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Research article

Pivotal-based inference for a Pareto distribution under the adaptive progressive Type-II censoring scheme

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Abstract: This paper proposes an inference approach based on a pivotal quantity under the adaptive progressive Type-II censoring scheme. To exemplify the proposed methodology, an extensively employed distribution, a Pareto distribution, is utilized. This distribution has limitations in estimating confidence intervals for unknown parameters from classical methods such as the maximum likelihood and bootstrap methods. For example, in the maximum likelihood method, the asymptotic variance-covariance matrix does not always exist. In addition, both classical methods can yield confidence intervals that do not satisfy nominal levels when a sample size is not large enough. Our approach resolves these limitations by allowing us to construct exact intervals for unknown parameters with computational simplicity. Aside from this, the proposed approach leads to closed-form estimators with properties such as unbiasedness and consistency. To verify the validity of the proposed methodology, two approaches, a Monte Carlo simulation and a real-world data analysis, are conducted. The simulation testifies to the superior performance of the proposed methodology as compared to the maximum likelihood method, and the real-world data analysis examines the applicability and scalability of the proposed methodology.

Keywords: adaptive progressive Type-II censored sample; Pareto distribution; pivotal quantity; weighted least squares method
Mathematics Subject Classification: 62F10

1. Introduction

In the experiments of various fields such as sciences, public health, and medicine, a proper censoring scheme is often considered because censored data arise commonly. The most famous censoring scheme is the progressive Type-II (PT-II) censoring scheme, where the observation number $m(\leq n)$ and the associated censoring scheme $\mathcal{R} = (R_1, \ldots, R_m)$ are pre-fixed, and the R_i surviving units are randomly withdrawn when the ith failure arises from the experiment. The PT-II censoring scheme is attractive in that cost and time are saved by withdrawing surviving units during the experiment. This advantage has given significant attention to this censoring scheme in the literature [1–3]. In particular, some authors developed inference methods based on a pivotal quantity as an alternative to the maximum likelihood method under the PT-II censoring scheme. Wang et al. [4] proved that a pivotal approach is superior to the maximum likelihood approach in parameter estimation of the two-parameter Weibull distribution for a small sample size. Seo and Kang [5] discussed a pivotal-based inference for estimating a scale parameter of the scaled half logistic distribution. They demonstrated the superiority of estimators based on a pivotal quantity for a small sample size, compared to the maximum likelihood estimator (MLE) and approximate MLEs. Seo et al. [6] studied an estimation method for unknown parameters of a Pareto distribution based on the regression-type framework using a pivotal quantity by extending the idea of Lu and Tao [7].

However, the PT-II censoring scheme has a drawback in that the total experiment time can still be long. To overcome this issue, Ng et al. [8] proposed the use of the ideal experiment time T as an adapted version, and adapted the PT-II censoring scheme as follows: If the mth failure arises before time T, then the experiment is terminated at the time $X_{m:m:n}$, which is the same as the PT-II censoring scheme. If the mth failure arises after time T, then surviving units are not withdrawn from the experiment by setting $R_{L+1} = \cdots = R_{m-1} = 0$, where $L(< m-1)$ is the number of observed failures until time T. At the time of the mth failure, all remaining $R_m = n - m - \sum_{i=1}^{L} R_i$ surviving units are withdrawn. This censoring scheme is called the adaptive PT-II (APT-II) censoring scheme. The basic idea of the APT-II censoring scheme is to finish the experiment as fast as possible when the experiment duration exceeds a predetermined time T. So, this censoring scheme can save both the total experiment time and cost, and increase the efficiency of statistical analysis. Due to this efficiency, the APT-II censoring scheme has been discussed by some authors such as Sobhi and Soliman [9], Ye et al. [10], and Mohan and Chacko [11].

This paper proposes an inference method based on a pivotal quantity under the APT-II censoring scheme. For an illustration of the proposed methodology, a Pareto distribution with the following cumulative distribution function (CDF) and probability density function is employed:

$$F(x; \lambda, \theta) = 1 - \left(\frac{\theta}{x}\right)^{\lambda} \tag{1.1}$$

and

$$f(x; \lambda, \theta) = \lambda \theta^{\lambda} x^{-(\lambda+1)}, \quad x > \theta,\ \lambda > 0,\ \theta > 0,$$

respectively, where λ is the shape parameter of interest and θ is the scale parameter. The Pareto distribution is one of the most widely used distributions to model a wide range of real-world cases in various fields such as economics, sociology, and engineering. For example, the distribution of income, internet traffic, and urban population is known to follow the Pareto distribution. To improve the estimation performance of the Pareto distribution, some authors have studied the inference for unknown parameters of this distribution by applying a pivotal quantity. Chen [12] used a pivotal quantity to obtain a joint confidence region for unknown parameters. Wu [13] introduced a pivotal-based method for obtaining a joint confidence region for unknown parameters under the doubly Type-II censoring scheme. Zhang [14] provided simplified versions of Chen [12] and Wu [13] to avoid computational difficulties. Kim et al. [15] proposed an estimation method based on the regression-type framework using a pivotal quantity, which provides a consistent estimator for the shape parameter. However, despite the superiority of the pivotal-based method, Mohie El-Din et al. [16] discussed a Bayesian approach for the Pareto distribution under the APT-II censoring scheme, and the approach has a substantial computational burden. Alternatively, by extending a pivotal approach to the APT-II censoring scheme, this paper proposes an inference method based on its excellent scalability, and the superiority and applicability of the proposed method are substantiated. Significantly, this study marks a novel approach by utilizing a pivotal-based method for the first time, aimed at estimating the unknown parameters of the Pareto distribution within the context of the APT-II censoring scheme.

For the Pareto distribution with CDF (1.1), classical methods such as the maximum likelihood and bootstrap methods have limitations in estimating confidence intervals (CIs) for λ and θ. The maximum likelihood method yields not exact but approximate CIs with the burden of computing the Fisher information matrix (FIM), and the approximate CIs do not ensure the satisfaction of the nominal levels when a sample size is not large enough. In addition, the asymptotic variance-covariance matrix (AVCM) of the MLEs suffers from a constraint. In the case of the bootstrap method, the CIs fail to satisfy nominal levels. These issues are expounded in Section 2. On the other hand, the proposed pivotal-based method not only easily leads to exact CIs (ECIs) for λ and θ without any conditions even in a situation where a sample size is not large enough, but also provides closed-form inference results.

Furthermore, this paper introduces the generalized pivotal quantity (GPQ) under the APT-II censoring scheme, which is applicable to the inference for functions with unknown parameters. As a specific example, we develop a method of generating the replicated data from the marginal distribution for the observed APT-II censored sample, inferring the distribution.

The rest of this paper is organized as follows: Section 2 proposes a pivotal-based estimation method for the unknown parameters of the Pareto distribution under the APT-II censoring scheme. Section 3 provides an algorithm, that generates the replicated data of the observed APT-II censored sample, based on GPQs. Section 4 conducts the Monte Carlo simulation and the real data analysis to assess the proposed methodology. Section 5 concludes this paper.

2. Pivotal-based inference

Let $X_{1:m:n} \leq \cdots \leq X_{m:m:n}$ be an APT-II censored sample with the censoring scheme

$$\mathcal{R}^* = \begin{cases} \mathcal{R}, & \text{if} \quad X_{m:m:n} < T, \\ \left(R_1, \ldots, R_L, 0^{*m-L-1}, n - m - \sum\limits_{i=1}^{L} R_i\right), & \text{if} \quad X_{m:m:n} > T, \end{cases} \tag{2.1}$$

where 0^{*m-L-1} denotes a vector of zeros of the size $m - L - 1$. The censoring scheme (2.1) means that the APT-II censoring scheme is the same as the PT-II censoring scheme for $X_{m:m:n} < T$, and does not allow the removal of experimental units by setting $R_{L+1} = \cdots = R_{m-1} = 0$ for $X_{m:m:n} > T$.

Suppose that the APT-II censored sample has a Pareto distribution with CDF (1.1). The corresponding likelihood and its logarithm functions are given by

$$L(\lambda, \theta) \propto \lambda^m \theta^{\lambda n} \prod_{i=1}^{m} x_{i:m:n}^{-\lambda(1+R_i)-1}$$

and

$$\log L(\lambda, \theta) \propto m \log \lambda + \lambda n \log \theta - \lambda \sum_{i=1}^{m} (1 + R_i) \log x_{i:m:n}, \tag{2.2}$$

respectively. Since the log-likelihood function (2.2) is a monotonically increasing function of θ, the MLE of θ is obtained as $\hat{\theta} = X_{1:m:n}$. Then, the MLE of λ is obtained as $\hat{\lambda} = m\Big/\left(\sum_{i=1}^{m} (1 + R_i) \log X_{i:m:n} - n \log X_{1:m:n}\right)$ by maximizing the log-likelihood function with $\theta = X_{1:m:n}$. Note that the MLEs $\hat{\lambda}$ and $\hat{\theta}$ are biased estimators. This is proved after stating Lemma 2.1. Moreover, constructing approximate CIs based on the MLEs is straightforward due to the asymptotic normality of the MLEs. However, this process necessitates the variance of the MLEs, which can be derived from the following FIM, calculated using the second partial derivatives of the negative log-likelihood function:

$$\begin{aligned} I(\lambda, \theta) &= \begin{bmatrix} E\left(\frac{-\partial^2 \log L(\lambda,\theta)}{\partial \lambda^2}\right) & E\left(\frac{-\partial^2 \log L(\lambda,\theta)}{\partial \lambda \partial \theta}\right) \\ E\left(\frac{-\partial^2 \log L(\lambda,\theta)}{\partial \theta \partial \lambda}\right) & E\left(\frac{-\partial^2 \log L(\lambda,\theta)}{\partial \theta^2}\right) \end{bmatrix} \\ &= \begin{pmatrix} m/\lambda^2 & -n/\theta \\ -n/\theta & n\lambda/\theta^2 \end{pmatrix}. \end{aligned} \tag{2.3}$$

By inverting the FIM (2.3), the AVCM of the MLEs $\hat{\lambda}$ and $\hat{\theta}$ is obtained as

$$\begin{aligned} \hat{\Sigma} &= \begin{pmatrix} m/\lambda^2 & -n/\theta \\ -n/\theta & n\lambda/\theta^2 \end{pmatrix}^{-1} \Bigg|_{(\lambda=\hat{\lambda},\ \theta=\hat{\theta})} \\ &= \frac{\hat{\theta}^2 \hat{\lambda}}{n\left(m - n\hat{\lambda}\right)} \begin{pmatrix} n\hat{\lambda}/\hat{\theta}^2 & n/\hat{\theta} \\ n/\hat{\theta} & m/\hat{\lambda}^2 \end{pmatrix}, \quad \hat{\lambda} < \frac{m}{n}. \end{aligned} \tag{2.4}$$

The diagonal elements of the AVCM (2.4) are the variances of the MLEs $\hat{\lambda}$ and $\hat{\theta}$, respectively. Note that the AVCM (2.4) does not always exist because of the constraint $\hat{\lambda} < m/n$. In other words, the construction of approximate CIs is not always possible. Alternatively, we propose inference based on the pivotal quantity under the APT-II censoring scheme.

Let

$$\begin{aligned} Y_{i:m:n} &= -\log\left(1 - F(x_{i:m:n}; \lambda, \theta)\right) \\ &= \lambda \log\left(\frac{X_{i:m:n}}{\theta}\right), \quad i = 1, \ldots, m. \end{aligned} \tag{2.5}$$

Then, $Y_{1:m:n} \leq \cdots \leq Y_{m:m:n}$ is an APT-II censored sample that has a standard exponential distribution with mean $E(Y_{i:m:n}) = \sum_{j=1}^{i} 1/\Gamma_j$, where $\Gamma_1 = n$ and $\Gamma_j = n - \sum_{k=1}^{j-1} (1 + R_k)$ for $j = 2, \ldots, m$. It is readily apparent that, as the quantity (2.5) is expressed as $Y_{i:m:n} \stackrel{d}{=} -\log(1 - U_{i:m:n})$, using the relationship $F(X_{i:m:n}; \lambda, \theta) \stackrel{d}{=} U_{i:m:n}$, where $\mathfrak{A} \stackrel{d}{=} \mathfrak{B}$ denotes that $\mathfrak{A}$ and $\mathfrak{B}$ have the same distribution, and $U_{i:m:n}$ is an ith APT-II censored order statistic from a standard uniform distribution. In addition, the quantity (2.5) induces the normalized spacings

$$Z_i = \Gamma_i \left(Y_{i:m:n} - Y_{i-1:m:n}\right)$$

$$= \lambda\Gamma_i \log\left(\frac{X_{i:m:n}}{X_{i-1:m:n}}\right), \quad i = 1, \ldots, m,$$

where $Y_{0:m:n} = 0$. Note that Z_i $(i = 1, \ldots, m)$ are independent standard exponential random variables [17], which leads to some pivotal quantities that play an important role in deriving estimation equations. These pivotal quantities are provided in Lemma 2.1. Before introducing Lemma 2.1, to simply express the frequently used distributions, we declare two notations as follows: χ^2_ν and $F(d_1, d_2)$ denote a χ^2 distribution with ν degrees of freedom and an F distribution with (d_1, d_2) degrees of freedom, respectively.

Lemma 2.1. *Let $X_{1:m:n} \leq \cdots \leq X_{m:m:n}$ be an APT-II censored sample with the censoring scheme $\mathcal{R}^*$ from the Pareto distribution with CDF* (1.1)*. Then,*

$$(a)\ \mathfrak{X}_1(\lambda) = 2\lambda\left(\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}\right),$$

$$(b)\ \mathfrak{F}(\theta) = \frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}}{n(m-1)\log\left(X_{1:m:n}/\theta\right)},$$

and

$$(c)\ \mathfrak{X}_2(\theta) = 2(m-1)\log\left(\sum_{j=1}^{m}\left(1+R_j\right)\log X_{j:m:n} - n\log\theta\right)$$
$$- 2\sum_{i=1}^{m-1}\log\left[\sum_{j=1}^{i}\left(1+R_j\right)\log X_{j:m:n} + \left(n - \sum_{j=1}^{i}\left(1+R_j\right)\right)\log X_{i:m:n} - n\log\theta\right],$$

which have $\chi^2_{2(m-1)}$, $F\left(2(m-1), 2\right)$, and $\chi^2_{2(m-1)}$ distributions, respectively.

Proof. (a) and (b) are clear from $\mathfrak{X}_1(\lambda) = 2\sum_{i=2}^{m} Z_i$ and $\mathfrak{F}(\theta) = \left(2\sum_{i=2}^{m} Z_i \Big/ [2(m-1)]\right) \Big/ (2Z_1/2)$ since Z_i $(i = 1, \ldots, m)$ are independent standard exponential random variables as mentioned earlier. In addition, the quantity $\sum_{j=1}^{i} Z_j \Big/ \sum_{j=1}^{m} Z_j$ $(i = 1, \ldots, m-1)$ are the order statistics from a standard uniform distribution with the sample size $m-1$. Then, (c) is proved from

$$\mathfrak{X}_2(\theta) = -2\sum_{i=1}^{m-1}\log\left(\frac{\sum_{j=1}^{i} Z_j}{\sum_{j=1}^{m} Z_j}\right).$$

This completes the proof. □

The following subsections provide inference results based on the pivotal quantities in Lemma 2.1.

2.1. Point estimation

From the pivotal quantity $\mathfrak{X}_1(\lambda)$ in Lemma 2.1, it is easily proved that the MLE $\hat{\lambda}$ is a biased estimator because it has an inverse gamma distribution with parameters $(m-1, \lambda m)$. Alternatively, an estimator of λ is obtained as

$$\hat{\lambda}_u = \frac{m-2}{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}}.$$

Theorem 2.2. *The estimator $\hat{\lambda}_u$ has unbiasedness and consistency.*

Proof. The estimator $\hat{\lambda}_u$ has an inverse gamma distribution with parameters $(m-1, \lambda(m-2))$ from the pivotal quantity $\mathfrak{X}_1(\lambda)$ in Lemma 2.1. This indicates $E\left(\hat{\lambda}_u\right) = \lambda$ and $Var\left(\hat{\lambda}_u\right) = \lambda^2/(m-3)$. In addition, the variance $Var\left(\hat{\lambda}_u\right)$ converges to zero in probability as $m \to \infty$. This completes the proof. □

We introduce another consistent estimator for λ under the APT-II censoring scheme, which is derived from a weighted least squares approach based on a regression-type framework using the quantity (2.5).

Let $D_{i:m:n} = Y_{i:m:n} - Y_{1:m:n}$ $(i = 2, \ldots, m)$. Then, its expectation is given by

$$\begin{aligned} E(D_{i:m:n}) &= E(Y_{i:m:n}) - E(Y_{1:m:n}) \\ &= \sum_{j=1}^{i}\frac{1}{\Gamma_j} - \frac{1}{\Gamma_1} \\ &= \sum_{j=2}^{i}\frac{1}{\Gamma_j}, \quad i = 2, \ldots, m. \end{aligned}$$

Using it, we can consider the following linear regression model:

$$E(D_{i:m:n}) = \lambda\log\left(\frac{X_{i:m:n}}{X_{1:m:n}}\right) + \varepsilon_i, \quad i = 2, \ldots, m, \tag{2.6}$$

where ε_i is the error term with $E(\varepsilon_i) = 0$. Note that the regression model (2.6) is a simple regression model with no intercept and does not depend on θ. Then, the weighted least squares estimator for λ is obtained from Eq (2.6) by minimizing the quantity $\sum_{i=2}^{m} w_{i:m:n}\left[E(D_{i:m:n}) - \lambda\log(X_{i:m:n}/X_{1:m:n})\right]^2$ with respect to λ as

$$\hat{\lambda}_w = \frac{\sum_{i=2}^{m} w_{i:m:n}E(D_{i:m:n})\log(X_{i:m:n}/X_{1:m:n})}{\sum_{i=2}^{m} w_{i:m:n}\left[\log(X_{i:m:n}/X_{1:m:n})\right]^2}, \tag{2.7}$$

where $w_{i:m:n}$ is the weight of each data point and assumes that it is not an identical constant.

Theorem 2.3. *Let $w_{i:m:n} = 1/Var(D_{i:m:n})$ in the estimator* (2.7)*, where $Var(D_{i:m:n}) = \sum_{j=2}^{i}\Gamma_j^{-2}$, $i = 2, \ldots, m$. Then, the estimator $\hat{\lambda}_w$ is a consistent estimator.*

Proof. The estimator $\hat{\lambda}_w$ with $w_{i:m:n} = 1/Var(D_{i:m:n})$ can be written as

$$\begin{aligned}\hat{\lambda}_w &= \lambda \frac{\sum\limits_{i=2}^{m} D_{i:m:n} E(D_{i:m:n}) / Var(D_{i:m:n})}{\sum\limits_{i=2}^{m} D_{i:m:n}^2 / Var(D_{i:m:n})} \\ &= \lambda \frac{\sum\limits_{i=2}^{m} Q_{1,i:m:n} + \sum\limits_{i=2}^{m} Q_{3,i:m:n}}{\sum\limits_{i=2}^{m} Q_{2,i:m:n} + \sum\limits_{i=2}^{m} Q_{3,i:m:n}} \\ &= \lambda \frac{\sum\limits_{i=2}^{m} Q_{1,i:m:n} \Big/ m^2 + \sum\limits_{i=2}^{m} Q_{3,i:m:n} \Big/ m^2}{\sum\limits_{i=2}^{m} Q_{2,i:m:n} \Big/ m^2 + \sum\limits_{i=2}^{m} Q_{3,i:m:n} \Big/ m^2},\end{aligned} \tag{2.8}$$

where

$$\begin{aligned}Q_{1,i:m:n} &= \frac{D_{i:m:n} E(D_{i:m:n}) - E^2(D_{i:m:n})}{Var(D_{i:m:n})}, \\ Q_{2,i:m:n} &= \frac{D_{i:m:n}^2 - E^2(D_{i:m:n})}{Var(D_{i:m:n})}, \\ Q_{3,i:m:m} &= \frac{E^2(D_{i:m:n})}{Var(D_{i:m:n})}.\end{aligned}$$

In Eq (2.8), the quantities $\sum\limits_{i=2}^{m} Q_{1,i:m:n} \Big/ m^2$ and $\sum\limits_{i=2}^{m} Q_{2,i:m:n} \Big/ m^2$ converge to zero in probability as $m \to \infty$. In addition, since the quantity $\sum\limits_{i=2}^{m} Q_{3,i:m:n} \Big/ m^2$ converges to a constant in probability as $m \to \infty$, the fraction term in Eq (2.8) converges to 1 in probability as $m \to \infty$. The results for these probability convergences can be easily shown according to Seo et al. [6]. The proof in Seo et al. [6] is restated to match our notation.

The quantities $\sum\limits_{i=2}^{m} Q_{1,i:m:n} \Big/ m^2$ and $\sum\limits_{i=2}^{m} Q_{2,i:m:n} \Big/ m^2$ converge in the mean to 0 since

$$E\left(\left|\frac{1}{m^2} \sum_{i=2}^{m} Q_{1,i:m:n}\right|\right) = 0$$

and

$$E\left(\left|\frac{1}{m^2} \sum_{i=2}^{m} Q_{2,i:m:n}\right|\right) = \frac{m-1}{m^2},$$

which implies convergence in probability [18]. In addition,

$$\frac{1}{m^2} \sum_{i=2}^{m} Q_{3,i:m:n} = \frac{1}{m^2} \sum_{i=2}^{m} \frac{\left\{\sum\limits_{j=2}^{i} \left[\sum\limits_{k=j}^{m} (1+R_k)\right]^{-1}\right\}^2}{\sum\limits_{j=2}^{i} \left[\sum\limits_{k=j}^{m} (1+R_k)\right]^{-2}}$$

$$\geq \frac{1}{m^2}\sum_{i=2}^{m}\frac{\left[\sum_{j=2}^{i}\left(m-1+\sum_{k=j}^{m}R_k\right)^{-1}\right]^2}{\sum_{j=2}^{i}(m-i+1)^{-2}}$$

$$= \frac{1}{m^2}\sum_{i=2}^{m}(i-1)\left(\frac{m-i+1}{m-1+\sum_{k=j}^{m}R_k}\right)^2$$

$$= \frac{1}{m^2\left(m-1+\sum_{k=j}^{m}R_k\right)^2}\left(m^2\sum_{j=1}^{m}j+\sum_{j=1}^{m}j^3-2m\sum_{j=1}^{m}j^2\right),$$

which converges to a constant as $m \to \infty$. This completes the proof. □

The estimation for θ can be accomplished by a similar argument to that used for obtaining the estimator $\hat{\lambda}_u$. To do this, we employ the pivotal quantity $\mathfrak{F}(\theta)$ in Lemma 2.1. The inverse of the pivotal quantity $\mathfrak{F}(\theta)$ is written as

$$\frac{1}{\mathfrak{F}(\theta)} = \frac{2Z_1/2}{2\sum_{i=2}^{m}Z_i\Big/[2(m-1)]}, \tag{2.9}$$

which has an $F(2, 2(m-1))$ distribution. By Slutsky's theorem [19], the distribution of $2/\mathfrak{F}(\theta)$ converges to a χ^2_2 distribution as $m \to \infty$ since the denominator term in (2.9) converges to 1 in probability as $m \to \infty$. Then, using the mean of a χ^2_2 distribution, an equation for θ is obtained as $2/\mathfrak{F}(\theta) = 2$. From the equation, an estimator of θ is derived as

$$\hat{\theta}_p = X_{1:m:n}\exp\left(-\frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}}{n(m-1)}\right).$$

Theorem 2.4. *The estimator $\hat{\theta}_p$ is an asymptotic unbiased and consistent estimator.*

Proof. Since the quantity $\left(\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}\right)\Big/[n(m-1)]$ has a gamma distribution with parameters $(m-1, n\lambda(m-1))$, by its moment generating function, we have

$$E\left[\exp\left(-\frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}}{n(m-1)}\right)\right] = \left[1+\frac{1}{n\lambda(m-1)}\right]^{-(m-1)} \tag{2.10}$$

and

$$E\left[\exp\left(-\frac{2\left(\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n}-n\log X_{1:m:n}\right)}{n(m-1)}\right)\right]=\left[1+\frac{2}{n\lambda(m-1)}\right]^{-(m-1)}. \tag{2.11}$$

In addition, the first and second moments of $X_{1:m:n}$ are

$$\begin{aligned}E\left(X_{1:m:n}\right)&=\int_{\theta}^{\infty}n\lambda\left(\frac{\theta}{x}\right)^{\lambda n}dx\\&=\frac{n\lambda\theta}{n\lambda-1},\quad \lambda>\frac{1}{n}\end{aligned} \tag{2.12}$$

and

$$\begin{aligned}E\left(X_{1:m:n}^{2}\right)&=\int_{\theta}^{\infty}n\lambda x\left(\frac{\theta}{x}\right)^{\lambda n}dx\\&=\frac{n\lambda\theta^{2}}{n\lambda-2},\quad \lambda>\frac{2}{n},\end{aligned} \tag{2.13}$$

respectively. Then, using (2.10) and (2.12), the mean of the estimator $\hat{\theta}_p$ is obtained as

$$\begin{aligned}E\left(\hat{\theta}_p\right)&=E\left[X_{1:m:n}\exp\left(-\frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n}-n\log X_{1:m:n}}{n(m-1)}\right)\right]\\&=E\left(X_{1:m:n}\right)E\left[\exp\left(-\frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n}-n\log X_{1:m:n}}{n(m-1)}\right)\right]\\&=\frac{n\lambda\theta}{(n\lambda-1)}\left[1+\frac{1}{n\lambda(m-1)}\right]^{-(m-1)}\end{aligned}$$

and using (2.10)–(2.13), the variance of the estimator $\hat{\theta}_p$ is obtained as

$$\begin{aligned}Var\left(\hat{\theta}_p\right)=&Var\left[X_{1:m:n}\exp\left(-\frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n}-n\log X_{1:m:n}}{n(m-1)}\right)\right]\\=&\left\{E\left[\exp\left(-\frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n}-n\log X_{1:m:n}}{n(m-1)}\right)\right]\right\}^{2}Var\left(X_{1:m:n}\right)\\&+Var\left[\exp\left(-\frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n}-n\log X_{1:m:n}}{n(m-1)}\right)\right]E\left(X_{1:m:n}^{2}\right)\end{aligned}$$

$$= -\left[1+\frac{1}{n\lambda(m-1)}\right]^{-2(m-1)}\frac{n^2\lambda^2\theta^2}{(n\lambda-1)^2}+\left[1+\frac{2}{n\lambda(m-1)}\right]^{-(m-1)}\frac{n\lambda\theta^2}{(n\lambda-2)}$$

$$= -\left[1+\frac{1}{\left(m+\sum_{i=1}^{m}R_i\right)\lambda(m-1)}\right]^{-2(m-1)}\frac{\left(m+\sum_{i=1}^{m}R_i\right)^2\lambda^2\theta^2}{\left[\left(m+\sum_{i=1}^{m}R_i\right)\lambda-1\right]^2}$$

$$+\left[1+\frac{2}{\left(m+\sum_{i=1}^{m}R_i\right)\lambda(m-1)}\right]^{-(m-1)}\frac{\left(m+\sum_{i=1}^{m}R_i\right)\lambda\theta^2}{\left[\left(m+\sum_{i=1}^{m}R_i\right)\lambda-2\right]}.$$

The mean $E\left(\hat{\theta}_p\right)$ converges to θ in probability as $m \to \infty$, and the variance $Var\left(\hat{\theta}_p\right)$ converges to zero in probability as $m \to \infty$. This completes the proof. □

Note that the estimator $\hat{\theta}_p$ is not unbiased because of $E\left(\hat{\theta}_p\right) \neq \theta$. So, we provide another estimator for θ that is improved in terms of the bias. From (2.12), an unbiased estimator of θ is obtained as $X_{1:m:n}\,[1 - 1/(n\lambda)]$ for known λ. Then, by substituting λ with the MLE $\hat{\lambda}$, an unbiased estimator of θ is derived as

$$\begin{aligned}\hat{\theta}_u &= X_{1:m:n}\left[1-\frac{m}{n(m-1)\hat{\lambda}}\right]\\ &= X_{1:m:n}\left[1-\frac{\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n}-n\log X_{1:m:n}}{n(m-1)}\right].\end{aligned}$$

The relationship $\hat{\theta}_p \geq \hat{\theta}_u$ is satisfied because of $\exp(-a) \geq 1 - a$ for all real numbers a which can be easily shown from the Maclaurin series $\exp(a) = 1 + a + \sum_{i=2}^{\infty} a^i/i!$ for all real numbers a.

Theorem 2.5. *The estimator $\hat{\theta}_u$ has unbiasedness and consistency.*

Proof. The inverse of MLE $\hat{\lambda}$ has a gamma distribution with parameters $(m-1, \lambda m)$, and its first and second moments are

$$E\left(\frac{1}{\hat{\lambda}}\right) = \frac{m-1}{\lambda m} \tag{2.14}$$

and

$$E\left(\frac{1}{\hat{\lambda}^2}\right) = \frac{m-1}{\lambda^2 m}, \tag{2.15}$$

respectively. Then, using (2.12) and (2.14), the mean of the estimator $\hat{\theta}_u$ is obtained as

$$E\left(\hat{\theta}_u\right) = E\left[X_{1:m:n}\left(1-\frac{m}{n(m-1)\hat{\lambda}}\right)\right]$$

$$
\begin{aligned}
&= E(X_{1:m:n}) - E(X_{1:m:n})\, E\left[\frac{m}{n(m-1)\hat{\lambda}}\right] \\
&= \frac{n\lambda\theta}{n\lambda - 1} - \frac{\theta}{n\lambda - 1} \\
&= \theta.
\end{aligned}
$$

Using (2.12)–(2.15), the variance of the estimator $\hat{\theta}_u$ is obtained as

$$
\begin{aligned}
Var\left(\hat{\theta}_u\right) &= Var\left[X_{1:m:n}\left(1 - \frac{m}{n(m-1)\hat{\lambda}}\right)\right] \\
&= \left\{E\left[1 - \frac{m}{n(m-1)\hat{\lambda}}\right]\right\}^2 Var(X_{1:m:n}) + Var\left[1 - \frac{m}{n(m-1)\hat{\lambda}}\right] E\left(X_{i:m:n}^2\right) \\
&= \frac{m\theta^2}{n\lambda(n\lambda - 2)(m-1)} \\
&= \frac{m\theta^2}{\left(m + \sum_{i=1}^{m} R_i\right)\lambda\left[\left(m + \sum_{i=1}^{m} R_i\right)\lambda - 2\right](m-1)},
\end{aligned}
$$

which converges to zero as $m \to \infty$. This completes the proof. □

On the other hand, the pivotal quantity $\mathfrak{X}_2(\theta)$ yields an estimation equation $\mathfrak{X}_2(\theta) = 2(m-2)$ from the fact that $\mathfrak{X}_2(\theta)/2(m-2)$ converges to 1 in probability as $m \to \infty$. However, the equation does not provide a closed form of solution, so it is not considered here.

2.2. *Interval estimation*

As mentioned earlier, the approximate CIs based on the MLEs are constructed when the condition $\hat{\lambda} < m/n$ is satisfied. Another classical method of constructing CIs for λ and θ is based on the bootstrap method. It is conducted through the following steps: First, the MLEs $\hat{\lambda}$ and $\hat{\theta}$ are calculated based on the original APT-II censored sample. Second, B bootstrap APT-II censored samples are generated from the marginal distribution with the MLEs $\hat{\lambda}$ and $\hat{\theta}$, which is denoted as $X_{1:m:n}^{(b)} \le \cdots \le X_{m:m:n}^{(b)},\ b = 1, \ldots, B$. Then, the MLEs for λ and θ are calculated based on the bootstrap APT-II censored sample $\left\{X_{1:m:n}^{(b)}, \ldots, X_{m:m:n}^{(b)}\right\}$, and it is denoted as $\hat{\lambda}^{(b)}$ and $\hat{\theta}^{(b)},\ b = 1, \ldots, B$, respectively. After obtaining $\left\{\hat{\lambda}^{(1)}, \ldots, \hat{\lambda}^{(B)}\right\}$ and $\left\{\hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(B)}\right\}$, we can construct the CIs for λ and θ using their percentiles. In other words, $100(1-\alpha)\%$ CIs for λ and θ are constructed as $\left(\hat{\lambda}^{([(\alpha/2)B])}, \hat{\lambda}^{([(1-\alpha/2)B])}\right)$ and $\left(\hat{\theta}^{([(\alpha/2)B])}, \hat{\theta}^{([(1-\alpha/2)B])}\right)$ for $0 < \alpha < 1$, where $\hat{\lambda}^{([\alpha B])}$ and $\hat{\theta}^{([\alpha B])}$ denote the $[\alpha B]$th smallest values of $\left\{\hat{\lambda}^{(1)}, \ldots, \hat{\lambda}^{(B)}\right\}$ and $\left\{\hat{\theta}^{(1)}, \ldots, \hat{\theta}^{(B)}\right\}$, respectively. However, the CI for θ does not satisfy nominal levels because $\hat{\theta}^{(b)} = X_{1:m:n}^{(b)},\ b = 1, \ldots, B$ is always greater than the true value of θ.

The pivotal-based interval inference we now provide can address these limitations, and it does not require complex mathematical calculations, unlike the FIM. Even for a small sample size, the estimated interval from the pivotal quantity satisfies nominal levels well. Here, ECIs for λ and θ are provided using the pivotal quantities in Lemma 2.1.

For λ, since the pivotal quantity $\mathfrak{X}_1(\lambda)$ in Lemma 2.1 has a $\chi^2_{2(m-1)}$ distribution, it follows that

$$1-\alpha = P\left[\chi^2_{1-\alpha/2,2(m-1)} < 2\lambda\left(\sum_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}\right) < \chi^2_{\alpha/2,2(m-1)}\right]$$

for $0 < \alpha < 1$, where $\chi^2_{\alpha,2(m-1)}$ denotes the upper α percentile of a $\chi^2_{2(m-1)}$ distribution. Then, an exact $100(1-\alpha)\%$ CI for λ is given by

$$\left[\frac{\chi^2_{1-\alpha/2,2(m-1)}}{2\left(\sum\limits_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}\right)}, \frac{\chi^2_{\alpha/2,2(m-1)}}{2\left(\sum\limits_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}\right)}\right].$$

For θ, since the pivotal quantity $\mathfrak{F}(\theta)$ in Lemma 2.1 has an $F\,(2(m-1),2)$ distribution, it follows that

$$1-\alpha = P\left[F_{1-\alpha/2(2(m-1),2)} < \frac{\sum\limits_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}}{n(m-1)\log\left(X_{1:m:n}/\theta\right)} < F_{\alpha/2(2(m-1),2)}\right]$$

for $0 < \alpha < 1$, where $F_{\alpha(2(m-1),2)}$ denotes the upper α percentile of an $F\,(2(m-1),2)$ distribution. Then, an exact $100(1-\alpha)\%$ CI for θ is given by

$$\left[X_{1:m:n}\exp\left(-F_{\alpha/2(2,2(m-1))}g(\mathbf{X})\right),\ X_{1:m:n}\exp\left(-F_{1-\alpha/2(2,2(m-1))}g(\mathbf{X})\right)\right],$$

where $g(\mathbf{X}) = \left(\sum\limits_{i=1}^{m}(1+R_i)\log X_{i:m:n} - n\log X_{1:m:n}\right)\Big/[n(m-1)]$.

3. Replication

Let $X^{\text{rep}}_{i:m:n}$ $(i = 1,\ldots,m)$ be the replicated data of the observed APT-II censored sample $\boldsymbol{x} = \{x_{1:m:n},\ldots,x_{m:m:n}\}$ with the censoring scheme $\mathcal{R}^*$. The replicated data $X^{\text{rep}}_{i:m:n}$ is generated by inferring the marginal distribution $F\left(X^{\text{rep}}_{i:m:n}|\boldsymbol{x};\lambda,\theta\right)$ based on the observed APT-II censored sample. To achieve it, GPQs for λ and θ are first defined as

$$\mathfrak{G}_1(\lambda) = \frac{\mathfrak{X}_1(\lambda)}{2\left(\sum\limits_{i=1}^{m}(1+R_i)\log x_{i:m:n} - n\log x_{1:m:n}\right)}$$

and

$$\mathfrak{G}_2(\theta) = \exp\left(\log x_{1:m:n} - \frac{\sum\limits_{i=1}^{m}(1+R_i)\log x_{i:m:n} - n\log x_{1:m:n}}{n(m-1)\mathfrak{F}(\theta)}\right),$$

respectively, according to the argument of Weerahandi [20]. The GPQs $\mathfrak{G}_1(\lambda)$ and $\mathfrak{G}_2(\theta)$ obviously have two properties: The distributions of the GPQs $\mathfrak{G}_1(\lambda)$ and $\mathfrak{G}_2(\theta)$ are free of unknown parameters, and the realization of the GPQs $\mathfrak{G}_1(\lambda)$ and $\mathfrak{G}_2(\theta)$ does not depend on the nuisance parameter.

The realization of the GPQs $\mathfrak{G}_1(\lambda)$ and $\mathfrak{G}_2(\theta)$ is obtained by using a pseudorandom sequence from a χ^2_2 distribution based on an idea on which the pivotal quantities $\mathfrak{X}_1(\lambda)$ and $\mathfrak{F}(\theta)$ in Lemma 2.1 are derived. Then, by substituting λ and θ in the marginal distribution $F\left(X^{\text{rep}}_{i:m:n}|\boldsymbol{x};\lambda,\theta\right)$ with the realization of the GPQs $\mathfrak{G}_1(\lambda)$ and $\mathfrak{G}_2(\theta)$, respectively, the replicated data $X^{\text{rep}}_{i:m:n}$ is generated. The detailed steps are given in the following algorithm:

Algorithm 1

(a) Generate $\zeta_1,\dots,\zeta_m$ from a χ^2_2 distribution.

(b) Compute $\mathfrak{X}^* = \sum_{i=2}^{m}\zeta_i$, followed by $\mathfrak{G}_1^*(\lambda) = \frac{\mathfrak{X}^*}{2\left(\sum_{i=1}^{m}(1+R_i)\log x_{i:m:n} - n\log x_{1:m:n}\right)}$.

(c) Compute $\mathfrak{F}^* = \sum_{i=2}^{m}\zeta_i/[2(m-1)]/(\zeta_1/2)$, followed by $\mathfrak{G}_2^*(\theta) = \exp\left(\log x_{1:m:n} - \frac{\sum_{i=1}^{m}(1+R_i)\log x_{i:m:n} - n\log x_{1:m:n}}{n(m-1)\mathfrak{F}^*}\right)$.

(d) Generate $X^{\text{rep}}_{i:m:n}$ from the sampling distribution $F\left(X^{\text{rep}}_{i:m:n}|\boldsymbol{x};\mathfrak{G}_1^*(\lambda),\mathfrak{G}_2^*(\theta)\right)$.

(e) Repeat $N(\geq 10,000)$ times (a)–(d).

From ***Algorithm 1***, a $100(1-\alpha)\%$ interval for the replicated data $X^{\text{rep}}_{i:m:n}$ is constructed as

$$\left(X^{\text{rep},([(\alpha/2)N])}_{i:m:n},\ X^{\text{rep},([(1-\alpha/2)N])}_{i:m:n}\right),$$

where $X^{\text{rep},([\alpha N])}_{i:m:n}$ denotes the $[\alpha N]$th smallest value of $\left\{X^{\text{rep},1}_{i:m:n},\dots,X^{\text{rep},N}_{i:m:n}\right\}$. This interval is employed to evaluate the uncertainty of the replicated data $X^{\text{rep}}_{i:m:n}$ in Section 4.2.

4. Application

The proposed methodology is evaluated using the Monte Carlo simulation technique. Moreover, its applicability and scalability are examined by performing real-world data analysis.

4.1. Simulation study

To verify the estimation performance of the proposed estimators in comparison with the MLEs, Monte Carlo simulations with 10,000 replications are conducted through the *R* software. For the Pareto distribution, the shape parameter of interest, λ, is assigned 0.5(0.5)1.5 to showcase the variations in results according to the values of λ, and the scale parameter θ is assigned 1 without loss of generality. In addition, the following censoring scheme is employed to emphasize the strength of the proposed methodology for a sample size that is not large enough in various situations:

$$\begin{aligned}\text{Scheme}\quad &\text{I}: n=20,\ m=8,\ \mathcal{R}=\left(1^{*3},0^{*4},9\right),\\ &\text{II}: n=20,\ m=8,\ \mathcal{R}=\left(2^{*2},0^{*5},8\right),\\ &\text{III}: n=20,\ m=6,\ \mathcal{R}=\left(1^{*2},0^{*3},12\right),\end{aligned}$$

$$\text{IV}: n=20,\ m=6,\ \mathcal{R}=\left(2,0^{*4},12\right),$$
$$\text{V}: n=40,\ m=16,\ \mathcal{R}=\left(1^{*7},0^{*8},17\right),$$
$$\text{VI}: n=40,\ m=16,\ \mathcal{R}=\left(2^{*6},0^{*9},12\right),$$
$$\text{VII}: n=40,\ m=12,\ \mathcal{R}=\left(1^{*5},0^{*6},23\right),$$
$$\text{VIII}: n=40,\ m=12,\ \mathcal{R}=\left(2^{*4},0^{*7},20\right).$$

For each censoring scheme, an APT-II censored sample is generated using the following steps:

(a) Generate an ordinary PT-II censored sample $X^*_{1:m:n},\ldots,X^*_{m:m:n}$ with the censoring scheme $\mathcal{R}$ from the algorithms of Balakrishnan and Sandhu [21] as follows:

(1) Generate m random numbers $W_1,\ldots,W_m$ from a standard uniform distribution.

(2) Compute $V_i = W_i^{1/(i+R_m+\cdots+R_{m-i+1})}$, $i=1,\ldots,m$.

(3) Compute $U_i = 1 - V_m \cdots V_{m-i+1}$, $i=1,\ldots,m$, where $U_1,\ldots,U_m$ is a PT-II censored sample of size m from a standard uniform distribution.

(4) Compute $X^*_{i:m:n} = \theta/(1-U_i)^{1/\lambda}$, $i=1,\ldots,m$, to obtain a PT-II censored sample from the Pareto distribution with CDF (1.1).

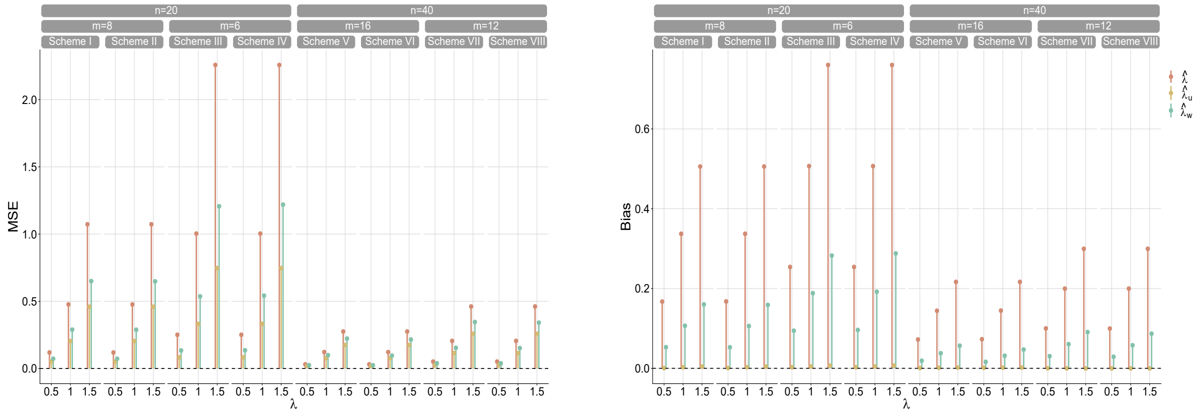
(b) Determine the value of L, where $X^*_{L:m:n} < T < X^*_{L+1:m:n}$.

(c) Generate the first $m-l-1$ order statistics from a truncated distribution $f(x)/[1-F(x_{l+1:m:n})]$ with sample size $n-\sum_{i=1}^{l}(1+R_i)-1$.

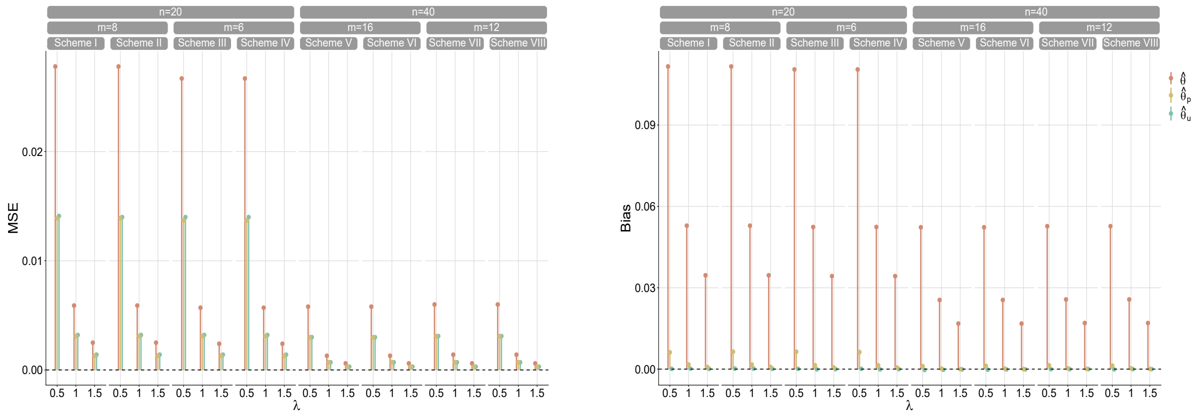
(d) Substitute $X^*_{l+2:m:n},\ldots,X^*_{m:m:n}$ with the first $m-l-1$ order statistics obtained in (c).

To ensure that the number of simulated APT-II censored samples observed until T is greater than 1, we assign $T=2.5$. Based on the generated APT-II censored samples, the mean squared errors (MSEs) and biases of the provided estimators are computed for each censoring scheme. The results are reported in Figure 1. In addition, the coverage probabilities (CPs) for the exact 95% CIs are provided in Figure 2. As mentioned earlier, the approximate and bootstrap CIs based on the MLEs suffer from constraints. Accordingly, only the results of the proposed ECIs for λ and θ are reported.

In Figure 1, the length of the line indicates the difference from the true value, so the shorter this is, the better the performance of the corresponding estimator. Based on this argument, the results are summarized as follows: For λ, $\hat{\lambda}_u$ has the best performance in terms of the MSE and bias, followed by $\hat{\lambda}_w$. For θ, $\hat{\theta}_p$ and $\hat{\theta}_u$ are more efficient than $\hat{\theta}$ in terms of the MSE and bias. To be specific, $\hat{\theta}_u$ has the smallest bias, as expected, while $\hat{\theta}_p$ is slightly better than $\hat{\theta}_u$ in terms of the MSE. Taken together, the proposed estimators have generally better performance than the corresponding MLEs in terms of MSE and bias. Furthermore, the use of the proposed estimators, especially $\hat{\lambda}_u$ and $\hat{\theta}_u$, is strongly recommended to infer the unknown parameters of the Pareto distribution with CDF (1.1) when the sample size is not large enough. In Figure 2, the length of the line represents the difference from the considered nominal level 0.95. According to this argument, the proposed intervals have a highly closer CP to the considered nominal level.



(a)



(b)

Figure 1. MSEs and biases of estimators for (a) λ and (b) θ.

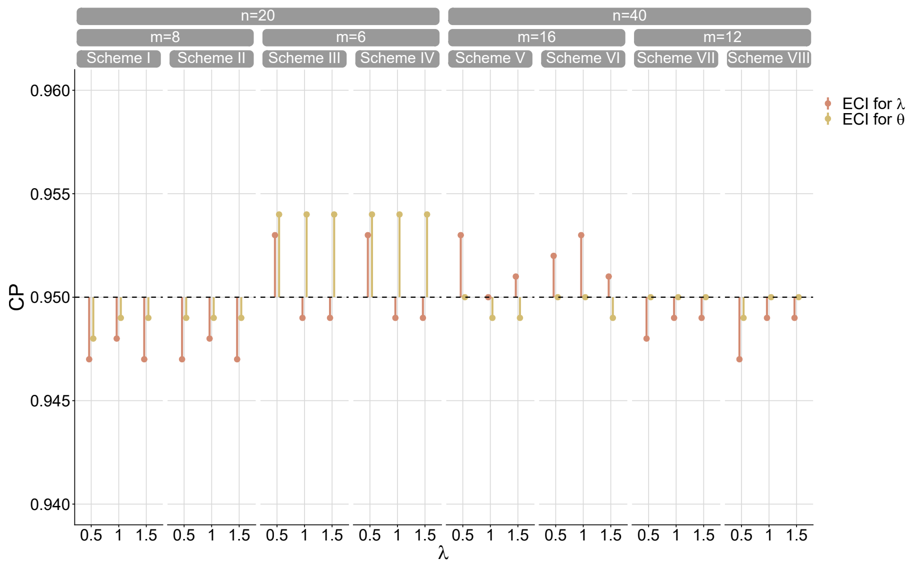


Figure 2. CPs of ECIs for λ and θ.

In conclusion, the results in Figures 1 and 2 reveal that the proposed estimation method yields more satisfactory results than the maximum likelihood estimation method in a situation where the sample size is not large enough.

4.2. Data analysis

The Pareto distribution is often employed for modeling a wide range of real-world datasets, such as insurance, reliability, engineering, and economics. In this subsection, the mortality rate of COVID-19 studied by Almetwally et al. [22] and Nik et al. [23] is analyzed to validate the practical application of the proposed approaches for a real-world dataset, where the mortality rate refers to the proportion of individuals who have died in a specific population or group affected by a particular disease. Nik et al. [23] applied a new Pareto-type distribution to the mortality rate of COVID-19 for 25 days, from 10 April to 4 May 2020, in Canada, and the data is as follows:

3.1091	3.3825	3.1444	3.2135	2.4946	3.5146	4.9274	3.3769	6.8686	3.0914	4.9378	3.1091	3.2823
3.8594	4.0480	4.1685	3.6426	3.2110	2.8636	3.2218	2.9078	3.6346	2.7957	4.2781	4.2202	

For analysis, an APT-II censored sample is generated from the above dataset by setting $\mathcal{R} = (2, 1^{*11})$ with $m = 12$ and $T = 3.15$. The generated APT-II censored sample and the analysis results are reported in Tables 1 and 2, respectively.

Table 1. An APT-II censored sample from the mortality rate of COVID-19.

i	1	2	3	4	5	6	7	8	9	10	11	12
$x_{i:12:25}$	2.4946	2.7957	2.8636	2.9078	3.1091	3.1091	3.2218	3.2823	3.3769	3.3825	3.6346	3.6426

Table 2. Estimates and 95% CIs of λ and θ.

Estimates						95% CIs	
$\hat{\lambda}$	$\hat{\lambda}_u$	$\hat{\lambda}_w$	$\hat{\theta}$	$\hat{\theta}_p$	$\hat{\theta}_u$	ECI for λ	ECI for θ
2.050	1.708	1.209	2.495	2.441	2.442	(0.938, 3.141)	(2.272, 2.493)

Additionally, to examine the applicability of ***Algorithm 1*** in Section 3, the 95% intervals for the replicated data $X^{\text{rep}}_{i:12:25}$ are obtained based on $N = 20000$. The resulting plot is presented in Figure 3, which shows that the observed APT-II censored sample lies well within the 95% intervals. This result reveals the applicability of the provided ***Algorithm 1*** for real-world data analysis. In addition, Figure 3 shows the goodness-of-fit test results obtained through the plug-in method. The plug-in method generates the replicated data $X^{\text{rep}}_{i:12:25}$ by substituting the unknown parameters λ and θ in the marginal distribution $F\left(X^{\text{rep}}_{i:12:25}|\boldsymbol{x}; \lambda, \theta\right)$ with their estimators. Here, 20,000 replicated data $X^{\text{rep}}_{i:12:25}$ are from $F\left(X^{\text{rep}}_{i:12:25}|\boldsymbol{x}; \hat{\lambda}, \hat{\theta}\right)$ and $F\left(X^{\text{rep}}_{i:12:25}|\boldsymbol{x}; \hat{\lambda}_u, \hat{\theta}_p\right)$, and the empirical mean is computed. From Figure 3, it can be seen that the APT-II censored sample reported in Table 1 has the Pareto distribution because the points are close to a straight line.

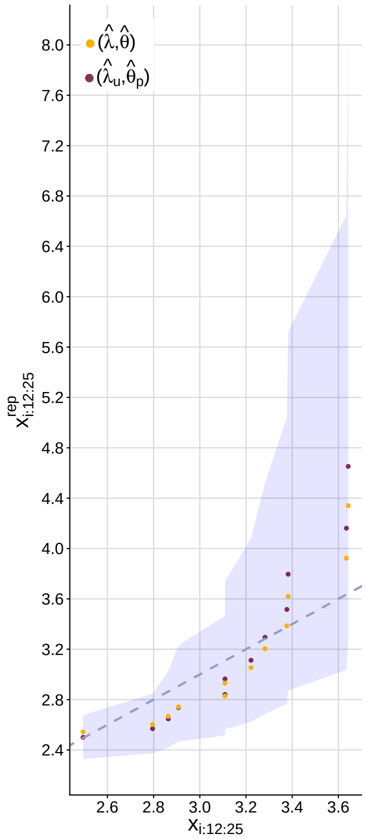


Figure 3. 95% intervals for $X^{\text{rep}}_{i:12:25}$ and the scatter plot between the APT-II censored sample $x_{i:12:25}$ and the empirical mean of $X^{\text{rep}}_{i:12:25}$.

5. Conclusions

The main goal of this paper is the proposal of an efficient inference method compared to classical methods under the APT-II censoring scheme. To achieve this goal, we employed the pivotal quantity. For illustration, the proposed method was applied to the Pareto distribution and provided closed-form estimators with excellent properties for the unknown parameters. In addition, the pivotal approach led to the exact inference results for the unknown parameters even when a sample size is not large enough, unlike the maximum likelihood and bootstrap methods which suffer from constraints. Additionally, an algorithm was proposed that generates the replicated data based on the GPQs.

The proposed methodology was evaluated through Monte Carlo simulations for small and middle sample sizes. Our results showed that the provided estimators are superior to the MLEs in terms of MSE and bias, and, especially, the unbiased and consistent estimators $\hat{\lambda}_u$ and $\hat{\theta}_u$ have the best performance. In addition, the proposed intervals are clearly exact for unknown parameters, and it is demonstrated that their CPs are highly close to the considered nominal level in simulation results. These results reveal the usefulness of the proposed method in a situation where the sample size is not large enough. Additionally, the mortality rate of COVID-19 in Canada was analyzed, through which the applicability of the proposed methodology for a real-world dataset is demonstrated.

In conclusion, for the Pareto distribution, the proposed methodology has superior performance when the sample size is not large enough, in comparison to the classical method. While this paper primarily provided insights into the Pareto distribution, future studies will delve into several critical aspects. First, the efficiency and accuracy of the proposed methodology will be assessed in various realistic scenarios, to extend its practical utility. Second, the applicability of our approach to various other probability distributions will be explored, broadening the scope of the study. In addition, a sensitivity analysis will also be conducted to ascertain the robustness of our methodology, especially in situations where the distribution estimation may be incorrect. Finally, subsequent studies will focus on validating the performance of our approach in real-world applications, and a deeper exploration of applicability in industrial settings will be undertaken.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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