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Research article

Semi-local convergence of Cordero's sixth-order method

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Abstract: In this paper, the semi-local convergence of the Cordero's sixth-order iterative method in Banach space was proved by the method of recursion relation. In the process of proving, the auxiliary sequence and three increasing scalar functions can be derived using Lipschitz conditions on the first-order derivatives. By using the properties of auxiliary sequence and scalar function, it was proved that the iterative sequence obtained by the iterative method was a Cauchy sequence, then the convergence radius was obtained and its uniqueness was proven. Compared with Cordero's process of proving convergence, this paper does not need to ensure that $\mathcal{G}(s)$ is continuously differentiable in higher order, and only the first-order Fréchet derivative was used to prove semi-local convergence. Finally, the numerical results showed that the recursion relationship is reasonable.

Keywords: semi-local convergence; Kurchatov's method; nonlinear equations; recurrence relation; domain of existence and uniqueness; iterative method
Mathematics Subject Classification: 65B99, 65H05

1. Introduction

Solving nonlinear equations in Banach space is an important task in the field of applied science. All sorts of questions can be turned into

$$\mathcal{G}(s) = 0. \tag{1.1}$$

Here, $\mathcal{G} : \Phi \subseteq \mathbb{B}_1 \to \mathbb{B}_2$ is a nonlinear sufficiently differentiable operator on an upper convex subset of $\mathbb{B}_1$, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces. For this kind of nonlinear Eq (1.1), it is difficult to solve it analytically. Moreover, in most practical problems, it is not necessary to require the exact solution of the equation, but only the approximate value, and the error of the approximate value and the exact solution should be limited to the acceptable range of the practical problem. This approximation can be obtained by numerical iteration.

The fixed-point iteration method is still the main numerical method to solve the nonlinear equation.

One of the most famous iteration methods is Newton's method [1], whose iteration scheme is

$$s^{(k+1)} = s^{(k)} - \Delta^{(k)}\mathcal{G}(s^{(k)}), \tag{1.2}$$

where $\Delta^{(k)} = \mathcal{G}'(s^{(k)})^{-1}$ for $k = 0, 1, 2, 3\cdots$. Because of its simple structure, small amount of computation, and fast convergence speed, Newton's method is still the most important iterative method for solving nonlinear equations in concrete calculation and application. However, its disadvantages are also obvious, such as the convergence speed is only second order. Therefore, in order to meet the need of high precision, scholars have proposed many high order convergence iterative methods [2–4] on the basis of Newton's method. Cordero et al. proposed an iterative method [5] of sixth-order convergence. The iteration format of this sixth-order method is

$$\begin{cases} t^{(k)} = s^{(k)} - \frac{1}{2}\Delta^{(k)}\mathcal{G}(s^{(k)}), \\ r^{(k)} = s^{(k)} + [\mathcal{G}'(s^{(k)}) + 2\mathcal{G}'(t^{(k)})]^{-1}[3\mathcal{G}(s^{(k)}) - 4\mathcal{G}(t^{(k)})], \\ s^{(k+1)} = r^{(k)} + [\mathcal{G}'(s^{(k)}) - 2\mathcal{G}'(t^{(k)})]^{-1}\mathcal{G}(r^{(k)}). \end{cases} \tag{1.3}$$

In the iterative method (1.3), three function values $\mathcal{G}(s^{(k)})$,$\mathcal{G}(r^{(k)})$,$\mathcal{G}(t^{(k)})$ and two Jacobian matrices $\mathcal{G}'(s^{(k)})$, $\mathcal{G}'(t^{(k)})$ need to be calculated and three LU factorizations (LU factorization is a type of matrix factorization that can decompose a matrix into the product of a lower trigonometric matrix and an upper trigonometric matrix) need to be performed.

Zhanlav et al. also proposed an iterative method [6] with sixth-order convergence , which is in the form of

$$\begin{cases} t^{(k)} = s^{(k)} - \Delta^{(k)}\mathcal{G}(s^{(k)}), \\ r^{(k)} = s^{(k)} - \Pi_k\Delta^{(k)}\mathcal{G}(s^{(k)}), \\ s^{(k+1)} = r^{(k)} - \Psi_k\Delta^{(k)}\mathcal{G}(r^{(k)}), \end{cases} \tag{1.4}$$

where $\Pi_k = (I - 4M_k)^{-1}(I - \frac{7}{2}M_k)$, $\Psi_k = (I + M_k)^{-1}(I + 2M_k - \frac{1}{2}M_k^2)$,$M_k = I - \Delta^{(k)}\mathcal{G}'(t^{(k)})$. In the iterative method (1.4), it is also necessary to compute three function values $\mathcal{G}(s^{(k)})$,$\mathcal{G}(t^{(k)})$,$\mathcal{G}(r^{(k)})$ and two Jacobian matrices $\mathcal{G}'(s^{(k)})$, $\mathcal{G}'(t^{(k)})$ and to perform three LU factorizations.

Cordero et al. also proposed an iterative method [7] of sixth-order convergence, whose iteration format is

$$\begin{cases} t^{(k)} = s^{(k)} - \Delta^{(k)}\mathcal{G}(s^{(k)}), \\ r^{(k)} = t^{(k)} - [2I - \mathcal{G}'(t^{(k)})\mathcal{G}'(s^{(k)})^{-1}]\Delta^{(k)}\mathcal{G}(t^{(k)}), \\ s^{(k+1)} = r^{(k)} - \mathcal{G}'(t^{(k)})^{-1}\mathcal{G}(r^{(k)}) \end{cases} \tag{1.5}$$

where $\Delta^{(k)} = \mathcal{G}'(s^{(k)})^{-1}$. Compared with the iterative method (1.3) and (1.4), which are also sixth-order converging, the iterative method (1.5) needs to compute three function values $\mathcal{G}(s^{(k)})$,$\mathcal{G}(t^{(k)})$,$\mathcal{G}(r^{(k)})$ and two Jacobian matrices $\mathcal{G}'(s^{(k)})$,$\mathcal{G}'(t^{(k)})$, and only needs to perform two LU factorizations. The computational cost of iterative methods (1.5) is lower than that of iterative methods (1.3) and (1.4).

At present, the most commonly used methods to prove semi-local convergence mainly include the majorizing sequence method [8,9] and recursion method [10,11]. In fact, both methods were proposed by Kantorovich [12], and their main idea was to prove them by induction. In the process of proving the semi-local convergence of the iterative method of the system of equations, we usually study the iterative method in one-dimensional space because the iterative method of solving the nonlinear equation in one-dimensional real number space can be generalized to Banach space [13].

This paper mainly uses the recursion method to analyze the semi-local convergence of Cordero's sixth-order convergence iterative method (1.5). In Cordero's proof of sixth-order convergence, the operator $\mathcal{G}$ is usually required to be a sufficiently differentiable function in the neighborhood of the solution to guarantee the continuity of the sixth-order derivative used to prove the convergence of the iterative method. Let's think about this function

$$\mathcal{G}(s) = \begin{cases} s^4 \ln s^2 + 5s^5 - 5s^4, & s \neq 0, \\ 0, & s = 0, \end{cases}$$

where $\mathcal{G} : \Phi \subset R \to R$ and $\Phi = [-1, 2]$. The root of this function is denoted by α, so we can observe that $\alpha = 1$ is the root of $\mathcal{G}(s)$ and $\mathcal{G}'''(s) = 24s \ln s^2 + 300s^2 - 68s$. It is obvious that $\mathcal{G}'''(s)$ is unbounded on Φ, so the previous analysis does not guarantee the convergence of method (1.5). Therefore, in order to avoid the use of higher derivatives, we apply Lipschitz conditions only to first-order Fréchet derivatives to prove the semi-local convergence of the iterative method (1.5).

This paper is divided into six parts. In Section 2 , we give three scalar functions and three auxiliary sequences to prove semi-local convergence, and we analyze the properties of the auxiliary sequences and scalar functions. In Section 3, the recursion relation used to prove the semi-local convergence of iterative method (1.5) is given. In Section 4, the semi-local convergence of method (1.5) and the uniqueness of the solution are both proven. The numerical example and results are shown in Sections 5 and 6, respectively.

2. Preparatory knowledge

Let $\mathcal{G} : \Phi \subseteq \mathbb{B}_1 \to \mathbb{B}_2$ be a differentiable nonlinear Fréchet operator in the open set Φ and let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Banach spaces. Suppose the inverse $\Delta_0 \in \mathbb{L}(\mathbb{B}_2, \mathbb{B}_1)$ of the Jacobian matrix of the first iteration in the iterative system (1.5) and s_0 satisfies $s_0 \in \Phi$, where $\mathbb{L}(\mathbb{B}_2, \mathbb{B}_1)$ is the set of linear operators from $\mathbb{B}_2$ to $\mathbb{B}_1$.

In addition, we use the Kantorovich condition [12] to obtain the semi-local convergence result of this iterative method (1.5).
(C_1) $\| \Delta_0 \| \leq \beta$,
(C_2) $\| \Delta_0 \mathcal{G}(s_0) \| \leq \eta$,
(C_3) $\| \mathcal{G}'(s) - \mathcal{G}'(t) \| \leq K \| s - t \|$,
where K, β, η are nonnegative real numbers. For simplicity of form, we denote $\eta_0 = \eta$, $\lambda_0 = K\beta\eta$, $\mu_0 = q(\lambda_0)p(\lambda_0)$, let $\lambda_0 < \sigma$ and $\sigma \approx 0.603 < 1$ be the smallest positive root of the scalar function $sh(s) - 1$, and define the sequences

$$\eta_{n+1} = \mu_n \eta_n, \tag{2.1}$$

$$\lambda_{n+1} = \lambda_n p(\lambda_n)^2 q(\lambda_n), \tag{2.2}$$

$$\mu_{n+1} = q(\lambda_{n+1}) p(\lambda_{n+1}), \tag{2.3}$$

where $n \geq 0$. The scalar functions are

$$h(s) = 1 + \frac{s}{2} + \frac{s^2}{2} + \frac{s^3(1+s)^2}{8(1-s)}, \tag{2.4}$$

$$p(s) = \frac{1}{1 - sh(s)}, \tag{2.5}$$

$$q(s) = \frac{s}{2}(h(s))^2 + h(s) - 1. \tag{2.6}$$

This is the key to study the semi-local convergence of iterative methods. The following is the interrelation between scalar functions defined by (2.4)–(2.6) and sequences defined by (2.1)–(2.3) by some lemmas, which we will use later in the derivation of recursive relations.

Lemma 2.1. *The functions $h(s)$, $p(s)$, and $q(s)$ are defined by (2.4)–(2.6), and some of their properties are as follows:*

(a) $h(s)$, $p(s)$, and $q(s)$ are increasing ,where $p(s) > 1$ and $h(s) > 1$ for $0 < s < \sigma$,
(b) $p(\lambda_0)q(\lambda_0) < 1$ for $\lambda_0 < 0.359$,
(c) $p(\lambda_0)^2 q(\lambda_0) < 1$ for $\lambda_0 < 0.297$.
Proof: Using the definition of increasing function, it is easy to prove (a). A numerical calculation is then performed to prove $(b), (c)$. As $p(\lambda_0)^2 q(\lambda_0) < 1$, then by constructing λ_n, it is a decreasing sequence. So, $\lambda_n < \lambda_0 \leq 0.297$ for all $n \geq 1$.

Lemma 2.2. *Let $p(s)$,$h(s)$, and $q(s)$ be the auxiliary functions defined by (2.4)–(2.6), and σ is the smallest positive root of the scalar function $sh(s) - 1$. If*

$$\lambda_0 < \sigma, p(\lambda_n)\mu_n < 1, \tag{2.7}$$

then,
(a) $p(\lambda_0) > 1$, $\mu_n < 1$ $(n \geq 0)$,
(b) the sequence $\{\lambda_n\}$,$\{\mu_n\}$, and $\{\eta_n\}$ are decreasing ,where $\lambda_n < 0.297$ for $n \geq 0$,
(c) $h(\lambda_n)\lambda_n < 1, p(\lambda_n)\mu_n < 1 (n \geq 0)$.
Proof: (a) From Lemma 2.1 and (2.7), we can see that $p(\lambda_0) > 1$ is true and $\mu_0 < 1$, so it is true when $n = 0$. When $n = 1$, the same reason $\mu_1 < 1$ is true, it can be obtained by mathematical induction that $\mu_n < 1$ is true.
(b) From the definition of the sequence (2.1)–(2.3) and (a), we can obtain $\mu_n < 1$, so $\eta_{n+1} < \eta_n$, and $\{\eta_n\}$ is a decreasing sequence. By Lemma 2.1, when $n = 0$, $p(\lambda_0)^2 q(\lambda_0) < 1$, so $\lambda_1 < \lambda_0$. By mathematical induction, $\{\lambda_n\}$ is a decreasing sequence.In the same way, $\mu_1 < \mu_0$ and $\{\mu_n\}$ is also a decreasing sequence.
(c) From Lemma 2.1 and the above results, we can see that $h(\lambda_1)\lambda_1 < h(\lambda_0)\lambda_0 < 1$ and $p(\lambda_1)\mu_1 < h(\lambda_0)\mu_0 < 1$ are true and (c) is established by induction.

3. Recursive relation

The required recursion relations and auxiliary functions are defined, and we begin to analyze the iterative method (1.5), which serves as the basis for later semi-local convergence analysis. We define $B(s,r) = \{t \in \mathbb{B}_1 :\| t - s \|< r\}$, $\overline{B(s,r)} = \{t \in \mathbb{B}_1 :\| t - s \|\leq r\}$. Under the assumption (C_1)–(C_3) in the previous section, the recursion relation that defines the iterative method in (1.5) is given below.

We expand the Taylor series of t_0 at $\mathcal{G}$ estimated near s_0 to

$$\mathcal{G}(t_0) = \mathcal{G}(s_0) + \mathcal{G}'(s_0)(t_0 - s_0) + \int_{s_0}^{t_0} (\mathcal{G}'(s) - \mathcal{G}'(t))ds.$$

From the first-step of the iterative method (1.5), the term $\mathcal{G}(s_0)+\mathcal{G}'(s_0)(t_0-s_0)$ is equal to zero. Using variable substitution $s = s_0 + v(t_0 - s_0)$, we get

$$\mathcal{G}(t_0) = \int_0^1 (\mathcal{G}'(s_0 + v(t_0 - s_0) - \mathcal{G}'(s_0))(t_0 - s_0)dv,$$

when $n = 0$. It is known that Δ_0 exists from the hypothesis (C_1)–(C_3), and it shows that t_0 also exists, thus, there is

$$\| t_0 - s_0 \| = \| \Delta_0 \mathcal{G}(s_0) \| \leq \eta_0. \tag{3.1}$$

This shows that $t_0 \in B(s_0, R\eta)$

$$\begin{aligned} r_0 - s_0 =& t_0 - s_0 - \mathcal{G}'(s_0)^{-1}[2I - \mathcal{G}'(t_0)\mathcal{G}'(s_0)^{-1}]\mathcal{G}(t_0) \\ =& t_0 - s_0 - \Delta_0[I + (\mathcal{G}'(s_0) - \mathcal{G}'(t_0))\Delta_0]\mathcal{G}(t_0) \\ =& t_0 - s_0 - \Delta_0 \int_0^1 (\mathcal{G}'(s_0 + v(t_0 - s_0) - \mathcal{G}'(s_0)))(t_0 - s_0)dv \\ &+ \Delta_0(\mathcal{G}'(t_0) - \mathcal{G}'(s_0))\Delta_0 \int_0^1 (\mathcal{G}'(s_0 + v(t_0 - s_0) - \mathcal{G}'(s_0)))(t_0 - s_0)dv. \end{aligned} \tag{3.2}$$

Take the norm (3.2) and apply the Lipschitz condition [14]. We obtain

$$\begin{aligned} \| r_0 - s_0 \| \leq& \| t_0 - s_0 \| + \| \Delta_0 \| \frac{K}{2} \| t_0 - s_0 \|^2 + \| \Delta_0 \|^2 K \| t_0 - s_0 \| \frac{K}{2} \| t_0 - s_0 \|^2 \\ \leq& \| t_0 - s_0 \| + \frac{K}{2} \| \Delta_0 \| \| t_0 - s_0 \|^2 + \frac{K^2}{2} \| \Delta_0 \|^2 \| t_0 - s_0 \|^3 \\ \leq& \eta + \frac{K}{2}\beta\eta^2 + \frac{K^2}{2}\beta^2\eta^3 \\ =& \eta(1 + \frac{1}{2}K\beta\eta + \frac{1}{2}K^2\beta^2\eta^2) \\ =& \eta(1 + \frac{\lambda_0}{2} + \frac{\lambda_0^2}{2}), \end{aligned} \tag{3.3}$$

so that

$$\| r_0 - s_0 \| \leq \eta(1 + \frac{\lambda_0}{2} + \frac{\lambda_0^2}{2}). \tag{3.4}$$

Similarly, we can get $r_0 - t_0$.

$$\begin{aligned} \| r_0 - t_0 \| =& -\mathcal{G}'(s_0)^{-1}[2I - \mathcal{G}'(t_0)\mathcal{G}'(s_0)^{-1}]\mathcal{G}(t_0) \\ =& -\mathcal{G}'(s_0)^{-1}[I + I - \mathcal{G}'(t_0)\mathcal{G}'(s_0)^{-1}]\mathcal{G}(t_0) \\ =& -\mathcal{G}'(s_0)^{-1}\mathcal{G}(t_0) - \mathcal{G}'(s_0)^{-1}(\mathcal{G}'(s_0) - \mathcal{G}'(t_0))\mathcal{G}'(s_0)^{-1}\mathcal{G}(t_0) \\ =& -\mathcal{G}'(s_0)^{-1} \int_0^1 (\mathcal{G}'(s_0 + v(t_0 - s_0) - \mathcal{G}'(s_0)))(t_0 - s_0)dv \\ &- \mathcal{G}'(s_0)^{-1}(\mathcal{G}'(s_0) - \mathcal{G}'(t_0))\mathcal{G}'(s_0)^{-1} \int_0^1 (\mathcal{G}'(s_0 + v(t_0 - s_0) - \mathcal{G}'(s_0)))(t_0 - s_0)dv, \end{aligned} \tag{3.5}$$

so that

$$\begin{aligned}\| r_0 - t_0 \| &\leq \| \Delta_0 \| \frac{K}{2} \| t_0 - s_0 \|^2 + \| \Delta_0 \|^2 K \| t_0 - s_0 \| \frac{K}{2} \| t_0 - s_0 \|^2 \\ &\leq \frac{1}{2} K\beta\eta^2 + \frac{1}{2} K^2\beta^2\eta^3 \\ &= \frac{\eta}{2}(K\beta\eta + K^2\beta^2\eta^2) \\ &= \frac{\eta}{2}(\lambda_0 + \lambda_0^2).\end{aligned} \tag{3.6}$$

Applying Banach's lemma [15], it follows that

$$\| I - \Delta_0 \mathcal{G}'(t_0) \| \leq \| \Delta_0 \| \| \mathcal{G}'(s_0) - \mathcal{G}'(t_0) \| \leq \beta K \| t_0 - s_0 \| \leq K\beta\eta = \lambda_0 < \sigma. \tag{3.7}$$

Thus, $\mathcal{G}'(t_0)^{-1}$ exists and

$$\| \mathcal{G}'(t_0)^{-1} \| \leq \frac{\beta}{1 - \lambda_0}. \tag{3.8}$$

The Taylor series expansion of $\mathcal{G}$ around t_0 evaluated in r_0 is

$$\mathcal{G}(r_0) = \int_0^1 (\mathcal{G}'(t_0 + v(r_0 - t_0) - \mathcal{G}'(t_0))(r_0 - t_0)dv. \tag{3.9}$$

Taking norms and applying Lipschitz condition, we obtain

$$\| \mathcal{G}(r_0) \| \leq \frac{K}{2} \| r_0 - t_0 \|^2 . \tag{3.10}$$

Thus,

$$\begin{aligned}\| s_1 - s_0 \| &\leq \| r_0 - s_0 \| + \| \mathcal{G}'(t_0)^{-1}\mathcal{G}(r_0) \| \\ &\leq \eta(1 + \frac{\lambda_0}{2} + \frac{\lambda_0^2}{2}) + \frac{\beta}{1 - \lambda_0} \frac{K}{2} \frac{\eta^2}{4} (\lambda_0 + \lambda_0^2)^2 \\ &= \eta(1 + \frac{\lambda_0}{2} + \frac{\lambda_0^2}{2} + \frac{\lambda_0^3}{8(1 - \lambda_0)}(1 + \lambda_0)^2) = \eta h(\lambda_0).\end{aligned} \tag{3.11}$$

Therefore,

$$\| s_1 - s_0 \| \leq \eta h(\lambda_0), \tag{3.12}$$

where $\lambda_0 = K\beta\eta$ and $h(s) = 1 + \frac{s}{2} + \frac{s^2}{2} + \frac{s^3(1+s)^2}{8(1-s)}$.

Apply the Banach lemma again, one has

$$\begin{aligned}\| I - \Delta_0 \mathcal{G}'(s_1) \| &= \| \Delta_0 \mathcal{G}'(s_0) - \Delta_0 \mathcal{G}'(s_1) \| \leq \| \Delta_0 \| \| \mathcal{G}'(s_0) - \mathcal{G}'(s_1) \| \leq K\beta \| s_1 - s_0 \| \\ &\leq K\beta\eta(1 + \frac{\lambda_0}{2} + \frac{\lambda_0^2}{2} + \frac{\lambda_0^3}{8(1 - \lambda_0)}(1 + \lambda_0)^2) = \lambda_0 h(\lambda_0) < 1,\end{aligned} \tag{3.13}$$

then, as far as $\lambda_0 h(\lambda_0) < 1$ (by taking $\lambda_0 < \sigma$), Banach's lemma guarantees that $(\Delta_0 \mathcal{G}'(s_1))^{-1} = \Delta_1 \Delta_0^{-1}$ exists and

$$\| \Delta_1 \| \leq \frac{1}{1 - \lambda_0 h(\lambda_0)} \| \Delta_0 \| = p(\lambda_0) \| \Delta_0 \| \tag{3.14}$$

being $p(s) = \frac{1}{1 - sh(s)}$.

Repeating the extrapolation process above, we can get the recurrence relationship given by the following lemma.

Lemma 3.1. *The following corollary is proved by induction when $n \geq 1$:*

$(I_n) \parallel \Delta_n \parallel \leq p(\lambda_{n-1}) \parallel \Delta_{n-1} \parallel$
$(II_n) \parallel t_n - s_n \parallel = \parallel \Delta_n \mathcal{G}(s_n) \parallel \leq \eta_n$
$(III_n) K \parallel \Delta_n \parallel \parallel t_n - s_n \parallel \leq \lambda_n$
$(IV_n) \parallel s_n - s_{n-1} \parallel \leq h(\lambda_{n-1})\eta_{n-1}$
Proof: Starting from $n = 1$, (I_1) has been proved in (3.14).

For (II_1), take the Taylor expansion of $\mathcal{G}(s_1)$ near t_0, and we get

$$\begin{aligned}\mathcal{G}(s_1) =& \mathcal{G}(t_0) + \mathcal{G}'(t_0)(s_1 - t_0) + \int_{t_0}^{s_1} (\mathcal{G}'(s) - \mathcal{G}'(t_0))ds \\ =& \mathcal{G}(t_0) + (\mathcal{G}'(t_0) - \mathcal{G}'(s_0))(s_1 - t_0) + \mathcal{G}'(s_0)(s_1 - t_0) \\ &+ \int_0^1 (\mathcal{G}'(t_0 + v(s_1 - t_0) - \mathcal{G}'(t_0))(s_1 - t_0)dv.\end{aligned} \tag{3.15}$$

Taking the norm of $\mathcal{G}(s_1)$,

$$\begin{aligned}\mathcal{G}(s_1) =& \parallel \mathcal{G}(t_0) \parallel + K \parallel t_0 - s_0 \parallel \parallel s_1 - t_0 \parallel + \parallel \mathcal{G}'(s_0)(s_1 - t_0) \parallel + \frac{K}{2} \parallel s_1 - t_0 \parallel^2 \\ \leq& \frac{K}{2} \parallel t_0 - s_0 \parallel^2 + K \parallel t_0 - s_0 \parallel \parallel s_1 - t_0 \parallel + \frac{1}{\beta} \parallel s_1 - t_0 \parallel + \frac{K}{2} \parallel s_1 - t_0 \parallel^2 .\end{aligned} \tag{3.16}$$

When one

$$\begin{aligned}\parallel s_1 - t_0 \parallel =& \parallel r_0 - t_0 - \mathcal{G}'(t_0)^{-1}\mathcal{G}(r_0) \parallel \\ \leq& \parallel r_0 - t_0 \parallel + \parallel \mathcal{G}'(t_0)^{-1}\mathcal{G}(r_0) \parallel \\ \leq& \frac{\eta}{2}(\lambda_0 + \lambda_0^2) + \parallel \mathcal{G}'(t_0)^{-1} \frac{K}{2} \parallel r_0 - t_0 \parallel^2 \\ \leq& \frac{\eta}{2}(\lambda_0 + \lambda_0^2) + \frac{\beta}{1-\lambda_0} \frac{K}{2} \frac{\eta^2}{4}(\lambda_0 + \lambda_0^2)^2 \\ =& \eta(\frac{\lambda_0}{2} + \frac{\lambda_0^2}{2} + \frac{\lambda_0^3}{8(1-\lambda_0)}(1+\lambda_0)^2) = \eta(h(\lambda_0 - 1)),\end{aligned} \tag{3.17}$$

then,

$$\mathcal{G}(s_1) \leq \frac{K}{2}\eta^2 + K\eta^2(h(\lambda_0) - 1) + \frac{1}{\beta}\eta(h(\lambda_0) - 1) + \frac{K}{2}\eta^2(h(\lambda_0) - 1)^2. \tag{3.18}$$

By applying (I_1), we can get

$$\begin{aligned}\parallel t_1 - s_1 \parallel =& \parallel \Delta_1 \mathcal{G}(s_1) \parallel \leq p(\lambda_0) \parallel \Delta_0 \parallel \parallel \mathcal{G}(s_1) \parallel \\ \leq& p(\lambda_0)\beta\eta(\frac{K}{2}\eta + K\eta(h(\lambda_0) - 1) + \frac{1}{\beta}(h(\lambda_0) - 1) + \frac{K}{2}\eta(h(\lambda_0) - 1)^2) \\ \leq& p(\lambda_0)\eta(\frac{\lambda_0}{2} + \lambda_0(h(\lambda_0) - 1) + h(\lambda_0) - 1 + \frac{\lambda_0}{2}(h(\lambda_0) - 1)^2) \\ =& (\frac{\lambda_0}{2}(h(\lambda_0))^2 + h(\lambda_0) - 1)p(\lambda_0)\eta \\ =& \mu_0\eta = \eta_1.\end{aligned} \tag{3.19}$$

Let

$$\| t_1 - s_1 \| \leq q(\lambda_0)p(\lambda_0)\eta = \eta_1, \tag{3.20}$$

where $\mu_0 = q(\lambda_0)p(\lambda_0)$ and

$$q(s) = \frac{s}{2}(h(s))^2 + h(s) - 1. \tag{3.21}$$

(III_1): Use (I_1) and (II_1) for $n = 1$ to prove

$$\begin{aligned} K \| \Delta_1 \| \| t_1 - s_1 \| \leq & Kp(\lambda_0) \| \Delta_0 \| \eta_1 \\ = & Kp(\lambda_0) \| \Delta_0 \| q(\lambda_0)p(\lambda_0)\eta \\ \leq & \lambda_0 (p(\lambda_0))^2 q(\lambda_0) = \lambda_1. \end{aligned} \tag{3.22}$$

(IV_1): This has been proven in (3.11), when $n = 1$.

4. Semi-local convergence analysis

In this section, we give the semi-local convergence theorem of the iterative method of sixth-order convergence. It is first necessary to prove that the sequence $\{s_n\}$ is a Cauchy sequence, because this guarantees that the sequence $\{s_n\}$ is convergent in the Banach space. According to the above analysis of recursive sequences $\{\lambda_n\}, \{\mu_n\}$ and auxiliary functions $h(x), p(x), q(x)$, we give the following preliminary results:

Theorem 4.1. *Let $\mathcal{G} : \Phi \subseteq \mathbb{B}_1 \to \mathbb{B}_2$ be a quadratic differentiable Fréchet nonlinear operator on the open set Φ, where $\mathbb{B}_1$ and $\mathbb{B}_2$ are Banach spaces. Let $s_0 \in \Phi$ and $\Delta_0 = [\mathcal{G}'(s_0)]^{-1}$ exist, and the condition $(C_1) - (C_3)$ is satisfied. Let $\lambda_0 = K\beta\eta$ and $\lambda_0 < \sigma$ and define $\eta_{n+1} = \mu_n\eta_n$, $\mu_{n+1} = q(\lambda_{n+1})p(\lambda_{n+1})$, $\lambda_0 < \sigma$, and $p(\lambda_0)\mu_0 < 1$, where σ is the smallest positive root of the scalar function $sh(s) - 1$. If $B_e(s_0, R\eta) = \{s \in X : \| s - s_0 \| < R\eta\} \subset \Phi$ and $R = \frac{h(\lambda_0)}{1-q(\lambda_0)p(\lambda_0)}$, then the iterated sequence s_0 defined at (1.5) converges from the initial point s_0 to the solution s^* of $\mathcal{G}(x) = 0$. In this case, the iterated sequences $\{s_n\}$ and $\{t_n\}$ are included in $B_e(s_0, R\eta)$ and $s^* \in B(s_0, R\eta)$, where s^* is the unique solution of the equation $\mathcal{G}(x) = 0$ in $B_n(s_0, \frac{2}{K\beta} - R\eta) \cap \Phi$.*

Proof: According to Lemma 2.1, we can write

$$\eta_n = q(\lambda_{n-1})p(\lambda_{n-1})\eta_{n-1} = \prod_{i=0}^{n-1}(q(\lambda_i)p(\lambda_i))\eta \leq (q(\lambda_0)p(\lambda_0))^n\eta. \tag{4.1}$$

Thus,

$$\sum_{i=0}^{n}\eta_i \leq \sum_{i=0}^{n}(q(\lambda_0)p(\lambda_0))^i\eta = \frac{1-(q(\lambda_0)p(\lambda_0))^{n+1}}{1-q(\lambda_0)p(\lambda_0)}\eta. \tag{4.2}$$

According to Lemmas 2.1 and 2.2, the functions $p(s)$ and $q(s)$ are increasing. So, we express $s_{n+1} - s_0$ in terms of partial sums of geometric series,

$$\begin{aligned} \| s_{n+1} - s_0 \| \leq & \sum_{i=0}^{n} \| s_{i+1} - s_i \| \leq \sum_{i=0}^{n} h(\lambda_i)\eta_i \leq h(\lambda_0)\sum_{i=0}^{n}\eta_i \\ \leq & h(\lambda_0)\eta\frac{1-(q(\lambda_0)p(\lambda_0))^{n+1}}{1-q(\lambda_0)p(\lambda_0)} < R\eta. \end{aligned} \tag{4.3}$$

Therefore, when $p(\lambda_0)q(\lambda_0) < 1$ of Lemma 2.1 holds, we can conclude that $\{s_n\}$ all belong to $\overline{B_e(s_0, R\eta)}$. From Lemmas 2.1 and 2.2, we know that $p(s)$, $q(s)$ and $h(s)$ increase and $\{\lambda_n\}$ decreases, and then we can show that $\{s_n\}$ is a Cauchy sequence.

$$\begin{aligned} \| s_{n+m} - s_n \| &\le \sum_{i=n}^{n+m-1} \| s_{i+1} - s_i \| \\ &\le \sum_{i=n}^{n+m-1} h(\lambda_i)\eta_i \le h(\lambda_0) \sum_{i=n}^{n+m-1} \eta_i \\ &\le h(\lambda_0)\eta \frac{1 - (q(\lambda_0)p(\lambda_0))^{n+m}}{1 - q(\lambda_0)p(\lambda_0)}. \end{aligned} \tag{4.4}$$

So, $\{s_n\}$ is a convergent Cauchy sequence. Therefore, there is s^*, such that $\lim_{n\to\infty} s_n = s^*$. In (4.3), let $n = 0, m \to \infty$, and we get $\| s^* - s_0 \| \le R\eta$, which shows that $\overline{B_e(s^*, R\eta)}$.

Finally, it is proven that we know the uniqueness of s^* in $B_n(s_0, \frac{2}{K\beta} - R\eta) \bigcap \Phi$.

$$\frac{2}{K\beta} - R\eta = (\frac{2}{\lambda_0} - R)\eta > \frac{1}{\lambda_0}\eta > R\eta, \tag{4.5}$$

so $\overline{B_e(s_0, R\eta)} \subset B_n(s_0, \frac{2}{K\beta} - R\eta) \cap \Phi$. Below, we assume that t^* is another solution of $\mathcal{G}(s) = 0$ in $B_n(s_0, \frac{2}{K\beta} - R\eta) \bigcap \Phi$ and prove that $s^* = t^*$. Let's first take the Taylor expansion of $\mathcal{G}$ around s^*,

$$\mathcal{G}(t^*) = \mathcal{G}(s^*) + \int_0^1 (\mathcal{G}(s^* + v(t^* - s^*))(t^* - s^*)dv,$$

so that

$$0 = \mathcal{G}(t^*) - \mathcal{G}(s^*) = (t^* - s^*) \int_0^1 (\mathcal{G}'(s^* + v(t^* - s^*))dv.$$

We need to prove that the operator $\int_0^1 (F'(x^* + t(y^* - x^*))$ is invertible, thus guaranteeing that $y^* - x^* = 0$. Then, applying hypothesis (C_3),

$$\begin{aligned} \| \Delta_0 \| \int_0^1 \| \mathcal{G}'(s^* + v(t^* - s^*) - \mathcal{G}'(s_0) \| dt &\le K\beta \int_0^1 \| s^* + v(t^* - s^*) - s_0 \| dv \\ &\le K\beta \int_0^1 ((1 - v) \| s - s_0 \| + v \| t^* - s_0 \|)dv \\ &< \frac{K\beta}{2}(R\eta + \frac{2}{K\beta} - R\eta) = 1. \end{aligned} \tag{4.6}$$

It follows from Banach's lemma that the operator $\int_0^1 (F'(x^* + t(y^* - x^*))$ is invertible and $\int_0^1 (F'(x^* + t(y^* - x^*)) \in L(\mathbb{B}_1, \mathbb{B}_2)$. The proof is completed by estimating $0 = \mathcal{G}(t^*) - \mathcal{G}(s^*) = (t^* - s^*) \int_0^1 (F'(x^* + t(y^* - x^*))$ to obtain $t^* = s^*$.

5. Numerical experiments

In this section, we will use the iterative method (1.5) to solve nonlinear systems, showing that the recursion relationship we derive is reasonable. In addition, we use the iterative method (1.5) to solve practical chemical problems to demonstrate its applicability.

Problem 1. Nonlinear integral equations appear in many branches of mathematical physics, such as fracture mechanics, hythermoelasticity, fluid mechanics, and so on. In this section, we introduce the nonlinear integral equation of Hammerstein type [16], which is a special form of Urysohn type Volterra integral equation, and then we use the obtained results to solve the Hammerstein type integral equation to prove the applicability of the theoretical results. The format of the Hammerstein equation is as follows:

$$s(x) = 1 + \frac{1}{3}\int_0^1 H(x,y)s(y)^3 dy, \tag{5.1}$$

where $s \in \mathbb{C}(0,1)$, $x \in [0,1]$, $y \in [0,1]$,with the kernel H as

$$H(x,y) = \begin{cases} (1-x)y & \text{if } y \le x, \\ x(1-y) & \text{if } x < y. \end{cases} \tag{5.2}$$

Equation (5.1) is solved by converting (5.1) into nonlinear equations through the discretization process. Next, GaussLegendre quadrature is used to approximate the integral in (5.1),

$$\int_0^1 x(y)dy \approx \sum_{i=1}^{7} \delta_i x(y_i) \tag{5.3}$$

with y_i and δ_i serving as the Gauss-Legendre polynomial's nodes and weights, respectively. Using the system of nonlinear equations, we estimate (5.1) after denoting the approximation of $s_i, i = 1,2,...,m$ as $s(y_i)$, where s_i approximated is

$$s_i = 1 + \frac{1}{3}\sum_{j=1}^{7} \theta_{ij} s_j^3, \tag{5.4}$$

where

$$\theta_{ij} = \begin{cases} \delta_j y_j (1-y_i) & \text{if } j \le i, \\ \delta_j y_i (1-y_j) & \text{if } j > i. \end{cases} \tag{5.5}$$

One way to rewrite the system is

$$\begin{aligned} \mathcal{G}(s) &= s - 1 - \frac{1}{3} M\gamma_s, \quad \gamma_s = (s_1^3, s_2^3, ..., s_7^3)^T, \\ \mathcal{G}'(s) &= I - MN(s), \quad N(s) = diag(s_1^2, s_2^2, ..., s_7^2), \end{aligned} \tag{5.6}$$

where $\mathcal{G}'$ is the Fréchet derivative of $\mathcal{G}$, a nonlinear operator in $L(\mathbb{R}^L, \mathbb{R}^L)$, and $\mathbb{R}^L$ is the Banach space. We shall apply it to solve the nonlinear systems in accordance with (1.5).

Using the infinite norm while taking $s_0 = (1.6, 1.6, 1.6, 1.6, 1.6, 1.6, 1.6)^T, L = 7$, we can get

$$\begin{aligned} \| \Delta_0 \| \leq \beta, &\qquad \beta \approx 1.3667, \\ \| \Delta_0 \mathcal{G}(s_0) \| \leq \eta, &\qquad \eta \approx 1.6301, \\ \| \mathcal{G}'(s) - \mathcal{G}'(t) \| \leq K \| s - t \|, &\qquad K \approx 0.1040, \\ \lambda_0 = K\beta\eta, &\qquad \lambda_0 \approx 0.2316, \\ \mu_0 = q(\lambda_0)p(\lambda_0), &\qquad \mu_0 \approx 0.4052. \end{aligned} \tag{5.7}$$

The above results satisfy the condition of semi-local convergence, so we apply this method to the system. In addition, the existence of the solution of s_0 in $B_e(s_0, 6.3578)$ and uniqueness in $B_n(s_0, 6.6419)$ are guaranteed by Theorem 4.1. In Table 1, we give the existence radius R_e and uniqueness radius R_n when the initial estimator s_0 with equal components takes different values. At the same time, we note that when $s_{0i} > 1.7, i = 1, 2, ..., 7$, the iterative method does not meet the convergence condition, so its convergence cannot be guaranteed.

When we use the iterative method (1.5) to solve Eq (5.2), the exact solution we get is

$$s^* = \{1.005, 1.021, 1.040, 1.048, 1.040, 1.021, 1.005\}^T .$$

In Table 1 , the values of relevant parameters in the conditions are given when different initial values are taken, and the existence radius R_e and uniqueness radius R_n are obtained when different initial values are taken. Table 2 shows the errors and function values corresponding to different initial values, and proves that the iterative method (1.5) is convergent of sixth-order. The results obtained in Tables 1 and 2 are similar. We can converge to a unique solution under the Kantorovich condition [12] by choosing different initial values, and the closer the initial value is to the root, the lower the error estimate. The proof of semi-local convergence, which guarantees the existence and uniqueness of the solution under certain assumptions, is especially valuable in the process where the existence of the solution cannot be proven.

Table 1. Problem 1 takes parameter values with different initial values.

s_{0i}	β	η	K	λ_0	μ_0	R_e	R_n
0	1	2.6458	0.0397	0.1051	0.1318	3.2250	47.1528
0.2	1.0042	2.1230	0.0479	0.1022	0.1273	2.5701	39.0089
0.4	1.0171	1.6094	0.0566	0.0927	0.1129	1.9011	32.8405
0.6	1.0392	1.0977	0.0661	0.0756	0.0886	1.2558	27.8600
0.8	1.0719	0.5889	0.0775	0.0489	0.0541	0.6386	23.4368
1.0	1.1171	0.0804	0.1148	0.0103	0.0150	0.0817	15.5137
1.2	1.1778	0.4689	0.0810	0.0448	0.0491	0.5047	20.4593
1.4	1.2586	1.0285	0.0942	0.1220	0.1592	1.3074	15.5617

Table 2. Experimental results of Problem 1.

s_{0i}	*iter*	$\parallel s_k - s_{k-1} \parallel$	$\parallel \mathcal{G}(s_k) \parallel$	ρ
0.2	4	9.755e-216	8.611e-1295	6
0.4	4	5.671e-228	3.324e-1368	6
0.6	4	2.622e-251	3.250e-1508	6
0.8	4	3.849e-297	3.247e-1783	6
1.0	4	2.135e-452	9.463e-2715	6
1.2	4	2.922e-314	6.215e-1886	6
1.4	4	5.294e-228	2.199e-1368	6
1.6	4	2.853e-176	4.300e-4096	6

Problem 2. The gas equation of the state problem is one of the most important problems in solving practical chemistry problems, and we apply the iterative method (1.5) to this problem. First, give the van der Waals equation

$$\mathcal{G}(V) = (p + \frac{an^2}{V^2})(V - nb) - nRT, \tag{5.8}$$

where $a = 4.17atm \cdot L/mol^2$, $b = 0.0371L/mol$. Let's consider the pressure of $945.36kPa(9.33atm)$, the temperature of $300.2K$, and the nitrogen of $2mol$, and then find the volume of the container. Finally, by substituting the data into Eq (5.8), we can get

$$\mathcal{G}(V) = 9.33V^3 - 96.9611V^2 + 16.68V - 1.23766.$$

Taking $s_0 = 1.1$ and the infinity norm, we get

$$\lambda_0 = K\beta\eta, \lambda \approx 1.2344,$$

$$\mu_0 = q(\lambda_0)p(\lambda_0), \mu_0 \approx 0.2263.$$

Therefore, the method satisfies the convergence condition, the solution exists in $B_e(x_0, 3.3975)$, and the uniqueness domain is $B_n(x_0, 7.2177)$. When the initial value satisfies the Kantorovich condition, the initial value in this range is taken to solve the nonlinear system. Using iterative method (1.5) to solve system (5.8) gives the root $s^* = 1.60917$. A similar result can be obtained in Table 3; that is, under the Kantorovich condition, convergence to a unique solution can be achieved by selecting different initial values. The closer the initial value is to the root, the smaller the error estimate.

Table 3. Experimental results of Problem 2.

s_{0i}	*iter*	$\parallel s_k - s_{k-1} \parallel$	$\parallel \mathcal{G}(s_k) \parallel$	ρ
1.1	4	3.4721e-422	1.2863e-420	6
1.2	4	2.254e-713	8.35e-712	6
1.3	4	1.7708e-1398	6.5602e-1397	6
1.4	4	7.3119e-1394	2.7088e-1392	6
1.5	4	2.3388e-847	8.6644e-846	6
1.6	4	1.1197e-614	4.1482e-613	6
1.7	4	2.6379e-475	9.7725e-474	6

6. Conclusions

In this paper, the semi-local convergence of Cordero's sixth-order iterative method (1.5) was proved by the recursive method. In order to study the semi-local convergence of Cordero's iterative method, first, we studied the properties of auxiliary sequences η_n, λ_n, μ_n and scalar functions $h(s), p(s), q(s)$. Second, the neighborhood $B(s_0, R)$ centered on the initial point was given, and then it was proved that the iterative sequence converges to $s^* \in \overline{B(s_0, R)}$, where s^* satisfies $\mathcal{G}(s^*) = 0$, and the radius of convergence R was obtained, thus proving the existence of a solution. Finally, the uniqueness of a solution was proved by using Banach's lemma. In the whole process of proving semi-local convergence, the Lipschitz condition of the first-order Fréchet derivative was used to prove the semi-local convergence of the Cordero's iterative method. The correctness of the theory was proved by numerical experiments.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflict of interest.

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