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Research article

Novel categories of spaces in the frame of fuzzy soft topologies

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Abstract: In the present paper, we introduce and discuss a new set of separation properties in fuzzy soft topological spaces called $FS\delta$-separation and $FS\delta$-regularity axioms by using fuzzy soft δ-open sets and the quasi-coincident relation. We provide a comprehensive study of their properties with some supporting examples. Our analysis includes more characterizations, results, and theorems related to these notions, which contributes to a deeper understanding of fuzzy soft separability properties. We show that the $FS\delta$-separation and $FS\delta$-regularity axioms are harmonic and heredity property. Additionally, we examine the connections between $FS\delta^*$-compactness and $FS\delta$-separation axioms and explore the relationships between them. Overall, this work offers a new perspective on the theory of separation properties in fuzzy soft topological spaces, as well as provides a robust foundation for further research in the transmission of properties from fuzzy soft topologies to fuzzy and soft topologies and vice-versa by swapping between the membership function and characteristic function in the case of fuzzy topology and the set of parameters and a singleton set in the case of soft topology.

Keywords: fuzzy soft set; fuzzy soft δ-open set; fuzzy soft topology; $FS\delta$-separation axioms; $FS\delta$-regularity axioms
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1. Introduction

Zadeh [50] was the first to come up with the unprecedented theory of fuzzy set (F-set) for dealing with some types of uncertainties where conventional tools fail. This theory brought a grand paradigmatic change in mathematics and offered a convenient framework to model a huge number of empirical problems. On the other hand, this theory has its inherent difficulties, which are possibly attributed to the inadequacy of the parameterization tool and pre-requirement of membership function, as pointed out by Molodtsov in his pioneering work [41]. He introduced the concept of soft sets (S-sets) as a remarkable mathematical tool for coping with vagueness that is free from the aforementioned difficulties. Then, the S-set theory has been applied in many fields by many authors [16, 35, 36]. One of these fields that attracted a lot of attention is the abstract topological structures that were displayed by Shabir-Naz [48] and Çağman et al. [20]. Some divergences between classical and soft topologies were illuminated in [7].

Over time, complicated issues have appeared that need combining parameterization of S-sets with the membership degree of F-sets. To tackle such dilemmas, Maji et al. [37] put forward a new paradigm known as a fuzzy soft set (FS-set) and demonstrated how this paradigm is applied [38]. Since then, the FS-set theory and its applications have been studied by several intellectuals [5, 19, 24, 25]. To cover more situations and expand the range of applications, the concept of FS-set was generalized to (a, b)-Fuzzy soft sets by [11]. Kharal and Ahmad [34] defined the concept of mappings of FS-classes. Subsequently, the study of topological structure over the family of FS-sets was started by Tanay-Kandemir [49]. Mukherjee et al. [42] introduced the notions of $FS\delta$-open and $FS\delta$-closed sets, $FS\delta$-closure and $FS\delta$-interior operators, and $FS\delta$-continuity. Kandil et al. provided the concepts of fuzzy soft connected and fuzzy soft hyperconnected spaces in [31,32], respectively. Various concepts in fuzzy soft settings have been considered, such as disjoint union of fuzzy soft topological structures [6] and filters [26].

Since the importance of separation axioms in topological spaces, it was investigated topologies over the different types of uncertainty spaces. Kandil and El-Etriby [29] structured separation axioms in the spaces of fuzzy topologies, then Kandil and El-Shafei [30] familiarized the axioms of regularity in fuzzy topologies and FR_i-proximities. Saleh et al. [45] displayed stronger types of separation and regularity axioms in the spaces of fuzzy topologies using fuzzy pre-open sets. In fuzzy soft topological spaces, separation axioms have been presented and discussed by many authors; see, for example, [1,2, 39,40]. Kandil et al. [33] scrutinized the characterizations of separation axioms and regularity inspired by quasi-coincident and neighborhood systems. Recently, Saleh et al. [46,47] have described another sorts of FS-separation axioms and regularity axioms. In soft setting, a wide class of separation axioms have been offered by Al-shami and his coauthors [12,13,17,21–23]. They successfully exploited these axioms to address some real-life situations as given in [8, 9]. Alcantud [3] conducted an interesting work to describe the relationships between topological structures in soft and fuzzy settings.

To go along this line of research, we are writing this paper, which contributes to the understanding of fuzzy soft separability properties and produces some categories of fuzzy soft topological spaces. It is well known that the environment of the current work widens other known generalizations such as fuzzy topology and soft topology; this means the results and relationships obtained in these frames are special cases of their counterparts investigated herein. This is attributed to that the frameworks of soft and fuzzy topologies are produced by "fuzzy soft topology" by replacing the membership function with the characteristic function in the case of fuzzy topology and restricting the set of parameters by a singleton set in the case of soft topology. Hence, the paper enhances the body of knowledge and provides a comprehensive insight to study the properties and characteristics of topological structures.

After this introduction, the reader may pursue the content of this research as follows. In Section 2, we requisition the definitions and findings that are needful to go along with the results obtained herein. In the next sections, Sections 3 and 4, we delve into the topic of separation properties in fuzzy soft topological spaces and propose a new set of axioms called $FS\delta$-separation (FS-δT_i, where $i = 0, 1, 2, 3, 4$) and $FS\delta$-regularity (FS-δR_i, where $i = 0, 1.2, 3$). These separations are structured by utilizing the ideas of fuzzy soft δ-open sets and the quasi-coincident relation. We provide various characterizations of these properties and present a range of results, theorems, and relationships related to these notions. In Section 5, we look at the interplay between $FS\delta^*$-compactness and $FS\delta$-separation axioms and analyze the relationships between them. In the end, we outline the master contributions of this manuscript and suggest a road map for future direction in Section 6.

2. Some basic definitions

Here, we recall the basic definitions that will be needed in this sequel. In the present work, U refers to the universe set, E is the set of all parameters for U, $I = [0, 1]$, and FS- refers to fuzzy soft.

Definition 2.1. *[50] An F-set A of U is a mapping $A : U \longrightarrow I$. I^U refers to the set of all F-sets on* U. *An F-point x_λ, $\lambda \in (0, 1]$ is an F-set in U given by $x_\lambda(y) = \lambda$ at $x = y$ and $x_\lambda(y) = 0$ for all $y \in U$. For $\alpha \in I$, $\underline{\alpha} \in I^U$ refers to the F-constant function where $\underline{\alpha}(x) = \alpha \ \forall\ x \in U$.*

Definition 2.2. *[37] An FS-set $h_E = (f, E)$ on U is the set of ordered pairs $h_E = \{(e, h(e) : e \in E, h(e) \in I^U\}$.*

In this content, $FSS(U_E)$ refers to the set of all FS-sets on U. $\widetilde{\alpha}_E \in FSS(U_E)$ defined by $\widetilde{\alpha}_E = \{(e, \underline{\alpha}) : e \in E\ ,\ \underline{\alpha} \in I^U\}$ is called an FS-constant set.

Definition 2.3. *[18,43] An FS-point x^e_α on U_E is an FS-set on U defined by $x^e_\alpha(e') = x_\alpha$ if $e' = e$ and $x^e_\alpha(e') = \underline{0}$* if *$e' \in E - \{e\}$, where x_α is the F-point in U. $FSP(U_E)$ refers to the set of all FS-points in U. An FS-point $x^e_\alpha \widetilde{\in} f_E$ if $\alpha \leq f(e)(x)$.*

Definition 2.4. *[43,49] The triplet (U, τ, E) is called a fuzzy soft topological space (briefly, an $FSTS$), where U is an initial universal set, E is a fixed set of parameters, and τ is a family of FS-sets on U such that τ is closed under arbitrary union and finite intersection and $0_E, 1_E$ belong to τ. The elements in τ are called fuzzy soft open sets (briefly, FSO-sets) and the complements of them are called fuzzy soft closed sets (briefly, FSC-sets).*

Definition 2.5. *[18] The FS-sets h_E and g_E are called quasi-coincident, denoted by $f_E q g_E$ if there are $e \in E$, $u \in U$ such that $h(e)(u) + g(e)(u) > 1$. If h_E is not quasi-coincident with g_E, then we write $h_E \tilde{q} g_E$.*

Proposition 2.1. *[18,46] Let $x^e_r, y^e_t \in FSP(U_E)$, f_E, g_E, $h_E \in FSS(U_E)$, and $\{f_{iE} : i \in J\} \subseteq FSS(U_E)$, then*

(i) $f_E \tilde{q} g_E \Leftrightarrow f_E \sqsubseteq g^c_E$ *and* $f_E \tilde{q} f^c_E$.

(ii) $f_E \sqcap g_E = 0_E \Rightarrow f_E \tilde{q} g_E$.

(iii) $f_E \tilde{q} g_E, h_E \sqsubseteq g_E \Rightarrow f_E \tilde{q} h_E$.

(iv) $f_E q g_E \Rightarrow x^e_r q g_E,\ for\ some\ x^e_r \widetilde{\in} f_E$.

(v) $f_E \sqsubseteq g_E \Leftrightarrow (x_r^e q f_E \Rightarrow x_r^e q g_E), \forall x_r^e$.

(vi) *If* $x_r^e q(\sqcap_{i\in J} f_{iE})$, *then* $x_r^e q f_{iE,}\ \forall\ i \in J$.

(vii) $x \neq y \Rightarrow x_r^e \tilde{q} y_t^e,\ \forall\ r,\ t \in I$.

(viii) $x_r^e \tilde{q} y_t^e \Leftrightarrow x \neq y$ *or* $(x = y$ *and* $r + t \leq 1)$.

Definition 2.6. *[42] An FS-set h_E in (U, τ, E) is called q-neighborhood (briefly, q-nbd) of x_α^e if there is an FSO-set g_E such that $x_\alpha^e q g_E \sqsubseteq h_E$.*

Definition 2.7. *[34] Let $FSS(U_E)$ and $FSS(V_K)$ be two classes of all FS-sets over U and V, respectively. Let $u : U \longrightarrow V$ and $p : E \longrightarrow K$ be two maps, then $f_{up} : FSS\,(U_E) \longrightarrow FSS\,(V_K)$ is called a fuzzy soft map (or an FS-map) for which:*

(i) *If $f_E \in FSS(U_E)$, then the image of f_E denoted by $f_{up}(f_E)$ is the FS-set on V given by $f_{up}\,(f_E)\,(k) = sup\{u(f\,(e)) : e \in p^{-1}\,(k)\}$ if $p^{-1}\,(k) \neq \emptyset$, and $f_{up}\,(f_E)\,(k) = \tilde{0}_V$ otherwise, for all $k \in K$.*

(ii) *If $g_K \in FSS(V_K)$, then the preimage of g_K denoted by $f_{up}^{-1}(g_K)$ is the FS-set on U defined by, $f_{up}^{-1}(g_K)(e) = u^{-1}(g\,(p\,(e)))$ for all $e \in E$.*

The FS-map f_{up} is called one-to-one (onto), if u and p are one-to-one (onto). For more details about the properties of FS-maps; see, [34].

Definition 2.8. *[46] Let (U, τ, E) be an $FSTS$ and $Y \subseteq U$. Let h_E^Y be an FS-set on Y_E such that $h_E^Y : E \longrightarrow I^Y$, $h_E^Y(e) \in I^Y$ and $h_E^Y(e)(x) = 1$ if $x \in Y$, $h_E\,(e)\,(x) = 0$ if $x \notin Y$. Let $\tau_Y = \left\{h_E^Y \sqcap g_E : g_E \in \tau\right\}$, then τ_Y is a fuzzy soft topology (in short, FST) on Y and (Y, τ_Y, E) is called an FS-subspace of (U, τ, E). If $h_E^Y \in \tau$ (resp., $h_E^Y \in \tau^c$), then (Y, τ_Y, E) is called an FS-open (resp., closed) subspace of (U, τ, E).*

Definition 2.9. *[18, 42] For an FS-set h_E in (U, τ, E), we have:*

(i) *The FS-closure $cl(h_E)$ of h_E is the intersection of all FSC-sets containing h_E, and the FS-interior $int(h_E)$ of h_E is the union of all FSO-sets contained in h_E.*

(ii) *h_E is said to be a fuzzy soft regular open set (FSRO-set) if $h_E = int(cl(h_E))$. The complement of an FSRO-set is called a fuzzy closed regular set (FSRC-set). $FSRO(U_E)$ refers to the set of all FSRO-sets and $FSRC(U_E)$ refers to the set of all FSRC-sets.*

(iii) *h_E is said to be a fuzzy soft δ-neighborhood (briefly, FSδ-nbd) of x_α^e if and only if there is FSRO q-nbd g_E of x_α^e such that $g_E \sqsubseteq f_E$.*

Definition 2.10. *[42] Let h_E be an FS-set in (U, τ, E), then:*

(i) *An FS-point x_α^e is called an FSδ-cluster point of h_E if and only if every FSRO q-nbd f_E of x_α^e, $f_E q h_E$. The set of all FSδ-cluster points of h_E is called the FSδ-closure of h_E, denoted by $cl_\delta(h_E)$; that is, $cl_\delta(h_E) = \sqcap\{g_E \in FSRC\,(U_E) : h_E \sqsubseteq g_E\}$.*

(ii) *An FS-set h_E is called a fuzzy soft δ-closed set (FSδC-set) if and only if $h_E = cl_\delta(h_E)$. The complement of an FSδC-set is called a fuzzy soft δ-open set (FSδO-set). $FS\delta C\,(U_E)$ refers to the set of all FSδC-sets and $FS\delta O(U_E)$ refers to the set of all FSδO-sets.*

(iii) *The $FS\delta$-interior $int_\delta(h_E)$ of h_E is defined by $int_\delta(h_E) = \tilde{1}_E - cl_\delta(h^c_E)$; that is, $int_\delta(h_E) = \sqcup\{g_E \in FSRO(U_E) : g_E \sqsubseteq h_E\}$. Consequently, h_E is $FS\delta$-open if and only if $h_E = int_\delta(h_E)$.*

Notation. For x^e_r in $FSP(U_E)$, $O_{x^e_r}$ refers to an $FS\delta O$-set containing x^e_r. In general, O_{h_E} refers to an $FS\delta O$-set containing an FS-set h_E.
Result 1. *[42]* Every $FSRO$-set is an $FS\delta O$-set and every $FS\delta O$-set is an FSO-set. Moreover, if h_E is an FS-semi open set in (U, τ, E), then $cl(h_E) = cl_\delta(h_E)$.
Result 2. *[42]* If h_E is an FSO-set in (U, τ, E), then $cl(h_E)$ is an $FSRC$-set; that is, $\{cl(g_E) : g_E \in \tau\} = \{h_E : h_E \in FSRC(U_E)\}$, and for any FS-set h_E in (U, τ, E), $cl_\delta(h_E) = \sqcap\{cl(g_E) : h_E \sqsubseteq g_E\ ,\ g_E \in \tau\}$.

Theorem 2.1. *[42] For any FS-sets f_E and g_E in (U, τ, E), we have:*

(i) *$cl_\delta(0_E) = 0_E$ and $cl_\delta(1_E) = 1_E$.*

(ii) *$cl_\delta(f_E)$ is an $FS\delta C$-set, that is, $cl_\delta(cl_\delta(f_E)) = cl_\delta(f_E)$.*

(iii) *$cl(f_E) \sqsubseteq cl_\delta(f_E)$ and if $f_E \in \tau$, then $cl(f_E) = cl_\delta(f_E)$.*

Result 3. *[42]* The $FS\delta$-closure operator on (U, τ, E) satisfies the Kuratowski closure axioms so that there is one topology on U. This topology is defined as follows:

The set of all $FS\delta O$-sets of (U, τ, E) forms an FS-topology, denoted by τ_δ. It is called an $FS\delta$-topology on U, and the triplet (U, τ_δ, E) is called an $FS\delta$-topological space. Moreover, $\tau_\delta \sqsubseteq \tau$.

Definition 2.11. *[42] An FS-map $f_{up} : (U, \tau, E) \longrightarrow (V, \delta, K)$ is called:*

(i) *$FS\delta$-open if $f_{up}(h_E)$ is an $FS\delta O$-set in V for all $FS\delta O$-sets h_E in U.*

(ii) *$FS\delta$-closed if $f_{up}(g_E)$ is an $FS\delta C$-set in V for all $FS\delta C$-sets g_E in U.*

Theorem 2.2. *[42] Let $f_{up} : (U, \tau, E) \longrightarrow (V, \delta, K)$ be an FS-map, then the next items are equivalent:*

(i) *f_{up} is $FS\delta$-continuous.*

(ii) *$f^{-1}_{up}(g_K)$ is an $FS\delta O$-set in (U, τ, E) for all $FS\delta O$-sets g_K in (V, δ, K).*

(iii) *$f^{-1}_{up}(g_K)$ is an $FS\delta C$-set in (U, τ, E) for all $FS\delta C$-sets g_K in (V, δ, K).*

Definition 2.12. *[46] An $FSTS$ (U, τ, E) is said to be:*

(i) *FST_0 if for any $x^e_r, y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} y^e_t$, then $x^e_r \tilde{q} cl(y^e_t)$ or $cl(x^e_r)\tilde{q} y^e_t$.*

(ii) *FST_1 if for any $x^e_r, y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} y^e_t$, then $x^e_r \tilde{q} cl(y^e_t)$ and $cl(x^e_r)\tilde{q} y^e_t$.*

(iii) *FST_2 if for any $x^e_r, y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} y^e_t$, there are FSO-sets $O_{x^e_r}, O_{y^e_t} \in \tau$ such that $O_{x^e_r}\tilde{q} O_{y^e_t}$.*

Definition 2.13. *[44] For $FSTS(U, \tau, E)$ and $h_E \in FSS(U_E)$, then:*

(i) *A family $\mathcal{A} = \{l_{iE} : i \in J\}$ of $FS\delta O$-sets is called an $FS\delta^*$- open cover of h_E if for all $x^e_r \widetilde{\in} h_E$ there is $i_0 \in J$ such that $x^e_r \widetilde{\in} f_{i_0 E}$.*

(ii) *h_E is called an $FS\delta^*$-compact set if every $FS\delta^*$-open cover of h_E has a finite $FS\delta^*$-open subcover. In general, (U, τ, E) is $FS\delta^*$-compact if 1_E itself is $FS\delta^*$-compact.*

3. Fuzzy soft δ-separation axioms

Here, we are going to give the definitions of a new class of separation axioms called $FS\delta$-separation axioms (or FS-δT_i , $i = 0, 1, 2$) and study some their properties.

Definition 3.1. *An $FSTS$ (U, τ, E) is said to be:*

(i) *FS-δT_0 if for any $x^e_r, y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} y^e_t$, there is an $FS\delta O$-set $O_{x^e_r}$ such that $O_{x^e_r} \tilde{q} y^e_t$, or there is an $FS\delta O$-set $O_{y^e_t}$ such that $O_{y^e_t} \tilde{q} x^e_r$.*

(ii) *FS-δT_1 if for any $x^e_r, y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} y^e_t$, there are $FS\delta O$-sets $O_{x^e_r}, O_{y^e_t}$ such that $y^e_t \tilde{q} O_{x^e_r}$ and $x^e_r \tilde{q} O_{y^e_t}$.*

(iii) *FS-δT_2 if for any $x^e_r, y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} y^e_t$, there are $FS\delta O$-sets $O_{x^e_r}, O_{y^e_t}$ such that $O_{x^e_r} \tilde{q} O_{y^e_t}$.*

Lemma 3.1. *For $FSTS$ (U, τ, E), $x^e_r \in FSP(U_E)$, and $h_E \in FSS(U_E)$, then:*

(i) *$x^e_r \widetilde{\in} int_\delta(h_E) \Leftrightarrow$ there is an $FS\delta O$-set $O_{x^e_r}$ such that $O_{x^e_r} \sqsubseteq h_E$.*

(ii) *$x^e_r q cl_\delta(h_E) \Leftrightarrow O_{x^e_r} q h_E$ for any $FS\delta O$-set $O_{x^e_r}$ in (U, τ, E).*

(iii) *$g_E q h_E \Leftrightarrow g_E q cl_\delta(h_E)$ for any $FS\delta O$-set g_E in (U, τ, E).*

In the next results, we give some characterizations of FS-δT_i space, $i = 0, 1, 2$.

Theorem 3.1. *An $FSTS$ (U, τ, E) is FS-δT_0 if and only if for any x^e_r, $y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} y^e_t$ implies $x^e_r \tilde{q} cl_\delta(y^e_t)$ or $cl_\delta(x^e_r) \tilde{q} y^e_t$.*

Proof. Let (U, τ, E) be FS-δT_0 and $x^e_r \tilde{q} y^e_t$ for any $x^e_r, y^e_t \in FSP(U_E)$. Then there is an $FS\delta O$-set $O_{x^e_r}$ such that $y^e_t q O_{x^e_r}$ or there is an $FS\delta O$-set $O_{y^e_t}$ such that $x^e_r \tilde{q} O_{y^e_t}$. From (ii) of the above lemma, we get $x^e_r \tilde{q} cl_\delta(y^e_t)$ or $cl_\delta(x^e_r) \tilde{q} y^e_t$.

Conversely, let $x^e_r \tilde{q} y^e_t$. By given $x^e_\alpha \tilde{q} cl_\delta(y^e_t)$ or $cl_\delta(x^e_r) \tilde{q} y^e_\beta$, and again from (ii) of the above lemma, there is an $FS\delta O$-set $O_{x^e_r}$ such that $O_{x^e_r} \tilde{q} y^e_t$ or there is $O_{y^e_t}$ with $O_{y^e_t} \tilde{q} x^e_r$. Hence, (U, τ, E) is FS-δT_0.

Remark 3.1. *Clearly, every FS-δT_0 is FST_0. The converse is not necessarily true.*

Example 3.1. *Let $U = [0, 1]$ and $E = \{e\}$, then the class $\tau = \{0_E, 1_E\} \cup \{f_{iE} : i \in N\}$ is an FST on U, where*

$$f_{iE}(e)(u) = \begin{cases} \underline{1}, & u = 0, \\ 1 - \frac{1}{i}, & 0 < u \leq \frac{1}{i}, \\ \underline{1}, & \frac{1}{i} < u \leq 1. \end{cases}$$

One can check that (U, τ, E) is FST_0. On other hand, clearly $\{cl(h_E) : h_E \in \tau\} = \{g_E : g_E \in FSRC(U_E)\}$, and for any l_E in (U, τ, E), we have $cl_\delta(l_E) = \sqcap\{g_E \in FSRC(U_E) : l_E \sqsubseteq g_E\}$. Since $cl(f_{iE}) = 1_E$ for all $i \in N$, $cl(1_E) = 1_E$, and $cl(0_E) = 0_E$, then $FSRC(U_E) = \{0_E, 1_E\}$. Therefore, $FS\delta C(U_E) = \{0_E, 1_E\} = FS\delta O(U_E)$. Thus, (U, τ, E) is not FS-δT_0.

Theorem 3.2. *Let $f_{up} : (U, \tau, E) \longrightarrow (V, \delta, K)$ be one-to-one and $FS\delta$-continuous. If (V, δ, K) is FS-δT_0, then (U, τ, E) also is FS-δT_0.*

Proof. Let $x_r^e\tilde{q}y_t^e$ for any $x_r^e, y_t^e \in FSP(U_E)$. Since f_{up} is one-to-one, then $f_{up}(x_r^e)\tilde{q}f_{up}(y_t^e)$. Since (V,δ,K) is FS-δT_0, there is an $FS\delta O$-set $O_{f_{up}(x_r^e)}$ such that $f_{up}(y_t^e)\tilde{q}O_{f_{up}(x_r^e)}$, or there is an $FS\delta O$-set $O_{f_{up}(y_t^e)}$ such that $f_{up}(x_r^e)\tilde{q}O_{f_{up}(y_t^e)}$. Since f_{up} is $FS\delta$-continuous, we have $f_{up}^{-1}(O_{f_{up}(x_r^e)})$ as an $FS\delta O$-set in (U,τ,E) with $y_t^e\tilde{q}f_{up}^{-1}(O_{f_{up}(x_r^e)})$, or there is an $FS\delta O$-set $f_{up}^{-1}(O_{f_{up}(y_t^e)})$ in (U,τ,E) with $x_r^e\tilde{q}f_{up}^{-1}(O_{f_{up}(y_t^e)})$. Hence, (U,τ,E) is FS-δT_0.

Theorem 3.3. *An $FSTS$ (U,τ,E) is FS-δT_1 if and only if for any $x_r^e, y_t^e \in FSP(U_E)$ with $x_r^e\widetilde{\mathrm{q}}y_t^e$ implies $x_r^e\tilde{q}cl_\delta(y_t^e)$ and $cl_\delta(x_r^e)\tilde{q}y_t^e$.*

Proof. By a similar way to that in Theorem 3.3.

Theorem 3.4. *For an $FSTS$ (U,τ,E), the next items are equivalent:*

(i) *(U,τ,E) is FS-δT_1.*

(ii) *$cl_\delta(x_r^e) = x_r^e$ for all $x_r^e \in FSP(U_E)$.*

Proof. $(i) \Longrightarrow (ii)$. Let (U,τ,E) be FS-δT_1 and $x_r^e, y_t^e \in FSP(U_E)$ with $x_r^e\tilde{q}y_t^e$, then there is an $FS\delta O$-set $O_{y_t^e}$ such that $x_r^e\tilde{q}O_{y_t^e}$. This implies $O_{y_t^e} \sqsubseteq (x_r^e)^c$; thus, $(x_r^e)^c$ is an $FS\delta O$-set and is, x_r^e is an $FS\delta C$-set for all $x_\alpha^e \in FSP(U_E)$. Hence, $cl_\delta(x_r^e) = x_r^e$.
$(ii) \Longrightarrow (i)$. Let $x_r^e, y_t^e \in FSP(U_E)$ with $x_r^e\tilde{q}y_t^e$, then $x_\alpha^e \sqsubseteq (y_\beta^e)^c$ and $y_\beta^e \sqsubseteq (x_\alpha^e)^c$(since FS-points x_r^e, y_t^e are $FS\delta C$-sets). Now, take $O_{x_r^e} = (y_\beta^e)^c$ and $O_{y_t^e} = (x_\alpha^e)^c$. Thus, there are $FS\delta O$-sets $O_{x_\alpha^e}$ and $O_{y_t^e}$ such that $x_r^e\tilde{q}(x_r^e)^c = O_{y_t^e}$ and $y_t^e\tilde{q}(y_t^e)^c = O_{x_r^e}$. The result holds.

Theorem 3.5. *If (V,δ,K) is an FS-δT_1 and $f_{up} : (U,\tau,E) \longrightarrow (V,\delta,K)$ are one-to-one and $FS\delta$-continuous, then so is (U,τ,E).*

Proof. Let (V,δ,K) be FS-δT_1 and $x_r^e\tilde{q}y_t^e$ for any x_r^e, $y_t^e \in FSS(U_E)$. Since f_{up} is one-to-one, we have $f_{up}(x_r^e)\tilde{q}f_{up}(y_t^e)$. Since (V,δ,K) is FS-δT_1, there are $FS\delta O$-sets $O_{f_{up}(x_r^e)}$, $O_{f_{up}(y_t^e)} \in \delta$ such that $f_{up}(y_t^e)\tilde{q}O_{f_{up}(x_r^e)}$ and $f_{up}(x_r^e)\tilde{q}O_{f_{up}(y_t^e)}$. Since f_{up} is $FS\delta$-continuous, we have $f_{up}^{-1}(O_{f_{up}(x_r^e)})$ and $f_{up}^{-1}(O_{f_{up}(y_t^e)})$ as $FS\delta O$-sets in (U,τ,E) with $y_t^e\tilde{q}f_{up}^{-1}(O_{f_{up}(x_r^e)})$ and $x_r^e\tilde{q}f_{up}^{-1}(O_{f_{up}(y_t^e)})$. Hence, (U,τ,E) is FS-δT_1 .

Theorem 3.6. *If $FSTS(U,\tau,E)$ is FS-δT_2, then $x_r^e = \sqcap\{cl_\delta(h_E) : x_r^e\widetilde{\in}h_E\}$.*

Proof. Let (U,τ,E) be FS-δT_2 and $x_r^e \in FSP(U_E)$, then for any $x_r^e\tilde{q}y_t^e$, there are $FS\delta O$-sets $h_E = O_{x_r^e}$ and $O_{y_t^e}$ such that $h_E\tilde{q}O_{y_t^e}$. From (ii) of Lemma 3.2, we have $y_t^e\tilde{q}cl_\delta(h_E)$ and $y_t^e\tilde{q} \sqcap \{cl_\delta(h_E) : x_r^e\widetilde{\in}h_E\}$. From (v) of Proposition 2.2, we have $\sqcap\{cl_\delta(h_E) : x_r^e\widetilde{\in}h_E\} \sqsubseteq x_r^e$, but $x_r^e\widetilde{\in} \sqcap \{cl_\delta(h_E) : x_r^e\widetilde{\in}h_E\}$. The result holds.

Proposition 3.1. *Every FS-δT_i is FS-δT_{i-1} , $i = 1, 2$.*

Proof. It is obvious.

The next example shows that the converse of the above proposition is not necessarily true.

Example 3.2. *Let $U = \{x,y\}$, $E = \{e\}$, and $\tau = \{0_E , 1_E\} \cup \{x_r^e : r \in (0,1]\}$, then τ is an FST on U. It is easy to check that (U,τ,E) is FS-δT_0. Indeed, all members in τ are $FSRO$-sets, so they are $FS\delta O$-sets. For any x_r^e, y_t^e with $x_r^e\tilde{q}y_t^e$, there is an $FS\delta O$-set $O_{x_r^e} = x_r^e$ such that $O_{x_r^e} = x_r^e\tilde{q}y_t^e$. On the other hand, the unique $FS\delta O$-set containing y_t^e is 1_E, but $1_E q x_r^e$. Therefore, (U,τ,E) is not FS-δT_1.*

Theorem 3.7. *Let (U,τ,E) be FS-δT_1 and h_E be any $FS\delta O$-set. If h_E^c is also $FS\delta O$-set in (U,τ,E), then (U,τ,E) is FS-δT_2.*

Proof. Let $x_r^e\tilde{q}y_t^e$ for any x_r^e,y_t^e∈$FSP(U_E)$. Since (U,τ,E) is FS-δT_1, there is an $FS\delta O$-set $O_{x_r^e}$ such that $O_{x_r^e}\tilde{q}y_t^e$, or there is an $FS\delta O$-set $O_{y_t^e}$ such that $O_{y_t^e}\tilde{q}x_r^e$. Let us assume that $O_{x_r^e}\tilde{q}y_t^e$, then $y_t^e \sqsubseteq (O_{x_r^e})^c$, which is an $FS\delta O$-set by assumption, and $O_{x_r^e}\tilde{q}(O_{x_r^e})^c$. This completes the proof.

Proposition 3.2. *If every crisp FS-point x_1^e is $FS\delta O$-set in (U,τ,E), then (U,τ,E) is FS-δT_2.*

Proof. It is obvious.

Theorem 3.8. *If (V,δ,K) is FS-δT_2 and $f_{up}:(U,\tau,E)\longrightarrow(V,\delta,K)$ is one-to-one and $FS\delta$-continuous, then (U,τ,E) is FS-δT_2.*

Proof. It follows by using a similar way to that in Theorem 3.9.

Theorem 3.9. *Every FS-subspace (Y,τ_Y,E) of FS-$\delta T_i(U,\tau,E)$ is FS-δT_i, $i=0,1,2$.*

Proof. As a sample, we prove the case $i=1$. The proof of the rest of the cases is similar. Let x_r^e, y_t^e∈$FSP(Y_E)$ with $x_r^e\widetilde{q}y_t^e$, then also x_r^e , y_t^e ∈$FSP(U_E)$ with $x_\alpha^e\tilde{q}y_\beta^e$. Since (U,τ,E) is FS-δT_1, there is $FS\delta O$-sets $O_{x_r^e}$, $O_{y_t^e}$ such that $y_t^e\tilde{q}O_{x_r^e}$ and $x_r^e\tilde{q}O_{y_t^e}$. Thus, $O_{x_r^e}\sqcap h_E^Y$ and $O_{y_\beta^e}\sqcap h_E^Y$ are $FS\delta O$-sets in (Y,τ_Y,E). Take $O^*_{x_r^e}=O_{x_r^e}\sqcap h_E^Y$ and $O^*_{y_t^e}=O_{y_t^e}\sqcap h_E^Y$, then $y_t^e\tilde{q}O^*_{x_r^e}$ and $x_r^e\widetilde{q}O^*_{y_t^e}$. Hence, the result holds.

4. Fuzzy soft δ-regularity axioms

Here, we introduce the definitions of a new class of regularity axioms, namely, $FS\delta$-regularity axioms (or FS-δR_i, $i=0,1,2,3$), and investigate some its properties.

Definition 4.1. *An $FSTS$ (U,τ,E) is said to be:*

(i) *FS-δR_0 if for any x_r^e, $y_t^e\in FSP(U_E)$ with $x_r^e\tilde{q}cl_\delta(y_t^e)$ implies $cl_\delta(x_r^e)\tilde{q}y_t^e$.*

(ii) *FS-δR_1 if for any x_r^e, $y_t^e\in FSP(U_E)$ with $x_r^e\tilde{q}cl_\delta(y_t^e)$, there are $FS\delta O$-sets $O_{x_r^e}$ and $O_{y_t^e}$ such that $O_{x_r^e}\widetilde{q}O_{y_t^e}$.*

In the next results, some descriptions of FS-δR_i spaces for $i=0,1$ are investigated.

Theorem 4.1. *In an $FSTS$ (U,τ,E), the next items are equivalent:*

(i) *(U,τ,E) is FS -δR_0 .*

(ii) *$cl_\delta(x_r^e)\sqsubseteq O_{x_r^e}$ for any $FS\delta O$-set $O_{x_r^e}$.*

(iii) *$cl_\delta(x_r^e)\sqsubseteq\sqcap\left\{O_{x_r^e}:O_{x_r^e}\in FS\delta OS(U_E)\right\}$ for all $x_r^e\in FSP(U_E)$.*

Proof. (i)⟹(ii) Let (U,τ,E) be FS-δR_0 and $y_t^eqcl_\delta(x_r^e)$, then $x_r^eqcl_\delta(y_t^e)$. From (ii) of Lemma 3.2, we have $y_t^eqO_{x_r^e}$, and by (v) of Proposition 2.2, we get $cl_\delta(x_r^e)\sqsubseteq O_{x_r^e}$ for any $FS\delta O$-set $O_{x_r^e}$. The result holds.
(ii)⟹ (iii) It is clear.
(iii)⟹(i) Let $cl_\delta(x_r^e)\sqsubseteq\sqcap\left\{O_{x_r^e}:O_{x_r^e}\in FS\delta O(U_E)\right\}\sqsubseteq O_{x_r^e}$ for any $O_{x_r^e}$ and let $x_r^e,y_t^e\in FSP(U_E)$ with $x_r^e\tilde{q}cl_\delta(y_t^e)$, then $x_\alpha^e\in[cl_\delta(y_t^e)]^c=O_{x_r^e}$, which is an $FS\delta O$-set containing x_r^e. So by hypothesis, $cl_\delta(x_r^e)\sqsubseteq O_{x_r^e}=[cl_\delta(y_t^e)]^c=int_\delta\left[(y_t^e)^c\right]\sqsubseteq(y_t^e)^c$. Thus, $cl_\delta(x_r^e)\tilde{q}y_t^e$. Hence, (U,τ,E) is FS-δR_0.

Theorem 4.2. *An $FSTS$ (U,τ,E) is FS-δR_0 if and only if h_E is an $FS\delta C$-set with $x^e_\alpha \tilde{q} h_E$, and there is an $FS\delta O$-set O_{h_E} containing h_E such that $x^e_\alpha \tilde{q} O_{h_E}$.*

Proof. Let (U,τ,E) be FS-δR_0 and $h_E \in FS\delta C(U_E)$ with $x^e_r \tilde{q} h_E$, then $x^e_r \in h^c_E = O_{x^e_r}$. From (ii) of Theorem 4.2, we have $cl_\delta(x^e_r) \sqsubseteq h^c_E = O_{x^e_\alpha}$ and $h_E \sqsubseteq [cl_\delta(x^e_r)]^c = O_{h_E}$. Since $x^e_r \sqsubseteq cl_\delta(x^e_r)$, then $[cl_\delta(x^e_r)]^c \sqsubseteq (x^e_r)^c$. Therefore, $x^e_r \tilde{q}[cl_\delta(x^e_r)]^c = O_{h_E}$. The result holds.
The converse part is obvious.

Theorem 4.3. *In an $FSTS$ (U,τ,E), the next properties are equivalent:*

(i) *(U,τ,E) is FS-δR_0 .*

(ii) *If g_E is $FS\delta C$-set with $x^e_r \tilde{q} g_E$, then $cl_\delta(x^e_r)\tilde{q} g_E$.*

(iii) *If $x^e_r \tilde{q} cl_\delta(y^e_t)$, then $cl_\delta(x^e_r)\tilde{q} cl_\delta(y^e_t)$.*

Proof. (i)$\Longrightarrow$(ii) Let g_E be an $FS\delta C$-set with $x^e_r \tilde{q} g_E$. Since (U,τ,E) is FS-δR_0, then by the above theorem there is an $FS\delta O$-set O_{g_E} such that $x^e_r \tilde{q} O_{g_E}$. From (ii) of Lemma 3.2, we have $cl_\delta(x^e_r)\tilde{q} g_E$.
(ii)$\Longrightarrow$ (iii) It is obvious.
(iii)$\Longrightarrow$ (i) Let $x^e_r, y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} cl_\delta\,(y^e_t)$. By given $cl_\delta(x^e_r)\tilde{q} cl_\delta(y^e_t)$, since $y^e_t \sqsubseteq cl_\delta(y^e_t)$, we have $cl_\delta(x^e_r)\tilde{q} y^e_t$. Thus (U,τ,E) is FS-δR_0.

Proposition 4.1. *An $FSTS$ (U,τ,E) is FS-δR_1 if and only if for any x^e_r , $y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} cl_\delta(y^e_t)$, there are $FS\delta O$-sets $O_{cl_\delta(x^e_r)}$ and $O_{cl_\delta(y^e_t)}$ such that $O_{cl_\delta(x^e_r)}\tilde{q} O_{cl_\delta(y^e_t)}$.*

Proof. It follows from that of the above theorem and from (ii) of Theorem 4.2.

Theorem 4.4. *Every FS-subspace (Y,τ_Y,E) of FS-δR_i is FS-δR_i, $i = 0,1$.*

Proof. As a sample, we prove the case $i = 1$. The proof of the rest case is similar.

Let x^e_r , y^e_t be FS-points in (Y_E) with $x^e_r \tilde{q} cl_\delta\,(y^e_t)$, then also x^e_r , $y^e_t \in FSP(U_E)$ and $x^e_r \widetilde{\mathrm{q}} cl_\delta\,(y^e_t)$. Since (U,τ,E) is FS-δR_1, there are $FS\delta O$-sets $O_{x^e_r}, O_{y^e_t}$ such that $O_{x^e_r}\tilde{q} O_{y^e_t}$. Take $O^*_{x^e_r} = O_{x^e_r} \sqcap h^Y_E$ and $O^*_{y^e_t} = O_{y^e_t} \sqcap h^Y_E$, then $O^*_{x^e_r}, O^*_{y^e_t}$ are $FS\delta O$-sets in (Y,τ_Y,E) and $O^*_{x^e_r}\widetilde{\mathrm{q}} O^*_{y^e_t}$. Hence, $(Y,\delta_Y E)$ is FS-δR_1.

Proposition 4.2. *For $FSTS$ (U,τ,E), every FS-δT_i is FS-δR_{i-1}, $i = 1,2$.*

Proof. It is obvious.

The next example shows that the converse of the above proposition is not necessarily true.

Example 4.1. *Let $U\{u\}$ and $E = \{e_1,e_2\}$. The family $\tau = \{0_{\mathrm{E}}, 1_{\mathrm{E}}, h_E = \{(e_1,u_{0.5}),(e_2,u_{0.5})\}$ is an FST on U. One can check that (U,τ,E) is FS-δR_0, but is not FS-δT_0. Indeed, for $x^{e_1}_{0.7}\tilde{q} x^{e_1}_{0.2}$, the unique $FS\delta O$-set containing $u^{\mathrm{e}_1}_{0.7}$ is 1_{E}, but $1_{\mathrm{E}}\mathrm{q}u^{\mathrm{e}_1}_{0.2}$.*

Theorem 4.5. *An $FSTS$ (U,τ,E) is FS-δT_i if and only if it is both FS-δT_{i-1} and FS-δR_{i-1}, $i = 1,2$.*

Proof. As a sample, we prove the case $i = 2$. The proof of the rest case is similar. Necessity follows from the Proposition 3.11 and 4.7.

Conversely, let (U,τ,E) be FS-δT_1 and FS-δR_1, and let $x^e_r, y^e_t \in FSP(U_E)$ with $x^e_r \tilde{q} y^e_t$. By Theorem 3.7, we have $x^e_r \tilde{q} cl(y^e_t)$. Since (U,τ,E) is FS-δR_1, there are $FS\delta O$-sets $O_{x^e_r}, O_{y^e_t}$ such that $O_{x^e_r}\tilde{q} O_{y^e_t}$. Therefore, (U,τ,E) is FS-δT_2.

Definition 4.2. *An $FSTS$ (U, τ, E) is said to be:*

(i) *$FS\delta$-regular(or FS-δR_2) if for any $FS\delta C$-set h_E and any FS-point x_r^e with $x_r^e \tilde{q} h_E$, there are $FS\delta O$-sets $O_{x_r^e}$ and O_{h_E} such that $O_{x_r^e} \tilde{q} O_{h_E}$.*

(ii) *$FS\delta$-normal(or FS-δR_3) if for any $FS\delta C$-sets h_E and g_E with $h_E \tilde{q} g_E$, there are $FS\delta O$-sets O_{h_E} and O_{g_E} such that $O_{h_E} \tilde{q} O_{g_E}$.*

(iii) *FS-δT_3(resp., FS-δT_4) if it is FS-δR_2(resp., FS- δR_3) and FS-δT_1.*

Theorem 4.6. *For an $FSTS$ (U, τ, E), the next items are equivalent:*

(i) *(U, τ, E) is FS-δR_2.*

(ii) *For any $x_r^e \in FSP(U_E)$ and any $FS\delta O$-set $O_{x_r^e}$, there is an $FS\delta O$-set $O^*_{x_r^e}$ containing x_r^e such that $cl_\delta(O^*_{x_r^e}) \sqsubseteq O_{x_r^e}$.*

Proof. $(i) \implies (ii)$ Let $x_r^e \in FSP(U_E)$ and $O_{x_r^e}$ be any $FS\delta O$-set containing x_r^e, then $O^c_{x_r^e} = h_E$ is an $FS\delta C$-set. Clearly, $O_{x_r^e} \tilde{q} O^c_{x_r^e}$ and$x_r^e \tilde{q} O^c_{x_r^e}$. Since (U, τ, E) is FS-δR_2, there are $FS\delta O$-sets $O^*_{x_r^e}, O_{O^c_{x_r^e}}$ such that $O^*_{x_r^e} \tilde{q} O_{O^c_{x_r^e}} = O_{h_E}$, then $O^*_{x_r^e} \sqsubseteq O^c_{h_E}$ and $cl_\delta(O^*_{x_r^e}) \sqsubseteq O^c_{h_E}$. Clearly, $O^c_{x_r^e} \sqsubseteq O_{O^c_{x_r^e}} = O_{h_E}$, so $O^c_{h_E} \sqsubseteq O_{x_r^e}$. Therefore, $cl_\delta(O^*_{x_r^e}) \sqsubseteq O_{x_r^e}$.
$(ii) \implies (i)$ Let $x_r^e \in FSP(U_E)$ and g_E be any $FS\delta C$-set with $x_r^e \tilde{q} g_E$, then $x_r^e \widetilde{\in} g^c_E = O_{x_r^e}$ which is an $FS\delta O$-set containing x_r^e. So there is an $FS\delta O$-set $O^*_{x_r^e}$ such that $cl_\delta(O^*_{x_r^e}) \sqsubseteq O_{x_r^e} = g^c_E$, which implies $g_E \sqsubseteq [cl_\delta(O^*_{x_r^e})]^c = O_{g_E}$. Clearly, $cl_\delta(O^*_{x_r^e}) \tilde{q} [cl_\delta(O^*_{x_r^e})]^c = O_{g_E}$ and $O^*_{x_r^e} \tilde{q} O_{g_E}$. Thus, the result holds.

Theorem 4.7. *An $FSTS$ (U, τ, E) is FS-δR_2 if and only if for any $FS\delta C$-set h_E with $x_r^e \widetilde{q} h_E$, there are $FS\delta O$-sets $O_{x_r^e}, O_{h_E}$ such that $cl(O_{x_r^e}) \widetilde{q} cl(O_{h_E})$.*

Proof. Let $x_r^e \in FSP(U_E)$ and h_E be an $FS\delta C$-set with $x_r^e \tilde{q} h_E$. Since (U, τ, E) is FS-δR_2, there are $FS\delta O$-sets $O^*_{x_r^e}, O_{h_E}$ such that $O_{h_E} \tilde{q} O^*_{x_r^e}$. From (iii) of Lemma 3.2, we obtain $cl(O_{h_E}) \tilde{q} O^*_{x_r^e}$, that is, $cl(O_{h_E}) \tilde{q} x_r^e$. Again (U, τ, E) is FS-δR_2, and there are $FS\delta O$-sets $O^{**}_{x_r^e}, O_{cl(O_{h_E})}$ such that $O^{**}_{x_r^e} \tilde{q} O_{cl(O_{h_E})}$. By (iii) of Lemma 3.2, we have $cl(O^{**}_{x_r^e}) \tilde{q} O_{cl(O_{h_E})}$. Now, put $O_{x_r^e} = O^*_{x_r^e} \sqcap O^{**}_{x_r^e}$. Since (U, τ, E) is FS-δR_2 and $O^*_{x_r^e}$ is an $FS\delta O$-set, then by the above theorem, there is an $FS\delta O$-set $O_{x_r^e}$ such that $cl_\delta(O_{x_r^e}) \sqsubseteq O^*_{x_r^e}$ that is, $cl(O_{x_r^e}) \sqsubseteq O^*_{x_r^e}$. Since $cl(O_{h_E}) \tilde{q} O^*_{x_r^e}$, then $cl(O_{h_E}) \tilde{q} cl(O_{x_r^e})$.

Conversely, it follows from hypothesis.

Theorem 4.8. *For an $FSTS$ (U, τ, E), the next items are equivalent:*

(i) *(U, τ, E) is FS-δR_3.*

(ii) *For any $FS\delta C$-set h_E and any $FS\delta O$-set O_{h_E}, there is an $FS\delta O$-set $O^*_{h_E}$ containing h_E such that $cl_\delta(O^*_{h_E}) \sqsubseteq O_{h_E}$.*

Proof. Let (U, τ, E) be FS-δR_3, h_E be an $FS\delta C$-set, and O_{h_E} be any $FS\delta O$-set containing h_E, then $O^c_{h_E}$ is an $FS\delta C$-set. Since $O_{h_E} \widetilde{q} O^c_{h_E}$, that is, $h_E \widetilde{q} O^c_{h_E}$, (U, τ, E) is FS-δR_3, there are $FS\delta O$-sets $O^*_{h_E}, O_{O^c_{h_E}}$ such that $O^*_{h_E} \widetilde{q} O_{O^c_{h_E}}$, then $O^*_{h_E} \sqsubseteq (O_{O^c_{h_E}})^c$ and $cl_\delta(O^*_{h_E}) \sqsubseteq (O_{O^c_{h_E}})^c$. Since $O^c_{h_E} \sqsubseteq O_{O^c_{h_E}}$, then $(O_{O^c_{h_E}})^c \sqsubseteq O_{h_E}$ and $cl_\delta(O^*_{h_E}) \sqsubseteq (O_{O^c_{h_E}})^c \sqsubseteq O_{h_E}$. The result holds.

Conversely, let f_E, g_E be two $FS\delta C$-sets with $f_E \widetilde{q} g_E$, then $f_E \sqsubseteq g^c_E = O_{f_E}$ which is an $FS\delta O$-set containing f_E. By hypothesis, there is an $FS\delta O$-set $O^*_{f_E}$ such that $cl_\delta(O^*_{f_E}) \sqsubseteq g^c_E = O_{f_E}$, then $g_E \sqsubseteq [cl_\delta(O^*_{f_E})]^c = O_{g_E}$. Since $cl_\delta(O^*_{f_E}) \widetilde{q} [cl_\delta(O^*_{f_E})]^c = O_{g_E}$, then $O_{g_E} \widetilde{q} O^*_{f_E}$. The result holds.

Theorem 4.9. *An $FSTS$ (U,τ,E) is FS-δR_3 if and only if for any two $FS\delta C$-sets h_E, g_E with $h_E\tilde{q}g_E$, there are $FS\delta O$-sets O_{h_E}, O_{g_E} such that $cl(O_{h_E})\tilde{q}cl(O_{g_E})$.*

Proof. It is analogous to that in Theorem 4.12.

5. More characterizations and relations

Saleh et. al [44] introduced and studied a new type of FS-compactness, namely, $FS\delta^*$-compactness. In this section, we study more properties and investigate the relations between $FS\delta^*$-compact and $FS\delta$-separation axioms, which are introduced in this work.

To begin we show that the axioms FS-δR_i, $i = 1,2,3$ and FS-δT_i, $i = 1,2,3,4$ are harmonic.

Theorem 5.1. *For $FSTS$ (U,τ,E), we have:*

$$FS-\delta R_3 \wedge FS-\delta R_0 \Rightarrow FS-\delta R_2 \Rightarrow FS-\delta R_1 \Rightarrow FS-\delta R_0.$$

Proof. Let (U,τ,E) be FS-δR_3, FS-$\delta R_0, x_r^e \in FSP(U_E)$ for any $FS\delta C$-set f_E with $x_r^e\tilde{q}h_E$. Since (U,τ,E) is FS-δR_0, then $cl_\delta(x_r^e)\tilde{q}f_E$ where $cl_\delta(x_r^e), h_E$ are $FS\delta C$-sets. Again, (U,τ,E) is FS-δR_3, so there are $FS\delta O$-sets $O_{cl_\delta(x_r^e)}$, O_{h_E} such that $O_{cl_\delta(x_r^e)}\tilde{q}O_{h_E}$. Put $O_{cl_\delta(x_r^e)}=O_{x_r^e}$, and we have $O_{x_r^e}\tilde{q}O_{h_E}$. Thus, (U,τ,E) is FS-δR_2. The proof for the rest of the cases is obvious.

Theorem 5.2. *For an $FSTS$ (U,τ,E), we have:*

$$FS-\delta T_4 \Rightarrow FS-\delta T_3 \Rightarrow FS-\delta T_2 \Rightarrow FS-\delta T_1 \Rightarrow FS-\delta T_0$$

Proof. Let (U,τ,E) be FS-δT_4, then it is both FS-δR_3 and FS-δT_1. From Proposition 4.7, we have (U,τ,E) is FSR_0. Let us assume that $x_r^e \in FSP(U_E), h_E$ is an $FS\delta C$-set with $x_r^e\tilde{q}h_E$, then by Theorem 4.4, $cl_\delta(x_r^e)\tilde{q}h_E$, where $cl_\delta(x_r^e), h_E$ are $FS\delta C$-sets. Since (U,τ,E) is FS-δR_3, there are $FS\delta O$-sets $O_{cl_\delta(x_r^e)}, O_{h_E}$ such that $O_{cl_\delta(x_r^e)}\tilde{q}O_{h_E}$. Take $O_{cl_\delta(x_r^e)} = O_{x_r^e}$, and we have $O_{x_r^e}\tilde{q}O_{h_E}$. Thus, (U,τ,E) is FS-δR_2. Hence, we obtain the result.

The proof of the rest of the cases follows from the above theorem and Proposition 3.11.

From the above theorems, Definition 4.10, and Proposition 4.7, we obtain the next result.

Corollary 5.1. *For an $FSTS$ (U,τ,E), the next implications hold.*

$$\begin{array}{ccccccccc}
FS-\delta T_4 & \Longrightarrow & FS-\delta T_3 & \Longrightarrow & FS-\delta T_2 & \Longrightarrow & FS-\delta T_1 & \Longrightarrow & FS-\delta T_0 \\
\Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\
FS-\delta R_3 \wedge FS-\delta R_0 & \Rightarrow & FS-\delta R_2 & \Rightarrow & FS-\delta R_1 & \Rightarrow & FS-\delta R_0. & &
\end{array}$$

Theorem 5.3. *Let (U,τ,E) be FS-δT_3 and g_E be an $FS\delta^*$-compact set, then for any $FS\delta C$-set h_E with $h_E\widetilde{q}g_E$, there are $FS\delta O$-sets O_{h_E}, O_{g_E} such that $O_{h_E}\tilde{q}O_{g_E}$.*

Proof. Let (U,τ,E) be FS-δT_3 and g_E be an $FS\delta^*$-compact set, then for any $FS\delta C$-set h_E with $h_E\tilde{q}g_E$, we have for any $y_t^e\widetilde{\in}g_E$, there are $FS\delta O$-sets $O_{y_t^e}, O_{h_E}$ such that $O_{y_t^e}\tilde{q}O_{h_E}$. Clearly, $\{O_{y_t^e} : y_t^e\widetilde{\in}g_E\}$ is $FS\delta^*$-open cover of g_E. Since g_E is $FS\delta^*$-compact, there is a finite $FS\delta^*$-open subcover of g_E, say, $\{O^i_{y_t^e} : i = 1,2,\ldots, n\}$. One can verify that $O_{g_E} = \sqcup_{i=1}^n O^i_{y_t^e}$ and $O_{h_E} = \sqcap_{i=1}^n O^i_{h_E}$ have the required property.

Theorem 5.4. *Let (U, τ, E) be FS-δT_2, $x_r^e \in FSP(U_E)$ and g_E be an $FS\delta^*$-compact set with $x_r^e \tilde{q} g_E$, then there are $FS\delta O$-sets $O_{x_r^e}$ and O_{g_E} such that $O_{x_r^e} \tilde{q} O_{g_E}$.*

Moreover, if h_E, g_E are $FS\delta^*$-compact sets with $h_E \tilde{q} g_E$, then there are $FS\delta O$-sets O_{h_E}, O_{g_E} such that $O_{h_E} \tilde{q} O_{g_E}$.

Proof. It follows by a similar way to that in the above theorem.

Theorem 5.5. *Every $FS\delta^*$-compact set in an FS-δT_2* space *is an $FS\delta C$-set.*

Proof. Let (U, τ, E) be FS-δT_2 and g_E be an $FS\delta^*$-compact set. From the above theorem for any $x_r^e \in FSP(U_E)$ with $x_r^e \tilde{q} g_E$, there is $FS\delta O$-set $O_{x_r^e}$ such that $O_{x_r^e} \tilde{q} g_E$; that is, for any $x_r^e \widetilde{\in} g_E^c$, there is $FS\delta O$-set $O_{x_r^e}$ such that $O_{x_r^e} \widetilde{\in} g_E^c$. Therefore, g_E^c is an $FS\delta O$-set in (U, τ, E). Thus, g_E is an $FS\delta C$-set.

Theorem 5.6. *Let (U, τ, E) be FS-δR_1, then (U, τ, E) is FS-δT_2 if and only if every $FS\delta^*$-compact set is an $FS\delta C$-set.*

Proof. The necessary parts follows directly from the above theorem. Conversely, if any $FS\delta^*$-compact set is an $FS\delta C$-set, then (U, τ, E) is an FS-δT_1 space. Since (U, τ, E) is FS-δR_1 and FS-δT_1, then by Theorem 4.9, we obtain that (U, τ, E) is FS-δT_2.

Theorem 5.7. *For $FSTS\,(U, \tau, E)$, every $FS\delta^*$-compact FS-δR_1 space is FS-δR_2 (FS-δR_3).*

Proof. Let (U, τ, E) be an $FS\delta^*$-compact, FS-δR_1 space and let h_E be an $FS\delta C$-set with $x_r^e \tilde{q} h_E$, then for any FS-point $y_t^e \widetilde{\in} h_E$, we have $x_r^e \tilde{q} cl_\delta(y_t^e)$. Since (U, τ, E) is FS-δR_1, there are $FS\delta O$-sets $O_{x_r^e}$, $O_{y_t^e}$ such that $O_{x_r^e} \tilde{q} O_{y_t^e}$ so that the family $\{O_{y_t^e} : y_t^e \widetilde{\in} h_E\}$ is an $FS\delta^*$-open cover of h_E. Since (U, τ, E) is $FS\delta^*$-compact, h_E is $FS\delta^*$-compact and there is a finite $FS\delta^*$-open subcover of h_E, say, $\{O^i_{y_t^e} : y_t^e \widetilde{\in} h_E\ ,\ i = 1, 2, \ldots, n\}$. Take $O^*_{x_r^e} = \sqcap_{i=1}^n O^i_{x_r^e}$ and $O_{h_E} = \sqcup_{i=1}^n O^i_{y_t^e}$, then $O^*_{x_r^e}$, O_{h_E} are $FS\delta O$-sets with $O^*_{x_r^e} \tilde{q} O_{h_E}$. The result holds.

The proof of the rest case is analogous.

Corollary 5.2. *For $FS\delta^*$-compact space (U, τ, E), the next items are equivalent:*

(i) *(U, τ, E) is FS-δR_1.*

(ii) *(U, τ, E) is FS-δR_2.*

(iii) *(U, τ, E) is FS-δR_0 and FS-δR_3.*

Proof. It is obvious.

Theorem 5.8. *(U, τ, E) is FS-$\delta T_i \Longleftrightarrow (U, \tau_\delta, E)$ is FST_i, $i = 0, 1, 2$.*

Proof. It follows directly from Result 3 and Definition 3.1.

6. Concluding remarks and future work

It is well known that separation axioms provide some categories for topological spaces and help to prove some interesting properties of compactness and connectedness. Therefore, we have written this article to shed light on the properties of separability in the framework of fuzzy soft topologies.

We have defined and studied a new set of separation properties in fuzzy soft topological spaces, namely, $FS\delta$-separation and regularity properties via $FS\delta O$-sets by using quasi-coincident relation for FS-points. Several basic desirable properties, relations, and results have been obtained with some necessary examples. The relationships between $FS\delta^*$-compact spaces and $FS\delta$-separation have been investigated as well. We have shown that the implications $FS\text{-}\delta T_4 \Rightarrow FS\text{-}\delta T_3 \Rightarrow FS\text{-}\delta T_2 \Rightarrow FS\text{-}\delta T_1 \Rightarrow FS\text{-}\delta T_0$ hold true, but we cannot get examples to show that the converse in these implications may not be true in general, except the case $FS\text{-}\delta T_0 \nRightarrow FS\text{-}\delta T_1$.

By and large, the results obtained in the manuscript frame "fuzzy soft topology" represent a wider view than that inspired by the frameworks of fuzzy and soft topologies, since these frames are created by replacing the membership function with the characteristic function in the case of fuzzy topology and restricting the set of parameters by a singleton set in the case of soft topology. The present results elucidate that the perspective on the theory of separation axioms adopted in this paper is very useful and will open up the door for further contributions. We plan in upcoming studies to generate fuzzy soft topologies by hybridizing F-set with the recent types of F-set like complemental fuzzy sets [4], $(2,1)$-fuzzy sets [10], (m,n)-fuzzy sets [15], n^{th} power root fuzzy sets [14, 27], and k_m^n-Rung picture fuzzy sets [28]. One may examine the current concepts and the previous ones in these hybridizations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

References

1. A. M. Abd El-latif, Fuzzy soft separation axioms based on fuzzy β-open soft sets, *Ann. Fuzzy Math. Inform.*, **11** (2016), 223–239.
2. A. M. Abd El-latif, Some fuzzy soft topological properties based on fuzzy b-open soft sets, *J. Indian Math. Soc.*, **83** (2016), 251–267. https://www.informaticsjournals.com/index.php/jims/article/view/6608
3. J. C. R. Alcantud, The relationship between fuzzy soft and soft topologies, *Int. J. Fuzzy Syst.*, **24** (2022), 1653–1668. https://doi.org/10.1007/s40815-021-01225-4

4. J. C. R. Alcantud, Complemental fuzzy sets: A semantic justification of q-rung orthopair fuzzy sets, *IEEE T. Fuzzy Syst.*, **31** (2023), 4262–4270, https://doi.org/10.1109/TFUZZ.2023.3280221
5. B. Ahmad, A. Kharal, On fuzzy soft sets, *Adv. Fuzzy Syst.*, **2009** (2009), 586507. https://doi.org/10.1155/2009/586507
6. A. Atay, Disjoint union of fuzzy soft topological spaces, *AIMS Math.*, **8** (2023), 10547–10557. https://doi.org/10.3934/math.2023535
7. T. M. Al-shami, Comments on some results related to soft separation axioms, *Afr. Mat.*, **31** (2020), 1105–1119. https://doi.org/10.1007/s13370-020-00783-4
8. T. M. Al-shami, On soft separation axioms and their applications on decision-making problem, *Math. Probl. Eng.*, **2021** (2021), 8876978. https://doi.org/10.1155/2021/8876978
9. T. M. Al-shami, Soft somewhat open sets: Soft separation axioms and medical application to nutrition, *Comput. Appl. Math.*, **41** (2022), 216. https://doi.org/10.1007/s40314-022-01919-x
10. T. M. Al-shami, (2,1)-Fuzzy sets: Properties, weighted aggregated operators and their applications to multi-criteria decision-making methods, *Complex Intell. Syst.*, **9** (2023), 1687–1705. https://doi.org/10.1007/s40747-022-00878-4
11. T. M. Al-shami, J. C. R. Alcantud, A. Mhemdi, New generalization of fuzzy soft sets: (a,b)-Fuzzy soft sets, *AIMS Math.*, **8** (2023), 2995–3025. https://doi.org/10.3934/math.2023155
12. T. M. Al-shami, J. C. R. Alcantud, A. A. Azzam, Two new families of supra-soft topological spaces defined by separation axioms, *Mathematics*, **10** (2022), 4488. https://doi.org/10.3390/math10234488
13. T. M. Al-shami, M. E. El-Shafei, On supra soft topological ordered spaces, *Arab J. Basic Appl. Sci.*, **26** (2019), 433–445. https://doi.org/10.1080/25765299.2019.1664101
14. T. M. Al-shami, H. Z. Ibrahim, A. Mhemdi, Radwan Abu-Gdairi, n^{th} power root fuzzy sets and its topology, *Int. J. Fuzzy Log. Inte.*, **22** (2022), 350–365. http://doi.org/10.5391/IJFIS.2022.22.4.350
15. T. M. Al-shami, A. Mhemdi, Generalized frame for orthopair fuzzy sets: (m,n)-Fuzzy sets and their applications to multi-criteria decision-making methods, *Information*, **14** (2023), 56. https://doi.org/10.3390/info14010056
16. T. M. Al-shami, A. Mhemdi, R. Abu-Gdairi, M. E. El-Shafei, Compactness and connectedness via the class of soft somewhat open sets, *AIMS math.*, **8** (2023), 815–840. https://doi.org/10.3934/math.2023040
17. T. M. Al-shami, A. Mhemdi, A. Rawshdeh, H. Al-jarrah, Soft version of compact and Lindelöf spaces using soft somewhere dense set, *AIMS Math.*, **6** (2021), 8064–8077. https://doi.org/10.3934/math.2021468
18. S. Atmaca, I. Zorlutuna, On fuzzy soft topological spaces, *Ann. Fuzzy Math. Inform.*, **5** (2013), 377–386.
19. A. Aygünoğlu, H. Aygün, Some notes on soft topological spaces, *Neural Comput. Appl.*, **21** (2012), 113–119. https://doi.org/10.1007/s00521-011-0722-3
20. N. Çağman, S. Karataş, S. Enginoglu, Soft topology, *Comput. Math. Appl.*, **62** (2011) 351–358. https://doi.org/10.1016/j.camwa.2011.05.016

21. M. E. El-Shafei, T. M. Al-shami, Applications of partial belong and total non-belong relations on soft separation axioms and dec ision-making problem, *Comput. Appl. Math.*, **39** (2020), 138. https://doi.org/10.1007/s40314-020-01161-3

22. M. E. El-Shafei, M. Abo-Elhamayel, T. M. Al-shami, Partial soft separation axioms and soft compact spaces, *Filomat*, **32** (2018), 4755–4771. https://doi.org/10.2298/FIL1813755E

23. S. A. El-Sheikh, R. A. Hosny, A. M. Abd El-latif, Characterizations of b-soft separation axioms in soft topological spaces, *Inf. Sci. Lett.*, **4** (2015), 125–133. https://doi.org/10.12785/isl/040303

24. F. Feng, Y. B. Jun, X. Liu, L. F. Li, An adjustable approach to fuzzy soft set based decision making, *J. Comput. Appl. Math.*, **234** (2010), 10–20. https://doi.org/10.1016/j.cam.2009.11.055

25. F. Feng, C. X. Li, B. Davvaz, M. I. Ali, Soft sets combined with fuzzy sets and rough sets: A tentative approach, *Soft Comput.*, **14** (2010), 899–911. https://doi.org/10.1007/s00500-009-0465-6

26. R. Gao, J. Wu, Filter with its applications in fuzzy soft topological spaces, *AIMS Math.*, **6** (2020), 2359–2368. https://doi.org/10.3934/math.2021143

27. H. Z. Ibrahim, T. M. Al-shami, A. Mhemdi, Applications of n^{th} power root fuzzy sets in multicriteria decision making, *J. Math.*, **2023** (2023), 1487724. https://doi.org/10.1155/2023/1487724

28. H.Z.Ibrahim, T. M. Al-shami, M. Arar, M. Hosny, k^n_m-Rung picture fuzzy information in a modern approach to multi-attribute group decision-making, *Complex Intell. Syst.*, 2023. https://doi.org/10.1007/s40747-023-01277-z

29. A. Kandil, A. M. El-Etriby, On separation axioms in fuzzy topological spaces, *Tamkang J. Math.*, **18** (1987), 49–59.

30. A. Kandil, M. E. El-Shafei, Regularity axioms in fuzzy topological spaces and FR_i-proximities, *Fuzzy Set. Syst.*, **27** (1988), 217–231. https://doi.org/10.1016/0165-0114(88)90151-0

31. A. Kandil, O. A. E. Tantawy, Fuzzy soft connected sets in fuzzy soft topological spaces, *J. Adv. Math.*, **12** (2016), 6473–6488. https://doi.org/10.24297/jam.v12i8.136

32. A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh, S. S. S. El-Sayed, Fuzzy Soft Hyperconnected spaces, *Ann. Fuzzy Math. Inform.*, **13** (2017), 689–702.

33. A. Kandil, O. A. E. Tantawy, S. A. El-Sheikh, Sawsan S. El-Sayed, Fuzzy soft topological spaces: Regularity and separation axioms, *South Asian J. Math.*, **8** (2018), 1–10.

34. B. Ahmad, A. Kharal, On fuzzy soft sets, *Adv. Fuzzy Syst.*, **2009** (2009), 586507. https://doi.org/10.1155/2009/586507

35. T. M. Al-shami, L. D. R. Kočinac, Nearly soft Menger spaces, *J. Math.*, **2020** (2020), 3807418. https://doi.org/10.1155/2020/3807418

36. T. M. Al-shami, L. D. R. Kočinac, Almost soft Menger and weakly soft Menger spaces, *Appl. Comput. Math.*, **21** (2022), 35–51. https://doi.org/10.30546/1683-6154.21.1.2022.35

37. P. K. Maji, R. Biswas, A. R. Roy, Fuzzy soft sets, *J. Fuzzy Math.*, **9** (2001), 589–602.

38. P. K. Maji, R. Biswas, A. R. Roy, A fuzzy soft set theoretic approach to decision making problems, *Comput. Math. Appl.*, **203** (2007), 412–418. https://doi.org/10.1016/j.cam.2006.04.008

39. J. Mahanta, P. K. Das, Results on fuzzy soft topological spaces, 2012. https://doi.org/10.48550/arXiv.1203.0634

40. S. Mishra, R. Srivastava, On T_0 and T_1 fuzzy soft topological spaces, *Ann. Fuzzy Math. Inform.*, **10** (2015), 591–605.

41. D. Molodtsov, Soft set theory-first results, *Comput. Math. Appl.*, **37** (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5

42. P. Mukherjee, R. P. Chakraborty, C. Park, On fuzzy soft δ-open sets and fuzzy soft δ-continuity, *Ann. Fuzzy Math. Inform.*, **11** (2016), 327–340.

43. S. Roy, T. K. Samanta, A note on fuzzy soft topological spaces, *Ann. Fuzzy Math. Inform.*, **3** (2012), 305–311.

44. S. Saleh, T. M. Al-shami, A. Mhmedi, On some new types of fuzzy soft compact spaces, *J. Math.*, **2023** (2023), 5065592. https://doi.org/10.1155/2023/5065592

45. S. Saleh, T. M. Al-shami, A. A. Azzam, M. Hosny, Stronger forms of fuzzy pre-Separation and regularity axioms via fuzzy topology, *Mathematics*, **11** (2023), 4801. https://doi.org/10.3390/math11234801

46. S. Saleh, A. M. Abd El-latif, A. Al-Salemi, On separation axioms in fuzzy soft topological spaces, *S. Asian J. Math.*, **8** (2018), 92–102.

47. S. Saleh, A. Al-Salemi, The R_0 and R_1 Properties in fuzzy soft topological spaces, *J. New Theory*, **24** (2018), 50–58.

48. M. Shabir, M. Naz, On soft topological spaces, *Comput. Math. Appl.*, **61** (2011), 1786–1799. https://doi.org/10.1016/j.camwa.2011.02.006

49. B. Tanay, M. B. Kandemir, Topological structures of fuzzy soft sets, *Comput. Math. Appl.*, **61** (2011), 2952–2957. https://doi.org/10.1016/j.camwa.2011.03.056

50. L. A. Zadeh, Fuzzy sets, *Inf. Control*, **8** (1965), 38–353. https://doi.org/10.1016/S0019-9958(65)90241-X