



Research article

Dynamic wave solutions for (2+1)-dimensional DJKM equation in plasma physics

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Abstract: In this paper, we attempt to obtain exact and novel solutions for Date-Jimbo-Kashiwara-Miwa equation (DKJM) via two different techniques: Lie symmetry analysis and generalized Kudryashov method (GKM). This equation has applications in plasma physics, fluid mechanics, and other fields. The Lie symmetry method is applied to reduce the governing equation to five different ordinary differential equations (ODEs). GKM is used to obtain general and various periodic solutions. These solutions have different behaviors such as kink wave, anti-kink wave, double soliton, and single wave solution. The physical behavior of the solutions was reviewed through 2-D and 3-D graphs.

Keywords: Lie symmetry analysis; generalization Kudryashov method; Date-Jimbo-Kashiwara-Miwa equation; exact solutions

Mathematics Subject Classification: 34A06, 35A09, 35A24

1. Introduction

1.1. Overview

Many physical phenomena, in the fields of sound and electrical waves, heat transfer, mechanics, and optics can be described through partial differential equations. Therefore, we can say that the study of mathematical physics depends primarily on the study of partial differential equations. During previous decades, many methods were developed to reduce and solve partial differential equations, including Symmetry method [1–4], scattering transformation [5], Tanh method [6], the homogeneous balance method [7], Aboodh transform decomposition method [8], $(G'/G, 1/G)$ -expansion method [9], Exp-function method [10], Generalized He's Exp-function method [11], Kudryashov method [12], and Generalized Kudryashov method [13]. Consider the nonlinear (2+1)-dimensional Date-Jimbo-

Kashiwara-Miwa (DJKM) equation [14, 15] in the following form:

$$\Phi_{xxxxxy} + 4\Phi_{xxy}\Phi_x + 2\Phi_{xxx}\Phi_y + 6\Phi_{xy}\Phi_{xx} + \Phi_{yyy} - 2\Phi_{xxt} = 0, \quad (1)$$

where $\Phi = \Phi(x, y, t)$. DJKM describes the wave diffusion of physical-quantity, and it has certain applications in plasma physics [16, 17]. This equation was presented by Jimbo and Miwa as second formulas for Kadomtsev-Petviashvili hierarchy.

1.2. Related literatures

There is a wealth of literature related to this governing equation such as: Bibi and Ahmad [14] who employed the symmetry method to reduce the governing equation to another equation of lower order and obtained its exact solutions. The $\exp(-\phi(\xi))$ -Expansion Method was utilized by Sajid and Akram [14] for obtaining traveling wave solutions for Eq (1). Halder et al. [18] used the Lie symmetry analysis and singularity method to study one type of the governing equation. Wazwaz [19] studied the integrability of this equation and obtained multiple soliton solutions by using Painlevé analysis and Hirota's method. Akram et al. [16] applied the extended direct algebraic method to obtain exact solutions for the governing equation.

1.3. Main idea

The previous literature led us to determine the study of the governing equation to apply the Lie symmetry method and the Generalized Kudryashov method. Our main objective of this paper is to convert the governing equation to several distinct ordinary differential equations (ODE). We then solve the resulting ODEs to obtain exact and novel wave solutions for the governing equation. As a result, we can show the dynamic behavior of wave solutions in 2-D and 3-D plots.

2. Lie symmetry method

In this part, the Lie symmetry method [20–22] is used to deduce the similarity reductions for the governing equation, so the one-parameter Lie symmetry of infinitesimal variables is shown as follows:

$$\begin{aligned} t^* &= t + \varepsilon A(t, x, y, \Phi) + o(\varepsilon^2), & x^* &= x + \varepsilon B(t, x, y, \Phi) + o(\varepsilon^2), \\ y^* &= y + \varepsilon C(t, x, y, \Phi) + o(\varepsilon^2), & \Phi^* &= \Phi + \varepsilon E(t, x, y, \Phi) + o(\varepsilon^2), \end{aligned} \quad (2)$$

where ε is the one parameter Lie group, A , B , C and E are infinitesimal variables and t , x and y are independent variables of the function Φ .

The vector generator χ is related to the above-mentioned group transformations which can be written as the following expression:

$$\chi = A \frac{\partial}{\partial t} + B \frac{\partial}{\partial x} + C \frac{\partial}{\partial y} + E \frac{\partial}{\partial \Phi}, \quad (3)$$

when the following invariance condition is satisfied:

$$\Gamma^{(3)}(\Delta) = 0, \quad (4)$$

where Δ represents Eq (1) and $\Gamma^{(3)}$ is the third order prolongation [23, 24] of the operator V

$$\begin{aligned} \Gamma^{(3)} = & \chi + E_{[x]} \frac{\partial}{\partial \Phi_x} + E_{[y]} \frac{\partial}{\partial \Phi_y} + E_{[xy]} \frac{\partial}{\partial \Phi_{xy}} + E_{[xxx]} \frac{\partial}{\partial \Phi_{xxx}} + E_{[xxy]} \frac{\partial}{\partial \Phi_{xxy}} \\ & + E_{[xxt]} \frac{\partial \Phi}{\partial \Phi_{xxt}} + E_{[xxy]} \frac{\partial}{\partial \Phi_{xxy}} + E_{[xxx]} \frac{\partial}{\partial \Phi_{xxx}} + E_{[yyy]} \frac{\partial}{\partial \Phi_{yyy}}, \end{aligned} \quad (5)$$

where the components $E_{[x]}$, $E_{[xx]}$, $E_{[xy]}$, $E_{[y]}$, $E_{[xy]}$ can be determined from the following expressions:

$$\begin{aligned} E_{[x]} &= D_x E - \Phi_t D_x A - \Phi_x D_x B - \Phi_y D_x C, \\ E_{[y]} &= D_y E - \Phi_t D_y A - \Phi_x D_y B - \Phi_y D_y C, \\ E_{[xx]} &= D_x E_{[x]} - \Phi_{xt} D_x A - \Phi_{xx} D_x B - \Phi_{xy} D_x C, \\ E_{[xy]} &= D_y E_{[x]} - \Phi_{xt} D_y A - \Phi_{xx} D_y B - \Phi_{xy} D_y C, \\ E_{[yy]} &= D_y E_{[y]} - \Phi_{yt} D_y A - \Phi_{yy} D_y B - \Phi_{xy} D_y C, \\ E_{[xxy]} &= D_y E_{[xxx]} - \Phi_{xxt} D_y A - \Phi_{xxx} D_y B - \Phi_{xxy} D_y C. \end{aligned} \quad (6)$$

Substituting (1) into Eq (4), yields an identity components $A_x, A_{xx}, B_t, B_x, \dots$ and adding the coefficients of $\Phi_{x,x,t}$, $\Phi_{y,y,y}$, ... and equating them to zero, then the infinitesimals A, B, C and E are obtained from the following equations:

$$\begin{aligned} A_x &= A_y = A_\Phi = 0, \\ B_y &= B_\Phi = 0, B_x = \frac{1}{4} A_t, \\ C_x &= B_\Phi = 0, C_z = \frac{1}{2} A_t, \\ E_x &= \frac{-1}{2} C_t, E_\Phi = \frac{-1}{4} A_t, E_y = \frac{-1}{2} B_t. \end{aligned} \quad (7)$$

solving the system of Eq (7), we get

$$\begin{aligned} A &= g_1(t), B = \frac{1}{4} g_1' x + g_3(t), C = \frac{1}{2} g_1' y + g_2(t), \\ E &= \frac{-1}{4} g_1' \Phi - \frac{1}{4} x y g_1'' - y g_2' - \frac{1}{2} x g_3' + g_4(t). \end{aligned} \quad (8)$$

The vector field operator V of Eq (8) can be presented as

$$V = V_1(c_1) + V_2(c_2) + V_3(c_3) + V_4(c_4), \quad (9)$$

where

$$\begin{aligned} V_1 &= g_1(t) \left(\frac{\partial}{\partial t} + \frac{1}{4} x \frac{\partial}{\partial x} + \frac{1}{2} y \frac{\partial}{\partial y} - \frac{1}{4} \Phi \frac{\partial}{\partial \Phi} \right) - \frac{1}{4} x y g_1'' \frac{\partial}{\partial \Phi}, \\ V_2 &= g_2(t) \frac{\partial}{\partial y} - y \alpha_2' \frac{\partial}{\partial \Phi}, \quad V_3 = g_3(t) \frac{\partial}{\partial x} - \frac{1}{2} g_3' \frac{\partial}{\partial \Phi}, \quad V_4 = g_4(t) \frac{\partial}{\partial \Phi}. \end{aligned} \quad (10)$$

The Lie-bracket $[V_i, V_j] = V_i V_j - V_j V_i$ is used for obtaining the commutator relations which are shown in Table 1, where

$$\begin{aligned} l_1 &= g_1 g_2' - \frac{1}{2} g_2 g_1', \quad l_2 = \frac{-1}{4} (g_2' g_1' + 2g_1 g_2'' - g_2 g_1'') x, \\ l_3 &= g_1 g_3' - \frac{1}{4} g_3 g_1', \quad l_4 = \frac{-1}{4} (g_3' g_1' + 4g_1 g_3'' - g_3 g_1'') y, \\ l_5 &= \frac{1}{4} (4g_1 g_4' + g_4 g_1'), \quad l_6 = \frac{-1}{2} (2g_2 g_3' - g_3 g_2'). \end{aligned} \quad (11)$$

Table 1. The commutator table.

	V_1	V_2	V_3	V_4
V_1	0	$l_1 \frac{\partial}{\partial y} + l_2 \frac{\partial}{\partial \Phi}$	$l_3 \frac{\partial}{\partial x} + a_4 \frac{\partial}{\partial \Phi}$	$l_5 \frac{\partial}{\partial \Phi}$
V_2	$-(l_1 \frac{\partial}{\partial y} + l_2 \frac{\partial}{\partial \Phi})$	0	$l_6 \frac{\partial}{\partial \Phi}$	0
V_3	$-(l_3 \frac{\partial}{\partial x} + l_4 \frac{\partial}{\partial \Phi})$	$-l_6 \frac{\partial}{\partial \Phi}$	0	0
V_4	$-l_5 \frac{\partial}{\partial \Phi}$	$-l_6 \frac{\partial}{\partial \Phi}$	0	0

3. Reductions and closed form solutions

To obtain new independent variables, we utilize the characteristic equation as follows:

$$\frac{dt}{A(t, x, y, \Phi)} = \frac{dx}{B(t, x, y, \Phi)} = \frac{dy}{C(t, x, y, \Phi)} = \frac{d\Phi}{E(t, x, y, \Phi)}. \quad (12)$$

Case I. Using V_1 where $g_1(t) = t^2$, Eq (12) takes the following form:

$$\frac{dt}{t^2} = \frac{dx}{\frac{1}{2}tx} = \frac{dy}{ty} = \frac{d\Phi}{\frac{-1}{2}t\Phi}. \quad (13)$$

The similarity solution of Eq (1) takes the form

$$\Phi(t, x, y) = \frac{1}{\sqrt{t}} F(\zeta, \eta) - \frac{xy}{2t}, \quad (14)$$

where $\zeta = \frac{x}{\sqrt{t}}$, $\eta = \frac{y}{t}$. Substituting Φ in Eq (1) we get

$$F_{\eta\eta\eta} + 6F_{\zeta\zeta} F_{\zeta\eta} + 2F_{\zeta\zeta\zeta} F_{\eta} + 4F_{\zeta\zeta\eta} F_{\zeta} + F_{\zeta\zeta\zeta\eta} = 0. \quad (15)$$

The ordinary differential equations of Eq (15) take the following expression:

$$h^3 F''''(\theta) + 6k^3 h F''^2(\theta) + 6k^3 h F'''(\theta) F'(\theta) + k^4 h F''''''(\theta) = 0, \quad (16)$$

where $\theta = k\zeta + h\eta$.

To obtain the exact solution for Eq (16), we utilize generalized Kudryashov method (GKM) [11], given as follows:

$$F(\theta) = A_0 + \sum_{i=1}^m \frac{A_i}{(1 + \phi(\theta))^i}, \quad (17)$$

where $\phi(\theta)$ is the solution for the following Riccati equation $\phi'(\theta) = A + B\phi(\theta) + C\phi^2(\theta)$. For simplicity, we take $i = 1$ and substitute Eq (17) into Eq (16) then to obtain

$$h = \pm \sqrt{4AC - B^2k^2}, \quad A_1 = 2k(C - B + A), \quad k \text{ and } A_0 \text{ are arbitrary constant.} \quad (18)$$

Substituting Eq (18) into Eqs (14) and (17), the exact solutions of Eq (1) can be expressed as Set 1. $A \neq 0, B \neq 0, C \neq 0$.

$$\Phi(t, x, y) = \frac{1}{\sqrt{t}} \left(A_0 + \frac{2k(C - B + A)}{\left[\frac{-B}{2C} + \frac{\sqrt{4AC - B^2}}{2C} \tan\left(\frac{1}{2}(\sqrt{4AC - B^2}\theta)\right) + 1 \right]} \right) - \frac{xy}{2t}, \quad (19)$$

where $\theta = k\frac{x}{\sqrt{t}} + \sqrt{4AC - B^2}k^2\frac{y}{t}$.

Set 2. $A = \frac{-1}{2}, B = 0, C = \frac{-1}{2}$.

$$\Phi(t, x, y) = \frac{1}{\sqrt{t}} \left(A_0 + \frac{-2k}{\cot(\theta) \pm \csc(\theta) + 1} \right) - \frac{xy}{2t}, \quad (20)$$

where $\theta = k\frac{x}{\sqrt{t}} + 2k^2\frac{y}{t}$.

Case II. Substituting V_1 in Eq (12) where $g_1(t) = t$, then Eq (12) is expressed as follows:

$$\frac{dt}{t} = \frac{dx}{\frac{1}{4}x} = \frac{dy}{\frac{1}{2}y} = \frac{d\Phi}{\frac{-1}{4}\Phi}. \quad (21)$$

Then the invariants are presented by $\zeta = \frac{x}{t^{\frac{1}{4}}}, \eta = \frac{y}{t^{\frac{1}{2}}}$ and $\Phi(t, x, y) = \frac{1}{t^{\frac{1}{4}}}F(\zeta, \eta)$. Further Eq (1) is reduced to the following form:

$$\frac{3}{2}F_{\zeta\zeta} + \frac{1}{2}\zeta F_{\zeta\zeta\zeta} + \eta F_{\eta\eta\eta} + F_{\eta\eta\eta} + 6F_{\zeta\zeta}F_{\zeta\eta} + 4F_{\zeta}F_{\zeta\zeta\eta} + 2F_{\zeta\zeta\zeta}F_{\eta} + F_{\zeta\zeta\zeta\eta} = 0. \quad (22)$$

For reducing Eq (22) to ODE we have two subcases:

Subcase (a). Using the scaling transformation in Eq (22)

$$F = e^{\epsilon} \bar{F}, \quad \zeta = e^{\epsilon_1} \bar{\zeta}, \quad \eta = e^{\epsilon_2} \bar{\eta}. \quad (23)$$

Substituting from Eq (23) into Eq (22) we have $2\epsilon_1 = \epsilon_2 = -2\epsilon$.

The characteristic is given as

$$\frac{2d\zeta}{\zeta} = \frac{d\eta}{\eta} = \frac{-2dF}{F}. \quad (24)$$

The invariant variables can be written as

$$\theta = \frac{\zeta^2}{\eta}, \quad f = \frac{1}{\zeta^2}F(\theta). \quad (25)$$

The ODE of Eq (22) is provided by

$$-\theta^2 f'''' - 6f'' - 6\theta f' - 48\theta f''^2 - 48\theta f f' - 48f' f'''' - 16f' f'''' - 80\theta f'''' - 60f'''' - 16\theta^2 f'''''' = 0. \quad (26)$$

To obtain the exact solutions of Eq (26), we suppose that the solution of Eq (26) has the form

$$f(\theta) = A_0 + \sum_{j=1}^n A_j \theta^j. \quad (27)$$

For simplicity, we take $j = 2$, substitute Eq (27) into Eq (26), then the closed form solution of Eq (1) takes the expression

$$\Phi(x, y) = A_0 + A_1 \frac{x}{y} - \frac{1}{16} \frac{x^3}{y^2}, \quad (28)$$

where A_0 and A_1 are arbitrary constant.

Subcase (b). Utilizing symmetry method for Eq (22), we obtained the infinitesimals of Eq (22) as follows:

$$\tau = \frac{1}{2} c_1 \zeta + c_2, \quad \beta = c_1 \eta + c_3, \quad \psi = \frac{-1}{2} (F + xy) c_1 - \frac{1}{4} \eta c_2 - \frac{1}{4} \zeta c_3. \quad (29)$$

The vector field of Eq (22) can take the form

$$\chi = \chi_1(c_1) + \chi_2(c_2) + \chi_3(c_3), \quad (30)$$

where

$$\chi_1 = \frac{1}{2} \zeta \frac{\partial}{\partial \zeta} + \frac{\partial}{\partial \eta} - \frac{1}{2} (F + xy) \frac{\partial}{\partial F}, \quad \chi_2 = \frac{\partial}{\partial \zeta} - \frac{1}{4} \eta \frac{\partial}{\partial F}, \quad \chi_3 = \frac{\partial}{\partial \eta} - \frac{1}{4} \zeta \frac{\partial}{\partial F}. \quad (31)$$

Using $\chi_2 + w\chi_3$, the characteristic equation is yielded

$$\frac{d\zeta}{1} = \frac{d\eta}{w} = \frac{dF}{-\frac{1}{4}\eta - \frac{w}{4}\zeta}. \quad (32)$$

The similarity solution for Eq (22) takes the form

$$F(\zeta, \eta) = f(\theta) - \frac{1}{4} \eta \zeta, \quad \text{where } \theta = \eta - w\zeta. \quad (33)$$

The reduced ODE can take the expression

$$f''''(\theta) - 6w^3 f''^2(\theta) - 6w^3 f'(\theta) f''''(\theta) + w^4 f''''''(\theta) = 0. \quad (34)$$

Utilizing GKM in the previous case, the closed-form solution of Eq (34) is shown as

$$f(\theta) = A_0 + \frac{A_1}{1 + \phi(\theta)} + \frac{A_2}{(1 + \phi(\theta))^2}. \quad (35)$$

Substituting Eq (35) into Eq (34), collecting the coefficients of derivative $\phi(\theta)$ and solving the algebraic equations, we obtain the values of A_0 and A_1

$$A_1 = \frac{-2(A + C - B)}{(4AC - B^2)^{\frac{1}{4}}}, \quad w = \frac{1}{(4AC - B^2)^{\frac{1}{4}}}, \quad A_0, A_2 \text{ are arbitrary.} \quad (36)$$

Set 1. $A \neq 0, B \neq 0, C \neq 0$.

$$\Phi(t, x, y) = \frac{1}{t^{\frac{1}{4}}} \left(A_0 + \frac{A_1}{\left[\frac{-B}{2C} + \frac{\sqrt{4AC - B^2}}{2C} \tan\left(\frac{1}{2}(\sqrt{4AC - B^2}\theta)\right) + 1 \right]} \right)$$

$$\left. + \frac{A_2}{\left[\frac{-B}{2C} + \frac{\sqrt{4AC-B^2}}{2C} \tan\left(\frac{1}{2}(\sqrt{4AC-B^2}\theta)\right) + 1 \right]^2} \right) - \frac{y}{4\sqrt{t}} \frac{x}{t^{\frac{1}{4}}}, \quad (37)$$

where $A_1 = \frac{-2(A+C-B)}{(4AC-B^2)^{\frac{1}{4}}}$, $\theta = \frac{y}{\sqrt{t}} + \frac{1}{(4AC-B^2)^{\frac{1}{4}}} \frac{x}{t^{\frac{1}{4}}}$.

Set 2. $A = \frac{1}{2}$, $B = 0$, $C = \frac{1}{2}$.

$$\Phi(t, x, y) = \frac{1}{t^{\frac{1}{4}}} \left(A_0 + \frac{A_1}{\csc(\theta) - \cot(\theta) + 1} + \frac{A_2}{(\csc(\theta) - \cot(\theta) + 1)^2} \right) - \frac{y}{4\sqrt{t}} \frac{x}{t^{\frac{1}{4}}}, \quad (38)$$

where $A_1 = \frac{-1}{2}$, $\theta = \frac{y}{\sqrt{t}} + \frac{x}{t^{\frac{1}{4}}}$.

Cases III. Using wV_2, V_3, V_4 where $g_2(t), g_3(t), g_4(t)$ are constants, we get

$$\frac{dt}{0} = \frac{dx}{w} = \frac{dy}{1} = \frac{d\Psi}{1}. \quad (39)$$

Using Eq (39), we obtain the invariant solutions

$$\Phi(t, x, y) = F(\zeta, \eta) + y, \quad \text{where } \zeta = t, \eta = x - wy. \quad (40)$$

Substituting Eq (40) into Eq (1) we get

$$2F_{\zeta\eta\eta} + w^3 F_{\eta\eta\eta} + 6wF_{\eta\eta}^2 + 6wF_{\eta\eta\eta}F_{\eta} - 2F_{\eta\eta\eta} + wF_{\eta\eta\eta\eta} = 0. \quad (41)$$

Utilizing transformation $\theta = k\zeta + h\eta$, Eq (41) is reduced to the next equation

$$h^2(2k + hw^3 - 2h)F''''(\theta) + 6wh^4F''^2(\theta) + 6wh^4F''''(\theta)F'(\theta) + wh^5F''''''(\theta) = 0. \quad (42)$$

The exact solutions of Eq (42) can be deduced by applying GKM in the previous method. Substituting Eq (17) into Eq (42) at $i = 1$ and collecting the coefficient of derivative $\phi(\theta)$, then equating this equations to zero, this equations is solved and the values of A_0, A_1, w, h and k are obtained as

$$k = \frac{1}{2}(4h^2wAC + 4 - w^3 - h^2wB^2), \quad A_1 = 2(C + A - B)h, \quad A_0, h, w \text{ are arbitrary.} \quad (43)$$

The closed-form solutions of Eq (1) can be expressed as

Set 1. $A \neq 0, B \neq 0, C = 0$.

$$\Phi(t, x, y) = A_0 + \frac{2(A - B)h}{\frac{-A}{B} + \frac{1}{B} \exp(B\theta) + 1} + y, \quad (44)$$

where $k = \frac{1}{2}(4 - w^3 - h^2wB^2)$, $\theta = kt + h(x - wy)$.

Set 2. $A = 0, B \neq 0, C \neq 0$.

$$\Phi(t, x, y) = A_0 + \frac{2(C - B)h(C \exp(B\theta) - 1)}{B \exp(B\theta) + C \exp(B\theta) - 1} + y, \quad (45)$$

where $k = \frac{1}{2}(4 - w^3 - h^2wB^2)$, $\theta = kt + h(x - wy)$.

Cases IV. Utilizing wV_2, V_3 , where $\alpha_2(t), \alpha_3(t)$ are constant, we obtain

$$\frac{dt}{0} = \frac{dx}{1} = \frac{dy}{w} = \frac{d\Psi}{0}. \quad (46)$$

Using Eq (46), we obtain the invariant solutions

$$\Phi(t, x, y) = F(\zeta, \eta), \quad \text{where } \zeta = t, \eta = x - wy. \quad (47)$$

Substituting Eq (47) into Eq (1) we get

$$2F_{\zeta\eta\eta} + w^3 F_{\eta\eta\eta} + 6wF_{\eta\eta}^2 + 6wF_{\eta\eta\eta}F_{\eta} + wF_{\eta\eta\eta\eta} = 0. \quad (48)$$

Taking the transformation $\theta = k\zeta + h\eta$, Eq (48) is reduced to the form

$$h^2(2k + hw^3 - 2h)F''''(\theta) + 6wh^4F''^2(\theta) + 6wh^4F''(\theta)F'(\theta) + wh^5F''''''(\theta) = 0. \quad (49)$$

The coveted exact solution of Eq (49) is given. Hence, by using Eq (17) in Eq (49), the solution for Eq (1) is provided as

Set 1. $A \neq 0, B \neq 0, C \neq 0$.

$$\Phi(t, x, y) = A_0 + \frac{2(C + A - B)h}{\left[\frac{-B}{2C} + \frac{\sqrt{4AC - B^2}}{2C} \tan\left(\frac{1}{2}(\sqrt{4AC - B^2}\theta)\right) + 1\right]}, \quad (50)$$

where $k = \frac{1}{2}(4h^2wAC + 4 - w^3 - h^2wB^2)$, $\theta = kt + h(x - wy)$.

Set 2. $A = \frac{1}{2}, B = 0, C = \frac{-1}{2}$.

$$\Phi(t, x, y) = A_0 - \frac{1}{2} \frac{h}{\coth(\theta) \pm \csc h(\theta) + 1}, \quad (51)$$

where $k = \frac{1}{2}(-h^2w + 4 - w^3)$, $\theta = kt + h(x - wy)$.

4. Results & discussion

In this section, we explain the dynamic behavior of the obtained solutions in Eqs (19), (20), (28), (37), (38), (44), (45) and compare these results with previous literatures.

The physical behavior of the solutions is illustrated in three-dimensional and two-dimensional forms. The solutions include trigonometric functions, hyperbolic functions, and expansion functions. The Maple software was used to review the solutions as follows:

In Figure 1, presents wave solution of Eq (19) in 3-D and 2-D shapes with perfect numerical values of $A_0 = 5, k = 0.1, A = 3, C = 2, B = -1$ at $y=1$.

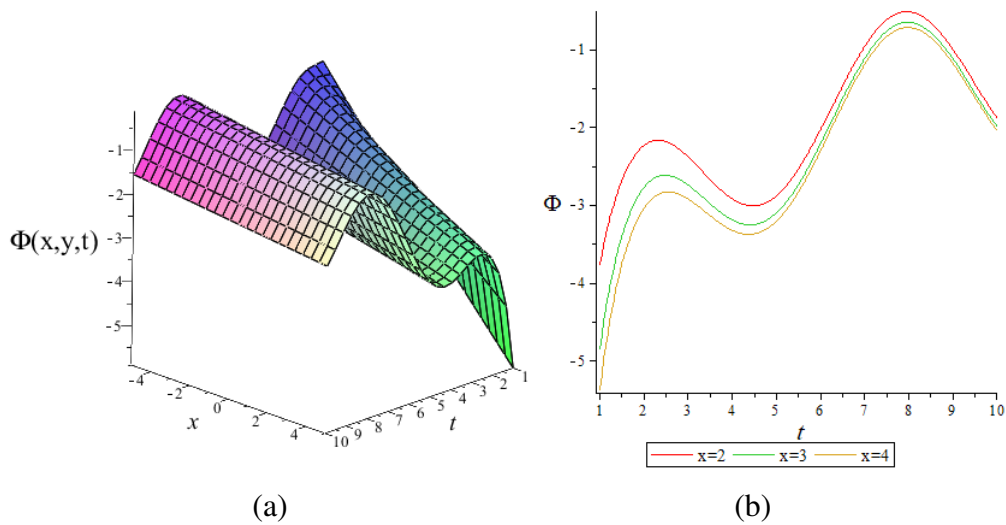


Figure 1. Wave solution of Eq (19) in 3-D and 2-D shapes.

In Figure 2, the solution of Eq (28) shows the behavior of a single wave and it is presented through 3-D and 2-D graphs. The constants are chosen as follows: $A_0 = 1, A_2 = 1, k = 0.1, A = 3, C = 2, B = 3$ when $y = 1$.

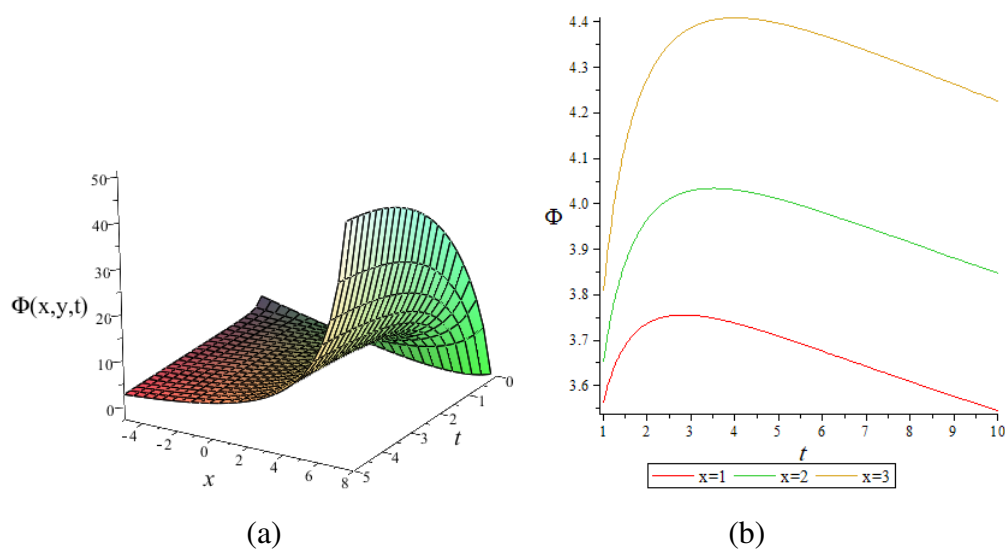


Figure 2. Single wave of Eq (28) in 3-D and 2-D graphs.

Figure 3 shows doubly soliton of Eq (37) in 3-D and 2-D shapes with perfect choice for values of $A_0 = 0, A_1 = 1$.

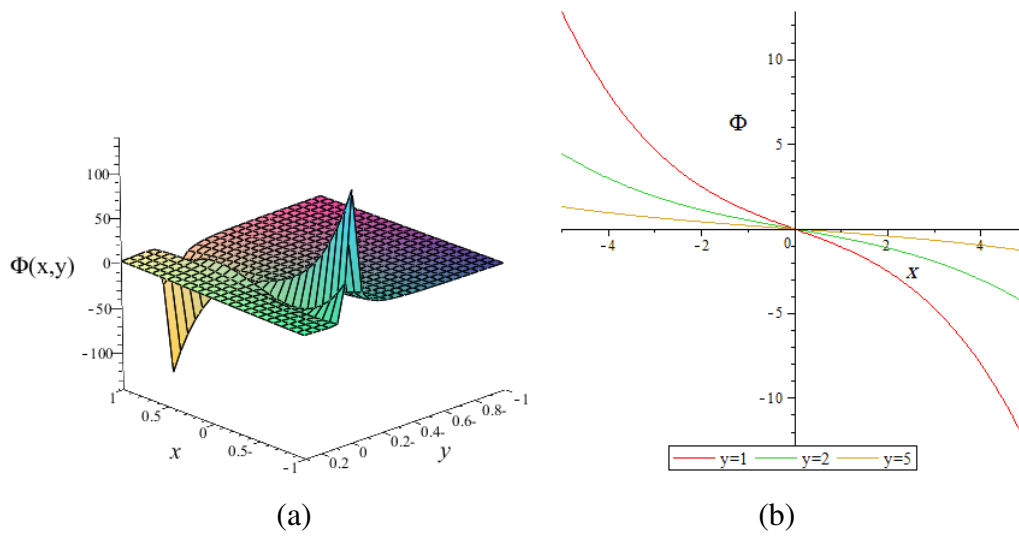


Figure 3. Doubly soliton of Eq (37) in 3-D and 2-D shapes.

Figure 4 depicts the view anti-kink wave solution of Eq (44) in 3-D and 2-D graphs with a suitable choice of values $A_0 = 3$, $w = -1$, $h = -3$, $A = 2$, $C = 0$, $B = 1$ with $y = 1$.

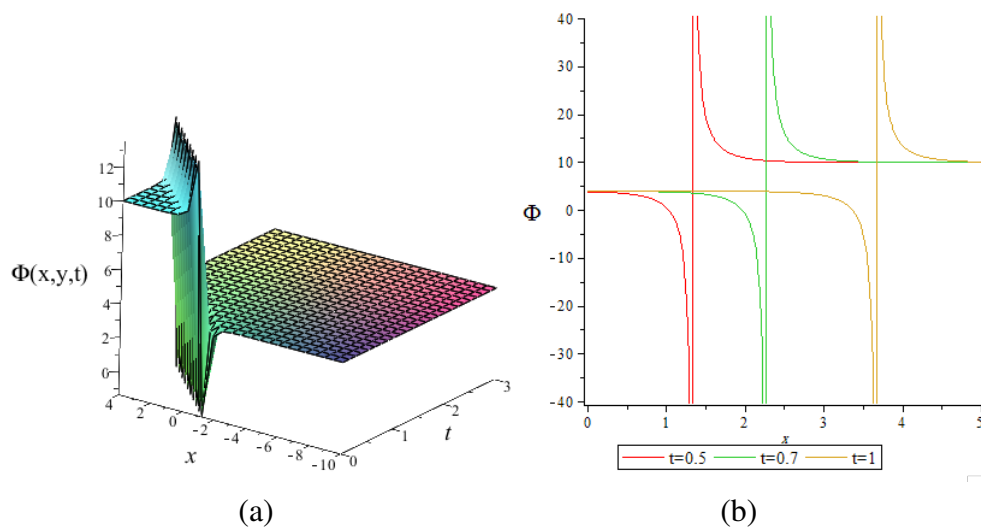


Figure 4. Anti-kink wave solution of Eq (44) in 3-D and 2-D graphs.

In Figure 5, the solution of Eq (45) represents a kink wave solution in 3-D and 2-D plots with perfect simulation of the values $A_0 = 1$, $h = 2$, $w = 1$, $z = 1$, $A = 0$, $C = -3$, $B = 1$ at $y = 1$.

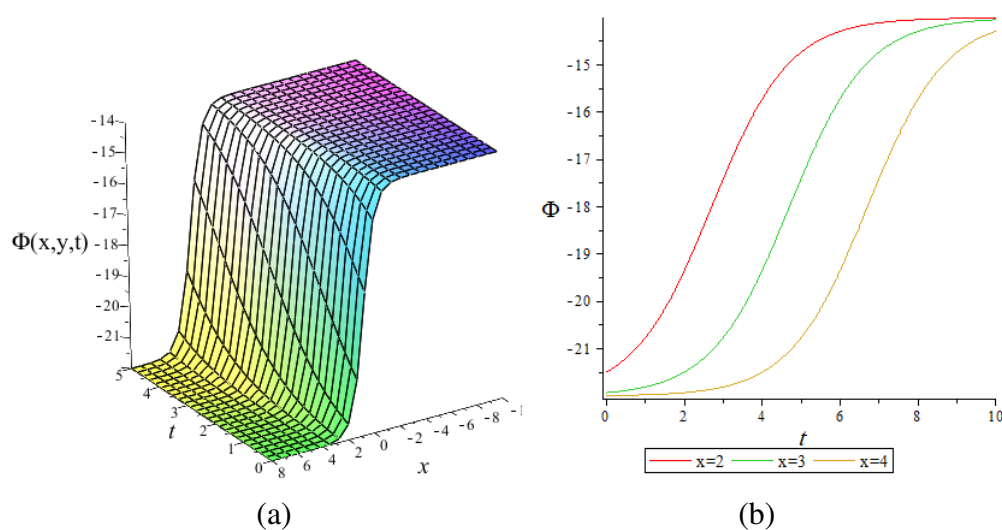


Figure 5. Kink wave solution of Eq (45) in 3-D and 2-D plots.

The authors in [13] used the symmetry method and obtained a subcase from our infinitesimals. Then, they reduced the governing equation to one reduction case, but we obtained five reduction cases.

The authors in [14] used two different traveling wave methods to obtain exact solutions. However, they reduced the governing equation through an algebraic transformation. When comparing these solutions with our obtained solutions, it turned out that our solutions are novel solutions and show different physical behavior from other solutions.

5. Conclusions

In this paper, we applied the Lie symmetry method to obtain infinitesimals and vector fields. This led to reduce the governing equation to a new form of partial differential equations (PDEs). Then, by utilizing the symmetry method again and similarity transformation: the new form of PDEs has been converted instead of reduced to five different ODEs. The general exact wave solutions are obtained by using GKM. These solutions are represented in 2-D and 3-D graphs to show the physical properties of the solutions.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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