AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 6073–6087.
DOI: 10.3934/math.2024297
Received: 22 November 2023
Revised: 31 December 2023
Accepted: 15 January 2024
Published: 01 February 2024

Research article

Convergence ball of a new fourth-order method for finding a zero of the derivative

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Abstract: There are numerous applications for finding zero of derivatives in function optimization. In this paper, a two-step fourth-order method was presented for finding a zero of the derivative. In the research process of iterative methods, determining the ball of convergence was one of the important issues. This paper discussed the radii of the convergence ball, uniqueness of the solution, and the measurable error distances. In particular, in contrast to Wang's method under hypotheses up to the fourth derivative, the local convergence of the new method was only analyzed under hypotheses up to the second derivative, and the convergence order of the new method was increased to four. Furthermore, different radii of the convergence ball was determined according to different weaker hypotheses. Finally, the convergence criteria was verified by three numerical examples and the new method was compared with Wang's method and the same order method by numerical experiments. The experimental results showed that the convergence order of the new method is four and the new method has higher accuracy at the same cost, so the new method is finer.

Keywords: iterative method; error estimate; convergence ball; center Lipschitz; Lipschitz condition
Mathematics Subject Classification: 65B99, 65H05

1. Introduction

The main aim of this study is to provide an approximate solution x_* for the following equation

$$f'(x) = 0 \tag{1.1}$$

where f denotes differentiable operator according to Fréchet, which is defined in a convex subset N in real space $\boldsymbol{R}$ (or complex space $\boldsymbol{C}$).

The above issue is crucial for many applications, particularly in function optimization (see [2, 7, 17, 18]). The K-T condition for the optimal problem of no restriction

$$\min f(x) \tag{1.2}$$

claims that if function f is a differentiable operator, the optimal solution of Eq (1.2) must be a solution of Eq (1.1). Since solutions of Eq (1.1) can only be discovered in closed form in certain cases, the majority of methods for solving Eq (1.1) are iterative methods.

Semi-local and local convergence analysis are the basis of the study of the convergence problem of iterative methods. The local convergence issue is to find estimates of the radii of the convergence ball based on the information surrounding a solution (see [1,8,10,13]), whereas the semi-local convergence issue is to provide criteria assuring the convergence of the iterative process based on the information around an initial point.

There are many iterative methods for finding a solution of (1.1). Iterative methods have many different properties [20,21]. Newton's method (see [4,5,9,15]) is defined by

$$x_{n+1} = x_n - \frac{f'(x_{n+1})}{f''(x_n)} \quad (n \geq 0)\ (x_0 \in N). \tag{1.3}$$

Newton's method converges quadratically under certain hypotheses. However, it needs to compute the derivative of second order, and this is frequently difficult in some cases. To avoid this, we can use the secant method (see [14,22]) instead, as follows

$$x_{n+1} = x_n - \frac{x_n - x_{n-1}}{f'(x_n) - f'(x_{n-1})} f'(x_n) \quad (n \geq 0)\ (x_0, x_{-1} \in N). \tag{1.4}$$

The order of convergence of the secant method is 1.618.... Wang [16] proposed an iterative method, that avoids computing the second derivative and keeps the convergence of the second order at the same time. Wang's method is defined by

$$x_{n+1} = x_n - \frac{f'(x_n)}{\delta(f; x_n, x_{n-1})} \quad (n \geq 0)\ (x_0, x_{-1} \in N) \tag{1.5}$$

where

$$\delta(f; x, y) = \frac{1}{x-y}\left(4f'(x) - 6\frac{f(x) - f(y)}{x-y} + 2f'(y)\right). \tag{1.6}$$

The convergence of the above method was studied under different conditions [9, 12]. In particular, a matter of concern is to estimate the radius of the convergence ball of Wang's method [23]. The convergence ball radius of Wang's method was given as

$$R_w = \frac{12}{\sqrt{81W^2 + 96Y} + 9W} \tag{1.7}$$

where

$$|f'''(x)f''(x_*)^{-1}| \leq W \quad (x \in N) \tag{1.8}$$

and

$$|f^{(IV)}(x)f''(x_*)^{-1}| \leq Y \quad (x \in N). \tag{1.9}$$

An open ball $B(x, r)$ stands for a convergence ball with center x and radius r.

Though the above method (1.5) may converge, $f^{(IV)}(x)$ in the condition (1.9) limits the applicability of this method. For example, function f in $N = [0, 1]$ is defined by

$$f(x) = \begin{cases} \frac{1}{4}\ln x - \frac{1}{12}x^6 + \frac{1}{20}x^5, & x \neq 0; \\ 0, & x = 0 \end{cases} \tag{1.10}$$

$f^{(IV)}(x)$ is unbounded on N and constant Y doesn't exist. Therefore, the results in [5, 6, 23] by applying Y cannot be used.

In order to make the convergence conditions weaker, suppose the second derivative of function f is Lipschitz continuous. Wang [11] proposed a local convergence theorem under the central Lipschitz condition, based on the new concept of restricted domain conditions (see [2, 3, 18]) as follows

$$|f''(x_*)^{-1}(f''(x) - f''(x_*))| \leq C_0|x - x_*| \quad (x \in N). \tag{1.11}$$

At the same time, the following is the restricted Lipschitz condition

$$|f''(x_*)^{-1}(f''(x) - f''(y))| \leq C|x - y| \quad (x, y \in N_0) \tag{1.12}$$

where,

$$N_0 = N \bigcap B(x_*, \frac{1}{3C_0}).$$

Under Lipschitz conditions (1.11) and (1.12), the radius of the convergence ball is obtained

$$\mathbb{R}_1 = \frac{6}{13C + 18C_0}.$$

Furthermore, Wang also proposed another restricted Lipschitz condition that could replace condition (1.12) as follows

$$|f''(x_*)^{-1}(f''(x) - f''(y))| \leq L_0|x - y| \quad (x, y \in S_0) \tag{1.13}$$

where,

$$S_0 = N \bigcap B(x_*, \frac{1}{C_0}).$$

Under Lipschitz conditions (1.11) and (1.13), the radius of the convergence ball is obtained

$$\mathbb{R}_2 = \frac{6}{17L_0 + 6C_0}.$$

In this study, based on Wang's method, a two-step four-order method is presented for analyzing a derivative's zero. This new method is defined for $x_0, y_{-1} \in Q$ and all $n = 0, 1, 2, \ldots$ as follows

$$\begin{cases} y_n = x_n - \dfrac{f'(x_n)}{\delta(f; x_n, y_{n-1})}, \\ x_{n+1} = y_n - \dfrac{f'(y_n)}{\delta(f; x_n, y_n)}. \end{cases} \tag{1.14}$$

The rest of the paper is laid out as follows: Section 2 discusses the local convergence analysis of the new method (1.14), including convergence ball radii, the measurable error distance, and uniqueness of the solution. In particular, different radius of convergence ball and the measurable error distances are determined according to different Lipschitz conditions. In Section 3, three examples are given, and the new method (1.14) is compared with Wang's method (1.5) and fourth-order method (3.1) by numerical experiments, so the new method (1.14) has a higher convergence order and higher accuracy. Section 4 is devoted to some conclusions.

2. Local convergence

In this section, the local convergence analysis of iterative method (1.14) is studied. Let us assume that f on the convex set N is twice differentiable. The definitions and attributes of $f[.,.]$ and $f[.,.,.]$ are found in [14], which are divided differences of order one and two, and the second order divided difference has the integral representations given as follows:

$$f[x,y,z] = \int_0^1 \int_0^1 (1-t)f''(tx + s(1-t)y + (1-s)(1-t)z)\,\mathrm{ds}\,\mathrm{dt} \quad (\text{for each } x,y,z \in N). \tag{2.1}$$

Assume $\delta(f;x,y)$ is given by (1.6). By applying the first-order and second-order divided difference definitions, we have

$$\delta(f;x,y) = 4f[x,x,y] - 2f[x,y,y]. \tag{2.2}$$

The following local convergence theorem of our method (1.14) is presented under conditions (1.11) and (1.12).

Theorem 2.1. *Suppose $f'(x_*) = 0, f''(x_*) \neq 0$, f is twice differentiable on N, and that Lipschitz continuous conditions* (1.11) *and* (1.12) *of the second-order derivative hold. Denote*

$$\mathbb{R}_1 = \frac{6}{13C + 18C_0}. \tag{2.3}$$

Given an initial point x_0 in ball $B(x_*, \mathbb{R}_1) \subseteq N$, the sequence $\{x_n\}$ produced by our method (1.14) *converges to its unique solution $x_* \in B(x_*, \frac{2}{C_0}) \subseteq N$. $B(x_*, \frac{2}{C_0})$ is larger than $B(x_*, \mathbb{R}_1)$. Furthermore, we get the following error estimates*

$$|x_{n+1} - x_*| \leq \frac{91C + 126C_0}{78}|y_n - x_*|^2 + \frac{13C + 18C_0}{13}|x_n - x_*||y_n - x_*|, \quad (n \geq 0) \tag{2.4}$$

where

$$|y_n - x_*| \leq \frac{13C + 18C_0}{26}|x_n - x_*|^2 + \frac{26C + 36C_0}{39}|x_n - x_*||y_{n-1} - x_*|, \quad (n \geq 0) \tag{2.5}$$

and

$$|x_{n+1} - x_*| \leq \frac{|x_n - x_*||y_n - x_*|}{\mathbb{R}_1}, \quad (n \geq 1). \tag{2.6}$$

Proof. By applying mathematical induction on n, we will prove that $x_n \in B(x_*, \mathbb{R}_1)$ is well defined, for $y_{-1}, x_0 \in B(x_*, \mathbb{R}_1)$, and error Eqs (2.4)–(2.6) hold under Lipschitz conditions (1.11) and (1.12).

Since $y_{-1}, x_0 \in B(x_*, \mathbb{R}_1)$, by (1.14) and (2.2), we have

$$\begin{aligned}
y_0 - x_* &= x_0 - x_* - \frac{f'(x_0)}{\delta(f;x_0,y_{-1})} = x_0 - x_* - \frac{f[x_0,x_0] - f[x_0,x_*] + f[x_0,x_*] - f[x_*,x_*]}{\delta(f;x_0,y_{-1})} \\
&= x_0 - x_* - \frac{f[x_0,x_0] - f[x_0,x_*] + f[x_0,x_*] - f[x_*,x_*]}{4f[x_0,x_0,y_{-1}] - 2f[x_0,y_{-1},y_{-1}]} \\
&= x_0 - x_* - \frac{f[x_0,x_0,x_*](x_0 - x_*) + f[x_0,x_*,x_*](x_0 - x_*)}{4f[x_0,x_0,y_{-1}] - 2f[x_0,y_{-1},y_{-1}]} \\
&= \frac{f''(x_*)^{-1}(4f[x_0,x_0,y_{-1}] - 2f[x_0,y_{-1},y_{-1}] - f[x_0,x_0,x_*] - f[x_0,x_*,x_*])}{f''(x_*)^{-1}(4f[x_0,x_0,y_{-1}] - 2f[x_0,y_{-1},y_{-1}])}(x_0 - x_*) \\
&= \frac{A_1}{B_1}(x_0 - x_*).
\end{aligned} \tag{2.7}$$

Using (2.2), (1.12) and (2.1), we have

$$\begin{aligned}
|A_1| =& 4f[x_0, x_0, y_{-1}] - 2f[x_0, y_{-1}, y_{-1}] \\
=& |f''(x_*)^{-1}(2(f[x_0, x_0, y_{-1}] - f[x_0, y_{-1}, y_{-1}]) + f[x_0, x_0, y_{-1}] \\
& - f[x_0, x_*, x_*] + f[x_0, x_0, y_{-1}] - f[x_0, x_*, x_*]| \\
=& |2\int_0^1 \int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_0 + s(1-t)x_0 + (1-s)(1-t)y_{-1}) \\
& - f''(tx_0 + s(1-t)y_{-1} + (1-s)(1-t)y_{-1}))\,ds\,dt \\
& + \int_0^1 \int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_0 + s(1-t)x_0 + (1-s)(1-t)y_{-1}) \\
& - f''(tx_0 + s(1-t)x_0 + (1-s)(1-t)x_*))\,ds\,dt \\
& + \int_0^1 \int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_0 + s(1-t)x_0 + (1-s)(1-t)y_{-1}) \\
& - f''(tx_0 + s(1-t)x_* + (1-s)(1-t)x_*))\,ds\,dt| \\
\leq& 2C\int_0^1 \int_0^1 s(1-t)^2|x_0 - y_{-1}|\,ds\,dt + C\int_0^1 \int_0^1 (1-s)(1-t)^2|x_* - y_{-1}|\,ds\,dt \\
& + C\int_0^1 \int_0^1 (s(1-t)^2|x_0 - x_*| + (1-s)(1-t)^2|x_* - y_{-1}|)\,ds\,dt \\
=& \frac{C}{3}|x_0 - y_{-1}| + \frac{C}{6}|x_0 - x_*| + \frac{C}{3}|x_* - y_{-1}| \\
\leq& \frac{C}{2}|x_0 - x_*| + \frac{2C}{3}|x_* - y_{-1}|.
\end{aligned} \tag{2.8}$$

Using (1.11) and (2.1), we get

$$\begin{aligned}
|1 - B_1| =& |1 - f''(x_*)^{-1}(4f[x_0, x_0, y_{-1}] - 2f[x_0, y_{-1}, y_{-1}])| \\
=& |f''(x_*)^{-1}(4(f[x_0, x_0, y_{-1}] - f[x_*, x_*, x_*]) - 2(f[x_0, y_{-1}, y_{-1}] - f[x_*, x_*, x_*])| \\
=& |4\int_0^1 \int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_0 + s(1-t)x_0 + (1-s)(1-t)y_{-1}) \\
& - f''(tx_* + s(1-t)x_* + (1-s)(1-t)x_*))\,ds\,dt \\
& - 2\int_0^1 \int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_0 + s(1-t)y_{-1} + (1-s)(1-t)y_{-1}) \\
& - f''(tx_* + s(1-t)x_* + (1-s)(1-t)x_*))\,ds\,dt| \\
\leq& 4C_0\int_0^1 \int_0^1 (1-t)(t|x_0 - x_*| + s(1-t)|x_0 - x_*| + (1-s)(1-t)|y_{-1} - x_*|)\,ds\,dt \\
& + 2C_0\int_0^1 \int_0^1 (1-t)(t|x_0 - x_*| + s(1-t)|x_0 - x_*| + (1-s)(1-t)|y_{-1} - x_*|)\,ds\,dt \\
=& \frac{5C_0}{3}|x_0 - x_*| + \frac{4C_0}{3}|y_{-1} - x_*| \\
<& 3C_0\mathbb{R}_1 = \frac{18C_0}{13C + 18C_0} < 1.
\end{aligned} \tag{2.9}$$

By Banach lemma [6], $B_1 \neq 0$ and

$$|B_1^{-1}| \leq \frac{1}{1 - \frac{5C_0}{3}|x_0 - x_*| - \frac{4C_0}{3}|y_{-1} - x_*|}. \tag{2.10}$$

Using (2.7), (2.8) and (2.10), $y_{-1}, x_0 \in B(x_*, \mathbb{R}_1)$. We have

$$\begin{aligned} |y_0 - x_*| \leq& \frac{\frac{L}{2}|x_0 - x_*| + \frac{2L}{3}|y_{-1} - x_*|}{1 - \frac{5C_0}{3}|x_0 - x_*| - \frac{4C_0}{3}|y_{-1} - x_*|}|x_0 - x_*| \leq \frac{\frac{7L}{6}\mathbb{R}_1}{1 - 3C_0\mathbb{R}_1}|x_0 - x_*| \\ =& \frac{7}{13}|x_0 - x_*| < \mathbb{R}_1. \end{aligned} \tag{2.11}$$

Additionally, we have

$$\begin{aligned} x_1 - x_* &= y_0 - x_* - \frac{f[y_0, y_0]}{\delta(f; x_0, y_0)} = y_0 - x_* - \frac{f[y_0, y_0] - f[y_0, x_*] + f[y_0, x_*] - f[x_*, x_*]}{4f[x_0, x_0, y_0] - 2f[x_0, y_0, y_0]} \\ &= y_0 - x_* - \frac{f[y_0, y_0, x_*](y_0 - x_*) + f[y_0, x_*, x_*](y_0 - x_*)}{4f[x_0, x_0, y_0] - 2f[x_0, y_0, y_0]} \\ &= \frac{f''(x_*)^{-1}(4f[x_0, x_0, y_0] - 2f[x_0, y_0, y_0] - f[y_0, y_0, x_*] - f[y_0, x_*, x_*])}{f''(x_*)^{-1}(4f[x_0, x_0, y_0] - 2f[x_0, y_0, y_0])}(y_0 - x_*) \\ &= \frac{C_1}{D_1}(y_0 - x_*). \end{aligned} \tag{2.12}$$

Using (2.2), (1.12) and ((2.1), we have

$$\begin{aligned} |C_1| =& |f''(x_*)^{-1}(4f[x_0, x_0, y_0] - 2f[x_0, y_0, y_0]) - f[y_0, y_0, x_*] - f[y_0, x_*, x_*)| \\ =& |f''(x_*)^{-1}(2(f[x_0, x_0, y_0] - f[x_0, y_0, y_0]) + f[x_0, x_0, y_0] - f[y_0, y_0, x_*] + f[x_0, x_0, y_0] - f[y_0, x_*, x_*]| \\ =& |2\int_0^1\int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_0 + s(1-t)x_0 + (1-s)(1-t)y_0) \\ & - f''(tx_0 + s(1-t)y_0 + (1-s)(1-t)y_0))\,ds\,dt \\ & + \int_0^1\int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_0 + s(1-t)x_0 + (1-s)(1-t)y_0) \\ & - f''(ty_0 + s(1-t)y_0 + (1-s)(1-t)x_*))\,ds\,dt \\ & + \int_0^1\int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_0 + s(1-t)x_0 + (1-s)(1-t)y_0) \\ & - f''(ty_0 + s(1-t)x_* + (1-s)(1-t)x_*))\,ds\,dt| \\ \leq& 2C\int_0^1\int_0^1 s(1-t)^2|x_0 - y_0|\,ds\,dt \\ & + C\int_0^1\int_0^1 (t(1-t)|x_0 - y_0| + s(1-t)^2|x_0 - y_0| + (1-s)(1-t)^2|y_0 - x_*|)\,ds\,dt \\ & + C\int_0^1\int_0^1 (1-t)(t|x_0 - x_*| + s(1-t)|x_0 - x_*| + (1-s)(1-t)|y_0 - x_*|)\,ds\,dt \\ =& \frac{5L}{6}|x_0 - y_0| + \frac{L}{3}|y_0 - x_*| + \frac{L}{6}|x_0 - x_*| \leq C|x_0 - x_*| + \frac{7C}{6}|y_0 - x_*|. \end{aligned} \tag{2.13}$$

Similar to an argument about (2.9),

$$|1 - D_1| \leq \frac{5C_0}{3}|x_0 - x_*| + \frac{4C_0}{3}|y_0 - x_*|. \tag{2.14}$$

By Banach lemma [6], $D_1 \neq 0$ and

$$|D_1^{-1}| \leq \frac{1}{1 - \frac{5C_0}{3}|x_0 - x_*| - \frac{4C_0}{3}|y_0 - x_*|}. \tag{2.15}$$

Using (2.7), (2.8) and (2.10), $y_0, x_0 \in B(x_*, \mathbb{R}_1)$. We have

$$|x_1 - x_*| \leq \frac{C|x_0 - x_*| + \frac{7C}{6}|y_0 - x_*|}{1 - \frac{5C_0}{3}|x_0 - x_*| - \frac{4C_0}{3}|y_0 - x_*|}|y_0 - x_*| \leq \frac{\frac{13C}{6}\mathbb{R}_1}{1 - 3C_0\mathbb{R}_1}|y_0 - x_*| = |y_0 - x_*| < \mathbb{R}_1. \tag{2.16}$$

Assume $y_l, x_l \in B(x_*, \mathbb{R}_1)(1 \leq l \leq n)$ are defined by (1.14). By similar process of (2.7)–(2.11), we obtain

$$\begin{aligned} y_n - x_* &= \frac{f''(x_*)^{-1}(4f[x_n, x_n, y_{n-1}] - 2f[x_n, y_{n-1}, y_{n-1}] - f[x_n, x_n, x_*] - f[x_n, x_*, x_*])}{f''(x_*)^{-1}(4f[x_n, x_n, y_{n-1}] - 2f[x_n, y_{n-1}, y_{n-1}])}(x_n - x_*) \\ &= \frac{A_{n+1}}{B_{n+1}}(x_n - x_*) \end{aligned} \tag{2.17}$$

$$|A_{n+1}| \leq \frac{C}{2}|x_n - x_*| + \frac{2C}{3}|x_* - y_{n-1}| \tag{2.18}$$

and

$$|1 - B_{n+1}| \leq \frac{5C_0}{3}|x_n - x_*| + \frac{4C_0}{3}|y_{n-1} - x_*| < 3C_0\mathbb{R}_1 = \frac{18C_0}{13C + 18C_0} < 1. \tag{2.19}$$

Applying Banach lemma [6], $B_{n+1} \neq 0$, so

$$|B_{n+1}^{-1}| \leq \frac{1}{1 - \frac{5C_0}{3}|x_n - x_*| - \frac{4C_0}{3}|y_{n-1} - x_*|}. \tag{2.20}$$

Using (2.17), (2.18), (1.12) and (2.20), we get

$$\begin{aligned} |y_n - x_*| &\leq \frac{\frac{C}{2}|x_n - x_*|^2 + \frac{2C}{3}|y_{n-1} - x_*||x_n - x_*|}{1 - \frac{5C_0}{3}|x_n - x_*| - \frac{4C_0}{3}|y_{n-1} - x_*|} \\ &\leq \frac{\frac{C}{2}|x_n - x_*|^2 + \frac{2C}{3}|y_{n-1} - x_*||x_n - x_*|}{1 - 3C_0\mathbb{R}_1} \\ &\leq \frac{\frac{7C}{6}(|x_n - x_*||y_{n-1} - x_*|}{1 - 3C_0\mathbb{R}_1} \\ &\leq |x_n - x_*| < \mathbb{R}_1, \end{aligned} \tag{2.21}$$

$$\begin{aligned} x_{n+1} - x_* &= \frac{f''(x_*)^{-1}(4f[x_n, x_n, y_n] - 2f[x_n, y_n, y_n] - f[y_n, y_n, x_*] - f[y_n, x_*, x_*])}{f''(x_*)^{-1}(4f[x_n, x_n, y_n] - 2f[x_n, y_n, y_n])}(y_n - x_*) \\ &= \frac{C_{n+1}}{D_{n+1}}(y_n - x_*), \end{aligned} \tag{2.22}$$

$$|C_{n+1}| \leq C|x_n - x_*| + \frac{7C}{6}|x_* - y_n|, \tag{2.23}$$

and

$$|1 - D_{n+1}| \leq \frac{5C_0}{3}|x_n - x_*| + \frac{4C_0}{3}|y_n - x_*| < 3C_0\mathbb{R}_1 = \frac{18C_0}{13C + 18C_0} < 1. \tag{2.24}$$

By Banach lemma [6], $D_{n+1} \neq 0$ and

$$|D_{n+1}^{-1}| \leq \frac{1}{1 - \frac{5C_0}{3}|x_n - x_*| - \frac{4C_0}{3}|y_n - x_*|}. \tag{2.25}$$

Using (2.22), (2.23), (1.12) and (2.25), we get

$$\begin{aligned}
|x_{n+1} - x_*| \leq & \frac{C|x_n - x_*||y_n - x_*| + \frac{7C}{6}|y_n - x_*|^2}{1 - \frac{5C_0}{3}|x_n - x_*| - \frac{4C_0}{3}|y_n - x_*|} \\
\leq & \frac{C|x_n - x_*||y_n - x_*| + \frac{7C}{6}|y_n - x_*|^2}{1 - 3C_0\mathbb{R}_1} \\
\leq & \frac{\frac{13C}{6}(|x_n - x_*||y_n - x_*|}{1 - 3C_0\mathbb{R}_1} \\
= & \frac{|x_n - x_*||y_n - x_*|}{\mathbb{R}_1} \\
\leq & |x_n - x_*| < \mathbb{R}_1,
\end{aligned} \tag{2.26}$$

which indicates x_{n+1} is well defined and $x_{n+1} \in B(x_*, \mathbb{R}_1)$. The sequence $\{x_n\}$ generated by (1.14), which starts from an initial point $x_0 \in B(x_*, \mathbb{R}_1)$, is clearly defined by induction, and $x_n, y_n \in B(x_*, \mathbb{R}_1) (n \geq 1)$. Moreover, for $n \geq 1$, (2.26) holds, which shows that error estimates (2.4) and (2.6) are true.

To prove the uniqueness of the solution x_* for method (1.7), suppose there is another solution $y_* \in B(x_*, \frac{2}{C_0})$ so that $f'(y_*) = 0$. Notice the operator $H = f'[x_*, y_*]$. If H is invertible, since $H(x_* - y_*) = f'(x_*) - f'(y_*) = 0$, $y_* = x_*$. In fact, combining (2.2) and (1.11), $y_* \in B(x_*, \frac{2}{C_0})$, and we obtain

$$\begin{aligned}
|1 - f''(x_*)^{-1}H| = & |f''(x_*)^{-1}(H - f''(x_*))| = |f''(x_*)^{-1}(\frac{f[x_*, x_*] - f[y_*, y_*]}{x_* - y_*} - f''(x_*)| \\
= & |f''(x_*)^{-1}(\frac{f[x_*, x_*] - f[x_*, y_*] + f[x_*, y_*] - f[y_*, y_*]}{x_* - y_*} - 2f[x_*, x_*, x_*]| \\
= & |f''(x_*)^{-1}(f[x_*, x_*, y_*] - f[x_*, x_*, x_*] + f[x_*, y_*, y_*] - f[x_*, x_*, x_*])| \\
= & \int_0^1 \int_0^1 (1 - t)f''(x_*)^{-1}(f''(tx_* + s(1 - t)x_* + (1 - s)(1 - t)y_*) \\
& - f''(tx_* + s(1 - t)x_* + (1 - s)(1 - t)x_*))\, ds\, dt \\
& + \int_0^1 \int_0^1 (1 - t)f''(x_*)^{-1}(f''(tx_* + s(1 - t)y_* + (1 - s)(1 - t)y_*) \\
& - f''(tx_* + s(1 - t)x_* + (1 - s)(1 - t)x_*))\, ds\, dt| \\
\leq & \frac{C_0}{2}|y_* - x_*| < 1.
\end{aligned} \tag{2.27}$$

Therefore, Banach's lemma proves that the operator H is invertible. We show easily that $B(x_*, \frac{2}{C_0})$ is larger than $B(x_*, \mathbb{R}_1)$ by (2.3). The proof is over. □

We can get another approximation for $|B_{n+1}^{-1}|$ under conditions (1.11) and (1.13) by similar method of Theorem 2.1. Thus, another local convergence theorem for our method (1.14) is established.

Theorem 2.2. *Suppose $f'(x_*) = 0, f''(x_*) \neq 0$, and f is twice differentiable on N. Let us assume that the Lipschitz conditions* (1.11) *and* (1.13) *hold. Denote*

$$\mathbb{R}_2 = \frac{6}{17L_0 + 6C_0}. \tag{2.28}$$

Given an initial point x_0 in ball $B(x_*, \mathbb{R}_2) \subseteq N$, the sequence $\{x_n\}$ generated by our method (1.14) converges to its unique solution $x_* \in B(x_*, \frac{2}{C_0}) \subseteq N$. $B(x_*, \frac{2}{C_0})$ is larger than $B(x_*, \mathbb{R}_2)$. Furthermore, we get the following error estimates

$$|x_{n+1} - x_*| \leq \frac{119L_0 + 42C_0}{78}|y_n - x_*|^2 + \frac{17L_0 + 6C_0}{13}|x_n - x_*||y_n - x_*|, \quad (n \geq 0) \tag{2.29}$$

where

$$|y_n - x_*| \leq \frac{17L_0 + 6C_0}{26}|x_n - x_*|^2 + \frac{34L_0 + 12C_0}{39}|x_n - x_*||y_{n-1} - x_*|, \quad (n \geq 0) \tag{2.30}$$

and

$$|x_{n+1} - x_*| \leq \frac{|x_n - x_*||y_n - x_*|}{\mathbb{R}_2}, \quad (n \geq 1). \tag{2.31}$$

Proof. By the similar proof to that of Theorem 2.1, applying conditions (1.11) and (1.13), for $n \geq 0$, we obtain

$$\begin{aligned}
|1 - B_{n+1}| =& |1 - f''(x_*)^{-1}(4f[x_n, x_n, y_{n-1}] - 2f[x_n, y_{n-1}, y_{n-1}])| \\
=& |f''(x_*)^{-1}(2(f[x_n, x_n, y_{n-1}] - f[x_n, y_{n-1}, y_{n-1}]) + 2(f[x_n, x_n, y_{n-1}] - f[x_*, x_*, x_*])| \\
=& |2\int_0^1\int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_n + s(1-t)x_n + (1-s)(1-t)y_{n-1}) \\
& - f''(tx_n + s(1-t)y_{n-1} + (1-s)(1-t)y_{n-1}))\,ds\,dt \\
& + 2\int_0^1\int_0^1 (1-t)f''(x_*)^{-1}(f''(tx_n + s(1-t)x_n + (1-s)(1-t)y_{n-1}) \\
& - f''(tx_* + s(1-t)x_* + (1-s)(1-t)x_*))\,ds\,dt| \\
\leq& 2L_0\int_0^1\int_0^1 s(1-t)^2|x_n - y_{n-1}|\,ds\,dt \\
& + 2C_0\int_0^1\int_0^1 (1-t)(t|x_n - x_*| + s(1-t)|x_n - x_*| + (1-s)(1-t)|y_{n-1} - x_*|)\,ds\,dt \\
=& \frac{L_0}{3}|x_n - y_{n-1}| + \frac{2C_0}{3}|x_n - x_*| + \frac{C_0}{3}|y_{n-1} - x_*| \\
\leq& \frac{L_0 + 2C_0}{3}|x_n - x_*| + \frac{L_0 + C_0}{3}|y_{n-1} - x_*| \\
<& \frac{2L_0 + 3C_0}{3}\mathbb{R}_2 = \frac{4L_0 + 6C_0}{17L_0 + 6C_0} < 1.
\end{aligned} \tag{2.32}$$

Applying Banach lemma [6], $B_{n+1} \neq 0$, so

$$|B_{n+1}^{-1}| \leq \frac{1}{1 - \frac{L_0+2C_0}{3}|x_n - x_*| - \frac{L_0+C_0}{3}|y_{n-1} - x_*|}. \tag{2.33}$$

Suppose $x_n, y_n \in B(x_*, \mathbb{R}_2)$ are defined by (1.14). Let us see that (2.17) holds. By the similar method to that of Theorem 2.1, we have

$$|A_{n+1}| \leq \frac{L_0}{2}|x_n - x_*| + \frac{2L_0}{3}|x_* - y_{n-1}|. \tag{2.34}$$

Using (2.17), (2.33), (1.12) and (2.34), we have in turn

$$\begin{aligned} |y_n - x_*| &\leq \frac{\frac{L_0}{2}|x_n - x_*|^2 + \frac{2L_0}{3}|y_{n-1} - x_*||x_n - x_*|}{1 - \frac{L_0+2C_0}{3}|x_n - x_*| - \frac{L_0+C_0}{3}|y_{n-1} - x_*|} \\ &\leq \frac{\frac{L_0}{2}|x_n - x_*|^2 + \frac{2L_0}{3}|y_{n-1} - x_*||x_n - x_*|}{1 - \frac{2L_0+3C_0}{3}\mathbb{R}_2} \\ &\leq \frac{\frac{7L_0}{6}(|x_n - x_*||y_{n-1} - x_*|)}{1 - \frac{2L_0+3C_0}{3}\mathbb{R}_2} \\ &< |x_n - x_*| < \mathbb{R}_2. \end{aligned} \tag{2.35}$$

On the other hand, let us see that (2.22) holds. By the similar method to that of Theorem 2.1, we have

$$|C_{n+1}| \leq L_0|x_n - x_*| + \frac{7L_0}{6}|x_* - y_n|. \tag{2.36}$$

$$\begin{aligned} |1 - D_{n+1}| =& \frac{L_0}{3}|x_n - y_n| + \frac{2C_0}{3}|x_n - x_*| + \frac{C_0}{3}|y_n - x_*| \\ \leq& \frac{L_0 + 2C_0}{3}|x_n - x_*| + \frac{L_0 + C_0}{3}|y_n - x_*| \\ <& \frac{2L_0 + 3C_0}{3}\mathbb{R}_2 = \frac{4L_0 + 6C_0}{17L_0 + 6C_0} < 1. \end{aligned} \tag{2.37}$$

By Banach lemma [6], $D_{n+1} \neq 0$ and

$$|D_{n+1}^{-1}| \leq \frac{1}{1 - \frac{L_0+2C_0}{3}|x_n - x_*| + \frac{L_0+C_0}{3}|y_n - x_*|}. \tag{2.38}$$

Using (2.22), (2.23), (1.12) and (2.38), we get

$$\begin{aligned} |x_{n+1} - x_*| \leq& \frac{L_0|x_n - x_*||y_n - x_*| + \frac{7L_0}{6}|y_n - x_*|^2}{1 - \frac{L_0+2C_0}{3}|x_n - x_*| + \frac{L_0+C_0}{3}|y_n - x_*|} \\ \leq& \frac{\frac{13L_0}{6}(|x_n - x_*||y_n - x_*|}{1 - \frac{2L_0+3C_0}{3}\mathbb{R}_2} \\ =& \frac{|x_n - x_*||y_n - x_*|}{\mathbb{R}_2} \\ \leq& |x_n - x_*| < \mathbb{R}_2 \end{aligned} \tag{2.39}$$

which denotes $x_{n+1} \in B(x_*, \mathbb{R}_2)$ as defined. The proof of the rest is similar to that of Theorem 2.1. The proof is over. □

Remark 2.1. *We find that if* $17L_0 \geq 12C_0 + 13C$*, then* $\mathbb{R}_1 \geq \mathbb{R}_2$*; otherwise, if* $12C_0 + 13C > 17L_0$*, then* $\mathbb{R}_1 < \mathbb{R}_2$ *by comparing* (2.3) *with* (2.28)*. We shall show by some examples that both options are possible. Additionally, denote* $L_0 = (C_0, S_0), C = (C_0, N_0)$ *and* $N_0 \subseteq S_0$*, so* $C \leq L_0$*.*

3. Numerical experiments

We apply following examples to prove the convergence conditions and compute the convergence ball radii of our method (1.14), and compare Wang's method (1.5) and fourth-order method (3.1) by numerical experiments.

Wang et al. in [19] proposed the following fourth-order method:

$$\begin{cases} y_n = x_n - \dfrac{1}{B} f(x_n), \\ x_{n+1} = y_n - (3 - \dfrac{2}{B} f[x_n, y_n]) \dfrac{1}{B} f(y_n) \end{cases} \tag{3.1}$$

where $B = f[w_n, s_n], w_n = x_n + f(x_n), s_n = x_n - f(x_n)$.

Example 3.1. *Set* $N = [-1, 1]$*. The function* f_1 *in* N *is defined by*

$$f_1(x) = e^x - x. \tag{3.2}$$

In addition, a root of $f_1'(x) = 0$ *is* $x_* = 0$*. According to condition* (1.8), (1.9) *and* (1.11)–(1.13), $W = Y = e, C = e^{\frac{1}{3(e-1)}}, L_0 = e^{\frac{1}{e-1}}, C_0 = e - 1$*. By* (2.3) *and* (2.28)*, the following radii are obtained*

$$\mathbb{R}_1 \approx 0.1112, \mathbb{R}_2 \approx 0.1473 \tag{3.3}$$

then

$$\mathbb{R}_1 < \mathbb{R}_2. \tag{3.4}$$

So, we can easily verify

$$12C_0 + 13C \approx 43.6704 > 17L_0 \approx 30.4227.$$

Example 3.2. *Function* f *in* $N = [0, 1]$ *is defined by* (1.13)*, then we easily obtain* $C_0 = C = L_0 \approx 1.0923$*. By* (2.3) *and* (2.28)*, the following radii are obtained*

$$\mathbb{R}_1 \approx 0.1772, \mathbb{R}_2 \approx 0.2388 \tag{3.5}$$

and

$$\mathbb{R}_2 > \mathbb{R}_1. \tag{3.6}$$

Example 3.3. *Set* $N = [-\frac{\pi}{2}, \frac{\pi}{2}]$*. The function* f *in* N *is defined by*

$$f(x) = \cos(cx) \tag{3.7}$$

where $c \geq \frac{2\sqrt{2}}{\pi}$ *is a constant. Additionally, a root of* $f'(x) = 0$ *is* $x_* = 0$*. According to conditions* (1.11), (1.12) *and* (1.14)*, we can obtain* $L_0 = \sin(\frac{4}{c^2\pi}), C_0 = \frac{c^2\pi}{4}$*, and* $C = \sin(\frac{4}{3c^2\pi})$*.*

Set $c = 0.88$ *and we have*

$$f_2(x) = \cos(0.88x). \tag{3.8}$$

By (2.3) *and* (2.28)*, the following radii are obtained*

$$\mathbb{R}_1 \approx 0.3386, \mathbb{R}_2 \approx 0.2912 \tag{3.9}$$

then

$$\mathbb{R}_2 < \mathbb{R}_1. \tag{3.10}$$

So, we can verify easily

$$12C_0 + 13C \approx 14.0719 > 17L_0 \approx 16.9543.$$

If set $c = 1$*, we have*

$$f_3(x) = \cos(x). \tag{3.11}$$

By (2.3) *and* (2.28)*, the following radii are obtained*

$$\mathbb{R}_1 \approx 0.3078, \mathbb{R}_2 \approx 0.2862 \tag{3.12}$$

then

$$\mathbb{R}_2 > \mathbb{R}_1. \tag{3.13}$$

So, we can easily verify

$$12C_0 + 13C \approx 14.7780 < 17L_0 \approx 16.2529.$$

Using the above three functions, Wang's method (1.5), fourth-order method (3.1), and our method (1.14) are compared by numerical experiments. In Table 1, $|x_n - x_*|$ is absolute errors, x_0 is initial point, and iter is number of iterations. The methods (1.5), (1.14) and (3.1) are iterated five times. The order of computational convergence ρ is shown. $E = \rho^{1/c}$ is efficiency index. It is used to judge the computational efficiency of the iterative method. c is computational cost.

Table 1. Numerical results of methods (1.5), (3.1) and (1.14).

Method	*Function*	x_0	*iter*	$\lvert x_n - x_*\rvert$	ρ	c	E
(1.5)	f_1	1.5	5	$1.72541e-30$	2.0	4	1.1892
(3.1)	f_1	1.5	5	$8.18368e-53$	4.0	4	1.4142
(1.14)	f_1	1.5	5	$4.00995e-352$	4.0	4	1.4142
(1.5)	f_2	5.0	5	0.000980133	2.0	4	1.1892
(3.1)	f_2	5.0	5	$3.79249e-82$	5.0	4	1.4953
(1.14)	f_2	5.0	5	$7.66302e-157$	4.0	4	1.4142
(1.5)	f_3	1.2	5	$5.57691e-23$	2.0	4	1.1892
(3.1)	f_3	1.2	5	$2.49293e-82$	5.0	4	1.4953
(1.14)	f_3	1.2	5	$3.8233e-322$	4.0	4	1.4142

In Table 1, the method (3.1) has the same order and cost as our method (1.14), but our method has higher accuracy. Our method has the same cost as Wang's method, but our method has a higher order of convergence and higher accuracy.

4. Conclusions

In this paper, the convergence ball of a new fourth-order method for finding a zero of a derivative was studied in cases that applied less restrictive assumptions that were not covered before. This study discussed the measurable error distances, radii of convergence ball, and uniqueness of the solution. The error estimates $|x_n - x_*|$ and the convergence order were established under these different assumptions about Lipschitz conditions. In addition, different radii of the convergence ball was determined according to different weaker hypotheses. In the experimental part, the convergence criteria was proved by three classical examples. The experimental results were consistent with theory. The new method (1.14) was compared with Wang's method (1.5) and fourth-order method (3.1) by numerical experiments. The experimental results showed that the convergence order of the new method (1.14) is twice as high as that of Wang's method (1.5), and the new method had higher accuracy, so the new method (1.14) is relatively better.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (No. 61976027), the Natural Science Foundation of Liaoning Province (Nos. 2022-MS-371, 2023-MS-296), Educational Commission Foundation of Liaoning Province of China (Nos. LJKMZ20221492, LJKMZ20221498), and the Key Project of Bohai University (No. 0522xn078).

Conflict of interest

The authors declare no conflict of interest.

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