Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 6088–6108.
DOI: 10.3934/math.2024298
Received: 12 December 2023
Revised: 18 January 2024
Accepted: 24 January 2024
Published: 02 February 2024

Research article

Construction of new Lie group and its geometric properties

Muhammad Asad Iqbal[1], Abid Ali[2,*], Ibtesam Alshammari[3] and Cenap Ozel[4]

[1] Department of Mathematics, Comsats University Islamabad, Pakistan; asadiqbal494@gmail.com
[2] Department of Mathematics, University of the Punjab, Lahore 54590, Pakistan
[3] University of Hafr Al Batin, Department of Mathematics, Hafr Al Batin, KSA; iealshamri@uhb.edu.sa
[4] University of King Abdulaziz, Department of Mathematics, Jeddah 21589, KSA; cozel@kau.edu.sa

* **Correspondence:** Email: abidsafi.math@gmail.com.

Abstract: In this paper, we constructed a novel Lie group by using oblate spheroidal coordinates. First, we took the metric tensor of oblate spheroidal coordinates, then found its Killing vectors by using the Killing equation. After solving a system of partial differential equations, we obtained the Killing vectors. With the help of these Killing vectors, we first constructed finite Lie algebra and then proved that Killing vectors form a Lie group. Also, we described the geometric properties in which this Lie group forms a regular surface, defined the differential map and differential of normal vector field, and found the gaussian and mean curvatures.

Keywords: Lie group G; Lie algebra; Lie symmetry; Killing vectors X^1, X^2 and X^3
Mathematics Subject Classification: 22Exx, 57Sxx, 58-XX

1. Introduction

Lie groups and Lie algebras have become fundamental to pure mathematics and theoretical physics. The theory of Lie groups was first introduced and investigated by Sophus Lie through transformation groups [1]. The Killing vectors are used to construct new Lie groups. To obtain the Killing vectors, we solve a system of partial differential equations using an equation. After finding the Killing vectors, we can determine the Lie algebra by solving the Lie bracket, and then we can find the Lie group.

A Lie group is a group that is also a differentiable manifold, meaning that it has a smooth structure compatible with the group operation. Lie groups are widely used in mathematics and physics to study geometric objects' symmetries, rigid bodies' rotations and translations, and the behavior of particles in quantum mechanics. One of their key properties is the Lie algebra, which is a vector space associated with the group that provides information about the group's infinitesimal behavior. They have since become a central object of study in modern mathematics and physics, with applications in fields ranging from algebraic geometry to string theory.

Lie groups are extensively applied in various fields of mathematics and physics. Lie groups have wide-ranging applications in various fields of mathematics and physics, including differential geometry. Lie groups provide a natural setting for studying the geometry of curved spaces, making them a powerful tool in the field. For instance, the group of rotations in Euclidean space is a Lie group that plays a crucial role in understanding the geometry of rigid bodies. Through the study of Lie groups, mathematicians and physicists can develop a deeper understanding of the underlying structures and symmetries in various mathematical and physical systems. The rich mathematical theory of Lie groups continues to inspire new discoveries and insights, making it an active area of research in many branches of mathematics and physics.

Moreover, Lie groups have broad and significant applications across various branches of mathematics and physics, such as group theory, physics, control theory, and quantum mechanics. In group theory, Lie groups play a crucial role in the study of properties of groups, including symmetry groups and permutation groups. In physics, Lie groups are commonly used to investigate symmetry and conservation laws, such as the Poincaré group, which is fundamental in studying special relativity. Control theory utilizes Lie groups to model complex dynamical systems like autonomous vehicles and robots. In the investigation of quantum mechanics, Lie groups play a crucial role in the theory of Lie algebras, which characterize the symmetry groups of physical systems. These symmetry groups are crucial in understanding the structure of these systems. Therefore, Lie groups have diverse and extensive applications in different fields of mathematics and physics, making them an essential tool for research and discovery.

2. Preliminaries

Differential geometry and mathematical physics use the term "Killing vector" to refer to a vector field on a smooth manifold that maintains the manifold's metric tensor. This vector field generates a symmetry transformation of the manifold's geometry. Specifically, Killing vectors are vector fields that preserve the geometry of a given space-time under an isometry (a certain type of transformation). In essence, a Killing vector generates a symmetry of the space-time while keeping its metric invariant. The Killing equation is the mathematical definition of a Killing vector as a vector field k that satisfies a specific condition

$$\nabla_\mu k_\nu + \nabla_\nu k_\mu = 0, \tag{2.1}$$

where ∇_μ is the covariant derivative with respect to the metric of the space-time. The Killing equation is a differential equation that guarantees the vector field preserves the metric tensor, which in turn maintains the space-time geometry. To be classified as a killing vector, a vector field must fulfill various other conditions apart from the killing equation, such as being a smooth and continuous function on the space-time. Killing vectors are crucial in multiple fields of physics, such as general relativity and classical mechanics. They are employed to describe symmetries and conservation laws. The direction in which the Lie derivative of metric tensor is zero is called the Killing vector, and the Killing equation is

$$L_\xi(g_{ab}) = 0 \tag{2.2}$$

and

$$g_{ab,c}\xi^c + g_{cb}\xi^c_{,a} + g_{ac}\xi^c_{,b} = 0, \tag{2.3}$$

where a comma (,) represents a partial derivative. The Lie derivative, is a concept in differential geometry that has become a fundamental tool in modern physics and mathematics. The concept of Lie derivative of a tensor field along a vector field is a crucial notion in differential geometry. It measures the change of a tensor field along the flow defined by another vector field, taking into account the non-commutativity of the vector field. This directional derivative is widely used in physics and plays an indispensable role in understanding the behavior of physical systems, especially in the context of general relativity.

The Lie derivative is also significant in defining the Lie algebra, which is the algebraic structure underlying the theory of Lie groups. Its far-reaching applications in various areas of physics and mathematics make it a crucial concept for examining the symmetries of physical systems and facilitating the development of mathematical models for various applications. Overall, the Lie derivative is a powerful mathematical operation that has become a cornerstone of modern physics and mathematics, providing insights into complex physical phenomena and helping to advance our understanding of the underlying mathematical structures.

Section 1 is the introduction. Section 2 presents some preliminaries. Section 3 covers the Killing vectors of metric tensor. Section 4 covers Lie algebras of Killing vectors. Section 5 covers Lie group. Section 6 proves the formation of a regular surface. Section 7 covers the differential map for Lie group. Section 8 presents the differential of normal vector field on Lie group. Section 9 presents Gaussian and mean curvatures of the Lie group [2–30]. Finally, Section 10 contains the conclusions.

3. The Killing vectors of metric tensor

This section focuses on the identification of the Killing vectors of the metric tensor of oblate spheroidal coordinates. Killing vectors are crucial concepts in differential geometry and physics that describe symmetries in space-time. They refer to vector fields that preserve the metric tensor of a manifold, a property that is significant in physics as it ensures the conservation of specific quantities such as energy and momentum along the flow of the Killing vector. The application of Killing vectors is seen in theories like general relativity and quantum field theory, where they help describe the symmetries of space-times such as black holes and the universe. Therefore, understanding the concept of Killing vectors is essential in helping physicists comprehend the symmetries of space-time and the fundamental laws of physics that govern it. Consider the well known metric tensor of oblate spheroidal coordinates [31],

$$ds^2 = a^2(sinh^2(\xi) + sin^2(\eta))d^2\xi + a^2(sinh^2(\xi) + sin^2(\eta))d^2\eta + a^2cosh^2(\xi)cos^2(\eta)d^2\phi, \tag{3.1}$$

$$g_{ab} = \begin{pmatrix} a^2(sinh^2(\xi) + sin^2(\eta)) & 0 & 0 \\ 0 & a^2(sinh^2(\xi) + sin^2(\eta)) & 0 \\ 0 & 0 & a^2cosh^2(\xi)cos^2(\eta) \end{pmatrix},$$

$$g_{ab} = \begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}.$$

Let

$$X = (X^1(\xi,\eta,\phi), X^2(\xi,\eta,\phi), X^3(\xi,\eta,\phi))$$

be the Killing vectors of oblate spheroidal coordinates. From Eq (2.3), for g_{11}

$$g_{11,c}X^c + g_{c1}X^c_{,1} + g_{1c}X^c_{,1} = 0. \tag{3.2}$$

Since $c = 1,2,3$,

$$g_{11,1}X^1 + g_{11}X^1_{,1} + g_{11}X^1_{,1} + g_{11,2}X^2 + g_{21}X^2_{,1} + g_{12}X^2_{,1} + g_{11,3}X^3 + g_{31}X^3_{,1} + g_{13}X^3_{,1} = 0, \tag{3.3}$$

$$\left(sinh^2(\xi) + sin^2(\eta)\right)\frac{\partial X^1}{\partial \xi} + (sinh(\xi)cosh(\xi))\,X^1 + (sin(\eta)cos(\eta))\,X^2 = 0. \tag{3.4}$$

For $g_{12} = g_{21}$,

$$g_{12,c}X^c + g_{c2}X^c_{,2} + g_{1c}X^c_{,2} = 0, \tag{3.5}$$

$$g_{12,1}X^1 + g_{12}X^1_{,1} + g_{11}X^1_{,2} + g_{12,2}X^2 + g_{22}X^2_{,1} + g_{12}X^2_{,2} + g_{12,3}X^3 + g_{32}X^3_{,1} + g_{13}X^3_{,2} = 0, \tag{3.6}$$

$$\frac{\partial X^1}{\partial \eta} + \frac{\partial X^2}{\partial \xi} = 0. \tag{3.7}$$

For $g_{13} = g_{31}$,

$$g_{13,c}X^c + g_{c3}X^c_{,1} + g_{1c}X^c_{,3} = 0, \tag{3.8}$$

$$g_{13,1}X^1 + g_{13}X^1_{,1} + g_{11}X^1_{,3} + g_{13,2}X^2 + g_{23}X^2_{,1} + g_{12}X^2_{,3} + g_{13,3}X^3 + g_{33}X^3_{,1} + g_{13}X^3_{,3} = 0, \tag{3.9}$$

$$\left(sinh^2(\xi) + sin^2(\eta)\right)\frac{\partial X^1}{\partial \phi} + \left(cosh^2(\xi)cos^2(\eta)\right)\frac{\partial X^3}{\partial \xi} = 0. \tag{3.10}$$

For g_{22},

$$g_{22,c}X^c + g_{c2}X^c_{,2} + g_{2c}X^c_{,2} = 0, \tag{3.11}$$

$$g_{22,1}X^1 + g_{12}X^1_{,2} + g_{21}X^1_{,2} + g_{22,2}X^2 + g_{22}X^2_{,2} + g_{22}X^2_{,2} + g_{22,3}X^3 + g_{32}X^3_{,2} + g_{23}X^3_{,2} = 0, \tag{3.12}$$

$$\left(sinh^2(\xi) + sin^2(\eta)\right)\frac{\partial X^1}{\partial \eta} + sinh(2\xi)X^1 + \left(sinh^2(\xi) + sin^2(\eta)\right)\frac{\partial X^2}{\partial \eta} + (sin(\eta)cos(\eta))\,X^2 = 0. \tag{3.13}$$

For $g_{23} = g_{32}$,

$$g_{23,c}X^c + g_{c3}X^c_{,2} + g_{2c}X^c_{,3} = 0, \tag{3.14}$$

$$g_{23,1}X^1 + g_{13}X^1_{,2} + g_{21}X^1_{,3} + g_{23,2}X^2 + g_{23,2}X^2 + g_{22}X^2_{,3} + g_{23,3}X^3_{,3} + g_{33}X^3_{,2} + g_{23}X^3_{,3} = 0, \tag{3.15}$$

$$\left(sinh^2(\xi) + sin^2(\eta)\right)\frac{\partial X^2}{\partial \phi} + \left(cosh^2(\xi)cos^2(\eta)\right)\frac{\partial X^3}{\partial \eta} = 0. \tag{3.16}$$

For g_{33},

$$g_{33,c}X^c + g_{c3}X^c_{,3} + g_{3c}X^c_{,3} = 0, \tag{3.17}$$

$$g_{33,1}X^1 + g_{13}X^1_{,3} + g_{31}X^1_{,3} + g_{33,2}X^2 + g_{23}X^2 + g_{32}X^2_{,3} + g_{33,3}X^3_{,3} + g_{33}X^3_{,3} + g_{33}X^3_{,3} = 0, \tag{3.18}$$

$$\frac{\partial X^3}{\partial \phi} - tan(\eta)X^2 + tanh(\xi)X^1 = 0. \tag{3.19}$$

From Eq (3.4),

$$\frac{\partial X^1}{\partial \xi} + \left(\frac{sinh(\xi)cosh(\xi)}{sinh^2(\xi) + sin^2(\eta)}\right) X^1 = 0. \tag{3.20}$$

Then, we have

$$X^1 = \frac{f(\eta, \phi)}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}. \tag{3.21}$$

From Eq (3.13),

$$\left(sinh^2(\xi) + sin^2(\eta)\right) \frac{\partial X^2}{\partial \eta} + (sin(\eta)cos(\eta))\, X^2 = 0, \tag{3.22}$$

$$\frac{\partial X^2}{\partial \eta} + \left(\frac{sinh(\eta)cos(\eta)}{sinh^2(\eta) + sin^2(\eta)}\right) X^2 = 0. \tag{3.23}$$

Then, we have

$$X^2 = \frac{g(\eta, \phi)}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}. \tag{3.24}$$

From Eq (3.16),

$$\left(sinh^2(\xi) + sin^2(\eta)\right) \frac{\partial}{\partial \phi}\left(\frac{g(\xi, \phi)}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}\right) + \left(cosh^2(\xi)cos^2(\xi)\right) \frac{\partial X^3}{\partial \eta} = 0, \tag{3.25}$$

$$\sqrt{sinh^2(\xi) + sin^2(\eta)}g'(\xi, \phi) + \left(cosh^2(\xi)cos^2(\xi)\right) \frac{\partial X^3}{\partial \eta} = 0, \tag{3.26}$$

$$\left(cosh^2(\xi)cos^2(\xi)\right) \frac{\partial X^3}{\partial \eta} = -\sqrt{sinh^2(\xi) + sin^2(\eta)}g'(\xi, \phi), \tag{3.27}$$

$$\frac{\partial X^3}{\partial \eta} = -\frac{\sqrt{sinh^2(\xi) + sin^2(\eta)}g'(\xi, \phi)}{(cosh^2(\xi)cos^2(\xi))}. \tag{3.28}$$

Then, we have

$$X^3 = \frac{-2(sinh^2(\xi) + sin^2(\eta))^{\frac{3}{2}}}{3(cosh^2(\xi)cos^2(\eta)sin^2(\eta))}. \tag{3.29}$$

From Eq (3.32)

$$\frac{\partial}{\partial \eta}\left(\frac{f(\eta, \phi)}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}\right) + \frac{\partial}{\partial \xi}\left(\frac{g(\eta, \phi)}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}\right) = 0. \tag{3.30}$$

After solving the previous equations and comparing coefficients, we get

$$f' + f = 0 \text{ and } g' - g = 0.$$

If

$$f(0, 0, 0) = 1 \text{ and } g(0, 0, 0) = 1,$$

then we obtain

$$f = e^{-\eta}, \quad g = e^{\xi}. \tag{3.31}$$

By putting the value of f and g in (3.21) and (3.24), we obtain the Killing vectors

$$X^1 = \frac{e^{-\eta}}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}, \tag{3.32}$$

$$X^2 = \frac{e^{\xi}}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}, \tag{3.33}$$

$$X^3 = \frac{-2(sinh^2(\xi) + sin^2(\eta))^{\frac{3}{2}} e^{\xi}}{3\left(cosh^2(\xi) cos^2(\eta) sin^2(\eta)\right)}, \tag{3.34}$$

X^1, X^2, and X^3 are Killing vectors of oblate spheroidal coordinates.

4. Lie algebra of Killing vectors

The main focus of this section to create the Lie algebra using oblate spheroidal coordinates' Killing vectors. To achieve this goal, the first step is to establish the Lie bracket of vector fields. The Lie bracket of Killing vectors is a critical concept in differential geometry that is crucial in the investigation of symmetries and conservation laws in physics. This concept provides information about the infinitesimal behavior of a Lie group and is utilized to determine the structure constants of the Lie algebra.

The Lie algebra linked with the group is a valuable tool in understanding the symmetries of geometric objects and has widespread applications in physics and mathematics. Its applications range from studying the behavior of particles in quantum mechanics to analyzing the rotations and translations of rigid bodies, and examining the symmetries of space-times. The concept of Killing vectors is essential in differential geometry and mathematical physics. It refers to a vector field on a manifold that generates an isometry, preserving the metric tensor and describing a symmetry of the manifold that leaves distances between points unchanged.

The Lie bracket of two Killing vectors is also a crucial object in the study of isometries of manifolds. It is defined as the commutator of their associated vector fields and has the property of being itself a Killing vector, which makes the space of Killing vectors on a given manifold a Lie algebra. This Lie algebra is a crucial tool in understanding the symmetries of a space. The Lie bracket of Killing vectors has applications in many areas of physics, including the study of conservation laws in general relativity, the construction of symmetries in quantum field theory, and the study of integrable systems in classical mechanics.

Additionally, it is a fundamental tool in the study of geometric structures such as holonomy and curvature. Overall, the Lie bracket of Killing vectors is a powerful concept that plays a central role in the study of symmetries and geometry, with a wide range of applications in theoretical physics and mathematics.

In the context of Lie algebras of Killing vectors, it is noted that the Lie bracket of two Killing vectors also results in a Killing vector. This is because Killing vectors uphold the metric tensor, and when a Killing vector is applied to the metric tensor, its Lie derivative is zero. Moreover, the Lie bracket operation on the space of Killing vectors satisfies the Jacobi identity, which is a fundamental property of Lie algebras. Specifically, for any three Killing vectors X, Y, and Z the Jacobi identity states that

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \tag{4.1}$$

where $[X, Y]$ denotes the Lie bracket of X and Y [32]. The Lie algebra of Killing vectors plays an important role in the study of symmetries and conserved quantities in physics. For example, in general relativity, the presence of a Lie algebra of Killing vectors can be used to construct conserved quantities associated with the symmetries of spacetime. Similarly, in supersymmetric field theories, the Lie algebra of Killing vectors can be used to construct supersymmetric charges that are conserved quantities.

Definition 4.1. [27] For two vector fields $f(x)$ and $g(x)$, the Lie bracket $[f, g](x)$ is another vector field defined by

$$[f, g](x) = \frac{\partial g}{\partial x} f(x) - \frac{\partial f}{\partial x} g(x), \tag{4.2}$$

which is the Lie bracket of g along f.

Now we find the Lie bracket of Killing vectors of oblate spheriodal coordinates.

$$[X^1, X^2](x) = \frac{\partial X^2}{\partial x} X^1(x) - \frac{\partial X^1}{\partial x} X^2(x), \tag{4.3}$$

$$\frac{\partial X^2}{\partial x} = \frac{e^{\xi}\left(2sinh^2(\xi) + 2sin^2(\eta) - sinh(2\xi) + sin(2\eta)\right)}{2\left(sinh^2(\xi) + sin^2(\eta)\right)^{\frac{3}{2}}}, \tag{4.4}$$

$$\frac{\partial X^2}{\partial x} X^1 = \frac{e^{\xi-\eta}\left(2sinh^2(\xi) + 2sin^2(\eta) - sinh(2\xi) + sin(2\eta)\right)}{2\left(sinh^2(\xi) + sin^2(\eta)\right)^2}, \tag{4.5}$$

$$\frac{\partial X^1}{\partial x} = \frac{e^{-\eta}\left(-2sinh^2(\xi) - 2sin^2(\eta) - sinh(2\xi) + sin(2\eta)\right)}{2\left(sinh^2(\xi) + sin^2(\eta)\right)^{\frac{3}{2}}}, \tag{4.6}$$

$$\frac{\partial X^1}{\partial x} X^2 = \frac{e^{\xi-\eta}\left(-2sinh^2(\xi) - 2sin^2(\eta) - sinh(2\xi) + sin(2\eta)\right)}{2\left(sinh^2(\xi) + sin^2(\eta)\right)^2}. \tag{4.7}$$

From Eqs (4.3), (4.5), and (4.7), we obtain

$$[X^1, X^2] = \frac{2e^{\xi-\eta}}{\left(sinh^2(\xi) + sin^2(\eta)\right)}, \tag{4.8}$$

$$[X^2, X^3](x) = \frac{\partial X^3}{\partial x} X^2(x) - \frac{\partial X^2}{\partial x} X^3(x), \tag{4.9}$$

$$\begin{aligned}\frac{\partial X^3}{\partial x} =& e^{\xi}\left(sinh^2(\xi) + sin^2(\eta)\right)^{\frac{1}{2}}\left(3\left(sinh^2(\xi)cosh^2(\eta)sin^2(\eta)\right)^2\right)^{-1} \\ &\times\Big[-3sinh^2(\xi)cosh^2(\eta)sin^2(\eta)\left(sinh(2\xi) - sin(2\eta)\right) \\ &+2\left(sinh^2(\xi) + sin^2(\eta)\right)\left(sinh(2\xi)sinh(2\eta)sin(2\eta)\right)\Big],\end{aligned} \tag{4.10}$$

$$\begin{aligned}\frac{\partial X^3}{\partial x} X^2 =& \frac{e^{2\xi}}{3(sinh^2(\xi)cosh^2(\eta)sin^2(\eta))^2} \times\Big[-3sinh^2(\xi)cosh^2(\eta)sin^2(\eta)(sinh(2\xi) - sin(2\eta)) \\ &+2(sinh^2(\xi) + sin^2(\eta))(sinh(2\xi)sinh(2\eta)sin(2\eta))\Big],\end{aligned} \tag{4.11}$$

$$\frac{\partial X^2}{\partial x}X^3 = \frac{-2e^{2\xi}(sinh^2(\xi) + sin^2(\eta) - sinh(\xi)cosh(\xi) + sin(\eta)cos(\eta))}{3(sinh^2(\xi)cosh^2(\eta)sin^2(\eta))}. \tag{4.12}$$

From Eqs (4.9), (4.11), and (4.12), we have

$$\begin{aligned}[X^2, X^3] =&e^{2\xi}|[\frac{-3\sinh^2(\xi)\cosh^2(\eta)\sin^2(\eta)|(\sinh(2\xi) - \sin(2\eta) + 2|(\sinh^2(\xi) + \sin^2(\eta)|)|)}{3\sinh^2(\xi)\cosh^2(\eta)\sin^2(\eta)}| \\ &|+\frac{2\sinh(2\xi)\sinh(2\eta)\sin(2\eta) + 2\sinh^2(\xi) + 2\sin^2(\eta) - \sinh(2\eta) + \sin(2\eta)}{3\sinh^2(\xi)\cosh^2(\eta)\sin^2(\eta)}|],\end{aligned} \tag{4.13}$$

$$[X^1, X^3](x) = \frac{\partial X^3}{\partial x}X^1(x) - \frac{\partial X^1}{\partial x}X^3(x), \tag{4.14}$$

$$\begin{aligned}\frac{\partial X^3}{\partial x}X^1 =&e^{\xi-\eta}|[\frac{-3\sinh^2(\xi)\cosh^2(\eta)\sin^2(\eta)(\sinh(2\xi) - \sin(2\eta))}{3\sinh^2(\xi)\cosh^2(\xi)\sin^2(\eta)}| \\ &|+\frac{2(\sinh^2(\xi) + \sin^2(\eta))(\sinh(2\xi)\sinh(2\eta)\sin(2\eta))}{3\sinh^2(\xi)\cosh^2(\xi)\sin^2(\eta)}|],\end{aligned} \tag{4.15}$$

$$\frac{\partial X^1}{\partial x}X^3 = \frac{e^{\xi-\eta}\left(-sinh^2(\xi) - sin^2(\eta) - sinh(\xi)cosh(\xi) + sin(\eta)cos(\eta)\right)}{(sinh^2(\xi) + sin^2(\eta))}. \tag{4.16}$$

From Eqs (4.14) and (4.16), we have

$$\begin{aligned}[X^1, X^3] =&e^{\xi-\eta}\left[\frac{-3\sinh^2(\xi)\cosh^2(\eta)\sin^2(\eta)(\sinh(2\xi) - \sin(2\eta))}{3\sinh^2(\xi)\cosh^2(\xi)\sin^2(\eta)}\right. \\ &\left.+\frac{2(\sinh^2(\xi) + \sin^2(\eta))(\sinh(2\xi)\sinh(2\xi)\sinh(2\eta)\sin(2\eta))}{3\sinh^2(\xi)\cosh^2(\xi)\sin^2(\eta)}\right].\end{aligned} \tag{4.17}$$

From Eqs (4.8), (4.13), and (4.17) we obtained a finite Lie algebras of Killing vectors.

5. Lie group

The objective of this section is to demonstrate that the Killing vectors being studied form a Lie group. A Lie group is a mathematical entity that merges the concepts of a group and a differentiable manifold. In other words, it is a combination of a group and a differentiable manifold, which allows it to be analyzed using calculus and characterized by smoothness or continuity. By analyzing the Killing vectors within the context of a Lie group, we can gain a better understanding of their properties and behavior.

To provide more detail, a Lie group is a set (G) that comes equipped with a binary operation, usually denoted by multiplication, that satisfies the group axioms such as associativity, identity element, and inverse element. Additionally, it is also a smooth manifold, which means that it has a collection of charts that can describe it locally by coordinates. The smooth structure must be compatible with the group structure in the sense that the group multiplication operation must be a smooth function in the coordinates provided by the charts.

Lie groups are essential mathematical objects that have many important applications in both mathematics and physics. One of their primary uses is in the study of symmetries of geometric objects and physical systems. For example, rotations, translations, and reflections in Euclidean space are all examples of Lie groups that arise from the study of continuous symmetries of geometric objects. Lie groups are abstract mathematical structures that have significant applications in both mathematics and physics. The general linear group and the special unitary group are examples of Lie groups that are commonly used in these fields. The study of Lie groups is a vast and complex subject with connections to various areas of mathematics and physics, including differential geometry, algebraic geometry, representation theory, and quantum field theory. Understanding the symmetries of physical systems and geometric objects is essential in both mathematics and physics, and Lie groups provide a fundamental tool to achieve this understanding. It is therefore crucial to study Lie groups for anyone interested in these fields.

Theorem 5.1. *The set of Killing vectors form a Lie group.*

Proof. Let

$$G = \{X^1, X^2, X^3\}$$

be set of Killing vectors, where X^1, X^2, and X^3 are defined in Eqs (3.32)–(3.34), respectively. To prove G forms a Lie group, we need to define a group operation that satisfies certain properties. For this, we need to find a map that satisfies the conditions for a Lie group. The map that defines a Lie group for G is

$$\phi : R^3 \longrightarrow R^3 \text{ by } \phi(x, y, z) = (x_1, y_1, z_1),$$

where

$$x_1 = -ln(X^1) + ln(X^2), \quad x_2 = ln(X^1) + ln(X^2), \quad z_1 = z.$$

We can show that G forms a Lie group under the map ϕ.

(1) Closure

The group operation is defined by the map ϕ, which takes a element $(x, y, z) \in R^3$ and maps it to another element $(x_1, y_1, z_1) \in R^3$. Since expressions X^1, X^2, and X^3 are continuous functions, the map ϕ is also continuous, which implies that the group operation is closed.

(2) Associativity

The group operation is defined by composition of maps which is associative.

(3) Identity

The identity element is the point $(0, 0, 0)$ which corresponds to $X^1 = 1, X^2 = 1$, and $X^3 = 0$.

(4) Inverse

The inverse of an element (x, y, z) is given by $(x_1, y_1, -z_1)$. Since

$$X^1 = \frac{e^{-y_1}}{\sqrt{sinh^2(x_1) + sin^2(y_1)}} = \frac{e^{-y}}{\sqrt{sinh^2(x) + sin^2(y)}} = (X^1)^{-1}, \tag{5.1}$$

$$X^2 = \frac{e^{x_1}}{\sqrt{sinh^2(x_1) + sin^2(y_1)}} = \frac{e^{x}}{\sqrt{sinh^2(x) + sin^2(y)}} = (X^2)^{-1}, \tag{5.2}$$

$$X^3 = \frac{-2e^x(sinh^2(x_1) + sin^2(y_1))^{\frac{3}{2}}}{3(sinh^2(x_1)cosh^2(y_1)sin^2(y_1)} = -X^3. \tag{5.3}$$

(5) Smoothness

The expressions X^1, X^2, and X^3 are differentiable functions of their arguments and the map ϕ is also differentiable which implies that G is a differentiable manifold. Therefore, G forms a Lie group under the map ϕ. □

Theorem 5.2. *Let X^1, X^2, and X^3 be the Killing vectors. Then, these Killing vectors form a Lie group by using an exponential map.*

Proof. To define a Lie group, we need to define a Lie algebra generated by the killing vectors X^1, X^2, and X^3 to a Lie group. One common way to do this is to use the exponential map which maps elements of the Lie algebra to elements of the Lie group. The exponential map is defined as exp: $g \to G$ by

$$exp(tX) = e^{tX}, \tag{5.4}$$

where t is a scalar parameter and X is a Killing vector. We can use this map to define the Lie group generated by the Killing vectors X^1, X^2, and X^3.

To do this, we need to express the Killing vectors X^1, X^2, and X^3 in terms of a basis of the Lie algebra. One possible basis for this Lie algebra is given by

$$[E_1, E_2, E_3] = [X^1, X^2, X^3]. \tag{5.5}$$

Using this basis, we can express any element of the Lie algebra as a linear combination of the basis elements

$$(t_1 E_1 + t_2 E_2 + t_3 E_3), \tag{5.6}$$

where t_1, t_2, and t_3 are scalar parameters. To apply the exponential map to an element of the Lie algebra, we need to first find its matrix representation. This can be done by calculating the commutators of the basis elements

$$[E_1, E_2] = 0, \tag{5.7}$$

$$[E_1, E_3] = -2E_3, \tag{5.8}$$

$$[E_2, E_3] = 2E_3. \tag{5.9}$$

These commutation relations can be written as a matrix

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -2 & 0 \end{pmatrix}.$$

The exponential map of an element of the Lie algebra can then be the found by exponentiating this matrix

$$\exp(t_1 E_1 + t_2 E_2 + t_3 E_3) = e^{(t_1 0 + t_2 0 + t_3 A)} = e^{t_3 A}, \tag{5.10}$$

where A is the matrix defined above. The resulting exponential map will define the elements of the Lie group generated by the Killing vectors X^1, X^2, and X^3. Therefore, the Lie group generated by the Killing vectors X^1, X^2, and X^3 can be defined using the exponential map as follows:

$$g(x, y, z) = \exp(xE_1 + yE_2 + zE_3), \tag{5.11}$$

where x, y, and z are scalar parameters and E_1, E_2, and E_3 are the basis elements of the Lie algebra defined above in terms of X^1, X^2, and X^3. □

6. Regular surface

In this section, we prove that Lie group G form a regular surface.

Definition 6.1. In differential geometry, a regular surface refers to a mathematical concept that describes a smooth two-dimensional object embedded in three-dimensional space. To be more specific, a regular surface is a collection of points in three-dimensional space, and every point in this surface has a neighborhood that can be described by a smooth map from a subset of the plane. This indicates that the surface is locally flat, and it has a well-defined tangent plane at each point. Spheres, cones, and tori are a few examples of regular surfaces. On the other hand, a surface that contains singularities, such as a cone with its tip or a sphere with a point removed, is not considered a regular surface.

A surface patch X: $U \to R^3$ is called regular if it is smooth and the vectors X_u and X_v are linearly independent at the point $(u, v) \in U$. Equivalently, X should be smooth and the vector product $(X_u \times X_v)$ should be non-zero at every point of U [33].

Theorem 6.1. *Let G be a Lie group defined in Theorem 5.1. This Lie group G forms a regular surface.*

Proof. Let

$$G = \{X^1, X^2, X^3\}$$

be a Lie group, where X^1, X^2, and X^3 are defined in Eqs (3.32)–(3.34), respectively. In Figure 1, the representation of a regular surface is depicted by using these Killing vectors.

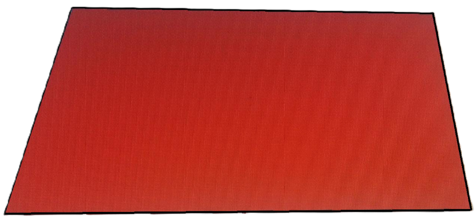

Figure 1. Regular surface.

$$G(\xi, \eta) = \left(\frac{e^{-\eta}}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}, \frac{e^{\xi}}{\sqrt{sinh^2(\xi) + sin^2(\eta)}}, \frac{-2(sinh^2(\xi) + sin^2(\eta))^{\frac{3}{2}} e^{\xi}}{3(cosh^2(\xi)cos^2(\eta)sin^2(\eta))} \right). \tag{6.1}$$

Partially differentiate with respect to ξ,

$$G_{\xi} = \left(\frac{-e^{-\eta}(sinh(\xi)cosh(\xi))}{(sinh^2(\xi) + sin^2(\eta))^{\frac{3}{2}}}, \frac{e^{\xi}(sinh^2(\xi) + sin^2(\eta) - sinh(\xi)cosh(\xi))}{(sinh^2(\xi) + sin^2(\eta))^{\frac{3}{2}}}, \frac{-2e^{\xi}\sqrt{(sinh^2(\xi) + sin^2(\eta))}V(\xi, \eta)}{3(sinh^2(\xi)cosh^2(\eta)sin^2(\eta))^2} \right), \tag{6.2}$$

$$\begin{aligned} V(\xi, \eta) =& (sinh^2(\xi)cosh^2(\eta)sin^2(\eta))(sinh^2(\xi) + sin^2(\eta)) + 3sinh^3(\xi)cosh(\xi)cosh^2(\eta)sin^2(\eta) \\ &- 2(sinh^2(\xi) + sin^2(\eta))(sinh(\xi)cosh(\xi)cosh^2(\eta)sin^2(\eta)). \end{aligned} \tag{6.3}$$

Take

$$G_\xi = (F_1(\xi,\eta), F_2(\xi,\eta), F_3(\xi,\eta)). \tag{6.4}$$

Partially differentiate with respect to η,

$$G_\eta = \left(\frac{e^{-\eta}(-\sinh^2(\xi) - \sin^2(\eta) + \sin(\eta)\cos(\eta))}{(\sinh^2(\xi) + \sin^2(\eta))^{\frac{3}{2}}}, \frac{e^{\xi}(\sin(\eta)\cos(\eta))}{(\sinh^2(\xi) + \sin^2(\eta))^{\frac{3}{2}}}, \right. \\ \left. \frac{-2e^{\xi}(\sinh^2(\xi) + \sin^2(\eta))^{\frac{1}{2}}(-3\sinh^2(\xi)\cosh^2(\eta)\sin^3(\eta)\cos(\eta) + (\sinh^2(\xi) + \sin^2(\eta))\sinh^2(\xi)\sinh(2\xi)\sin(2\eta))}{3(\sinh^2(\xi)\cosh^2(\eta)\sin^2(\eta))^2} \right). \tag{6.5}$$

Take

$$G_\eta = (G_1(\xi,\eta), G_2(\xi,\eta), G_3(\xi,\eta)), \tag{6.6}$$

$$G_\xi \times G_\eta = (F_2G_3 - F_3G_2, F_3G_1 - F_1G_3, F_1G_2 - F_2G_1). \tag{6.7}$$

Since

$$G_\xi \times G_\eta \neq 0, \ \ \forall(\xi,\eta) \in R^2 \tag{6.8}$$

and G is also smooth manifold. Therefore, Lie group G forms a regular surface. □

7. Differential map for Lie group

In the field of mathematics, a differential map refers to a function that connects two distinct manifolds, linking each point in the initial manifold to a linear transformation in the tangent spaces of the manifolds at that same point. Differential geometry is an area of mathematics that explores the study of geometric configurations on differential manifolds. The differential map is a key notion in differential geometry, representing a linear mapping between the tangent spaces of distinct differentiable manifolds.

The study of differential geometry, which deals with the geometric characteristics of curved objects, relies heavily on the notion of a tangent space. At any given point on a manifold, the tangent space is defined as the collection of all possible velocities of curves that pass through that point. This allows us to comprehend the behavior of the manifold in a local context. Additionally, the differential map, also known as the derivative map or the pushforward map, is a potent tool for examining manifolds. It is a linear transformation that takes the tangent space at one point and maps it to the tangent space at another point. By contrasting the tangent spaces at various points on the manifold using the differential map, we can gain valuable insights into the geometry of the manifold. Furthermore, the characteristics of the differential map can reveal the local and global structure of the manifold, making it a crucial concept in the realm of differential geometry.

The fundamental idea in differentiable manifolds is the concept of the differential map. This map characterizes the linearization of a differentiable function $f\colon M \to N$ between two manifolds, M and N, at a point p. Essentially, the differential map is a linear transformation that maps the tangent space T_pM of M at p to the tangent space $T_{f(p)}N$ of N at $f(p)$. It describes the relationship between the tangent spaces of M and N through the map f, enabling us to relate geometric concepts between manifolds that are connected through differentiable maps. Geometrically, the differential map maps a tangent vector $v \in T_pM$ to a tangent vector $df_p(v) \in T_{f(p)}N$, indicating the image of the tangent vector v under the influence of f at p. This makes the differential map a powerful tool for comprehending the local geometry of maps between manifolds and comprehending the structure and behavior of differentiable maps.

The differential map is an essential tool in differential geometry, as it allows us to compare geometric structures on differentiable manifolds. By using the differential map, we can define the concept of a smooth map between two manifolds, M and N. A smooth map is a differentiable function $f\colon M \to N$ that preserves the tangent structure of the manifold. In other words, f is a smooth map if it maps tangent vectors in M to tangent vectors in N in a smooth and consistent manner. The differential map is employed to define smoothness rigorously by assessing how well the map preserves the tangent spaces of the manifolds. Therefore, the differential map is a crucial concept in differential geometry that plays a significant role in studying the local and global properties of differentiable manifolds.

To summarize, the differential map is a crucial concept in differential geometry that serves as a key link between the geometry of differentiable manifolds. By mapping tangent vectors from one manifold to another, the differential map enables us to compare the local geometry of the manifolds and explore their interrelationships. Additionally, the differential map is employed to define essential concepts in differential geometry, such as smooth maps and map differentials. Ultimately, the differential map is a potent tool that aids us in comprehending the structure of differentiable manifolds and their connections through linear maps between their tangent spaces.

Definition 7.1. [33] Let $f\colon S \to R^n$ be a differentiable map and $p \in S$, $v \in T_p(S)$. Then, there exists a curve $\alpha\colon (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$, $\alpha'(0) = v$ and map $df_p\colon T_p(S_1) \to T_f(p)(S_2)$

$$df_p(v) = \frac{d}{dt}(f * \alpha)(t), \quad t = 0, \tag{7.1}$$

$$df_p(v) = \frac{d}{dt}(f(\alpha(t)), \quad t = 0. \tag{7.2}$$

Theorem 7.1. *If $f\colon S \to R^n$ is a differentiable map. Then, there exists a differential of f for the Lie group G defined above in Theorem 5.1 is $df_p(v)$.*

Proof. Let

$$\alpha(t) = (a_1(t), a_2(t), a_3(t))$$

be the curve passing through a point p on the surface G. Take differentiate with respect to t,

$$\alpha'(t) = (a_1'(t), a_2'(t), a_3'(t))\,. \tag{7.3}$$

Define a map $\alpha\colon (-\epsilon, \epsilon) \to S$ such that $\alpha(0) = p$, $\alpha'(0) = v$

$$v = \alpha'(0) = (a_1'(0), a_2'(0), a_3'(0)), \tag{7.4}$$

so,

$$df_p(v) = \frac{d}{dt}(f * \alpha)(t), \quad t = 0, \tag{7.5}$$

$$df_p(v) = \frac{d}{dt}(f(\alpha(t))) \tag{7.6}$$

$$df_p(v) = \frac{d}{dt}(f(a_1(t), a_2(t), a_3(t)), \tag{7.7}$$

$$df_p(v) = \frac{d}{dt}\left(\frac{e^{-a_1(t)}}{\sqrt{sinh^2(a_1(t) + sin^2(a_1(t)}}, \frac{e^{a_2(t)}}{\sqrt{sinh^2(a_2(t)) + sin^2(a_2(t))}}, \frac{-2\left(sinh^2(a_3(t)) + sin^2(a_3(t))\right)^{\frac{3}{2}} e^{a_3(t)}}{3(sinh^2(a_3(t)cosh^2(a_3(t))sin^2(a_3(t))}\right), \tag{7.8}$$

$$\begin{aligned} df_p(v) = &\left(\frac{a_1'(t)e^{-a_1(t)}(-2\sinh^2(a_1(t)) - 2\sin^2(a_1(t)) - \sinh(2a_1(t)) + \sin(2a_1(t)))}{2(\sinh^2(a_1(t)) + \sin^2(a_1(t)))^{\frac{3}{2}}}\right. \\ &\frac{a_2'(t)e^{a_2(t)}(2\sinh^2(a_2(t)) + 2\sin^2(a_2(t)) - \sinh(2a_2(t)) + \sin(2a_2(t)))}{2(\sinh^2(a_2(t)) + \sin^2(a_2(t)))^{\frac{3}{2}}} \\ &\left.\frac{-3a_3'(t)e^{a_3(t)}(\sinh^2(a_3(t))\cosh^2(a_3(t))\sin^2(a_3(t)))U(t)}{3(\sinh^2(a_3(t))\cosh^2(a_3(t))\sin^2(a_3(t)))^2}\right), \end{aligned} \tag{7.9}$$

where

$$\begin{aligned} U(t) = &\sqrt{\sinh^2(a_3(t)) + \sin^2(a_3(t))(\sinh(2a_3(t)) - \sin(2a_3(t)))} \\ &+ 2(\sinh^2(a_3(t)) + \sin^2(a_3(t)))^{\frac{3}{2}}(\sinh(2a_3(t))\sinh(2a_3(t))\sin(2a_3(t))), \end{aligned}$$

which is required differential map for Lie group G. □

8. Differential of normal vector field on Lie group

The differential of a unit normal vector field is an important concept in differential geometry that helps us to understand how the normal vector to a surface varies as we traverse the surface. More specifically, if N is a unit normal vector field on a smooth surface S in three-dimensional space, the differential of N at a point p on S is a vector field that describes the infinitesimal changes in the normal vector as we move around on the surface.

This concept is vital in differential geometry since it gives us valuable information about the geometry and curvature of the surface. The differential of a normal vector field is employed in various applications, including the study of geometric flows, surface dynamics, and physical laws that involve the behavior of surfaces in space. Therefore, the differential of a normal vector field is a significant concept in differential geometry, offering crucial insights into the local and global properties of surfaces in three-dimensional space.

Definition 8.1. [33] Let a map N: $S \to S^2$ which is a Gauss map. The differential of N is defined by

$$dN_p : T_p(S) \to T_N(p)(S^2) = T_p(S),$$

where N is a normal vector field on surface S

$$dN_p(v) = \frac{d}{dt}(N * \alpha)(t), \quad t = 0, \tag{8.1}$$

where α is a curve passing through the point on the surface.

Theorem 8.1. *Let N be a unitary normal vector field. Then, the differential of normal vector field of the surface which is a Lie group is*

$$dN_p(v) = \frac{v}{R}, \quad t = 0. \tag{8.2}$$

Proof. Since the unit normal vector field is

$$N = \frac{(F_2G_3 - F_3G_2, F_3G_1 - F_1G_3, F_1G_2 - F_2G_1)}{\sqrt{(F_2G_3 - F_3G_2)^2 + (F_3G_1 - F_1G_3)^2 + (F_1G_3 - F_2G_1)^2}}. \tag{8.3}$$

Let

$$\alpha(t) = (a_1(t), a_2(t), a_3(t))$$

be the curve passing through a point p on the surface G. Take differentiate with respect to t

$$\alpha'(t) = (a_1'(t), a_2'(t), a_3'(t)). \tag{8.4}$$

Define a map α: $(-\epsilon, \epsilon) \to S$, such that $\alpha(0) = p$, $\alpha'(0) = v$,

$$v = \alpha'(0) = (a_1'(0), a_2'(0), a_3'(0)), \tag{8.5}$$

$$dN_p(v) = \frac{d}{dt}(N * \alpha)(t), \quad t = 0, \tag{8.6}$$

$$dN_p(v) = \frac{d}{dt}(N(\alpha(t)), \tag{8.7}$$

$$dN_p(v) = \frac{d}{dt}(N(a_1(t), a_2(t), a_3(t))), \tag{8.8}$$

$$\begin{aligned} dN_p(v) =& \frac{d}{dt}\left(\frac{(F_2(a_1(t))G_3(a_1(t)) - G_2(a_1(t))F_3(a_1(t)))}{R}\right) \\ & \frac{d}{dt}\left(\frac{(G_1(a_2(t))F_3(a_2(t)) - F_1(a_2(t))G_3(a_2(t)))}{R}\right) \\ & \frac{d}{dt}\left(\frac{(F_1(a_3(t))G_2(a_3(t)) - F_2(a_3(t))G_1(a_3(t)))}{R}\right), \end{aligned} \tag{8.9}$$

where

$$R = \sqrt{(F_2G_3 - F_3G_2)^2 + (F_3G_1 - F_1G_3)^2 + (F_1G_3 - F_2G_1)^2},$$

$$\begin{aligned} dN_p(v) =& \frac{1}{R}\Big[F_2'(a_1(t))G_3(a_1(t)) + F_2(a_1(t))G_3'(a_1(t)) - G_2'(a_1(t))F_3(a_1(t)) - G_2(a_1(t))F_3'(a_1(t)) \\ & G_1'(a_2(t))F_3(a_2(t)) + G_1(a_2(t))F_3'(a_2(t)) - F_1'(a_2(t))G_3(a_2(t)) - F(a_2(t))G_3'(a_2(t)) \\ & F_1'(a_3(t))G_2(a_3(t)) + F_1(a_3(t))G_2'(a_3(t)) - F_2'(a_3(t))G_1(a_3(t)) - F_2(a_3(t))G_1'(a_3(t))\Big], \end{aligned} \tag{8.10}$$

$$dN_p(v) = \frac{v}{R}, \quad t = 0, \tag{8.11}$$

which is required. □

9. Gaussian (K_g) and mean (H) curvatures of a Lie group

In this section, the main objective is to determine the K_g and H of a surface that is classified as a Lie group. The methodology used for this task involves calculating the first fundamental form (FFF) and second fundamental form (SFF), which allow us to obtain the coefficients needed to determine the curvatures. It is important to note that this particular approach is specific to surfaces that fall under the category of Lie groups, and cannot be applied to other types of surfaces. By employing this method, we can gain a comprehensive understanding of the surface's geometry and properties. Overall, this process of determining curvatures serves as a valuable tool for further analysis and investigation of surfaces.

9.1. FFF

The FFF is a crucial mathematical tool that is extensively used to analyze surfaces in three-dimensional space. It enables the measurement of the intrinsic geometry of a surface, independent of any particular embedding in three-dimensional space. To define the FFF, the dot product of the tangent vectors to a surface at a given point is computed. This measures the inner product of the surface's partial derivatives with respect to its two parameters. The dot product provides essential information about the angles and lengths of curves on the surface, which are fundamental in comprehending the surface's geometry. Researchers can gain a better understanding of surfaces and their properties by utilizing the FFF.

The FFF is an essential concept in the field of differential geometry, which plays a vital role in the calculation of different geometric properties of surfaces. Mathematicians use this concept to determine surface curvature, length of curves on the surface, and other related properties. It is a mathematical tool represented by the symbol I, and it is defined as the dot product of tangent vectors to a surface at a particular point. The application of the FFF is not limited to the study of surfaces but also encompasses the investigation of minimal surfaces that minimize their surface area subject to certain constraints. This mathematical concept is crucial in understanding the local geometry of a surface in three-dimensional Euclidean space. In conclusion, the FFF is a fundamental tool in differential geometry that plays a critical role in the study of surfaces and is represented by an equation [33]

$$I = Edu^2 + 2Fdudv + Gdv^2, \tag{9.1}$$

where u and v are the parameters used to parameterize the surfac; and E, F, and G are functions of u and v that describe the local geometry of the surface. The coefficients of FFF are a vital aspect of surface analysis. In particular, the terms E, F, and G correspond to the lengths of the edges of the tangent plane to the surface and play a critical role in understanding the surface's geometry. Specifically, E and G represent the squares of the lengths of the two tangent vectors, $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$, respectively. On the other hand, F represents their dot product. By utilizing these coefficients, mathematicians can better understand the local geometry of a surface, and determine its curvature and other geometric properties. Therefore, understanding the meaning and significance of E, F, and G is fundamental in the study of surfaces and their properties. In other words, the first fundamental form describes how the surface bends and curves locally, and it is an essential tool for studying properties of surfaces, such as their curvature and area. The coefficients of FFF are

$$E = G_u.G_u = \parallel G_u \parallel^2, \tag{9.2}$$

$$E = F_1^2 + F_2^2 + F_3^2, \tag{9.3}$$

$$F = G_u.G_v = F_1G_1 + F_2G_2 + F_3G_3, \tag{9.4}$$

$$G = G_v.G_v = G_1^2 + G_2^2 + G_3^2. \tag{9.5}$$

FFF is

$$I = Edu^2 + 2Fdudv + Gdv^2, \tag{9.6}$$

$$I = (F_1^2 + F_2^2 + F_3^3)du^2 + 2(F_1G_1 + F_2G_2 + F_3G_3)dudv + (G_1^2 + G_2^2 + G_3^2)dv^2. \tag{9.7}$$

9.2. SFF

The L, M, and N coefficients play a crucial role in calculating different geometric quantities related to a surface. For instance, they can be used to determine K_g and H of the surface. The mean curvature, in particular, can be obtained by computing the trace of the SFF, which is equivalent to adding up the L, M, and N coefficients. It is important to note that these coefficients are independent of the coordinate system used and can provide valuable insights into the shape and behavior of a surface.

Understanding the SFF is important in many areas of mathematics and physics, such as the study of minimal surfaces, the theory of relativity, and the mechanics of surfaces under stress. The coefficients of SFF are

$$G_{uu} = \left(F_1'(u,v), F_2'(u,v), F_3'(u,v)\right), \tag{9.8}$$

$$G_{uv} = \left(H_1(u,v), H_2(u,v), H_3(u,v)\right), \tag{9.9}$$

$$G_{vv} = \left(G_1'(u,v), G_2'(u,v), G_3'(u,v)\right). \tag{9.10}$$

The unitary normal vector is

$$P = \frac{(F_2G_3 - F_3G_2, F_3G_1 - F_1G_3, F_1G_2 - F_2G_1)}{\sqrt{(F_2G_3 - F_3G_2)^2 + (F_3G_1 - F_1G_3)^2 + (F_1G_3 - F_2G_1)^2}}, \tag{9.11}$$

so

$$L = G_{uu}.P, \tag{9.12}$$

$$L = \frac{1}{R}[F_1'(F_2G_3 - F_3G_2) + F_2'(F_3G_1 - F_1G_3) + F_3'(F_1G_2 - F_2G_1)], \tag{9.13}$$

where (') represents the derivative.

$$M = G_{uv}.P, \tag{9.14}$$

$$M = \frac{1}{R}[H_1(F_2G_3 - G_2F_3) + H_2(G_1F_3 - F_1G_3) + H_3(F_1G_2 - G_1F_2)], \tag{9.15}$$

$$N = G_{vv}.P, \tag{9.16}$$

$$N = \frac{1}{R}[G_1'(F_2G_3 - G_2F_3) + G_2'(G_1F_3 - F_1G_3) + G_3'(F_1G_2 - G_1F_2)]. \tag{9.17}$$

The SFF is

$$II = Ldu^2 + 2Mdudv + Ndv^2, \tag{9.18}$$

$$II = \frac{1}{R}\left[F_1'(F_2G_3 - G_2F_3) + F_2'(G_1F_3 - F_1G_3) + F_3'(F_1G_2 - G_1F_2)\right]du^2 + \frac{2}{R}\left[H_1(F_2G_3 - G_2F_3) + H_2(G_1F_3 - F_1G_3) + H_3(F_1G_2 - G_1F_2)\right]dudv + \frac{1}{R}\left[G_1'(F_2G_3 - G_2F_3) + G_2'(G_1F_3 - F_1G_3) + G_3'(F_1G_2 - G_1F_2)\right]dv^2. \quad (9.19)$$

9.3. Gaussian (K_g) and mean (H) curvatures of Lie group

The K_g and H of a surface are geometric quantities that can be computed using the FFF and SFF coefficients of the surface. The FFF coefficients of a surface are the coefficients of the metric tensor, which determine the lengths of tangent vectors on the surface. The SFF coefficients describe how the surface curves in the ambient space. Specifically, they are related to the shape of the surface in the normal direction. The measurement of how much a surface curves in two dimensions is determined by its K_g. This curvature is calculated as the product of the surface's principal curvatures and can be computed using the coefficients of the FFF and SFF. To be precise, the K_g can be derived from the FFF coefficients, denoted as E, F, and G and the SFF coefficients (L, M, and N) [33]

$$K_g = (LN - M^2)/(EG - F^2). \quad (9.20)$$

The mean curvature of a surface is a measure of how much the surface curves in one dimension. It is defined as the average of the principal curvatures of the surface, which can also be computed using the FFF and SFF coefficients. Specifically, the mean curvature is given by

$$H = (1/2)((EN + GL - 2FM)/(EG - F^2)), \quad (9.21)$$

so, the K_g and H of a surface can be computed using the FFF and SFF coefficients.

$$LN - M^2 = \frac{1}{R^2}[(F_1'(F_2G_3 - F_3G_2) + F_2'(G_1F_3 - F_1G_3) + F_3'(F_1G_2 - G_1F_2)) + G_1'(F_2G_3 - F_3G_2) + G_2'(F_3G_1 - F_1G_3) + G_3'(F_1G_2 - F_2G_1) - (H_1(F_2G_3 - F_3G_2) + H_2(F_3G_1 - F_1G_3) + H_3(F_1G_2 - F_2G_1))^2]. \quad (9.22)$$

Take

$$LN - M^2 = Q(F_1, F_2, F_3, G_1, G_2, G_3, H_1, H_2, H_3, F_1', F_2', F_3', G_1', G_2', G_3'), \quad (9.23)$$

$$GE - F^2 = (G_1^2 + G_2^2 + G_3^3)(F_1^2 + F_2^2 + F_3^2) - (F_1G_1 + F_2G_2 + F_3G_3)^2. \quad (9.24)$$

Take

$$GE - F^2 = W(F_1, F_2, F_3, G_1, G_2, G_3), \quad (9.25)$$

so,

$$K_g = \frac{Q(F_1, F_2, F_3, G_1, G_2, G_3, H_1, H_2, H_3, F_1', F_2', F_3', G_1', G_2', G_3')}{W(F_1, F_2, F_3, G_1, G_2, G_3)}. \quad (9.26)$$

The mean curvature is

$$H = \frac{GL + NE - 2MF}{2(GE - F^2)}, \quad (9.27)$$

$$GL + NE - 2MF = \frac{1}{R}\Big((G_1^2 + G_2^2 + G_3^2)[F_1'(F_2G_3 - F_3G_2) + F_2'(G_1F_3 - F_1G_3) \\ +F_3'(F_1G_2 - F_2G_1)] + (F_1^2 + F_2^2 + F_3^2)[G_1'(F_2G_3 - F_3G_2) \\ +G_2'(F_3G_1 - F_1G_3) + G_3'(F_1G_2 - F_2G_1)] - 2(F_1G_1 + F_2G_2 + F_3G_3) \\ \times[H_1(F_2G_3 - F_3G_2) + H_2(F_3G_1 - F_1G_3) + H_3(F_1G_2 - F_2G_1)]\Big). \tag{9.28}$$

Take

$$GL + NE - 2MF = Z(F_1, F_2, F_3, G_1, G_2, G_3, H_1, H_2, H_3, F_1', F_2', F_3', G_1', G_2', G_3'), \tag{9.29}$$

so,

$$H = \frac{Z(F_1, F_2, F_3, G_1, G_2, G_3, H_1, H_2, H_3, F_1', F_2', F_3', G_1', G_2', G_3')}{2W(F_1, F_2, F_3, G_1, G_2, G_3)}. \tag{9.30}$$

10. Conclusions

In this paper, we constructed a new Lie group with the help of Killing vectors. First, we find the Killing vectors and then constructed the Lie algebra by using the Lie bracket on Killing vectors. After we constructed the Lie algebra, we proved that these Killing vectors form a Lie group by using an exponential map. Then, we discussed its geometric properties. We defined the differential map and differential of normal vector on the Lie group and also calculated its Gaussian and mean curvatures.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors thank to Dr. Ibtesam Eid Almasoudi Alshammari, Vice dean of E- Learning and Digital Transformation, Associate Professor, Faculty of Science, University of Hafr Al Batin for funding this research work.

Conflict of interest

The authors declare no conflicts of interest.

References

1. B. C. Hall, *Lie groups, Lie algebras, and representations*, Springer Science & Business Media, 2013. https://doi.org/10.1007/978-3-319-13467-3
2. V. Chiek, Geodesic on surfaces of constant Gaussian curvature, *Theses Digitization Proj.*, 2006, 3045.
3. H. Jafari, N. Kadkhoda, D. Baleanu, *Lie group theory for nonlinear fractional $K(m, n)$ type equation with variable coefficients*, Springer Science & Business Media, 2022, 207–227. https://doi.org/10.1007/978-3-030-77169-0_8

4. S. Deshmukh, A. Ishan, S. B. Al-Shaikh, C. Özgür, A note on Killing calculus on Riemannian manifolds, *Mathematics*, **9** (2021), 307. https://doi.org/10.3390/math9040307

5. E. Celledoni, A. Iserles, Approximating the exponential from a Lie algebra to a Lie group, *Math. Comput.*, **69** (2000), 1457–1480. https://doi.org/10.1090/S0025-5718-00-01223-0

6. A. Hasic, Representations of Lie groups, *Adv. Linear Algebra Matrix Theory*, **11** (2021), 117–134.

7. A. Hasic, Introduction to Lie algebras and their representations, *Adv. Linear Algebra Matrix Theory*, **11** (2021), 67–91. https://doi.org/10.4236/alamt.2021.113006

8. J. Yang, J. Meng, X. Xin, Lie symmetry analysis, optimal system and exact solutions for variable-coefficients Boiti-Leon-Manna-Pempinelli equation, *Phys. Scr.*, **99** (2024), 025233. https://doi.org/10.1088/1402-4896/ad1a32

9. A. Liaqat, I. Hussain, Gravitational mass and approximate Lie symmetries for the charged Hayward black hole, *New Astron.*, **107** (2024), 102152. https://doi.org/10.1016/j.newast.2023.102152

10. C. Chevalley, *Theory of Lie groups*, Courier Dover Publications, 2018. https://doi.org/10.2307/3610752

11. J. F. Adams, *Lectures on Lie groups*, University of Chicago Press, 1982.

12. M. Rahioui, E. H. E. Kinani, A. Ouhadan, Lie symmetries, invariant subspace method, and conservation laws for a time fractional generalized Broer-Kaup system, *Comput. Appl. Math.*, **43** (2024), 36. https://doi.org/10.1007/s40314-023-02556-8

13. M. A. Naimark, *Linear representations of the Lorentz group*, Elsevier, 2014.

14. R. W. Carter, *Simple groups of Lie type*, John Wiley & Sons, Inc., 1989.

15. A. W. Knapp, *Lie groups beyond an introduction*, Birkhäuser, 1996. https://doi.org/10.1007/978-1-4757-2453-0

16. D. Prinz, A. Schmeding, Lie theory for asymptotic symmetries in general relativity: the BMS group, *Class. Quantum Grav.*, **39** (2022), 065004. https://doi.org/10.1088/1361-6382/ac4ae2

17. F. Schwarz, *Algorithmic Lie theory for solving ordinary differential equations*, CRC Press, 2007. https://doi.org/10.1201/9781584888901

18. S. Silvestrov, E. Paal, V. Abramov, A. Stolin, *Generalized Lie theory in mathematics, physics and beyond*, Springer Science & Business Media, 2009. https://doi.org/10.1007/978-3-540-85332-9

19. V. Dobrev, *Lie theory and its applications in physics*, Springer Science & Business Media, 2013. https://doi.org/10.1007/978-4-431-54270-4

20. A. L. Onishchik, V. Gorbatsevich, E. Vinberg, *Foundations of Lie theory, Lie transformation groups*, Springer-Verlag, 1993.

21. S. Kumar, D. Kumar, H. Kharbanda, Lie symmetry analysis, abundant exact solutions and dynamics of multisolitons to the (2+1)-dimensional KP-BBM equation, *Pramana*, **95** (2021), 33. https://doi.org/10.1007/s12043-020-02057-x

22. S. Kumar, D. Kumar, Lie symmetry analysis and dynamical structures of soliton solutions for the (2+1)-dimensional modified CBS equation, *Int. J. Mod. Phys. B*, **34** (2020), 2050221. https://doi.org/10.1142/S0217979220502215

23. S. Kumar, S. Rani, Lie symmetry reductions and dynamics of soliton solutions of (2+1)-dimensional Pavlov equation, *Pramana*, **94** (2020), 116. https://doi.org/10.1007/s12043-020-01987-w

24. S. Kumar, H. Almusawa, A. Kumar, Some more closed-form invariant solutions and dynamical behavior of multiple solitons for the (2+1)-dimensional rdDym equation using the Lie symmetry approach, *Results Phys.*, **24** (2021), 104201. https://doi.org/10.1016/j.rinp.2021.104201

25. S. Kumar, D. Kumar, A. M. Wazwaz, Lie symmetries, optimal system, group-invariant solutions and dynamical behaviors of solitary wave solutions for a (3+1)-dimensional KdV-type equation, *Eur. Phys. J. Plus*, **136** (2021), 531. https://doi.org/10.1140/epjp/s13360-021-01528-3

26. D. Kumar, S. Kumar, Some new periodic solitary wave solutions of (3+1)-dimensional generalized shallow water wave equation by Lie symmetry approach, *Comput. Math. Appl.*, **78** (2019), 857–877. https://doi.org/10.1016/j.camwa.2019.03.007

27. S. L. Shen, C. J. Song, Disturbation to Lie symmetry for constrained Hamiltonian system within Agrawal's operators, *AIP Adv.*, **14** (2024), 0187329. https://doi.org/10.1063/5.0187329

28. E. Feleqi, F. Rampazzo, Iterated Lie brackets for nonsmooth vector fields, *Nonlinear Differ. Equations Appl. NoDEA*, **24** (2017), 61. https://doi.org/10.1007/s00030-017-0484-4

29. D. J. Cirilo-Lombardo, N. G. Sanchez, Quantum-spacetime symmetries: a principle of minimum group representation, *Universe*, **10** (2024), 22. https://doi.org/10.3390/universe10010022

30. L. Ju, Y. Zhang, F. Afzal, B. Feng, Conservation law and Lie symmetry analysis of the Zakharov-Kuznetsov equation, *Mod. Phys. Lett.*, **38** (2024), 2350254. https://doi.org/10.1142/S0217984923502548

31. S. Lipschutz, D. Spellman, M. R. Spiegel, *Vector analysis and an introduction to tensor analysis*, 2 Eds., McGraw-Hill, 2009.

32. U. Camci, K. Saifullah, Conformal symmetries of the energy-momentum tensor of spherically symmetric static spacetimes, *Symmetry*, **14** (2022), 647. https://doi.org/10.3390/sym14040647

33. A. N. Pressley, *Elementary differential geometry*, Springer Science & Business Media, 2011. https://doi.org/10.1007/978-1-84882-891-9