AIMS Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 5925–5936.
DOI: 10.3934/math.2024289
Received: 26 October 2023
Revised: 11 December 2023
Accepted: 10 January 2024
Published: 31 January 2024

Research article

Smoothing algorithm for the maximal eigenvalue of non-defective positive matrices

Na Li [1] **and Qin Zhong** [2,*]

[1] School of Intelligence Technology, Geely University of China, Chengdu 641423, China

[2] Department of Mathematics, Sichuan University Jinjiang College, Meishan 620860, China

* **Correspondence:** Email: zhongqin@scujj.edu.cn.

Abstract: This paper introduced a smoothing algorithm for calculating the maximal eigenvalue of non-defective positive matrices. Two special matrices were constructed to provide monotonically increasing lower-bound estimates and monotonically decreasing upper-bound estimates of the maximal eigenvalue. The monotonicity and convergence of these estimations was also proven. Finally, the effectiveness of the algorithm was demonstrated with numerical examples.

Keywords: non-defective; positive matrix; maximal eigenvalue; upper bound; lower bound
Mathematics Subject Classification: 15A42, 15A18

1. Introduction

A matrix A with all elements greater than zero is referred to as a positive matrix and is denoted by $A > 0$. Positive matrices possess several significant properties. Perron [1] first proposed that if A is a positive matrix, then the spectral radius is an eigenvalue of A. This eigenvalue, denoted by $\rho(A)$, is called the maximal eigenvalue of A or the Perron root of A, and $\rho(A)$ dominates all other eigenvalues in modulus.

If $A = (a_{ij})$, then A is called nonnegative if $a_{ij} \geq 0$ and is denoted by $A \geq 0$. Nonnegative matrices are frequently encountered in real-life applications. Frobenius [1] extended Perron's theory to nonnegative matrices and nonnegative irreducible matrices, leading to the rapid development of the nonnegative matrix theory. The famous Perron-Frobenius theorem is widely used in both theory and practice. Estimating the range of maximal eigenvalue of positive matrices is a popular topic in the nonnegative matrix theory and has been extensively and thoroughly studied in [2–10].

For any positive integer n, let $\langle n\rangle = \{1,2, \cdots, n\}$. Given $A = (a_{ij})_{n\times n} \geq 0$, we define

$$r_i(A) = \sum_{k=1}^{n} a_{ik}, c_j(A) = \sum_{k=1}^{n} a_{kj}, \min_i r_i(A) = r(A), \max_i r_i(A) = R(A), i, j \in \langle n\rangle .$$

Frobenius [1] obtained the following classical conclusion:

$$r(A) \leq \rho(A) \leq R(A). \tag{1}$$

Additionally, equality in (1) is achieved when the sum of each row of the matrix A is equal. A similar result holds for the column with the transpose matrix A^{T} in place of A. The above inequality implies that the maximal eigenvalue $\rho(A)$ of a nonnegative square matrix A is between the smallest row sum $r(A)$ and the largest row sum $R(A)$. This observation provides a convenient and efficient method for estimating the maximal eigenvalue using the elements of A.

The class of positive matrices, which is the subclass of nonnegative matrices, shares similar properties with nonnegative matrices. The generalization of the result in (1) was presented in [2–4] to improve the bounds of the maximal eigenvalues of positive matrices.

Minc [5] made improvements to the bounds in (1) for nonnegative matrices with nonzero row sums, resulting in the following:

$$\min_i \left(\frac{1}{r_i}\sum_{t=1}^{n} a_{it}r_t\right) \leq \rho(A) \leq \max_i \left(\frac{1}{r_i}\sum_{t=1}^{n} a_{it}r_t\right).$$

Liu [6] generalized the above result further and obtained the following conclusion:

$$\min_i \left[\frac{r_i\left(A^{k+m}\right)}{r_i\left(A^k\right)}\right]^{\frac{1}{m}} \leq \rho\left(A\right) \leq \max_i \left[\frac{r_i\left(A^{k+m}\right)}{r_i\left(A^k\right)}\right]^{\frac{1}{m}}, \tag{2}$$

where k is any nonnegative integer and m is any positive integer. The same is true for column sums.

Based on Eq (2), Liu et al. [7] gave an innovative result as follows:

$$\min_i \left[\frac{r_i\left(A^m B^k\right)}{r_i\left(B^k\right)}\right]^{\frac{1}{m}} \leq \rho\left(A\right) \leq \max_i \left[\frac{r_i\left(A^m B^k\right)}{r_i\left(B^k\right)}\right]^{\frac{1}{m}}, \tag{3}$$

where $B = (A + I)^{n-1}$ and I is the identity matrix of order n. The same is true for column sums.

If we set m=1 in (2), we will obtain

$$\min_i \frac{r_i\left(A^{k+1}\right)}{r_i\left(A^k\right)} \leq \rho\left(A\right) \leq \max_i \frac{r_i\left(A^{k+1}\right)}{r_i\left(A^k\right)}. \tag{4}$$

The boundaries are further generalized in [8] as follows:

$$\min_i \frac{r_i\left(AB\right)}{r_i\left(B\right)} \leq \rho\left(A\right) \leq \max_i \frac{r_i\left(AB\right)}{r_i\left(B\right)}, \tag{5}$$

where B is an arbitrary matrix that has positive row sums.

The following is the concept of a non-defective matrix. In linear algebra, a square matrix that lacks a full basis of eigenvectors and cannot be diagonalized is referred to as a defective matrix. Specifically, a matrix of order n is considered non-defective if it contains n linearly independent eigenvectors.

This paper is dedicated to the estimation and calculation of the maximal eigenvalue of a positive matrix. Initially, we present monotonically increasing lower-bound estimations and monotonically decreasing upper-bound estimations for the maximal eigenvalue of a positive matrix. Additionally, we rigorously prove the monotonicity and convergence of these estimations. Notably, if the positive matrix is non-defective, we provide a smoothing algorithm to calculate the maximal eigenvalue of such a non-defective positive matrix.

2. Main results

To derive our conclusions, we will recall some essential lemmas as follows.

Lemma 1. [5] Let λ be an eigenvalue of the square matrix A of order n and let $U = (u_1, u_2, \cdots, u_n)^{\mathrm{T}}$ and $V = (v_1, v_2, \cdots, v_n)^{\mathrm{T}}$ be eigenvectors corresponding to λ of A^{T} and A, respectively, then

$$\lambda \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} u_i r_i(A),$$

$$\lambda \sum_{j=1}^{n} v_j = \sum_{j=1}^{n} v_j c_j(A).$$

Lemma 2. [5] If $q_t > 0, t \in \langle n \rangle$, then for any real numbers $p_t, t \in \langle n \rangle$, the following inequality holds:

$$\min_t \frac{p_t}{q_t} \le \frac{\sum\limits_{t=1}^{n} p_t}{\sum\limits_{t=1}^{n} q_t} \le \max_t \frac{p_t}{q_t}.$$

Now, we present the upper and lower bounds on the maximal eigenvalue of a positive matrix.

Theorem 1. Given a positive matrix $A = \left(a_{ij}\right)_{n\times n}$, let $B_1 = (A + \alpha I)^{n-1}, B_2 = (A - \beta I)^{n-1}$, where $\alpha = \max\limits_i \{a_{ii}\}, \beta = \min\limits_i \{a_{ii}\}$. If $r_i (AB_1B_2) \neq 0$, $c_i (AB_1B_2) \neq 0$, $i \in \langle n \rangle$, then for any positive integer k, we have

$$\min_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}} \le \rho(A) \le \max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}, \tag{6}$$

$$\min_i \sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^kB_1B_2\right)}} \le \rho(A) \le \max_i \sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^kB_1B_2\right)}}. \tag{7}$$

Proof. First, we prove $r_i\left(A^kB_1B_2\right) > 0$ with the condition $r_i(AB_1B_2) \neq 0$. For any positive integer $k \ge 2$, the element of the matrix $A^{k-1}B_1B_2$ located in the t-th row and the j-th column is denoted by $\left(A^{k-1}B_1B_2\right)_{tj}$. Note that $A^kB_1B_2 = AA^{k-1}B_1B_2$. We have

$$r_i\left(A^kB_1B_2\right) = r_i\left(AA^{k-1}B_1B_2\right)$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \sum_{t=1}^{n} a_{it} \left(A^{k-1} B_1 B_2\right)_{tj} \\
&= \sum_{t=1}^{n} \sum_{j=1}^{n} a_{it} \left(A^{k-1} B_1 B_2\right)_{tj} \\
&= \sum_{t=1}^{n} a_{it} \sum_{j=1}^{n} \left(A^{k-1} B_1 B_2\right)_{tj} \\
&= \sum_{t=1}^{n} a_{it} r_t (A^{k-1} B_1 B_2). \qquad (8)
\end{aligned}
$$

It is evident that $B_1 = (A + \alpha I)^{n-1}$ and $B_2 = (A - \beta I)^{n-1}$ are nonnegative. Therefore, AB_1B_2 is nonnegative and $r_i(AB_1B_2) > 0$ holds under the restriction that $r_i(AB_1B_2) \neq 0$. As A is a positive matrix, that is, $a_{ij} > 0$, $i, j \in \langle n \rangle$, we immediately obtain $r_i\left(A^k B_1 B_2\right) > 0$ from Eq (8). By employing the same approach, we also obtain $c_i\left(A^k B_1 B_2\right) > 0$ if $c_i(AB_1B_2) \neq 0$, $i \in \langle n \rangle$. These assertions ensure that the expressions in Eqs (6) and (7) are valid.

Now, we assume $X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}} > 0$ is the eigenvector of the matrix A^{T} corresponding to $\rho(A)$, that is, $A^{\mathrm{T}} X = \rho(A) X$. Clearly, the maximal eigenvalues of the matrix polynomials

$$A^{k+2} B_1 B_2 = A^{k+2} (A + \alpha I)^{n-1} (A - \beta I)^{n-1}$$

and

$$A^{k} B_1 B_2 = A^{k} (A + \alpha I)^{n-1} (A - \beta I)^{n-1}$$

are $\rho^{k+2}(A)\,[\rho(A) + \alpha]^{n-1} [\rho(A) - \beta]^{n-1}$ and $\rho^{k}(A)\,[\rho(A) + \alpha]^{n-1} [\rho(A) - \beta]^{n-1}$, respectively. Therefore, we obtain

$$\left(A^{k+2} B_1 B_2\right)^{\mathrm{T}} X = \rho^{k+2}(A)\,[\rho(A) + \alpha]^{n-1} [\rho(A) - \beta]^{n-1} X$$

and

$$\left(A^{k} B_1 B_2\right)^{\mathrm{T}} X = \rho^{k}(A)\,[\rho(A) + \alpha]^{n-1} [\rho(A) - \beta]^{n-1} X.$$

Based on Lemma 1, we have

$$\rho^{k+2}(A)\,[\rho(A) + \alpha]^{n-1} [\rho(A) - \beta]^{n-1} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i r_i \left(A^{k+2} B_1 B_2\right) \qquad (9)$$

and

$$\rho^{k}(A)\,[\rho(A) + \alpha]^{n-1} [\rho(A) - \beta]^{n-1} \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i r_i \left(A^{k} B_1 B_2\right). \qquad (10)$$

Moreover, we must have

$$\rho^{k}(A)\,[\rho(A) + \alpha]^{n-1} [\rho(A) - \beta]^{n-1} > 0.$$

This is guaranteed by the previous proof $r_i\left(A^k B_1 B_2\right) > 0$ and

$$\rho^{k}(A)\,[\rho(A) + \alpha]^{n-1} [\rho(A) - \beta]^{n-1} \geq \min_i r_i \left(A^k B_1 B_2\right) > 0, i \in \langle n \rangle .$$

Therefore, according to Eqs (9) and (10), we can get

$$\rho^2(A) = \frac{\rho^{k+2}(A)\left[\rho(A)+\alpha\right]^{n-1}\left[\rho(A)-\beta\right]^{n-1}\sum\limits_{i=1}^{n} x_i}{\rho^{k}(A)\left[\rho(A)+\alpha\right]^{n-1}\left[\rho(A)-\beta\right]^{n-1}\sum\limits_{i=1}^{n} x_i} = \frac{\sum\limits_{i=1}^{n} x_i r_i\left(A^{k+2}B_1B_2\right)}{\sum\limits_{i=1}^{n} x_i r_i\left(A^{k}B_1B_2\right)}.$$

It follows from Lemma 2 that

$$\min_i \frac{x_i r_i\left(A^{k+2}B_1B_2\right)}{x_i r_i\left(A^{k}B_1B_2\right)} \le \rho^2(A) \le \max_i \frac{x_i r_i\left(A^{k+2}B_1B_2\right)}{x_i r_i\left(A^{k}B_1B_2\right)},$$

that is,

$$\min_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^{k}B_1B_2\right)}} \le \rho(A) \le \max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^{k}B_1B_2\right)}}.$$

Therefore, inequality (6) is proved. Similarly, inequality (7) holds.

Remark 1. From the formula

$$r_i\left(A^kB_1B_2\right) = \sum_{t=1}^{n} a_{it} r_t(A^{k-1}B_1B_2),$$

in (8), we can observe that $r_i\left(A^{k+2}B_1B_2\right), r_i\left(A^{k}B_1B_2\right)$ can be calculated by induction. In addition, a new matrix multiplication technique known as the semitensor product of matrices (STP) has been developed in recent years, which offers more powerful functionality compared to traditional matrix multiplication [11–13]. This implies that it is not difficult to compute the upper and lower bounds of $\rho(A)$.

In the following, we prove the convergence of the upper and lower bound expressions in Theorem 1 when the positive integer $k \to \infty$.

Theorem 2. Under the assumptions of Theorem 1, the following limits

$$\lim_{k\to\infty}\min_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^{k}B_1B_2\right)}}, \lim_{k\to\infty}\max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^{k}B_1B_2\right)}},$$

$$\lim_{k\to\infty}\min_i \sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^{k}B_1B_2\right)}}, \lim_{k\to\infty}\max_i \sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^{k}B_1B_2\right)}},$$

exist, and $\rho(A)$ satisfies the following inequalities:

$$\lim_{k\to\infty}\min_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^{k}B_1B_2\right)}} \le \rho(A) \le \lim_{k\to\infty}\max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^{k}B_1B_2\right)}},$$

$$\lim_{k\to\infty}\min_i \sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^{k}B_1B_2\right)}} \le \rho(A) \le \lim_{k\to\infty}\max_i \sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^{k}B_1B_2\right)}}.$$

Proof. For any positive integer $k \geq 2$, according to (8) and Lemma 2, we have

$$\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)} = \frac{\sum\limits_{t=1}^{n} a_{it} r_t\left(A^{k+1}B_1B_2\right)}{\sum\limits_{t=1}^{n} a_{it} r_t\left(A^{k-1}B_1B_2\right)} \leq \max_i \frac{r_i\left(A^{k+1}B_1B_2\right)}{r_i\left(A^{k-1}B_1B_2\right)} = \max_i \frac{r_i\left(A^{(k-1)+2}B_1B_2\right)}{r_i\left(A^{k-1}B_1B_2\right)}.$$

The above inequality shows that

$$\max_i \frac{r_i\left(A^{(k-1)+2}B_1B_2\right)}{r_i\left(A^{k-1}B_1B_2\right)} \geq \max_i \frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}.$$

Therefore, we acquire

$$\max_i \sqrt{\frac{r_i\left(A^{(k-1)+2}B_1B_2\right)}{r_i\left(A^{k-1}B_1B_2\right)}} \geq \max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}.$$

That is to say, the following sequence

$$\left\{\max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}\right\},$$

is monotonically decreasing with respect to k. On the other hand, by Theorem 1 we know that the sequence has a lower bound $\rho(A)$. Based on the monotonicity bounded criterion, we conclude that

$$\lim_{k\to\infty} \max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}},$$

exists and is not less than $\rho(A)$. Similarly, the sequence

$$\left\{\min_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}\right\},$$

increases monotonically with respect to k and has an upper bound $\rho(A)$ according to (6) in Theorem 1. Therefore, we derive that

$$\lim_{k\to\infty} \min_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}},$$

exists and is not more than $\rho(A)$. Using the same approach, we can establish analogous results for column rows when $c_i(AB_1B_2) \neq 0$, $i \in \langle n \rangle$.

Theorem 2 discusses the upper and lower bounds on the maximum eigenvalue of a positive matrix and proves that the sequence $\left\{\max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}\right\}$ decreases monotonically and has a lower bound $\rho(A)$, while the sequence $\left\{\min_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}\right\}$ increases monotonically and has an upper bound

$\rho(A)$. It is natural to wonder whether $\lim\limits_{k\to\infty}\max\limits_{i}\sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}$ or $\lim\limits_{k\to\infty}\min\limits_{i}\sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}$ is then equal to $\rho(A)$? In general, it is difficult to prove

$$\lim_{k\to\infty}\min_{i}\sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}=\rho(A)=\lim_{k\to\infty}\max_{i}\sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}.$$

Indeed, in special cases we have some elegant conclusions. The following are the results for a non-defective positive matrix.

Theorem 3. Under the assumptions of Theorem 1, if the positive matrix $A=\left(a_{ij}\right)_{n\times n}$ is non-defective, then we have

$$\lim_{k\to\infty}\min_{i}\sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}=\rho(A)=\lim_{k\to\infty}\max_{i}\sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}},\tag{11}$$

$$\lim_{k\to\infty}\min_{i}\sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^kB_1B_2\right)}}=\rho(A)=\lim_{k\to\infty}\max_{i}\sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^kB_1B_2\right)}}.\tag{12}$$

Proof. Without loss of generality, it may be assumed that $\lambda_1,\lambda_2,\cdots,\lambda_{n-1},\rho(A)$ are the eigenvalues of the matrix A such that

$$|\lambda_1|\le|\lambda_2|\le\cdots\le|\lambda_{n-1}|\le\rho(A).$$

The corresponding eigenvector family is $X_1,X_2,\cdots,X_{n\text{-}1},X_n$, in which

$$X_j=\left(x_{1j},x_{2j},\cdots,x_{nj}\right)^{\mathrm{T}},\ j=1,2,\cdots,n.$$

Since A is non-defective, the eigenvector family $X_1,X_2,\cdots,X_{n\text{-}1},X_n$ forms a basis of the n-dimensional vector space. Moreover, we have

$$\begin{aligned}r_i\left(A^{k+2}B_1B_2\right)&=r_i\left(A^{k+1}AB_1B_2\right)\\&=\sum_{j=1}^{n}\sum_{t=1}^{n}\left(A^{k+1}\right)_{it}(AB_1B_2)_{tj}\\&=\sum_{t=1}^{n}\sum_{j=1}^{n}\left(A^{k+1}\right)_{it}(AB_1B_2)_{tj}\\&=\sum_{t=1}^{n}\left(A^{k+1}\right)_{it}\sum_{j=1}^{n}(AB_1B_2)_{tj}\\&=\sum_{t=1}^{n}\left(A^{k+1}\right)_{it}r_t(AB_1B_2),\end{aligned}\tag{13}$$

where $\left(A^{k+1}\right)_{it}$ denotes the element of the matrix A^{k+1} located in the i-th row and the t-th column and $(AB_1B_2)_{tj}$ denotes the element of the matrix AB_1B_2 located in the t-th row and the j-th column, respectively. Similarly, we have

$$r_i\left(A^kB_1B_2\right)=\sum_{t=1}^{n}\left(A^{k-1}\right)_{it}r_t\left(AB_1B_2\right),k\ge2.\tag{14}$$

Now, we define a special vector as follows:

$$Y = \left(r_1\left(AB_1B_2\right), r_2\left(AB_1B_2\right), \cdots, r_n\left(AB_1B_2\right)\right)^{\mathrm{T}}.$$

It is obvious that Y is a positive vector since $r_i\left(AB_1B_2\right) \neq 0$ (more precisely, $r_i\left(AB_1B_2\right) > 0$), $i \in \langle n\rangle$.

Thus, Y can be expressed as

$$Y = C_1X_1 + C_2X_2 + \cdots + C_{n-1}X_{n-1} + C_nX_n, \tag{15}$$

where $C_1, C_2, \cdots, C_{n-1}, C_n$ are not all zero. Together with Eqs (13) and (14), one can get

$$\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)} = \frac{\sum\limits_{t=1}^{n}\left(A^{k+1}\right)_{it} r_t\left(AB_1B_2\right)}{\sum\limits_{t=1}^{n}\left(A^{k-1}\right)_{it} r_t\left(AB_1B_2\right)}, k \geq 2.$$

On the other hand, we have

$$\sum_{t=1}^{n}\left(A^{k+1}\right)_{it} r_t\left(AB_1B_2\right) = \left[A^{k+1}\left(r_1\left(AB_1B_2\right), r_2\left(AB_1B_2\right), \cdots, r_n\left(AB_1B_2\right)\right)^{\mathrm{T}}\right]_i = \left(A^{k+1}Y\right)_i \tag{16}$$

and

$$\sum_{t=1}^{n}\left(A^{k-1}\right)_{it} r_t\left(AB_1B_2\right) = \left[A^{k-1}\left(r_1\left(AB_1B_2\right), r_2\left(AB_1B_2\right), \cdots, r_n\left(AB_1B_2\right)\right)^{\mathrm{T}}\right]_i = \left(A^{k-1}Y\right)_i, \tag{17}$$

in which $\left(A^{k+1}Y\right)_i$ and $\left(A^{k-1}Y\right)_i$ denote the i-th coordinates of the vectors $A^{k+1}Y$ and $A^{k-1}Y$, respectively. Combined with Eqs (15)–(17), we obtain

$$\begin{aligned}
\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)} &= \frac{\left(A^{k+1}Y\right)_i}{\left(A^{k-1}Y\right)_i} \\
&= \frac{\left[A^{k+1}\left(C_1X_1 + C_2X_2 + \cdots + C_{n-1}X_{n-1} + C_nX_n\right)\right]_i}{\left[A^{k-1}\left(C_1X_1 + C_2X_2 + \cdots + C_{n-1}X_{n-1} + C_nX_n\right)\right]_i} \\
&= \frac{\left(C_1A^{k+1}X_1 + C_2A^{k+1}X_2 + \cdots + C_{n-1}A^{k+1}X_{n-1} + C_nA^{k+1}X_n\right)_i}{\left(C_1A^{k-1}X_1 + C_2A^{k-1}X_2 + \cdots + C_{n-1}A^{k-1}X_{n-1} + C_nA^{k-1}X_n\right)_i} \\
&= \frac{C_1\lambda_1^{k+1}x_{i1} + C_2\lambda_2^{k+1}x_{i2} + \cdots + C_{n-1}\lambda_{n-1}^{k+1}x_{i(n-1)} + C_n\rho^{k+1}(A)\,x_{in}}{C_1\lambda_1^{k-1}x_{i1} + C_2\lambda_2^{k-1}x_{i2} + \cdots + C_{n-1}\lambda_{n-1}^{k-1}x_{i(n-1)} + C_n\rho^{k-1}(A)\,x_{in}}.
\end{aligned} \tag{18}$$

Take the limit on both sides of Eq (18), and we acquire

$$\begin{aligned}
\lim_{k\to\infty}\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)} &= \lim_{k\to\infty}\frac{C_1\lambda_1^{k+1}x_{i1} + C_2\lambda_2^{k+1}x_{i2} + \cdots + C_{n-1}\lambda_{n-1}^{k+1}x_{i(n-1)} + C_n\rho^{k+1}(A)\,x_{in}}{C_1\lambda_1^{k-1}x_{i1} + C_2\lambda_2^{k-1}x_{i2} + \cdots + C_{n-1}\lambda_{n-1}^{k-1}x_{i(n-1)} + C_n\rho^{k-1}(A)\,x_{in}} \\
&= \lim_{k\to\infty}\frac{\dfrac{C_1\lambda_1^{k+1}x_{i1} + C_2\lambda_2^{k+1}x_{i2} + \cdots + C_{n-1}\lambda_{n-1}^{k+1}x_{i(n-1)} + C_n\rho^{k+1}(A)\,x_{in}}{\rho^{k+1}(A)}}{\dfrac{C_1\lambda_1^{k-1}x_{i1} + C_2\lambda_2^{k-1}x_{i2} + \cdots + C_{n-1}\lambda_{n-1}^{k-1}x_{i(n-1)} + C_n\rho^{k-1}(A)\,x_{in}}{\rho^{k-1}(A)}\cdot\dfrac{1}{\rho^2(A)}}
\end{aligned}$$

$$= \rho^2 (A).$$

Therefore, the equality in (11) holds. Similarly, we can prove that the corresponding result holds for the column.

Remark 2. Equations (11) and (12) in Theorem 3 show that for a non-defective positive matrix A, the limits of the maximum and minimum values of

$$\sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}\left(\sqrt{\frac{c_i\left(A^{k+2}B_1B_2\right)}{c_i\left(A^kB_1B_2\right)}}\right),$$

are equal to the maximum eigenvalue of A when k tends to infinity.

3. Algorithm

Based on Theorems 1–3, we can derive the algorithm for determining the maximum eigenvalue of a non-defective positive matrix.

Step 0. Given a non-defective positive matrix $A = \left(a_{ij}\right)_{n\times n}$ and a sufficiently small positive number $\varepsilon > 0$;

Step 1. Let $\alpha = \max_i \{a_{ii}\}, \beta = \min_i \{a_{ii}\}$. Compute $B_1 = (A + \alpha I)^{n-1}, B_2 = (A - \beta I)^{n-1}$;

Step 2. Let $k = 0$;

Step 3. Compute $T = \max_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}, t = \min_i \sqrt{\frac{r_i\left(A^{k+2}B_1B_2\right)}{r_i\left(A^kB_1B_2\right)}}$;

Step 4. If $T - t < \varepsilon$, go to Step 5; otherwise, set $k = k + 1$ and go back to Step 3;

Step 5. Output k and $\rho(A) = \frac{T+t}{2}$, stop.

Remark 3. Replace the row sums in the algorithm with the corresponding column sums. The algorithm is still valid.

Remark 4. In the above algorithm, the upper bound of $\rho(A)$ is decreasing, while the lower bound of $\rho(A)$ is increasing. This behavior exhibits a smoothing tendency and, as a result, we refer to this algorithm as a smoothing algorithm.

4. Numerical examples

In this section, we consider two examples to demonstrate our findings.

Example 1. Consider positive matrix:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 3 & 3 \\ 4 & 1 & 1 \end{pmatrix}.$$

The comparisons between the estimation results of [1–8] and Theorem 1 of this paper regarding the maximal eigenvalue of matrix A are presented in Table 1.

Table 1. Bounds for the maximal eigenvalue of A.

	By row	By column
(1)	$4.0000 < \rho(A) < 8.0000$	$5.0000 < \rho(A) < 7.0000$
Ledermann [2]	$4.1547 < \rho(A) < 7.8661$	$5.0800 < \rho(A) < 6.9259$
Ostrowski [3]	$4.5275 < \rho(A) < 7.6547$	$5.2247 < \rho(A) < 6.8165$
Brauer [4]	$4.8284 < \rho(A) < 7.4642$	$5.3722 < \rho(A) < 6.7016$
Minc [5]	$5.0000 < \rho(A) < 6.2500$	$5.6000 < \rho(A) < 5.8572$
$(4)(k = 2)$	$5.5833 \leq \rho(A) \leq 5.8667$	$5.7143 \leq \rho(A) \leq 5.7805$
$(2)(m = k = 2)$	$5.6789 \leq \rho(A) \leq 5.7735$	$5.7259 \leq \rho(A) \leq 5.7615$
$(3)(m = k = 2)$	$5.6836 \leq \rho(A) \leq 5.8539$	$5.6975 \leq \rho(A) \leq 6.3087$
$(5)(B = (A + I)^2)$	$5.1429 \leq \rho(A) \leq 6.4444$	$5.5000 \leq \rho(A) \leq 6.0000$
Theorem $1(k = 1)$	$5.7292 \leq \rho(A) \leq 5.7581$	$5.7408 \leq \rho(A) \leq 5.7428$
Theorem $1(k = 2)$	$5.7367 \leq \rho(A) \leq 5.7455$	$5.7413 \leq \rho(A) \leq 5.7419$
Theorem $1(k = 3)$	$5.7405 \leq \rho(A) \leq 5.7432$	$5.7416 \leq \rho(A) \leq 5.7418$

Indeed, $\rho(A) = 5.74165738\cdots$. The computational results in Table 1 demonstrate that the conclusions obtained from Theorem 1 in this paper improve upon the existing related results.

Example 2. Calculate the maximal eigenvalue of the non-defective positive matrix B using the algorithm in Section 3. The results are presented in Table 2.

$$B = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 & 4 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 & 5 & 5 & 5 \\ 1 & 2 & 3 & 4 & 5 & 6 & 6 & 6 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix}.$$

Table 2. Estimation for the maximal eigenvalue of B.

ε	Iteration numbers	$\rho(B)$
10^{-2}	1	29.37
10^{-4}	1	29.3653
10^{-6}	1	29.36529789
10^{-8}	1	29.36529789434
10^{-10}	2	29.36529789436882

5. Conclusions

In this paper, we have introduced monotonically increasing lower-bound estimators and monotonically decreasing upper-bound estimators for the maximal eigenvalue of a positive matrix.

These estimators are constructed using two special matrices associated with the positive matrix. The advantage of these estimators is that they are straightforward to compute, as they solely depend on the elements of the positive matrix. Furthermore, we have rigorously proven the monotonicity and convergence of both the upper and lower bound estimations for the maximal eigenvalue of positive matrices.

Additionally, we have developed a smoothing algorithm specifically designed to calculate the approximate value of the maximal eigenvalue for a non-defective positive matrix. This algorithm serves as an effective tool to obtain a reasonably accurate estimate for the maximal eigenvalue in such cases.

Overall, our findings provide valuable insights and practical tools for estimating and computing the maximal eigenvalue of positive matrices, with special attention given to non-defective positive matrices.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was financially supported by Sichuan University Jinjiang College Cultivation Project of Sichuan Higher Education Institutions of Double First-class Construction Gongga Plan.

Conflict of interest

The authors declare that they have no competing interests.

References

1. A. Berman, R. J. Plemmons, *Nonnegative matrices in the mathematical sciences*, Philadelphia: Society for Industrial and Applied Mathematics, 1994. https://doi.org/10.1137/1.9781611971262
2. W. Ledermann, Bounds for the greatest latent root of a positive matrix, *J. Lond. Math. Soc.*, **s1-25** (1950), 265–268. https://doi.org/10.1112/jlms/s1-25.4.265
3. A. Ostrowski, Bounds for the greatest latent root of a positive matrix, *J. Lond. Math. Soc.*, **s1-27** (1952), 253–256. https://doi.org/10.1112/jlms/s1-27.2.253
4. A. Brauer, The theorems of Ledermann and Ostrowski on positive matrices, *Duke Math. J.*, **24** (1957), 265–274. https://doi.org/10.1215/S0012-7094-57-02434-1
5. H. Minc, *Nonnegative Matrices*, New York: Wiley, 1988.
6. S. L. Liu, Bounds for the greatest characteristic root of a nonnegative matrix, *Linear Algebra Appl.*, **239** (1996), 151–160. https://doi.org/10.1016/S0024-3795(96)90008-7
7. L. M. Liu, T. Z. Huang, X. Q. Liu, New bounds for the greatest eigenvalue of a nonnegative matrix, *J. Univ. Electron. Sci. Techn. China*, 2007, 343–345.
8. P. Liao, Bounds for the Perron root of nonnegative matrices and spectral radius of iteration matrices, *Linear Algebra Appl.*, **530** (2017), 253–265. https://doi.org/10.1016/j.laa.2017.05.021

9. H. Cheriyath, N. Agarwal, On the Perron root and eigenvectors associated with a subshift of finite type, *Linear Algebra Appl.*, **633** (2022), 42–70. https://doi.org/10.1016/j.laa.2021.10.003

10. P. Liao, Bounds for the Perron root of positive matrices, *Linear Multilinear A.*, **71** (2023), 1849–1857. https://doi.org/10.1080/03081087.2022.2081310

11. Y. Y. Yan, D. Z. Cheng, J. E. Feng, H. T. Li, J. M. Yue, Survey on applications of algebraic state space theory of logical systems to finite state machines, *Sci. China Inform. Sci.*, **66** (2023), 111201. https://doi.org/10.1007/s11432-022-3538-4

12. D. Z. Cheng, H. S. Qi, Y. Zhao, *An introduction to semi-tensor product of matrices and its applications*, World Scientific, 2012. https://doi.org/10.1142/8323

13. J. M. Yue, Y. Y. Yan, H. Deng, Matrix approach to formulate and search k-ESS of graphs using the STP theory, *J. Math.*, **2021** (2021), 1–12. https://doi.org/10.1155/2021/7230661