\mathrm{w}\Big(x-\frac{t}{v}\Big)\Big)=0 \tag{2.50}
$$

Equating the coefficients of the linearly independent terms to zero, we deduce the coefficients $A, B, \mathrm{w}, v, k, \omega, a, b_1, b_2, b_3$ in the form:

Family I:

$$
\begin{aligned}
&A=0,\quad B=\pm i\sqrt{\frac{b_2}{3b_3}},\quad b_1=\frac{27b_3^2\omega+2b_2^3-27ab_3^2k^2}{9b_2b_3},\\
&\mathrm{w}=\pm\frac{1}{3b_3}\sqrt{\frac{27b_3^2\omega-b_2^3-27ab_3^2k^2}{6a}},\quad v=\frac{1}{2ak}.
\end{aligned}
\tag{2.51}
$$

The sufficient conditions for dark solitary wave solution stability are

$$
\frac{b_2}{b_3}<0,\quad b_2b_3\neq 0,\quad ak\neq 0,\quad \frac{27b_3^2\omega-b_2^3-27ab_3^2k^2}{a}>0. \tag{2.52}
$$

Family II:

$$
\begin{aligned}
&A=0,\quad B=\pm i\sqrt{\frac{b_2}{3b_3}},\quad \mathrm{w}=\pm\frac{1}{3b_3}\sqrt{\frac{3b_1b_2b_3-b_2^3}{2a}},\quad v=\frac{1}{2ak},\\
&\omega=\frac{27ab_3^2k^2-2b_2^3+9b_1b_3b_2}{27b_3^2},
\end{aligned}
\tag{2.53}
$$

provided that

$$
\frac{b_2}{b_3}<0,\quad \frac{3b_1b_2b_3-b_2^3}{a}>0,\quad ak\neq 0,\quad b_3\neq 0. \tag{2.54}
$$

Family III:

$$
\begin{aligned}
&A=0,\quad B=\pm\sqrt[6]{\frac{k^2-2kv\omega+2\mathrm{w}^2}{2b_3kv}},\quad a=\frac{1}{2kv},\\
&b_1=\frac{\sqrt[3]{b_3}\left(3k^2-6kv\omega+4\mathrm{w}^2\right)}{\sqrt[3]{4k^2v^2\left(k^2-2kv\omega+2\mathrm{w}^2\right)}},\quad b_2=-3\sqrt[3]{\frac{b_3^2\left(k^2-2kv\omega+2\mathrm{w}^2\right)}{2kv}},
\end{aligned}
\tag{2.55}
$$

such that

$$
\frac{k^2-2kv\omega+2\mathrm{w}^2}{b_3kv}>0,\quad kv\neq 0,\quad 4k^2v^2\left(k^2-2kv\omega+2\mathrm{w}^2\right)\neq 0. \tag{2.56}
$$

Family IV:

$$A = 0, \quad B = \pm\sqrt[6]{\frac{ak^2 + 2a\mathrm{w}^2 - \omega}{b_3}},$$
$$b_1 = \left(3ak^2 + 4a\mathrm{w}^2 - 3\omega\right)\sqrt[3]{\frac{b_3}{ak^2 + 2a\mathrm{w}^2 - \omega}}, \tag{2.57}$$
$$b_2 = -3\sqrt[3]{b_3^2\,(ak^2 + 2a\mathrm{w}^2 - \omega)}, \quad v = \frac{k^2 + 2\mathrm{w}^2}{2k\,(ak^2 + 2a\mathrm{w}^2)},$$

whenever

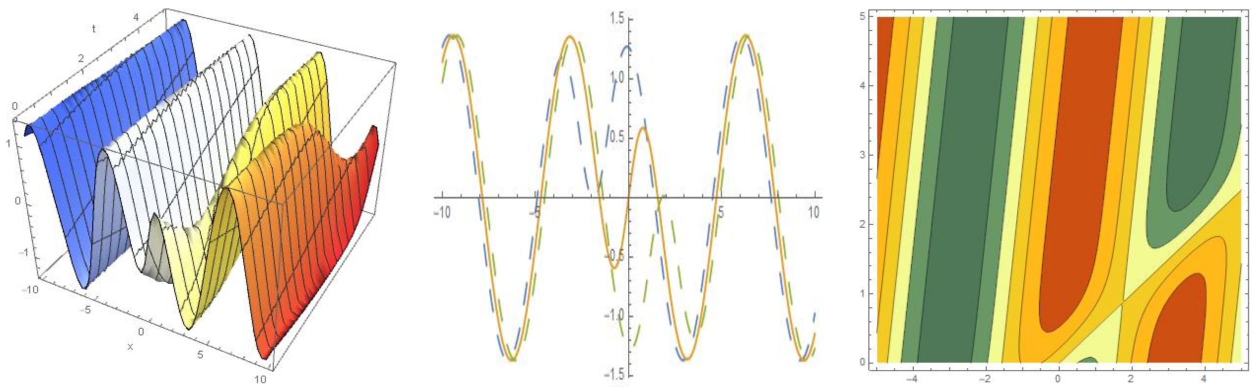
$$\frac{ak^2 + 2a\mathrm{w}^2 - \omega}{b_3} > 0, \quad ak^2 + 2a\mathrm{w}^2 - \omega \neq 0, \quad k\left(ak^2 + 2a\mathrm{w}^2\right) \neq 0. \tag{2.58}$$

Then, the dark soliton solutions of the NLSE with the polynomial nonlinear law Eq (2.1) are (see Figure 6):

$$\Gamma_{21}(x,t) = \pm i\sqrt{\frac{b_2}{3b_3}}\tanh\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right)e^{i(kx-\omega t)}, \tag{2.59}$$

$$\Gamma_{23}(x,t) = \pm\sqrt[6]{\frac{k^2 - 2kv\omega + 2\mathrm{w}^2}{2b_3kv}}\tanh\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right)e^{i(kx-\omega t)}, \tag{2.60}$$

$$\Gamma_{24}(x,t) = \pm\sqrt[6]{\frac{ak^2 + 2a\mathrm{w}^2 - \omega}{b_3}}\tanh\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right)e^{i(kx-\omega t)}. \tag{2.61}$$



(a) **(b)** **(c)**

Figure 6. Representation of solitary wave solution Γ_{24}. These figures are obtained by $a = 0.7$, $b_3 = 0.25$, $k = 1$, $v = 0.5$, $\mathrm{w} = 0.9$, $\omega = 0.2$, (6a) is plotted in 3D while (6b) is plotted in 2D at different positions, and (6c) plotted as contour.

3. Nonlinear Schrödinger equation with polynomial law nonlinearity model II

3.1. Formulation of the variational principle for the nonlinear Schrödinger equation with polynomial law nonlinearity

The model II of the NLSE with polynomial nonlinearity is given by [49]:

$$i\Psi_t + a\Psi_{xx} + b\Psi_{xt} + (k_1|\Psi|^2 + k_2|\Psi|^4 + k_3|\Psi|^6)\Psi = 0, \tag{3.1}$$

where $\Psi(x,t)$ denotes the complex valued function. The coefficients a and b indicate group velocity dispersion (GVD) and spatio-temporal dispersion (STD), respectively. The first term represents the linear evolution of pulses in nonlinear optical fibers. The final term represents the nonlinearity of the non-Kerr law. Since $\Psi(x,t)$ is a complex function on the form $\Psi(x,t) = u(x,t) + iv(x,t)$ such that u and v are real functions of x and t, also $|\Psi|^2 = (u+iv)(u-iv)$. We will use the variational technique to find solutions for the Eq (3.1), where we derive Ψ with respect to x and t to obtain the Lagrangian of this equation, which is expressed in the following way:

Let M and N are functionals in u and v,

$$\begin{aligned} M(u,v) =& \frac{\partial u}{\partial t} + a\frac{\partial^2 v}{\partial x^2} + b\frac{\partial^2 v}{\partial x \partial t} + k_1 v u^2 + k_1 v^3 + k_2 v u^4 + 2k_2 u^2 v^3 + k_2 v^5 \\ &+ k_3 v u^6 + 3k_3 v^3 u^4 + 3k_3 u^2 v^5 + k_3 v^7, \end{aligned} \tag{3.2}$$

$$\begin{aligned} N(u,v) =& -\frac{\partial v}{\partial t} + a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^2 u}{\partial x \partial t} + k_1 u^3 + k_1 u v^2 + k_2 u^5 + 2k_2 v^2 u^3 + k_2 u v^4 \\ &+ k_3 u^7 + 3k_3 v^2 u^5 + 3k_3 u^3 v^4 + k_3 u v^6. \end{aligned} \tag{3.3}$$

The system of Eqs (3.2) and (3.3) satisfies the conditions mentioned in [52, 53], then a functional integral $J(u,v)$ can be written down using the formula given by Tonti [53]:

$$\begin{aligned} J(u,v) = \int_\omega \Big[& \frac{1}{2}auu_{xx} + \frac{1}{2}avv_{xx} + \frac{1}{2}buu_{xt} + \frac{1}{2}bvv_{xt} + \frac{1}{2}k_3v^2u^6 + \frac{3}{4}k_3u^4v^4 \\ &+ \frac{1}{2}k_2v^2u^4 + \frac{1}{2}k_3u^2v^6 + \frac{1}{2}k_2u^2v^4 + \frac{1}{2}k_1u^2v^2 + \frac{1}{8}k_3u^8 + \frac{1}{6}k_2u^6 \\ &+ \frac{1}{4}k_1u^4 + \frac{1}{8}k_3v^8 + \frac{1}{6}k_2v^6 + \frac{1}{4}k_1v^4 + \frac{1}{2}vu_t - \frac{1}{2}uv_t \Big]\, d\omega, \end{aligned} \tag{3.4}$$

where $d\omega = dxdt$.

Then the Lagrangian L is given by

$$\begin{aligned} L(u,v) =& \frac{1}{2}auu_{xx} + \frac{1}{2}avv_{xx} + \frac{1}{2}buu_{xt} + \frac{1}{2}bvv_{xt} + \frac{1}{2}k_3v^2u^6 + \frac{3}{4}k_3u^4v^4 \\ &+ \frac{1}{2}k_2v^2u^4 + \frac{1}{2}k_3u^2v^6 + \frac{1}{2}k_2u^2v^4 + \frac{1}{2}k_1u^2v^2 + \frac{1}{8}k_3u^8 + \frac{1}{6}k_2u^6 \\ &+ \frac{1}{4}k_1u^4 + \frac{1}{8}k_3v^8 + \frac{1}{6}k_2v^6 + \frac{1}{4}k_1v^4 + \frac{1}{2}vu_t - \frac{1}{2}uv_t. \end{aligned} \tag{3.5}$$

As a necessary check to our calculations, we use the value of L in the Euler-Lagrange equations:

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial u_t}\right) + \frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial u_{xx}}\right) + \frac{\partial^2}{\partial x \partial t}\left(\frac{\partial L}{\partial u_{xt}}\right) = 0,$$

$$\frac{\partial L}{\partial v}-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial v_t}\right)+\frac{\partial^2}{\partial x^2}\left(\frac{\partial L}{\partial v_{xx}}\right)+\frac{\partial^2}{\partial x\partial t}\left(\frac{\partial L}{\partial v_{xt}}\right)=0,$$

which yields the system of Eqs (3.2) and (3.3).

3.1.1. The rectangular box

We provide an example of applying this technique by using a box-shaped initial pulse and a single nontrivial variational parameter based on linear Jost functions in three cases.

Case 1: We use the following ansatz for $u(x,t)$ and $v(x,t)$ functions:

$$u(x,t)=\begin{cases}100\exp(-\mu(t-5)(x-5)), & \text{at} \quad x>5,\ t>5,\\ (t+5)(x+5), & \text{at} \quad |x|<5,\ |t|<5,\\ 0, & \text{at} \quad x<-5,\ t<-5,\end{cases} \tag{3.6}$$

$$v(x,t)=\begin{cases}0, & \text{at} \quad x>5,\ t>5,\\ (5-t)(5-x), & \text{at} \quad |x|<5,\ |t|<5,\\ 100\exp(\mu(t+5)(x+5)), & \text{at} \quad x<-5,\ t<-5.\end{cases} \tag{3.7}$$

We found the values of the Lagrangian L from substituting Eqs (3.6) and (3.7) into Eq (3.5), and by using Mathematica program we get the value of $J(u,v)$ at interval $-5<x<5,\ -5<t<5$ as

$$J(u,v)=2500b+2.05556\times10^{8}k_1+6.89342\times10^{11}k_2+3.10406\times10^{15}k_3+8333.33. \tag{3.8}$$

Case 2: Assume $u(x,t)$ and $v(x,t)$ for the Jost function as the following:

$$u(x,t)=\begin{cases}20\exp(-\mu(t-5)(x-5)), & \text{at} \quad x>5,\ t>5,\\ t+x+10, & \text{at} \quad |x|<5,\ |t|<5,\\ 0, & \text{at} \quad x<-5,\ t<-5,\end{cases} \tag{3.9}$$

$$v(x,t)=\begin{cases}0, & \text{at} \quad x>5,\ t>5,\\ 10-t-x, & \text{at} \quad |x|<5,\ |t|<5,\\ 20\exp(\mu(t+5)(x+5)), & \text{at} \quad x<-5,\ t<-5.\end{cases} \tag{3.10}$$

We found the values of the Lagrangian L from substituting by Eqs (3.9) and (3.10) into Eq (3.5) (see Figure 7), then we have

$$J(u,v)=\frac{1000}{63}\left(88200k_1+14580000k_2+2812000000k_3+63\right). \tag{3.11}$$

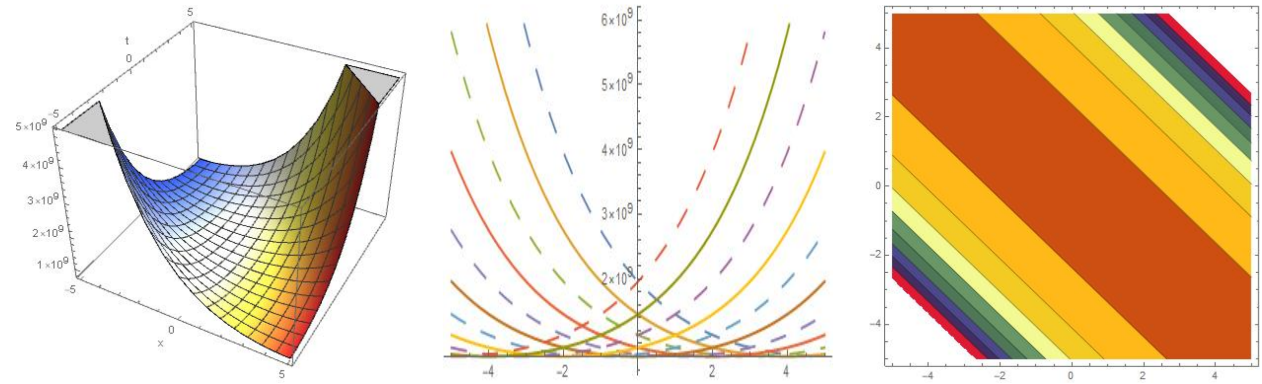


Figure 7. Show an example of the Lagrangian $L(x,t)$ with Eq (3.5), in the interval $-5 < x < 5$ and $-5 < t < 5$, by choosing the trial functions (3.9) and (3.10) at $a = 2,\ b = 6,\ k_1 = 2,\ k_2 = 3,\ k_3 = 4$.

Case 3:

$$u(x,t) = \begin{cases} \frac{1}{2}(\exp(6\pi - t - x) - \exp(-2\pi - t - x)), & \text{at} \quad x > \pi,\ t > \pi, \\ \sinh(t + x + 2\pi), & \text{at} \quad |x| < \pi,\ |t| < \pi, \\ 0, & \text{at} \quad x < -\pi,\ t < -\pi, \end{cases} \tag{3.12}$$

$$v(x,t) = \begin{cases} 0, & \text{at} \quad x > \pi,\ t > \pi, \\ \sinh(2\pi - t - x), & \text{at} \quad |x| < \pi,\ |t| < \pi, \\ \frac{1}{2}(\exp(6\pi + t + x) - \exp(-2\pi + t + x)), & \text{at} \quad x < -\pi,\ t < -\pi. \end{cases} \tag{3.13}$$

We found the values of the Lagrangian L from substituting by Eqs (3.12) and (3.13) into Eq (3.5), then we get

$$\begin{aligned} J(u,v) = & 1.02783 \times 10^{10} a + 1.02783 \times 10^{10} b + 2.64108 \times 10^{19} k_1 \\ & + 1.60864 \times 10^{29} k_2 + 1.39506 \times 10^{39} k_3 + 2.83012 \times 10^{6}. \end{aligned} \tag{3.14}$$

3.1.2. The two-box potential

Case 1: We try the following Jost functions:

$$u(x,t) = \begin{cases} \frac{1}{2}\Big(\exp(\pi^2(3 + 2\alpha + \alpha^2) - \pi t - \pi x) & \\ - \exp(\pi^2(1 - 2\alpha - \alpha^2) - \pi t - \pi x)\Big), & \text{at} \quad x > \pi,\ t > \pi, \\ \sinh((\pi + \alpha t)(\pi + \alpha x)), & \text{at} \quad 0 < x < \pi,\ 0 < t < \pi, \\ \sinh((t + \pi)(x + \pi)), & \text{at} \quad -\pi < x < 0,\ -\pi < t < 0, \\ 0, & \text{at} \quad x < -\pi,\ t < -\pi, \end{cases} \tag{3.15}$$

$$v(x,t) = \begin{cases} 0, & \text{at} \quad x > \pi,\ t > \pi, \\ \sinh((\pi - t)(\pi - x)), & \text{at} \quad 0 < x < \pi,\ 0 < t < \pi, \\ \sinh((\pi - \alpha t)(\pi - \alpha x)), & \text{at} \quad -\pi < x < 0,\ -\pi < t < 0, \\ \frac{1}{2}\Big(\exp(\pi^2(3 + 2\alpha + \alpha^2) + \pi t + \pi x) & \\ - \exp(\pi^2(1 - 2\alpha - \alpha^2) + \pi t + \pi x)\Big), & \text{at} \quad x < -\pi,\ t < -\pi. \end{cases} \tag{3.16}$$

This ansatz now contains nontrivial variational parameter α. Substituting Eqs (3.15) and (3.16) into Eq (3.5), then we obtain the functional integral at $\alpha = -1$ as follows:

$$\begin{aligned} J(u,v) =& 4.43568\times 10^{7}a + 4.92254\times 10^{7}b + 1.13554\times 10^{14}k_1 \\ &+ 6.23167\times 10^{21}k_2 + 4.89178\times 10^{29}k_3. \end{aligned} \tag{3.17}$$

Also, we can use $\alpha = 2$, then we get

$$\begin{aligned} \int_{-\pi}^{0}\int_{-\pi}^{0} L\,dxdt = \int_{0}^{\pi}\int_{0}^{\pi} L\,dxdt =& 4.42653\times 10^{75}a + 4.47679\times 10^{75}b + 5.59314\times 10^{148}k_1 \\ &+ 5.89626\times 10^{224}k_2 + 8.85443\times 10^{300}k_3 - 2.66975\times 10^{36}. \end{aligned}$$

The functional integral at $\alpha = 2$ becomes

$$\begin{aligned} J(u,v) =& 1.77562\times 10^{76}a + 1.78568\times 10^{76}b + 4.12756\times 10^{150}k_1 \\ &+ 4.35525\times 10^{226}k_2 + 6.5433\times 10^{302}k_3 - 5.33949\times 10^{36}. \end{aligned} \tag{3.18}$$

Case 2: We now try the following Jost function:

$$u(x,t) = \begin{cases} \frac{1}{2}\Big(\exp(4\pi - t - x + 2\pi\alpha) & \\ -\exp(-t - x - 2\pi\alpha)\Big), & \text{at} \quad x > \pi,\ t > \pi, \\ \sinh(2\pi + \alpha t + \alpha x), & \text{at} \quad 0 < x < \pi,\ 0 < t < \pi, \\ \sinh(2\pi + t + x), & \text{at} \quad -\pi < t < 0,\ -\pi < x < 0, \\ 0, & \text{at} \quad x < -\pi,\ t < -\pi, \end{cases} \tag{3.19}$$

$$v(x,t) = \begin{cases} 0, & \text{at} \quad x > \pi,\ t > \pi, \\ \sinh(2\pi - t - x), & \text{at} \quad 0 < x < \pi,\ 0 < t < \pi, \\ \sinh(2\pi - \alpha t - \alpha x), & \text{at} \quad -\pi < x < 0,\ -\pi < t < 0, \\ \frac{1}{2}\Big(\exp(4\pi + t + x + 2\pi\alpha) & \\ -\exp(t + x - 2\pi\alpha)\Big), & \text{at} \quad x < -\pi,\ t < -\pi. \end{cases} \tag{3.20}$$

This ansatz contains nontrivial variational parameter α. Substituting Eqs (3.19) and (3.20) into Eq (3.5) (see Figure 8), then we get the functional integral at $\alpha = -0.2$ as follows:

$$\begin{aligned} J(u,v) =& 28473.8a + 28473.8b + 4.44353\times 10^{9}k_1 + 1.50805\times 10^{14}k_2 \\ &+ 7.59137\times 10^{18}k_3 + 38071.3. \end{aligned} \tag{3.21}$$

Also, we can take $\alpha = 0.5$ then we obtain

$$\begin{aligned} \int_{-\pi}^{0}\int_{-\pi}^{0} Ldxdt = \int_{0}^{\pi}\int_{0}^{\pi} Ldxdt =& 4.40169\times 10^{6}a + 4.40169\times 10^{6}b + 9.17623\times 10^{13}k_1 \\ &+ 1.04745\times 10^{21}k_2 + 1.6966\times 10^{28}k_3 + 135281. \end{aligned}$$

The functional integral at $\alpha = 0.5$ becomes

$$\begin{aligned} J(u,v) =& 1.84004\times 10^{7}a + 1.84004\times 10^{7}b + 2.29576\times 10^{14}k_1 + 2.6187\times 10^{21}k_2 \\ &+ 4.24151\times 10^{28}k_3 + 270561. \end{aligned} \tag{3.22}$$

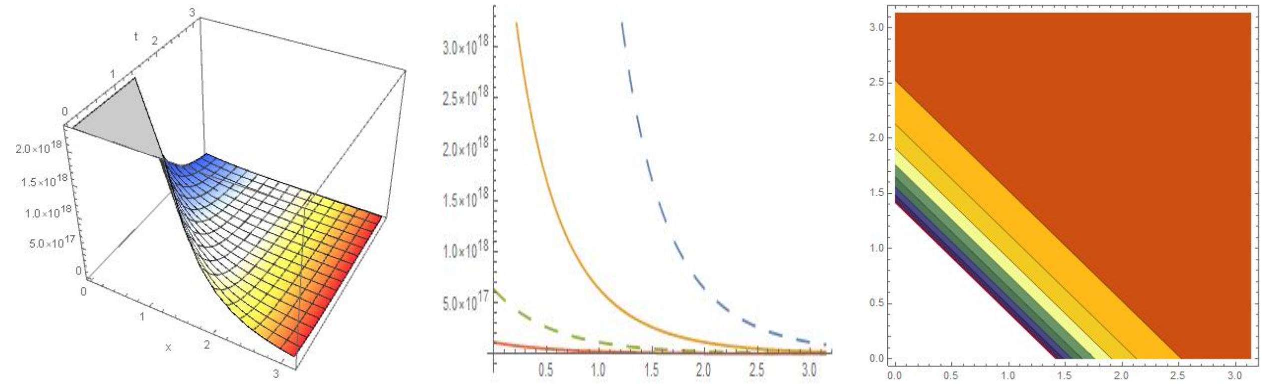


Figure 8. Show an example of the Lagrangian $L(x, t)$ with Eq (3.5), in the interval $0 < x < \pi$ and $0 < t < \pi$, by choosing the trial functions (3.19) and (3.20) at the parameter $\alpha = -0.2$ and $a = 2$, $b = 6$, $k_1 = 2$, $k_2 = 3$, $k_3 = 4$.

3.1.3. The Quadratic polynomials

We assume the Jost function by quadratic polynomials at interval $|x| < 5, |t| < 5$:

Case 1: We use the following Jost functions:

$$u(x,t) = \begin{cases} (100 + 10000\alpha)\exp(-\mu(t-5)(x-5)), & \text{at} \quad x > 5,\ t > 5, \\ (t+5)(x+5) + \alpha(5+t)^2(5+x)^2, & \text{at} \quad |x| < 5,\ |t| < 5, \\ 0, & \text{at} \quad x < -5,\ t < -5, \end{cases} \tag{3.23}$$

$$v(x,t) = \begin{cases} 0, & \text{at} \quad x > 5,\ t > 5, \\ (5-t)(5-x) + \alpha(5-t)^2(5-x)^2, & \text{at} \quad |x| < 5,\ |t| < 5, \\ (100 + 10000\alpha)\exp(\mu(t+5)(x+5)), & \text{at} \quad x < -5,\ t < -5. \end{cases} \tag{3.24}$$

We found the values of Lagrangian calculation from Eqs (3.23) and (3.24) into Eq (3.5) (see Figure 9), and we get

$$\begin{aligned} J(u,v) =& 5.55556 \times 10^6\alpha^2 + 1.33333 \times 10^7 a\alpha^2 + 250000a\alpha + 555556\alpha \\ &+ 2.5 \times 10^7\alpha^2 b + 555556\alpha b + 2500b + 8.65053 \times 10^{30}\alpha^8 k_3 \\ &+ 7.81255 \times 10^{29}\alpha^7 k_3 + 1.97241 \times 10^{23}\alpha^6 k_2 + 3.11121 \times 10^{28}\alpha^6 k_3 \\ &+ 1.38897 \times 10^{22}\alpha^5 k_2 + 7.14358 \times 10^{26}\alpha^5 k_3 + 6.1741 \times 10^{15}\alpha^4 k_1 \\ &+ 4.13321 \times 10^{20}\alpha^4 k_2 + 1.03576 \times 10^{25}\alpha^4 k_3 + 3.12755 \times 10^{14}\alpha^3 k_1 \\ &+ 6.67171 \times 10^{18}\alpha^3 k_2 + 9.72777 \times 10^{22}\alpha^3 k_3 + 6.14172 \times 10^{12}\alpha^2 k_1 \\ &+ 6.18569 \times 10^{16}\alpha^2 k_2 + 5.79228 \times 10^{20}\alpha^2 k_3 + 5.61111 \times 10^{10}\alpha k_1 \\ &+ 3.14172 \times 10^{14}\alpha k_2 + 2.00529 \times 10^{18}\alpha k_3 + 2.05556 \times 10^8 k_1 \\ &+ 6.89342 \times 10^{11} k_2 + 3.10406 \times 10^{15} k_3 + 8333.33. \end{aligned} \tag{3.25}$$

We put $a = 2$, $b = 6$, $k_1 = 2$, $k_2 = 3$, and $k_3 = 4$, then the functional integral $J(u, v)$ becomes

$$\begin{aligned} J(u,v) =& 3.46021 \times 10^{31}\alpha^8 + 3.12502 \times 10^{30}\alpha^7 + 1.24449 \times 10^{29}\alpha^6 \\ &+ 2.85747 \times 10^{27}\alpha^5 + 4.14318 \times 10^{25}\alpha^4 + 3.89131 \times 10^{23}\alpha^3 \\ &+ 2.3171 \times 10^{21}\alpha^2 + 8.02211 \times 10^{18}\alpha + 1.24183 \times 10^{16}. \end{aligned} \tag{3.26}$$

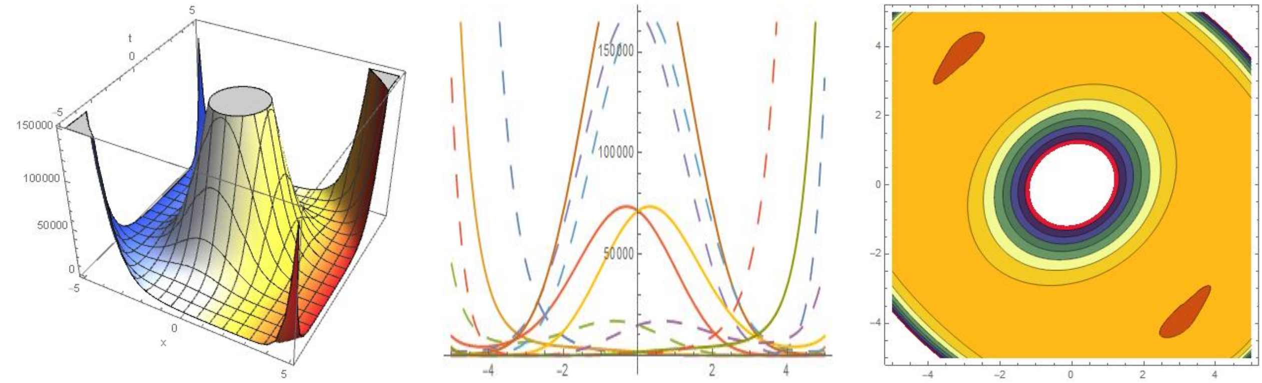


Figure 9. Show an example of the Lagrangian $L(x,t)$ with Eq (3.5), in the interval $-5 < x < 5$ and $-5 < t < 5$, by choosing the trial functions (3.28) and (3.29) at the parameter $\alpha = -0.131961$ and $a = 3$, $b = -2$, $k_1 = 2$, $k_2 = 4$, $k_3 = 6$.

Derive Eq (3.26) with respect to α, then values of α are:

$$\alpha = -0.0136235, \quad \alpha = -0.0129514 - 0.00218597i, \quad \alpha = -0.0129514 + 0.00218597i,$$
$$\alpha = -0.0110962 - 0.00364064i, \quad \alpha = -0.0110962 + 0.00364064i,$$
$$\alpha = -0.00865263 - 0.00394263i, \quad \alpha = -0.00865263 + 0.00394263i.$$

We substitute the exact roots of α into Eq (3.26) give the following analytical equations:

$$\begin{aligned}
J(u,v) &= 1.15381 \times 10^{12},\\
J(u,v) &= 9.16876 \times 10^{11} - 1.08685 \times 10^{12} i,\\
J(u,v) &= 9.16876 \times 10^{11} + 1.08685 \times 10^{12} i,\\
J(u,v) &= -7.64569 \times 10^{10} - 2.53327 \times 10^{12} i,\\
J(u,v) &= -7.64569 \times 10^{10} + 2.53327 \times 10^{12} i,\\
J(u,v) &= -4.47645 \times 10^{12} - 4.88381 \times 10^{12} i,\\
J(u,v) &= -4.47645 \times 10^{12} + 4.88381 \times 10^{12} i.
\end{aligned} \tag{3.27}$$

Case 2: We try the following Jost functions:

$$u(x,t) = \begin{cases} (20 + 200\alpha)\exp(-\mu(t-5)(x-5)), & \text{at} \quad x > 5,\ t > 5,\\ (10 + t + x) + \alpha(t+5)^2 + \alpha(x+5)^2, & \text{at} \quad |x| < 5,\ |t| < 5,\\ 0, & \text{at} \quad x < -5,\ t < -5, \end{cases} \tag{3.28}$$

$$v(x,t) = \begin{cases} 0, & \text{at} \quad x > 5,\ t > 5,\\ (10 - t - x) + \alpha(5-t)^2 + \alpha(5-x)^2, & \text{at} \quad |x| < 5,\ |t| < 5,\\ (20 + 200\alpha)\exp(\mu(t+5)(x+5)), & \text{at} \quad x < -5,\ t < -5. \end{cases} \tag{3.29}$$

We found the values of Lagrangian calculation from Eqs (3.28) and (3.29) into Eq (3.5), and we have

$$\begin{aligned} J(u,v) =& 50000\alpha^2 + 13333.3a\alpha^2 + 2000a\alpha + 15000\alpha + 9.90698\times10^{17}\alpha^8k_3 \\ &+ 9.04639\times10^{17}\alpha^7k_3 + 5.97273\times10^{13}\alpha^6k_2 + 3.65565\times10^{17}\alpha^6k_3 \\ &+ 4.25597\times10^{13}\alpha^5k_2 + 8.55415\times10^{16}\alpha^5k_3 + 4.78095\times10^{9}\alpha^4k_1 \\ &+ 1.28615\times10^{13}\alpha^4k_2 + 1.27048\times10^{16}\alpha^4k_3 + 2.39683\times10^{9}\alpha^3k_1 \\ &+ 2.11556\times10^{12}\alpha^3k_2 + 1.22953\times10^{15}\alpha^3k_3 + 4.62698\times10^{8}\alpha^2k_1 \\ &+ 2.00368\times10^{11}\alpha^2k_2 + 7.59375\times10^{13}\alpha^2k_3 + 4.08889\times10^{7}\alpha k_1 \\ &+ 1.03937\times10^{10}\alpha k_2 + 2.7454\times10^{12}\alpha k_3 + 1.4\times10^{6}k_1 \\ &+ 2.31429\times10^{8}k_2 + 4.46349\times10^{10}k_3 + 1000. \end{aligned} \tag{3.30}$$

We put $a = 3$, $b = -2$, $k_1 = 2$, $k_2 = 4$, and $k_3 = 6$, then the functional integral $J(u, v)$ becomes

$$\begin{aligned} J(u,v) =& 5.94419\times10^{18}\alpha^8 + 5.42784\times10^{18}\alpha^7 + 2.19363\times10^{18}\alpha^6 \\ &+ 5.13419\times10^{17}\alpha^5 + 7.62802\times10^{16}\alpha^4 + 7.38564\times10^{15}\alpha^3 \\ &+ 4.56427\times10^{14}\alpha^2 + 1.6514\times10^{13}\alpha + 2.68738\times10^{11}. \end{aligned} \tag{3.31}$$

Derive Eq (3.31) with respect to α, then values of α are:

$$\begin{aligned} &\alpha = -0.131961, \quad \alpha = -0.129142 - 0.0174743i, \quad \alpha = -0.129142 + 0.0174743i, \\ &\alpha = -0.117682 - 0.0366515i, \quad \alpha = -0.117682 + 0.0366515i, \\ &\alpha = -0.0866909 - 0.0518096i, \quad \alpha = -0.0866909 + 0.0518096i. \end{aligned}$$

We substitute the exact roots of α into Eq (3.31) give the following analytical equations:

$$\begin{aligned} J(u,v) &= 3.53554\times10^{6}, \\ J(u,v) &= -1.84827\times10^{6} - 5.13825\times10^{6}i, \\ J(u,v) &= -1.84827\times10^{6} + 5.13825\times10^{6}i, \\ J(u,v) &= -2.46359\times10^{7} + 1.0125\times10^{7}i, \\ J(u,v) &= -2.46359\times10^{7} - 1.0125\times10^{7}i, \\ J(u,v) &= -2.31965\times10^{8} + 3.16423\times10^{8}i, \\ J(u,v) &= -2.31965\times10^{8} - 3.16423\times10^{8}i. \end{aligned} \tag{3.32}$$

3.2. Exact solitary wave solutions of the nonlinear Schrödinger equation with polynomial law nonlinearity model II

We survey here the exact solutions of the nonlinear Schrödinger equation with polynomial law nonlinearity Eq (3.1).

Case 1: We suppose the ansatz function of the NLSE with polynomial law nonlinearity is in the form of a bright solitary wave solution.

$$h_1(x,t) = A\,\mathrm{sech}\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right),\ \Psi(x,t) = A\,\mathrm{sech}\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right)e^{i(\mu x-\omega t)}, \tag{3.33}$$

where A, w and v are the amplitude, the pulse width, and velocity of soliton in normalized unites. Substituting from Eq (3.33) in Eq (3.1), and separating the real and imaginary parts, we obtain

$$
\begin{aligned}
&av\mathrm{w}^2 + b\mu v\omega - a\mu^2 v - b\mathrm{w}^2 + v\omega + \Big(-2av\mathrm{w}^2 + A^2 k_1 v + 2b\mathrm{w}^2 \Big) \\
&\times \operatorname{sech}^2 \left(\mathrm{w} \left(x - \frac{t}{v} \right) \right) + A^4 k_2 v \operatorname{sech}^4 \left(\mathrm{w} \left(x - \frac{t}{v} \right) \right) + A^6 k_3 v \operatorname{sech}^6 \left(\mathrm{w} \left(x - \frac{t}{v} \right) \right) = 0, \qquad (3.34) \\
&\Big(-2a\mu v\mathrm{w} + bv\mathrm{w}\omega + b\mu\mathrm{w} + \mathrm{w} \Big) \tanh \left(\mathrm{w} \left(x - \frac{t}{v} \right) \right) = 0. \qquad (3.35)
\end{aligned}
$$

Equating the coefficients of the linearly independent terms to zero, we obtain the dynamical system in $A, \mathrm{w}, v, \mu, \omega, a, b, k_1, k_2, k_3$ by solving this system we get:

Family I:

$$
A = \pm \sqrt{\frac{2\left(\mu \mathrm{w}^2 - v\mathrm{w}^2\omega\right)}{k_1\mu^2 v + k_1 v\mathrm{w}^2}}, \quad a = \frac{-v^2\omega^2 - \mathrm{w}^2}{v\left(\mu^2 + \mathrm{w}^2\right)\left(v\omega - \mu\right)}, \quad b = \frac{2\mu v\omega + \mathrm{w}^2 - \mu^2}{\left(\mu^2 + \mathrm{w}^2\right)\left(\mu - v\omega\right)}. \qquad (3.36)
$$

The sufficient conditions for solitary wave solution stability are:

$$
\frac{\mu \mathrm{w}^2 - v\mathrm{w}^2\omega}{k_1\mu^2 v + k_1 v\mathrm{w}^2} > 0, \quad v\left(\mu^2 + \mathrm{w}^2\right)\left(v\omega - \mu\right) \neq 0, \quad \left(\mu^2 + \mathrm{w}^2\right)\left(\mu - v\omega\right) \neq 0. \qquad (3.37)
$$

Family II:

$$
\begin{aligned}
A &= \pm \sqrt{\frac{b\mu^2 - b\mu v\omega + \mu - 2v\omega}{k_1 v}}, \quad a = \frac{b\mu + bv\omega + 1}{2\mu v}, \\
\mathrm{w} &= \pm \sqrt{\frac{b\mu^3 - b\mu^2 v\omega + \mu^2 - 2\mu v\omega}{bv\omega + 1 - b\mu}}, \qquad (3.38)
\end{aligned}
$$

provided that

$$
\frac{b\mu^2 - b\mu v\omega + \mu - 2v\omega}{k_1 v} > 0, \quad \frac{b\mu^3 - b\mu^2 v\omega + \mu^2 - 2\mu v\omega}{bv\omega + 1 - b\mu} > 0, \quad \mu v \neq 0. \qquad (3.39)
$$

Family III:

$$
\begin{aligned}
A &= \pm \mathrm{w} \sqrt{\frac{2\omega}{bk_1\mu^3 + bk_1\mu\mathrm{w}^2 + k_1\mu^2 - k_1\mathrm{w}^2}}, \\
a &= \frac{\omega\left(b^2\mu^2 + b^2\mathrm{w}^2 + 2b\mu + 1\right)}{b\mu^3 + b\mu\mathrm{w}^2 + \mu^2 - \mathrm{w}^2}, \quad v = \frac{b\mu^3 + b\mu\mathrm{w}^2 + \mu^2 - \mathrm{w}^2}{\omega\left(b\mu^2 + b\mathrm{w}^2 + 2\mu\right)}, \qquad (3.40)
\end{aligned}
$$

whenever

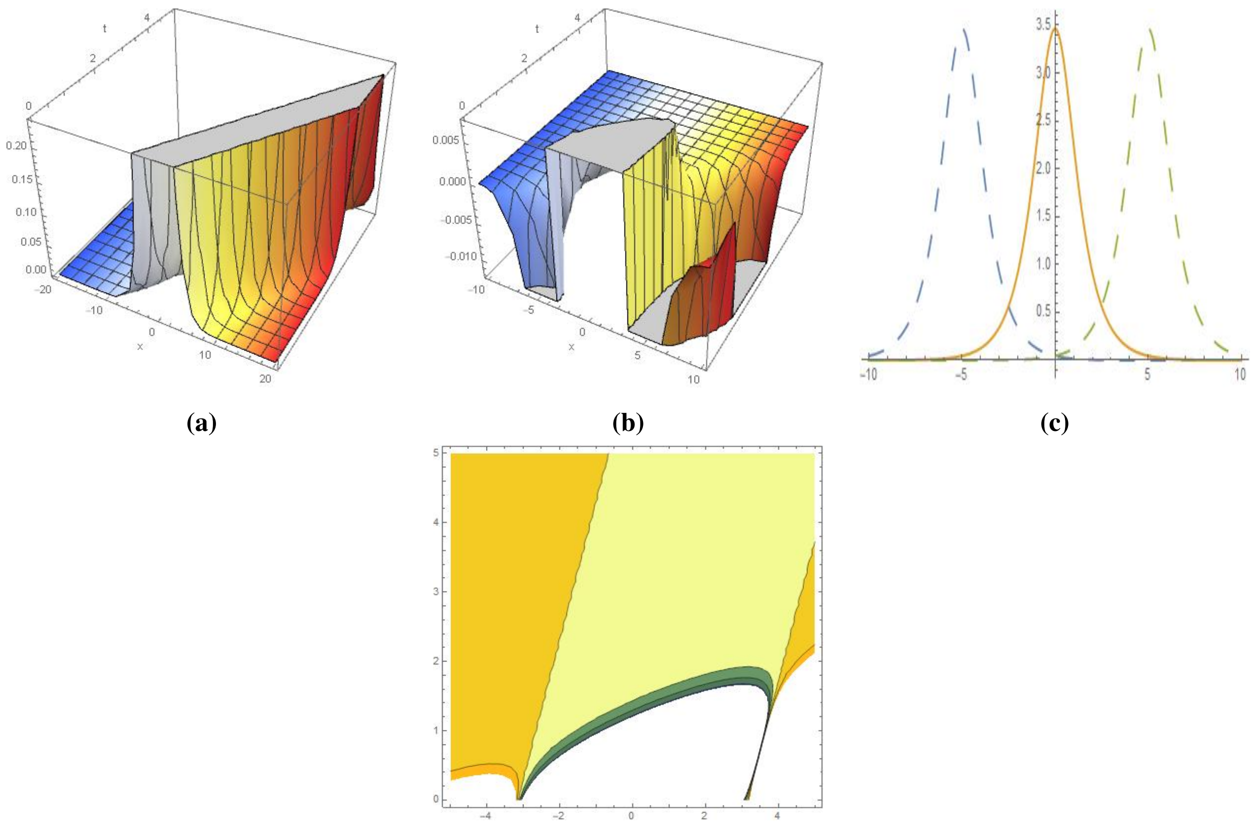
$$
\begin{aligned}
&\frac{\omega}{bk_1\mu^3 + bk_1\mu\mathrm{w}^2 + k_1\mu^2 - k_1\mathrm{w}^2} > 0, \quad b\mu^3 + b\mu\mathrm{w}^2 + \mu^2 - \mathrm{w}^2 \neq 0, \\
&\omega\left(b\mu^2 + b\mathrm{w}^2 + 2\mu\right) \neq 0. \qquad (3.41)
\end{aligned}
$$

Then, the solutions of the NLSE with polynomial law nonlinearity as bright solitary wave solutions are (see Figures 10 and 11):

$$\Psi_{11}(x,t) = \pm\sqrt{\frac{2\left(\mu \mathrm{w}^2 - v\mathrm{w}^2\omega\right)}{k_1\mu^2 v + k_1 v\mathrm{w}^2}}\,\mathrm{sech}\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(\mu x-\omega t)}, \tag{3.42}$$

$$\Psi_{12}(x,t) = \pm\sqrt{\frac{b\mu^2 - b\mu v\omega + \mu - 2v\omega}{k_1 v}}\,\mathrm{sech}\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(\mu x-\omega t)}, \tag{3.43}$$

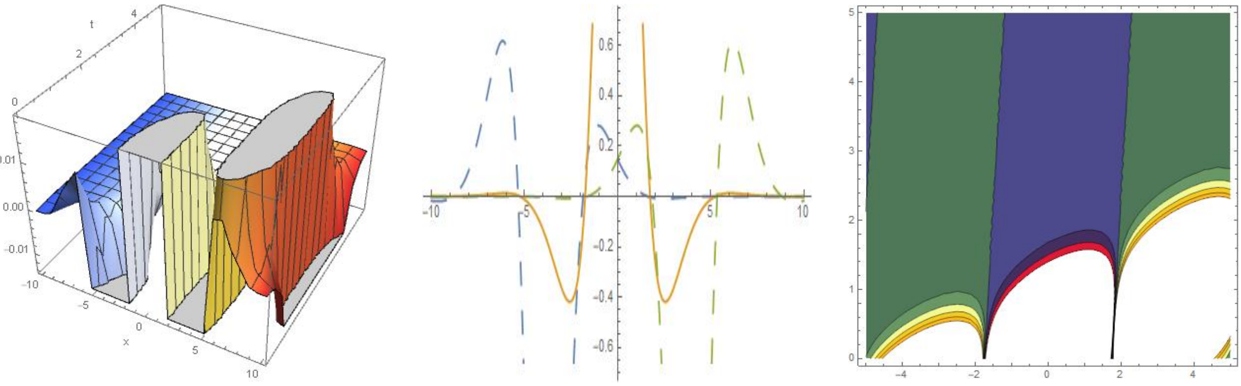
$$\Psi_{13}(x,t) = \pm\mathrm{w}\sqrt{\frac{2\omega}{bk_1\mu^3 + bk_1\mu\mathrm{w}^2 + k_1\mu^2 - k_1\mathrm{w}^2}}\,\mathrm{sech}\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(\mu x-\omega t)}. \tag{3.44}$$



(a) (b) (c)

(d)

Figure 10. Representation of solitary wave solution Ψ_{11}. These figures are obtained by $k_1 = 0.3$, $\mu = 0.5$, $v = 0.2$, w = 1, $\omega = 0.25$, (10a) and (10b) are plotted in 3D while (10c) is plotted in 2D at different positions, and (10d) plotted as contour.



(a) **(b)** **(c)**

Figure 11. Representation of solitary wave solution Ψ_{12}. These figures are obtained by $b = 0.5$, $k_1 = 0.3$, $\mu = 0.9$, $v = 0.25$, $\mathrm{w} = 1$, $\omega = 0.1$, (11a) is plotted in 3D while (11b) is plotted in 2D at different positions, and (11c) plotted as contour.

Case 2: Another choice of the dark solitary wave solution of the NLSE with polynomial law nonlinearity is

$$\Psi(x,t) = \left(A + B\tanh\left(\mathrm{w}\left(x - \frac{t}{v}\right)\right)\right)e^{i(\mu x-\omega t)}. \tag{3.45}$$

By replacement from Eq (3.45) in Eq (3.1) and separating the real and imaginary parts

$$\begin{aligned}
&A^7k_3 - aA\mu^2 + 21A^5B^2k_3 + A^5k_2 + 35A^3B^4k_3 + 10A^3B^2k_2 + A^3k_1\\
&+Ab\mu\omega + 7AB^6k_3 + 5AB^4k_2 + 3AB^2k_1 + A\omega + \Big(-21A^5B^2k_3\\
&-70A^3B^4k_3 - 10A^3B^2k_2 - 21AB^6k_3 - 10AB^4k_2 - 3AB^2k_1\Big)\\
&\times\mathrm{sech}^2\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right) + \left(35A^3B^4k_3 + 21AB^6k_3 + 5AB^4k_2\right)\\
&\times\mathrm{sech}^4\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right) - 7AB^6k_3\,\mathrm{sech}^6\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right) + \Big(-aB\mu^2\\
&+7A^6Bk_3 + 35A^4B^3k_3 + 5A^4Bk_2 + 21A^2B^5k_3 + 10A^2B^3k_2 + 3A^2Bk_1\\
&+bB\mu\omega + B^7k_3 + B^5k_2 + B^3k_1 + B\omega\Big)\tanh\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)\\
&+\Big(-2aB\mathrm{w}^2 - 35A^4B^3k_3 - 42A^2B^5k_3 - 10A^2B^3k_2 + \frac{2bB\mathrm{w}^2}{v}\\
&-3B^7k_3 - 2B^5k_2 - B^3k_1\Big)\tanh\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)\mathrm{sech}^2\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)\\
&+\left(21A^2B^5k_3 + 3B^7k_3 + B^5k_2\right)\tanh\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)\mathrm{sech}^4\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)\\
&-B^7k_3\tanh\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)\mathrm{sech}^6\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right) = 0,
\end{aligned} \tag{3.46}$$

$$\left(2aB\mu\mathrm{w} - \frac{bB\mu\mathrm{w}}{v} - bB\mathrm{w}\omega - \frac{B\mathrm{w}}{v}\right)\mathrm{sech}^2\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right) = 0. \tag{3.47}$$

Equating the coefficients of the linearly independent terms to zero, we deduce the coefficients $A, B, \mathrm{w}, v, \mu, \omega, a, b, k_1, k_2, k_3$ in the form:

Family I:

$$A=0,\quad B=\pm i\sqrt{\frac{k_2}{3k_3}},\quad a=\frac{k_2^3v^2\omega+k_2^3\mu v+54k_3^2\mathrm{w}^2-27k_3^2v^2\omega^2}{27k_3^2v\left(\mu^2-2\mathrm{w}^2\right)\left(v\omega-\mu\right)},$$

$$b=\frac{54k_3^2\mu v\omega-27k_3^2\mu^2-2k_2^3\mu v-54k_3^2\mathrm{w}^2}{27k_3^2\left(\mu^2-2\mathrm{w}^2\right)\left(\mu-v\omega\right)},\tag{3.48}$$

$$k_1=\frac{3k_2^3\mu^2v-54k_3^2v\mathrm{w}^2\omega-4k_2^3v\mathrm{w}^2+54k_3^2\mu\mathrm{w}^2}{9k_2k_3v\left(\mu^2-2\mathrm{w}^2\right)}.$$

The sufficient conditions for dark solitary wave solution stability are

$$\frac{k_2}{k_3}<0,\quad k_3^2v\left(\mu^2-2\mathrm{w}^2\right)\left(v\omega-\mu\right)\neq 0,\quad k_2k_3v\left(\mu^2-2\mathrm{w}^2\right)\neq 0.\tag{3.49}$$

Family II:

$$A=0,\quad B=\pm i\sqrt{\frac{k_2}{3k_3}},$$

$$a=\frac{27b^2k_3^2\mu^2-54b^2k_3^2\mathrm{w}^2+54bk_3^2\mu+bk_2^3v+27k_3^2}{27k_3^2v\left(b\mu^2-2b\mathrm{w}^2+2\mu\right)},$$

$$k_1=\frac{3bk_2^3\mu^2v-4bk_2^3v\mathrm{w}^2+6k_2^3\mu v+54k_3^2\mathrm{w}^2}{9k_2k_3v\left(b\mu^2-2b\mathrm{w}^2+2\mu\right)},\tag{3.50}$$

$$\omega=\frac{27bk_3^2\mu^3-54bk_3^2\mu\mathrm{w}^2+27k_3^2\mu^2+2k_2^3\mu v+54k_3^2\mathrm{w}^2}{27k_3^2v\left(b\mu^2-2b\mathrm{w}^2+2\mu\right)},$$

provided that

$$\frac{k_2}{k_3}<0,\quad k_2k_3v\left(b\mu^2-2b\mathrm{w}^2+2\mu\right)\neq 0,\quad k_3^2v\left(b\mu^2-2b\mathrm{w}^2+2\mu\right)\neq 0.\tag{3.51}$$

Family III:

$$A=0,\quad B=\pm i\sqrt{\frac{k_2}{3k_3}},\quad a=\frac{b\mu+bv\omega+1}{2\mu v},$$

$$k_1=\frac{27bk_3^2\mu v\omega-27bk_3^2\mu^2-27k_3^2\mu+54k_3^2v\omega+4k_2^3v}{18k_2k_3v},\tag{3.52}$$

$$\mathrm{w}=-\sqrt{\frac{\mu\left(27bk_3^2\mu v\omega-27k_3^2\mu+54k_3^2v\omega-2k_2^3v-27bk_3^2\mu^2\right)}{54bk_3^2v\omega+54k_3^2-54bk_3^2\mu}},$$

whenever

$$\frac{k_2}{k_3} < 0, \quad \mu v \neq 0, \quad k_2 k_3 v \neq 0, \quad \frac{\mu\left(27bk_3^2\mu v\omega - 27k_3^2\mu + 54k_3^2 v\omega - 2k_2^3 v - 27bk_3^2\mu^2\right)}{54bk_3^2 v\omega + 54k_3^2 - 54bk_3^2\mu} > 0. \tag{3.53}$$

Family IV:

$$\begin{aligned}
B &= \pm\sqrt[6]{\frac{a\mu^2v^2\omega - 2av^2\mathrm{w}^2\omega - a\mu^3 v + 2a\mu v\mathrm{w}^2 + v^2\omega^2 - 2\mathrm{w}^2}{v\left(2\mathrm{w}^2-\mu^2\right)(v\omega-\mu)\left(\frac{k_3(\mu+v\omega)}{\left(2\mathrm{w}^2-\mu^2\right)(\mu-v\omega)}\right)}},\\
A &= 0, \quad b = \frac{2a\mu v - 1}{\mu + v\omega},\\
k_1 &= \frac{1}{v^{2/3}(\mu+v\omega)\sqrt[3]{2\mathrm{w}^2(av(v\omega-\mu)+1) - v\left(a\mu^2(v\omega-\mu)+v\omega^2\right)}}\\
&\quad\times\left(\sqrt[3]{\left(\frac{k_3(\mu+v\omega)}{\left(2\mathrm{w}^2-\mu^2\right)(\mu-v\omega)}\right)\left(2\mathrm{w}^2-\mu^2\right)(v\omega-\mu)}\right.\\
&\quad\left.\times\left(3v\left(a\mu^2(v\omega-\mu)+v\omega^2\right) - 4\mathrm{w}^2\left(av(v\omega-\mu)+1\right)\right)\right),\\
k_2 &= \frac{3k_3\sqrt[3]{2\mathrm{w}^2(av(v\omega-\mu)+1) - v\left(a\mu^2(v\omega-\mu)+v\omega^2\right)}}{\sqrt[3]{v\left(2\mathrm{w}^2-\mu^2\right)(v\omega-\mu)\left(\frac{k_3(\mu+v\omega)}{\left(2\mathrm{w}^2-\mu^2\right)(\mu-v\omega)}\right)}},
\end{aligned} \tag{3.54}$$

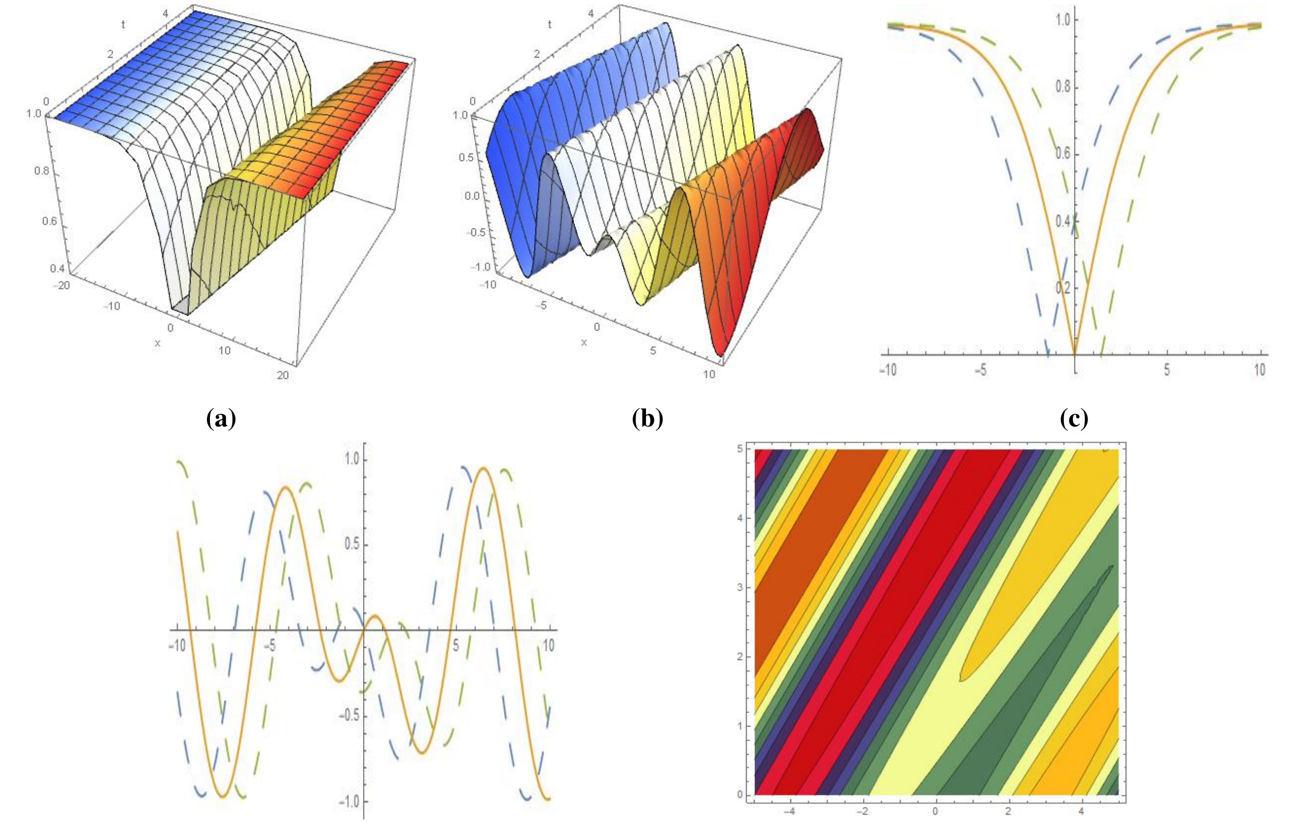
such that

$$\begin{aligned}
&\frac{a\mu^2v^2\omega - 2av^2\mathrm{w}^2\omega - a\mu^3 v + 2a\mu v\mathrm{w}^2 + v^2\omega^2 - 2\mathrm{w}^2}{v\left(2\mathrm{w}^2-\mu^2\right)(v\omega-\mu)\left(\frac{k_3(\mu+v\omega)}{\left(2\mathrm{w}^2-\mu^2\right)(\mu-v\omega)}\right)} > 0, \quad \mu + v\omega \neq 0,\\
&v^{2/3}(\mu+v\omega)\sqrt[3]{2\mathrm{w}^2(av(v\omega-\mu)+1) - v\left(a\mu^2(v\omega-\mu)+v\omega^2\right)} \neq 0,\\
&\sqrt[3]{v\left(2\mathrm{w}^2-\mu^2\right)(v\omega-\mu)\left(\frac{k_3(\mu+v\omega)}{\left(2\mathrm{w}^2-\mu^2\right)(\mu-v\omega)}\right)} \neq 0.
\end{aligned} \tag{3.55}$$

Then, the dark soliton solutions of the NLSE with polynomial law nonlinearity Eq (3.1) are (see Figure 12)

$$\Psi_{21}(x,t) = \pm i\sqrt{\frac{k_2}{3k_3}}\tanh\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(\mu x-\omega t)}, \tag{3.56}$$

$$\begin{aligned}
\Psi_{24}(x,t) &= \pm\sqrt[6]{\frac{a\mu^2v^2\omega - 2av^2\mathrm{w}^2\omega - a\mu^3 v + 2a\mu v\mathrm{w}^2 + v^2\omega^2 - 2\mathrm{w}^2}{v\left(2\mathrm{w}^2-\mu^2\right)(v\omega-\mu)\left(\frac{k_3(\mu+v\omega)}{\left(2\mathrm{w}^2-\mu^2\right)(\mu-v\omega)}\right)}}\\
&\quad\times\tanh\left(\mathrm{w}\left(x-\frac{t}{v}\right)\right)e^{i(\mu x-\omega t)}.
\end{aligned} \tag{3.57}$$



(a) (b) (c)

(d) (e)

Figure 12. Representation of solitary wave solution Ψ_{24}. These figures are obtained by $a = 0.5$, $k_3 = 0.25$, $\mu = 0.9$, $\nu = 0.7$, $\mathrm{w} = 0.3$, $\omega = 1$, (12a) and (12b) are plotted in 3D while (12c) and (12d) are plotted in 2D at different positions, and (12e) plotted as contour.

4. Conclusions

In this paper, we discussed the two models of the nonlinear Schrödinger equation (NLSE) with polynomial law nonlinearity by effective and understandable techniques, such as the variational principle method based on the Lagrangian and the amplitude ansatz method. We found the functional integral and the Lagrangian of these models. Meanwhile, the solutions of the proposed equations were extracted by choice of different ansatz functions based on the Jost function, and they are continuous at all intervals. Firstly, the Jost function was approximated by piecewise linear ansatz function with a single nontrivial variational parameter in three cases from a region of a rectangular box. Then, the Jost function was approximated by the piecewise ansatz function containing the hyperbolic function in two cases of the two-box potential and was also approximated by quadratic polynomials with two free parameters rather than a piecewise linear ansatz function. Finally, this trial function had been approximated by the tanh function. Besides, we applied the amplitude ansatz method to obtain the new soliton solutions of the offered equations that contain bright soliton, dark soliton, bright-dark solitary wave solutions, rational dark-bright solutions, and periodic solitary wave solutions. The conditions for the stability of solutions were conducted. Graphical models, such as 2D, 3D, and contour plots, were induced using appropriate parameter values. These solutions have vital applications in applied sciences and might provide valuable support for investigators and physicists to solve more complex physical phenomena.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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