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Research article

Boyd-Wong type functional contractions under locally transitive binary relation with applications to boundary value problems

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Abstract: The area of metric fixed point theory applied to relational metric spaces has received significant attention since the appearance of the relation-theoretic contraction principle. In recent times, a number of fixed point theorems addressing the various contractivity conditions in the relational metric space has been investigated. Such results are extremely advantageous in solving a variety of boundary value problems, matrix equations, and integral equations. This article offerred some fixed point results for a functional contractive mapping depending on a control function due to Boyd and Wong in a metric space endued with a local class of transitive relations. Our findings improved, developed, enhanced, combined and strengthened several fixed point theorems found in the literature. Several illustrative examples were delivered to argue for the reliability of our findings. To verify the relevance of our findings, we conveyed an existence and uniqueness theorem regarding the solution of a first-order boundary value problem.

Keywords: fixed points; functional contractions; control functions; binary relations; boundary value problems
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1. Introduction

Metric fixed point theory occupies an important role in nonlinear functional analysis. The strength of metric fixed point theory lies in its wide range of applications to various domains. For recent works related to applications of metric fixed point theory, readers are referred to [1–12]. In fact, the domain of metric fixed point theory commenced in 1922, when the classical BCP (abbreviation of 'Banach contraction principle') appeared. This core result is one of the most prominent fixed point results and nowadays, even the researchers of metric fixed point theory are inspired by this result.

In 2015, Alam and Imdad [13] laid out an innovative and intuitive variant of BCP, wherein metric space has been assigned with a relation such that the relevant mapping endures this relation. The relation-theoretic contraction principle [13] has been developed and built upon by various researchers, e.g., [14–22].

Many variants of BCP involve more comprehensive contractivity conditions on (ambient) metric space $(\mho, \mathsf{d})$ containing the displacement $\mathsf{d}(u, v)$ (where $u, v \in \mho$) on R.H.S. (abbreviation of 'right hand side'). On the other hand, various contraction conditions subsume $\mathsf{d}(u, v)$ along with the displacements of $u, v \in \mho$ under the map Q: $\mathsf{d}(u, Qu), \mathsf{d}(v, Qv), \mathsf{d}(u, Qv), \mathsf{d}(v, Qu)$. Such kinds of contractions are referred to as 'functional contractions'. We express such a contraction as:

$$\mathsf{d}(Qu, Qv) \leq \mathcal{F}(\mathsf{d}(u, v), \mathsf{d}(u, Qu), \mathsf{d}(v, Qv), \mathsf{d}(u, Qv), \mathsf{d}(v, Qu)), \quad \forall\, u, v \in \mho,$$

for an adequate selection of the function $\mathcal{F} : [0, \infty)^5 \to [0, \infty)$. In regards of suitable choices of $\mathcal{F}$, the readers are suggested to refer to the work contained in [23–38].

Very recently, Alharbi and Khan [21] proved some fixed point theorems under Boyd-Wong type strict almost contractions in the setting of relational metric space and utilized them to find a unique solution of a boundary value problem (abbreviated as: BVP). On the other hand, Ansari et al. [17] established some fixed point theorems under certain functional contractions involving a (c)-comparison function in a relational metric space and utilized the same to determine the unique solution of certain integral equation. Motivated by these two results, we'll demonstrate some fixed point theorems in a metric space comprising with a particular type of transitive relations subject to the following relational functional contraction:

$$\mathsf{d}(Qu, Qv) \leq \varrho(\mathsf{d}(u, v)) + \theta(\mathsf{d}(u, Qu), \mathsf{d}(v, Qv), \mathsf{d}(u, Qv), \mathsf{d}(v, Qu))$$

where $\varrho : [0, \infty) \to [0, \infty)$ is a control function in the sense of Boyd and Wong [39]. We'll also propose a variety of illustrative examples demonstrating the main results.

One important feature of the relational metric fixed-point theorems is that the encompassed contraction conditions in these findings merely hold for those elements that are connected through a relation. Relational functional contractions, in particular, are therefore comparatively weaker than ordinary functional contractions. Furthermore, such limitations enable these results to be applied in the areas of typical BVP, matrix equations and integral equations satisfying certain additional hypotheses, whereby the standard fixed point theorems cannot be implemented. Owing to the intention of limiting, we'll conclude the existence and uniqueness of a solution of a BVP corresponding to a first-order ODE (abbreviation of 'ordinary differential equation') as a manifestation of our findings.

2. Notations and abbreviations

Across this work, we'll adopt the following standard notations and abbreviations of mathematical logics.

$:=$	equal by definition
$\mathbb{N}$	the set of natural numbers
$\mathbb{N}_0$	the set of whole numbers
$\mathbb{R}$	the set of reals
$\mathbb{R}^+$	the set of nonnegative reals
$C[a,b]$	the class of continuous real functions on $[a,b]$
$C'[a,b]$	the class of continuously differentiable real functions on $[a,b]$
$\in, \notin$	belongs to and does not belong to
$\forall$	for all
$\exists$	there exists
$\Leftrightarrow$	logical equivalence
$\xrightarrow{\mathsf{d}}$	convergence in metric space $(\mho, \mathsf{d})$
i.e.	Latin phrase 'id est' which means: that is
e.g.	Latin phrase 'exempli gratia' which means: for example

3. Preliminaries

On a set $\mho$, a relation $\Im$ is defined as any subset of $\mho^2$. In subsequent notions, we'll take $\mho$ as an ambient set, d as a metric on $\mho$, $Q : \mho \to \mho$ as a map and $\Im$ as a relation on $\mho$.

Definition 3.1. *[13] $u, v \in \mho$ are called $\Im$-comparative, represented by $[u, v] \in \Im$, if*

$$(u, v) \in \Im \quad or \quad (v, u) \in \Im.$$

Definition 3.2. *[40] $\Im^{-1} := \{(u, v) \in \mho^2 : (v, u) \in \Im\}$ is called an inverse of $\Im$.*

Definition 3.3. *[40] The relation $\Im^s := \Im \cup \Im^{-1}$ is called a symmetric closure of $\Im$.*

Proposition 3.1. *[13] $(u, v) \in \Im^s \Leftrightarrow [u, v] \in \Im$.*

Definition 3.4. *[40] A relation on $\mathcal{V} \subseteq \mho$ described by*

$$\Im|_{\mathcal{V}} := \Im \cap \mathcal{V}^2$$

is called a restriction of $\Im$ on $\mathcal{V}$.

Definition 3.5. *[13] $\Im$ is named as Q-closed if*

$$(Qu, Qv) \in \Im, \quad \forall\ u, v \in \mho;\ (u, v) \in \Im.$$

Proposition 3.2. *[15] $\Im$ is Q^ι-closed for every $\iota \in \mathbb{N}$, whenever $\Im$ is Q-closed.*

Definition 3.6. *[13] $\{u_\iota\} \subset \mho$ is named as an $\Im$-preserving sequence if $(u_\iota, u_{\iota+1}) \in \Im$, $\forall\ \iota \in \mathbb{N}_0$.*

Definition 3.7. *[14] $(\mho, \mathsf{d})$ is named as an $\Im$-complete metric space if each $\Im$-preserving Cauchy sequence converges.*

Definition 3.8. *[14] Q is named as $\Im$-continuous at $u \in \mho$ if for every $\Im$-preserving sequence $\{u_\iota\} \subset \mho$ verifying $u_\iota \xrightarrow{d} u$,*

$$Q(u_\iota) \xrightarrow{d} Q(u).$$

Definition 3.9. *[14] Q is named as $\Im$-continuous whenever it is $\Im$-continuous at every point of $\mho$.*

Definition 3.10. *[13] $\Im$ is named as d-self-closed if the convergence limit of any $\Im$-preserving convergent sequence in $(\mho, d)$ is $\Im$-comparative with each term of a subsequence.*

Definition 3.11. *[41] A subset $\mathcal{V} \subseteq \mho$ is called $\Im$-directed if for any $u, v \in \mathcal{V}$, $\exists\, w \in \mho$ with $(u, w) \in \Im$ and $(v, w) \in \Im$.*

Definition 3.12. *[15] $\Im$ is named as locally Q-transitive if for any $\Im$-preserving sequence $\{u_\iota\} \subset Q(\mho)$ (with range $\mathcal{V} = \{u_\iota : \iota \in \mathbb{N}\}$), $\Im|_{\mathcal{V}}$ remains transitive.*

Proposition 3.3. *[15] If $\Im$ is Q-closed, then for each $\iota \in \mathbb{N}_0$, $\Im$ is Q^ι-closed.*

Lemma 3.1. *[42] Given a metric space $(\mho, d)$, if a sequence $\{u_\iota\} \subset \mho$ is not Cauchy, then $\exists\, \varepsilon_0 > 0$ and $\exists$ subsequences $\{u_{\iota_k}\}$ and $\{u_{l_k}\}$ of $\{u_\iota\}$ that verify*

- $k \leq l_k < \iota_k, \ \forall\, k \in \mathbb{N}$,
- $d(u_{l_k}, u_{\iota_k}) > \varepsilon_0, \ \forall\, k \in \mathbb{N}$,
- $d(u_{l_k}, u_{\iota_{k-1}}) \leq \varepsilon_0, \ \forall\, k \in \mathbb{N}$.

Additionally, if $\lim_{\iota \to \infty} d(u_\iota, u_{\iota+1}) = 0$, then

- $\lim_{k \to \infty} d(u_{l_k}, u_{\iota_k}) = \varepsilon_0$,
- $\lim_{k \to \infty} d(u_{l_k}, u_{\iota_k+1}) = \varepsilon_0$,
- $\lim_{k \to \infty} d(u_{l_k+1}, u_{\iota_k}) = \varepsilon_0$,
- $\lim_{k \to \infty} d(u_{l_k+1}, u_{\iota_k+1}) = \varepsilon_0$.

Following Boyd and Wong [39], Φ refers to the class of functions $\varrho : \mathbb{R}^+ \to \mathbb{R}^+$ verifying $\varrho(s) < s, \ \forall\, s > 0$ and $\limsup_{t \to s^+} \varrho(t) < s, \ \forall\, s > 0$.

Following Jleli et al. [42], Θ refers to the class of continuous functions $\theta : \mathbb{R}^{+^4} \to \mathbb{R}^+$ verifying $\theta(s_1, s_2, s_3, s_4) = 0 \Leftrightarrow s_i = 0$, for at least one $i \in \{1, 2, 3, 4\}$.

Proposition 3.4. *For any $\varrho \in \Phi$ and $\theta \in \Theta$, (I) and (II) are equivalent, where*

(I) $d(Qu, Qv) \leq \varrho(d(u, v)) + \theta(d(u, Qu), d(v, Qv), d(u, Qv), d(v, Qu)),$
$\forall\, u, v \in \mho$ *with* $(u, v) \in \Im$,

(II) $d(Qu, Qv) \leq \varrho(d(u, v)) + \theta(d(u, Qu), d(v, Qv), d(u, Qv), d(v, Qu)),$
$\forall\, u, v \in \mho$ *with* $[u, v] \in \Im$.

Proof. The result is followed by the symmetry of d. □

4. Main results

We'll prove the fixed point results employing the certain relational functional contraction condition.

Theorem 4.1. *It is assumed that* $(\mho, d)$ *is a metric space endued with a relation* $\Im$ *and* $Q : \mho \to \mho$ *continues to be map. Also,*

(a) $\Im$ *is locally* Q*-transitive and* Q*-closed,*
(b) $(\mho, d)$ *is* $\Im$*-complete,*
(c) $\exists\, u_0 \in \mho$ *for which* $(u_0, Qu_0) \in \Im$*,*
(d) Q *is* $\Im$*-continuous or* $\Im$ *is* d*-self-closed,*
(e) $\exists\, \varrho \in \Phi$ *and* $\theta \in \Theta$ *satisfying*

$$\begin{aligned} d(Qu, Qv) &\leq \varrho(d(u, v)) + \theta(d(u, Qu), d(v, Qv), d(u, Qv), d(v, Qu)), \\ &\forall \quad u, v \in \mho \text{ with } (u, v) \in \Im. \end{aligned}$$

Then, Q *possesses a fixed point.*

Proof. Keeping in mind that $u_0 \in \mho$, define the sequence $\{u_\iota\} \subset \mho$ such that

$$u_\iota = Q^\iota(u_0) = Q(u_{\iota-1}), \quad \forall\, \iota \in \mathbb{N}. \tag{4.1}$$

Utilizing the hypotheses (a) and (c) and Proposition 3.3, we conclude

$$(Q^\iota u_0, Q^{\iota+1} u_0) \in \Im$$

so that

$$(u_\iota, u_{\iota+1}) \in \Im, \quad \forall\, \iota \in \mathbb{N}. \tag{4.2}$$

This follows that $\{u_\iota\}$ is a $\Im$-preserving sequence.

Denote $\mathsf{d}_\iota := \mathsf{d}(u_\iota, u_{\iota+1})$. If for some $\iota_0 \in \mathbb{N}_0$, $\mathsf{d}_{\iota_0} = \mathsf{d}(u_{\iota_0}, u_{\iota_0+1}) = 0$, then by (4.1), we get $Q(u_{\iota_0}) = u_{\iota_0}$, i.e., u_{ι_0} remains a fixed point of Q. Hence we are through.

Otherwise, if $\mathsf{d}_\iota > 0$, $\forall\, \iota \in \mathbb{N}_0$, then utilizing assumption (e) for (4.2) and by property of θ, we find

$$\begin{aligned} \mathsf{d}_\iota &= \mathsf{d}(u_\iota, u_{\iota+1}) = \mathsf{d}(Qu_{\iota-1}, Qu_\iota) \leq \varrho(\mathsf{d}(u_{\iota-1}, u_\iota)) \\ &+ \theta(\mathsf{d}(Qu_\iota, u_\iota), \mathsf{d}(Qu_{\iota-1}, u_{\iota-1}), \mathsf{d}(Qu_{\iota-1}, u_\iota), \mathsf{d}(Qu_\iota, u_{\iota-1})), \\ &\leq \varrho(\mathsf{d}_{\iota-1}) + \theta(\mathsf{d}(u_{\iota+1}, u_\iota), \mathsf{d}(u_\iota, u_{\iota-1}), 0, \mathsf{d}(u_{\iota+1}, u_{\iota-1})), \ \forall\, \iota \in \mathbb{N} \end{aligned}$$

so that

$$\mathsf{d}_\iota \leq \varrho(\mathsf{d}_{\iota-1}), \quad \forall\, \iota \in \mathbb{N}_0. \tag{4.3}$$

Utilizing one of the axioms of Φ in (4.3), we get

$$\mathsf{d}_\iota \leq \varrho(\mathsf{d}_{\iota-1}) < \mathsf{d}_{\iota-1}, \quad \forall\, \iota \in \mathbb{N}$$

which is showing that $\{\mathsf{d}_\iota\}$ is a decreasing sequence in $\mathbb{R}^+$. Further, $\{\mathsf{d}_\iota\}$ is also bounded below, therefore $\exists\, \xi \geq 0$ such that

$$\lim_{\iota \to \infty} \mathsf{d}_\iota = \xi. \tag{4.4}$$

If possible, let us assume $\xi > 0$. Taking upper limit in (4.3) and utilizing (4.4) and one of the axioms of Φ, we get

$$\xi = \limsup_{\iota\to\infty} \mathsf{d}_\iota \leq \limsup_{\iota\to\infty} \varrho(\mathsf{d}_{\iota-1}) = \limsup_{\mathsf{d}_\iota\to\xi^+} \varrho(\mathsf{d}_{\iota-1}) < \xi,$$

which arises a contradiction. This concludes that $\xi = 0$, so

$$\lim_{\iota\to\infty} \mathsf{d}_\iota = 0. \tag{4.5}$$

We assert that $\{\mathsf{u}_\iota\}$ is Cauchy. On contrary, assume $\{\mathsf{u}_\iota\}$ is not Cauchy. Using Lemma 3.1, $\exists\, \varepsilon_0 > 0$ and $\exists$ subsequences $\{\mathsf{u}_{\iota_k}\}$ and $\{\mathsf{u}_{l_k}\}$ of $\{\mathsf{u}_\iota\}$,

$$k \leq l_k < \iota_k,\ \mathsf{d}(\mathsf{u}_{l_k}, \mathsf{u}_{\iota_k}) > \varepsilon_0 \geq \mathsf{d}(\mathsf{u}_{l_k}, \mathsf{u}_{\iota_{k-1}}), \quad \forall\, k \in \mathbb{N}.$$

Denote $\mathsf{D}_k := \mathsf{d}(\mathsf{u}_{l_k}, \mathsf{u}_{\iota_k})$. As $\{\mathsf{u}_\iota\}$ is $\mathfrak{I}$-preserving and $\{\mathsf{u}_\iota\} \subset Q(\mho)$, using locally Q-transitivity of $\mathfrak{I}$, we find $(\mathsf{u}_{l_k}, \mathsf{u}_{\iota_k}) \in \mathfrak{I}$. By assumption (e), we obtain

$$\begin{aligned}\mathsf{d}(\mathsf{u}_{l_k+1}, \mathsf{u}_{\iota_k+1}) &= \mathsf{d}(Q\mathsf{u}_{l_k}, Q\mathsf{u}_{\iota_k}) \\ &\leq \varrho(\mathsf{d}(\mathsf{u}_{l_k}, \mathsf{u}_{\iota_k})) + \theta(\mathsf{d}(\mathsf{u}_{l_k}, Q\mathsf{u}_{l_k}), \mathsf{d}(\mathsf{u}_{\iota_k}, Q\mathsf{u}_{\iota_k}), \mathsf{d}(\mathsf{u}_{l_k}, Q\mathsf{u}_{\iota_k}), \mathsf{d}(\mathsf{u}_{\iota_k}, Q\mathsf{u}_{l_k}))\end{aligned}$$

so that

$$\mathsf{d}(\mathsf{u}_{l_k+1}, \mathsf{u}_{\iota_k+1}) \leq \varrho(\mathsf{D}_k) + \theta(\mathsf{D}_{l_k}, \mathsf{D}_{\iota_k}, \mathsf{d}(\mathsf{u}_{l_k}, \mathsf{u}_{\iota_k+1}), \mathsf{d}(\mathsf{u}_{\iota_k}, \mathsf{u}_{l_k+1})). \tag{4.6}$$

Taking upper limit in (4.6) and using the continuity of θ and Lemma 3.1, one gets

$$\varepsilon_0 = \limsup_{k\to\infty} \mathsf{d}(\mathsf{u}_{l_k+1}, \mathsf{u}_{\iota_k+1}) \leq \limsup_{k\to\infty} \varrho(\mathsf{d}_k) + \theta(0, 0, \varepsilon_0, \varepsilon_0),$$

which, by employing the property of θ and one of the axioms of Φ, gives rise to

$$\varepsilon_0 \leq \limsup_{k\to\infty} \varrho(\mathsf{d}_k) = \limsup_{s\to\varepsilon_0^+} \varrho(s) < \varepsilon_0,$$

which arises a contradiction. This shows that $\{\mathsf{u}_\iota\}$ remains Cauchy. Since $\{\mathsf{u}_\iota\}$ is also $\mathfrak{I}$-preserving and owing to $\mathfrak{I}$-completeness of $(\mho, \mathsf{d})$, one can determine $\mathsf{u}^* \in \mho$ such that $u_\iota \xrightarrow{\mathsf{d}} \mathsf{u}^*$.

We'll employ assumption (d) verifying that u^* is the required fixed point of Q. To begin, we assume that Q is $\mathfrak{I}$-continuous. As $\{u_\iota\}$ is $\mathfrak{I}$-preserving such that $u_\iota \xrightarrow{\mathsf{d}} \mathsf{u}^*$, due to $\mathfrak{I}$-continuity of Q, we find $u_{\iota+1} = Q(u_\iota) \xrightarrow{\mathsf{d}} Q(\mathsf{u}^*)$. Consequently, we have $Q(\mathsf{u}^*) = \mathsf{u}^*$.

Otherwise, we assume that $\mathfrak{I}$ remains d-self closed. In this case, we determine a subsequence $\{u_{\iota_k}\}$ of $\{\mathsf{u}_\iota\}$ satisfying $[u_{\iota_k}, u^*] \in \mathfrak{I}$, $\forall\, k \in \mathbb{N}$. Employing the assumption (e), Proposition 3.4 and $[u_{\iota_k}, u^*] \in \mathfrak{I}$, we conclude

$$\begin{aligned}\mathsf{d}(u_{\iota_k+1}, Qu^*) &= \mathsf{d}(Qu_{\iota_k}, Qu^*) \\ &\leq \varrho(\mathsf{d}(u_{\iota_k}, u^*)) + \theta(\mathsf{d}(u^*, Qu^*), \mathsf{d}(u_{\iota_k}, u_{\iota_k+1}), \mathsf{d}(u^*, u_{\iota_k+1}), \mathsf{d}(u_{\iota_k}, Qu^*)).\end{aligned}$$

Now, it is claimed that

$$\mathsf{d}(u_{\iota_k+1}, Qu^*) \leq \mathsf{d}(u_{\iota_k}, u^*) + \theta(\mathsf{d}(u^*, Qu^*), \mathsf{d}(u_{\iota_k}, u_{\iota_k+1}), \mathsf{d}(u^*, u_{\iota_k+1}), \mathsf{d}(u_{\iota_k}, Qu^*)). \tag{4.7}$$

If for some $k_0 \in \mathbb{N}$, $\mathsf{d}(\mathsf{u}_{\iota_{k_0}}, u^*) = 0$ then we have $\mathsf{d}(Q\mathsf{u}_{\iota_{k_0}}, Qu^*) = 0$ so that $\mathsf{d}(\mathsf{u}_{\iota_{k_0}+1}, Qu^*) = 0$ and hence (4.7) holds for such $k_0 \in \mathbb{N}$. On the other hand, if $\mathsf{d}(\mathsf{u}_{\iota_k}, u^*) > 0 \;\; \forall\, k \in \mathbb{N}$, then utilizing one of the axioms of Φ, one finds $\varrho(\mathsf{d}(\mathsf{u}_{\iota_k}, u^*)) < \mathsf{d}(\mathsf{u}_{\iota_k}, u^*)\ \forall\, k \in \mathbb{N}$. Hence (4.7) is verified for every $k \in \mathbb{N}$. Computing the limit of (4.7) and by $\mathsf{u}_{\iota_k} \xrightarrow{\mathsf{d}} u^*$, we find $\mathsf{u}_{\iota_k+1} \xrightarrow{\mathsf{d}} Q(u^*)$. This implies that $Q(u^*) = u^*$. Therefore in both cases, u^* continues to be a fixed point of Q. □

Theorem 4.2. *Combined to the proposals of Theorem 4.1, if $Q(\mho)$ is $\Im^s$-directed, then Q owns a unique fixed point.*

Proof. In lieu of Theorem 4.1, if $\exists\ \breve{u}, \check{v} \in \mho$, which satisfy

$$Q(\breve{u}) = \breve{u} \text{ and } Q(\check{v}) = \check{v}. \tag{4.8}$$

Using the fact $\breve{u}, \check{v} \in Q(\mho)$ and the given hypothesis, $\exists\ \omega \in \mho$, which verifies

$$[\breve{u}, \omega] \in \Im \quad \text{and} \quad [\check{v}, \omega] \in \Im. \tag{4.9}$$

Write $\delta_\iota := \mathsf{d}(\breve{u}, Q^\iota \omega)$. By (4.8), (4.9), assumption (e) and Proposition 3.4, we obtain

$$\begin{aligned}
\delta_\iota = \mathsf{d}(\breve{u}, Q^\iota \omega) &= \mathsf{d}(Q\breve{u}, Q(Q^{\iota-1}\omega)) \\
&\leq \varrho(\mathsf{d}(\breve{u}, Q^{\iota-1}\omega)) + \theta(0, \mathsf{d}(Q^{\iota-1}\omega, Q^\iota\omega), \mathsf{d}(\breve{u}, Q^\iota\omega), \mathsf{d}(Q^{\iota-1}\omega, \breve{u})) \\
&= \varrho(\delta_{\iota-1}),
\end{aligned}$$

i.e.,

$$\delta_\iota \leq \varrho(\delta_{\iota-1}). \tag{4.10}$$

Assume that $\delta_{\iota_0} = 0$ for some $\iota_0 \in \mathbb{N}$, then we find $\delta_{\iota_0} \leq \delta_{\iota_0-1}$. When $\delta_\iota > 0,\ \forall\, \iota \in \mathbb{N}$, employing the property of ϱ, (4.10) becomes $\delta_\iota < \delta_{\iota-1}$. Thus in both cases, we conclude

$$\delta_\iota \leq \delta_{\iota-1}.$$

Proceeding the proof of Theorem 4.1, the above relation yields that

$$\lim_{\iota\to\infty} \delta_\iota = \lim_{\iota\to\infty} \mathsf{d}(\breve{u}, Q^\iota\omega) = 0. \tag{4.11}$$

Similarly, we have

$$\lim_{\iota\to\infty} \mathsf{d}(\check{v}, Q^\iota\omega) = 0. \tag{4.12}$$

Utilizing (4.11) and (4.12), we conclude

$$\mathsf{d}(\breve{u}, \check{v}) = \mathsf{d}(\breve{u}, Q^\iota\omega) + \mathsf{d}(Q^\iota\omega, \check{v}) \to 0 \quad \text{as} \quad \iota \to \infty$$

therefore yielding $\breve{u} = \check{v}$. This shows the uniqueness of fixed point. □

Remark 4.1. *Particularly for $\theta(s_1, s_2, s_3, s_4) = 0$, Theorems 4.1 and 4.2 deduce the main results of Alam and Imdad [15].*

Remark 4.2. *Under the restriction $\theta(s_1, s_2, s_3, s_4) = L.\min\{s_1, s_2, s_3, s_4\}$ (where $L \geq 0$), Theorems 4.1 and 4.2 reduce to the recent results of Alharbi and Khan [21].*

For universal relation $\Im = \mho^2$, Theorem 4.2 leads to the following result under the Boyd-Wong type functional contraction.

Corollary 4.1. *It is assumed that* $(\mho, d)$ *remains a metric space while* $Q : \mho \to \mho$ *continues to be a map. If* $\exists\ \varrho \in \Phi$ *and* $\theta \in \Theta$ *that satisfy*

$$d(Qu, Qv) \leq \varrho(d(u, v)) + \theta(d(u, Qu), d(v, Qv), d(u, Qv), d(v, Qu)), \quad \forall\ u, v \in \mho,$$

then Q *admits a unique fixed point.*

5. Examples

To illuminate our results, we'll offer the following examples.

Example 5.1. *Take* $\mho = [0, \infty)$ *with standard metric* d*. Define the map* $Q : \mho \to \mho$ *by* $Q(u) = u/(u+1)$*. Let* $\Im := \{(u, v) \in \mho^2 : u - v > 0\}$ *be a relation on* $\mho$*. Then* $(\mho, d)$ *is* $\Im$*-complete while* Q *is* $\Im$*-continuous. Moreover,* $\Im$ *is locally* Q*-transitive and* Q*-closed relation. Consider the functions*

$$\varrho(s) = s/(1 + s)$$

and

$$\theta(s_1, s_2, s_3, s_4) = \min\{s_1, s_2, s_3, s_4\}.$$

Thus, $\varrho \in \Phi$ *and* $\theta \in \Theta$*. Now, for all* $(u, v) \in \Im$*, one has*

$$\begin{aligned}
d(Qu, Qv) &= \left|\frac{u}{u+1} - \frac{v}{v+1}\right| = \left|\frac{u-v}{1+u+v+uv}\right| = \frac{u-v}{1+u+v+uv} \quad (as\ u, v \geq 0,\ u - v > 0)\\
&= \frac{u-v}{1+(u-v)+(2v+uv)} \leq \frac{u-v}{1+(u-v)} \quad (as\ 2v + uv \geq 0)\\
&= \frac{d(u,v)}{1+d(u,v)} \leq \phi(d(u, v)) + \theta(d(u, Qu), d(v, Qv), d(u, Qv), d(v, Qu)).
\end{aligned}$$

Thus, we have verified assumption (e) of Theorem 4.1. Similarly, left over the assumptions of Theorems 4.1 and 4.2 hold. It turns out that Q *owns a unique fixed point:* $\check{u} = 0$.

Example 5.2. *Assume that* $\mho = [1, 3]$ *with standard metric* d*. Consider a relation* $\Im = \{(1, 1), (1, 2), (2, 1), (2, 2), (1, 3)\}$ *on* $\mho$*. Let* $Q : \mho \to \mho$ *be a map defined by*

$$Q(u) = \begin{cases} 1 & \text{if } 1 \leq u \leq 2 \\ 2 & \text{if } 2 < u \leq 3. \end{cases}$$

The metric space $(\mho, d)$ *is* $\Im$*-complete while* ϱ *is* Q*-closed. Define the functions* $\varrho(s) = s/3$ *and* $\theta(s_1, s_2, s_3, s_4) = s_1 s_2 s_3 s_4$*, then* $\varrho \in \Phi$ *and* $\theta \in \Theta$*. The contractivity condition (e) of Theorem 4.1 can be readily verified.*

Assume $\{u_\iota\} \subset \mho$ *is a* $\Im$*-preserving sequence such that* $u_\iota \xrightarrow{d} u$ *which implies that* $(u_\iota, u_{\iota+1}) \in \Im$, $\forall\ \iota \in \mathbb{N}$*. As* $(u_\iota, u_{\iota+1}) \notin \{(1, 3)\}$*, we have* $(u_\iota, u_{\iota+1}) \in \{(1, 1), (1, 2), (2, 1), (2, 2)\}$, $\forall\ \iota \in \mathbb{N}$ *and hence* $\{u_\iota\} \subset \{1, 2\}$*. Using closedness of* $\{1, 2\}$*, we find* $[u_\iota, u] \in \varrho$*. This shows that* $\Im$ *is* d*-self-closed. Similarly, left over the assumptions of Theorems 4.1 and 4.2 hold. It turns out that* Q *owns a unique fixed point:* $\check{u} = 1$.

Example 5.3. *Consider $\mho = [0,2]$ with standard metric d. Define a map $Q : \mho \to \mho$ by*

$$Q(u) = \begin{cases} u/2, & \text{if } 0 \le u < 1 \\ 2, & \text{if } 1 \le u \le 2. \end{cases}$$

Consider a relation $\Im = \{(0,1),(0,2)\}$ on $\mho$. Clearly, $(\mho, d)$ is $\Im$-complete. Also, $\Im$ forms a locally Q-transitive, Q-closed and d-self-closed relation. Moreover, Q fulfills the contractivity condition (e) *for the auxiliary functions $\varrho(s) = s/2$ and $\theta(s_1, s_2, s_3, s_4) = 3s_4$. All the other requirements of Theorem 4.1 are also met.*

Since there isn't an element in $\mho$ that is at the same time $\Im$-comparative with $Q(4/5) = 2/5$ and $Q(1) = 2$, $Q(\mho)$ is not $\Im^s$-directed. It turns out that the present example fails to satisfy Theorem 4.2. Indeed, Q admits two fixed points: $\check{u} = 0$ and $\check{v} = 2$.

Remark 5.1. *Example 5.3 cannot be covered by the main results of Alam and Imdad [15] and Alharbi and Khan [21] for if we take $u = 0$ and $v = 1$, then the contraction condition of Alharbi and Khan [21] becomes*

$$2 \le \varrho(1) + L.\min\{0, 1, 2, 1\} < 1,$$

which is incorrect. Similarly, the contraction condition of Alam and Imdad [15] is also never satisfied. This substantiates the utility and novelty of Theorems 4.1 and 4.2 over the corresponding results of Alam and Imdad [15] as well as that of Alharbi and Khan [21].

6. An application to BVPs

In this section, an application for previous theorems is dealt with to discuss the existence and uniqueness of solution of the following first-order periodic BVP satisfying certain additional hypotheses.

$$\begin{cases} x'(\rho) = F(\rho, x(\rho)), & \rho \in [0, c] \\ x(0) = x(c) \end{cases} \tag{6.1}$$

where $F : [0, c] \times \mathbb{R} \to \mathbb{R}$ is continuous.

We indicate Ω to denote the class of increasing continuous functions $\varrho : \mathbb{R}^+ \to \mathbb{R}^+$ satisfying $\varrho(s) < s, \ \forall\ s > 0$. Naturally, we have $\Omega \subset \Phi$.

A function $\tilde{x} \in C'[0, c]$ is termed as a lower solution of (6.1) if

$$\begin{cases} \tilde{x}'(\rho) \le F(\rho, \tilde{x}(\rho)), & \rho \in [0, c] \\ \tilde{x}(0) \le \tilde{x}(c). \end{cases}$$

Theorem 6.1. *In conjunction to Problem* (6.1), *if $\exists\ \sigma > 0$ and $\varrho \in \Omega$ that satisfy*

$$0 \le [F(\rho, \upsilon) + \sigma\upsilon] - [F(\rho, \tau) + \sigma\tau] \le \sigma\varrho(\upsilon - \tau), \quad \forall\ \tau, \upsilon \in \mathbb{R} \text{ with } \tau \le \upsilon \tag{6.2}$$

then the Problem (6.1) *enjoys a unique solution whenever it admits a lower solution.*

Proof. Equation (6.1) can be written as

$$\begin{cases} x'(\rho) + \sigma x(\rho) = F(\rho, x(\rho)) + \sigma x(\rho), & \forall\, \rho \in [0, c] \\ x(0) = x(c) \end{cases}$$

which is equivalent to

$$x(\rho) = \int_0^c \Upsilon(\rho,\zeta)[F(\zeta, x(\zeta)) + \sigma x(\zeta)]d\zeta,$$

with Green function $\Upsilon(\rho,\zeta)$ described by

$$\Upsilon(\rho,\zeta) = \begin{cases} \frac{e^{\sigma(c+\zeta-\rho)}}{e^{\sigma c}-1}, & 0 \le \zeta < \rho \le c \\ \frac{e^{\sigma(\zeta-\rho)}}{e^{\sigma c}-1}, & 0 \le \rho < \zeta \le c. \end{cases}$$

Set $\mho := C[0, c]$. Define the map $Q : \mho \to \mho$ by

$$(Qx)(\rho) = \int_0^c \Upsilon(\rho,\zeta)[F(\zeta, x(\zeta)) + \sigma x(\zeta)]d\zeta, \quad \forall\, \rho \in [0, c]. \tag{6.3}$$

Consider the relation $\Im$ on $\mho$ given by

$$\Im = \{(u, v) \in \mho \times \mho : u(\rho) \le v(\rho),\ \forall\, \rho \in [0, c]\}. \tag{6.4}$$

Assume that $\tilde{x} \in C'[0, c]$ forms a lower solution of (6.1), then we'll verify that $(\tilde{x}, Q\tilde{x}) \in \Im$ and

$$\tilde{x}'(\rho) + \sigma\tilde{x}(\rho) \le F(\rho, \tilde{x}(\rho)) + \sigma\tilde{x}(\rho), \quad \forall\, \rho \in [0, c].$$

When multiplying the above inequality with $e^{\sigma\rho}$, we get

$$(\tilde{x}(\rho)e^{\sigma\rho})' \le [F(\rho, \tilde{x}(\rho)) + \sigma\tilde{x}(\rho)]e^{\sigma\rho}, \quad \forall\, \rho \in [0, c]$$

which implies

$$\tilde{x}(\rho)e^{\sigma\rho} \le \tilde{x}(0) + \int_0^\rho [F(\zeta, \tilde{x}(\zeta)) + \sigma\tilde{x}(\zeta)]e^{\sigma\zeta}d\zeta, \quad \forall\, \rho \in [0, c]. \tag{6.5}$$

Making use of $\tilde{x}(0) \le \tilde{x}(c)$, we get

$$\tilde{x}(0)e^{\sigma c} \le \tilde{x}(c)e^{\sigma c} \le \tilde{x}(0) + \int_0^c [F(\zeta, \tilde{x}(\zeta)) + \sigma\tilde{x}(\zeta)]e^{\sigma\zeta}d\zeta$$

so that

$$\tilde{x}(0) \le \int_0^c \frac{e^{\sigma\zeta}}{e^{\sigma c}-1}[F(\zeta, \tilde{x}(\zeta)) + \sigma\tilde{x}(\zeta)]d\zeta. \tag{6.6}$$

By (6.5) and (6.6), we get

$$\begin{aligned} \tilde{x}(\rho)e^{\sigma\rho} &\le \int_0^c \frac{e^{\sigma\zeta}}{e^{\sigma c}-1}[F(\zeta, \tilde{x}(\zeta)) + \sigma\tilde{x}(\zeta)]d\zeta + \int_0^\rho e^{\sigma\zeta}[F(\zeta, \tilde{x}(\zeta)) + \sigma\tilde{x}(\zeta)]d\zeta \\ &= \int_0^\rho \frac{e^{\sigma(c+\zeta)}}{e^{\sigma c}-1}[F(\zeta, \tilde{x}(\zeta)) + \sigma\tilde{x}(\zeta)]d\zeta + \int_\rho^c \frac{e^{\sigma\zeta}}{e^{\sigma c}-1}[F(\zeta, \tilde{x}(\zeta)) + \sigma\tilde{x}(\zeta)]d\zeta \end{aligned}$$

yielding

$$\begin{aligned}\tilde{x}(\rho) &\leq \int_0^\rho \frac{e^{\sigma(c+\zeta-\rho)}}{e^{\sigma c}-1}[F(\zeta,\tilde{x}(\zeta))+\sigma\tilde{x}(\zeta)]d\zeta + \int_\rho^c \frac{e^{\sigma(\zeta-\rho)}}{e^{\sigma c}-1}[F(\zeta,\tilde{x}(\zeta))+\sigma\tilde{x}(\zeta)]d\zeta \\ &= \int_0^c \Upsilon(\rho,\zeta)[F(\zeta,\tilde{x}(\zeta))+\sigma\tilde{x}(\zeta)]d\zeta \\ &= (\mathcal{Q}\tilde{x})(\rho), \quad \forall\, \rho\in[0,c]\end{aligned}$$

which shows that $(\tilde{x},\mathcal{Q}\tilde{x})\in\mathfrak{I}$.

Now, we'll prove that $\mathfrak{I}$ is $\mathcal{Q}$-closed. Take $u,v\in\mho$; $(u,v)\in\mathfrak{I}$. Using (6.2), we get

$$F(\rho,u(\rho))+\sigma u(\rho)\leq F(\rho,v(\rho))+\sigma v(\rho), \quad \forall\, \rho\in[0,c]. \tag{6.7}$$

Using (6.3), (6.7) and $\Upsilon(\rho,\zeta)>0$, $\forall\, \rho,\zeta\in[0,c]$, we find

$$\begin{aligned}(\mathcal{Q}u)(\rho) &= \int_0^c \Upsilon(\rho,\zeta)[F(\zeta,u(\zeta))+\sigma u(\zeta)]d\zeta \\ &\leq \int_0^c \Upsilon(\rho,\zeta)[F(\zeta,v(\zeta))+\sigma v(\zeta)]d\zeta \\ &= (\mathcal{Q}v)(\rho), \quad \forall\, \rho\in[0,c],\end{aligned}$$

which using (6.4) gives rise to $(\mathcal{Q}u,\mathcal{Q}v)\in\mathfrak{I}$ and we are through.

On $\mho$, consider the following metric

$$\mathsf{d}(u,v)=\sup_{\rho\in[0,c]}|u(\rho)-v(\rho)|, \quad \forall\, u,v\in\mho, \tag{6.8}$$

then $(\mho,\mathsf{d})$ is $\mathfrak{I}$-complete. Take $u,v\in\mho$ with $(u,v)\in\mathfrak{I}$. Using (6.2), (6.3) and (6.8), we get

$$\begin{aligned}\mathsf{d}(\mathcal{Q}u,\mathcal{Q}v) &= \sup_{\rho\in[0,c]}|(\mathcal{Q}u)(\rho)-(\mathcal{Q}v)(\rho)| = \sup_{\rho\in[0,c]}((\mathcal{Q}v)(\rho)-(\mathcal{Q}u)(\rho)) \\ &\leq \sup_{\rho\in[0,c]}\int_0^c \Upsilon(\rho,\zeta)[F(\zeta,v(\zeta))+\sigma v(\zeta)-F(\zeta,u(\zeta))-\sigma u(\zeta)]d\zeta \\ &\leq \sup_{\rho\in[0,c]}\int_0^c \Upsilon(\rho,\zeta)\sigma\varrho(v(\zeta)-u(\zeta))d\zeta.\end{aligned} \tag{6.9}$$

Using the fact $0\leq v(\zeta)-u(\zeta)\leq \mathsf{d}(u,v)$ and by increasing property of ϱ, we find

$$\varrho(v(\zeta)-u(\zeta))\leq\varrho(\mathsf{d}(u,v)).$$

Therefore, (6.9) becomes

$$\begin{aligned}\mathsf{d}(\mathcal{Q}u,\mathcal{Q}v) &\leq \sigma\varrho(\mathsf{d}(u,v))\sup_{\rho\in[0,c]}\int_0^c \Upsilon(\rho,\zeta)d\zeta \\ &= \sigma\varrho(\mathsf{d}(u,v))\sup_{\rho\in[0,c]}\frac{1}{e^{\sigma c}-1}\left[\frac{1}{\sigma}e^{\sigma(c+\zeta-\rho)}\Big|_0^\rho+\frac{1}{\sigma}e^{\sigma(\zeta-\rho)}\Big|_\rho^c\right]\end{aligned}$$

$$
\begin{aligned}
&= \sigma\varrho(\mathsf{d}(u,v))\frac{1}{\sigma(e^{\sigma c}-1)}(e^{\sigma c}-1)\\
&= \varrho(\mathsf{d}(u,v)),
\end{aligned}
$$

so

$$
\begin{aligned}
\mathsf{d}(Qu,Qv) \leq \varrho(\mathsf{d}(u,v)) + \theta(\mathsf{d}(u,Qu),\mathsf{d}(v,Qv),\mathsf{d}(u,Qv),\mathsf{d}(v,Qu)),\\
\forall\ u,v \in \mho \text{ satisfying } (u,v)\in\mathfrak{I}
\end{aligned}
$$

for an arbitrary $\theta \in \Omega$.

Assume that $\{x_\iota\} \subset \mho$ is a $\mathfrak{I}$-preserving sequence converging to $\overline{x} \in \mho$. Consequently, we find $x_\iota(\rho) \leq \overline{x}(\rho)$, $\forall\ \iota \in \mathbb{N}$, and $\forall\ \rho \in [0,c]$. Using (6.4), we get $(x_\iota, \overline{x}) \in \mathfrak{I}$, $\forall\ \iota \in \mathbb{N}$. This shows that $\mathfrak{I}$ is d-self-closed. Henceforth, the assumptions (a)-(e) of Theorem 4.1 hold. It turns out that Q possesses a fixed point.

Choose $u, v \in \mho$ arbitrarily; then $Q(u), Q(v) \in Q(\mho)$. Denote $\varpi := \max\{Qu, Qv\}$, yielding $(Qu, \varpi) \in \mathfrak{I}$ and $(Qv, \varpi) \in \mathfrak{I}$. Thus, $Q(\mho)$ forms a $\mathfrak{I}^s$-directed set, and by Theorem 4.2, Q has a unique fixed point, which remains the unique solution of (6.1). □

7. Conclusions

Ansari et al. [17] recently examined the fixed point results by implementing an amorphous relation through functional contraction that included a (c)-comparison function. In this work, we availed placement for yet another functional contraction that included a control function due to Boyd and Wong [39]. The underlying relation adopted in our results remains locally Q-transitive, which is restricted; nevertheless, the functional contraction condition is comparatively weaker. As an application of our findings, we presented an existence and uniqueness theorem for certain BVP if there exists a lower solution.

We built three examples to illuminate our findings. Examples 5.1 and 5.2 demonstrate Theorems 4.1 and 4.2, which respectively validate two separate alternative assertions (that is, Q is $\mathfrak{I}$-continuous, or $\mathfrak{I}$ is d-self-closed). Example 5.3 fails to demonstrate Theorem 4.2 even though it meets the criteria of Theorem 4.1. This substantiates the worth of Theorems 4.1 and 4.2 as compared to the recent results of Alam and Imdad [15] and Alharbi and Khan [21]. Furthermore, we derived a consequence in abstract metric space that extends the classical fixed point theorem of Boyd and Wong.

As some possible future works, Theorems 4.1 and 4.2 can further be extended in the following directions, which are highly relevant and prominent areas on their own.

- For locally finitely Q-transitive relation under a functional contraction depending on a control function of Alam et al. [18];
- to a variety of metrical frameworks, such as: semi-metric space, quasi metric space, dislocated space, partial metric space, and fuzzy metric space equipped with locally Q-transitive relation;
- to prove coincidence and common fixed point theorems;
- to prove an analogous of Theorem 6.1 for solving BVP (6.1) in the presence of an upper solution rather than the presence of a lower solution.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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