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Research article

Best proximity points for proximal Górnicki mappings and applications to variational inequality problems

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Abstract: We introduce a large class of mappings called proximal Górnicki mappings in metric spaces, which includes Górnicki mappings, enriched Kannan mappings, enriched Chatterjea mappings, and enriched mappings. We prove the existence of the best proximity points in metric spaces and partial metric spaces. Moreover, we utilize appropriate examples to illustrate our results, and we verify the convergence behavior. As an application of our result, we prove the existence and uniqueness of a solution for the variational inequality problems. The obtained results generalize the existing results in the literature.

Keywords: fixed points; best proximity points; Górnicki mapping; enriched contraction; partial metric space; variational inequality
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1. Introduction and preliminaries

The well-known Banach contraction mapping principle was introduced by Stefan Banach in the year 1922 [1]. The result indicates that every contraction mapping on a complete metric space has a unique fixed point. After that, metric fixed-point theory evolved by generalizing Banach's contraction principle. In the same way, here, we generalize a class of mapping called generalized enriched contractions [2].

In the year 2019, a large class of Picard operators called enriched contractions was introduced by Berinde and Pacurar [3] after that, Górnicki and Bisht [4] generalized the result of enriched contractions by using averaged mappings. In 2021, Alexandra Marchis [5] proved common fixed-point theorems by applying the enriched-type conditions for two single-valued mappings satisfying the weak commutativity conditions.

Considering this direction, the following class of enriched mappings were introduced and studied by various authors in recent years.

Enriched contractions [3], enriched Kannan mappings [6], enriched Chatterjea mappings [7], enriched almost contractions [8], semi-groups of enriched non-expansive mappings [9] are proven in Hilbert space, enriched strictly pseudo contractive operators [10], enriched Ciric-Reich-Rus contraction [11], enriched contraction mappings with rational contraction [12], and enriched non-expansive mappings [13] proven in Banach space. Another type of enriched non-expansive mappings [13] have been proven in Hilbert space.

Definition 1.1. *[3]* Let $(\Omega, \|.\|)$ be a normed linear space. A mapping $\phi : \Omega \to \Omega$ is said to be an enriched contraction if there exists $\kappa \in [0, +\infty)$ and $\sigma \in [0, \kappa + 1)$ such that, for every $\xi, \zeta \in \Omega$,

$$\|\kappa(\xi - \zeta) + \phi\xi - \phi\zeta\| \leq \sigma\|\xi - \zeta\|. \tag{1.1}$$

Example. *[3]* Any contraction ϕ with a constant c is an enriched contraction. Let $\kappa = 0$ and $\sigma = c \in [0, 1)$; then, ϕ becomes a contraction.

Example. *[3]* Let $\Omega = [0, 1]$ be endowed with the usual norm, and let $\phi : \Omega \to \Omega$ be defined by $\phi\xi = 1 - \xi$ for all $\xi \in [0, 1]$. Then, ϕ is non-expansive. Here, let $k \in (0, 1)$ and $\sigma = 1 - k$;then, ϕ is not a contraction, but ϕ is an enriched contraction

$$\|(\kappa - 1)(\xi - \zeta)\| \leq \sigma\|\xi - \zeta\| \ \ , \sigma \in [0, \kappa + 1).$$

Theorem 1.1. *[3] Let $(\Omega, \|.\|)$ be a Banach space and $\phi : \Omega \to \Omega$ a (κ, σ)-enriched contraction. Then*

(1) fix $(\phi) = p$;

(2) there exists $\lambda \in (0, 1]$ such that the iterative method (ξ_n), given by

$$\xi_{n+1} = (1 - \lambda)\xi_n + \lambda\phi\xi_n, \quad n \geq 0,$$

converges strongly to p, for any $\xi_0 \in \Omega$;

(3) the estimate

$$\|\xi_{n+i-1} - p\| \leq \frac{c^i}{1 - c}\|\xi_n - \xi_{n-1}\|$$

holds for every n, $i \in \{1, 2, 3, ...\}$, where $c = \frac{\sigma}{\kappa+1}$.

In 2021, Popescu [2] introduced a new class of Picard operators that generalizes the class of enriched contractions, enriched Kannan mappings, and enriched Chatterjea mappings and proved some fixed-point theorems. The new class of Picard operators called Górnicki mappings is a more general class of mappings that includes discontinuous functions.

Definition 1.2. *[2]* Let (Ω, d) be a complete metric space, and let $\phi : \Omega \to \Omega$ be a self-mapping. We say that ϕ is a Górnicki mapping if ϕ satisfies

$$d(\phi\xi, \phi\zeta) \leq M[d(\xi, \phi\xi) + d(\zeta, \phi\zeta) + d(\xi, \zeta)]$$

with $M < 1$ and there exists non-negative real constants α,β with $\alpha \leq 1$ such that, for arbitrary $\xi \in \Omega$, there exists $\mu \in \Omega$ with $d(\mu, \phi\mu) \leq \alpha d(\xi, \phi\xi)$ and $d(\mu, \xi) \leq \beta d(\xi, \phi\xi)$.

The following theorems were proved by Popescu [2] for Górnicki mappings including the class of enriched contractions.

Theorem 1.2. *[2] Let* (Ω, d) *be a complete metric space, and let* $\phi : \Omega \to \Omega$ *be a Górnicki mapping. Then* ϕ *has a fixed point.*

Theorem 1.3. *[2] Let* $(\Omega, \|.\|)$ *be a linear normed space and* $\phi : \Omega \to \Omega$ *a* (κ, σ)*-enriched contraction. Then,* ϕ *is a Górnicki mapping.*

Theorem 1.4. *[2] Let* $(\Omega, \|.\|)$ *be a linear normed space and* $\phi : \Omega \to \Omega$ *a* (κ, σ)*-enriched Kannan contraction. Then,* ϕ *is a Górnicki mapping.*

Theorem 1.5. *[2] Let* $(\Omega, \|.\|)$ *be a linear normed space and* $\phi : \Omega \to \Omega$ *a* (κ, σ)*-enriched Chatterjea contraction. Then,* ϕ *is a Górnicki mapping.*

Let Δ be a non-empty subset of a metric space (Ω, d), and let $\phi : \Delta \to \Omega$ represent a mapping. Suppose that ξ is a solution to the equation $\phi\xi = \xi$ if and only if ξ is a fixed point of ϕ. Hence, the condition $\phi(\Delta) \cap \Delta \neq \emptyset$ is necessary for the existence of fixed points for the operator ϕ. When this necessary condition is not satisfied, it implies that, for any $\xi \in \Delta$, $d(\xi, \phi\xi) > 0$, and as a result, the mapping $\phi : \Delta \to \Omega$ does not have any fixed points; this means that the equation $\phi\xi = \xi$ has no solutions. Consequently, we have to find an element ξ such that, the distance between ξ and $\phi\xi$ is minimized. The best approximation theorem and best proximity point theorem have been developed in this field of work. The references for best proximity points are as follows [14–27].

Definition 1.3. *[28]* Let Δ and Γ be two non-empty subsets of a metric space (Ω, d) and consider a mapping $\phi : \Delta \to \Gamma$. We say that $\eta \in \Delta$ is a best proximity point of ϕ if

$$d(\eta, \phi\eta) = d(\Delta, \Gamma) := inf\{d(\xi, \zeta) : \xi \in \Delta, \zeta \in \Gamma\}.$$

In 1994, Matthews [29] introduced partial metric spaces which generalized the usual metric spaces by relaxing the metric condition that the self-distance must be zero. It has significant application in computer science particularly in research on the denotational semantics of data flow networks [29]. Since this finding many authors have worked in that area. A few references are [30–32].

Definition 1.4. *[29]* A partial metric space on a non-empty set Ω is a function $p : \Omega \times \Omega \to [0, \infty)$ such that

(1) $\xi = \zeta \Leftrightarrow p(\xi, \xi) = p(\xi, \zeta) = p(\zeta, \zeta)$;

(2) $p(\xi, \xi) \leq p(\xi, \zeta)$ (self-distance);

(3) $p(\xi, \zeta) = p(\zeta, \xi)$ (symmetry);

(4) $p(\xi, \zeta) \leq p(\xi, \eta) + p(\eta, \zeta) - p(\eta, \eta)$ (triangular inequality).

It is clear that, if $p(\xi, \zeta) = 0$, then, from (1) and (2), $\xi = \zeta$. But, if $\xi = \zeta$, $p(\xi, \zeta)$ may not be 0.

Example. Consider that $\Omega = [0, \infty)$, with a partial metric $p : \Omega \times \Omega \to [0, \infty)$, is defined by

$$p(\xi, \zeta) = max\{\xi, \zeta\}$$

for all $\xi, \zeta \in \Omega$. It is easy to verify that (Ω, p) is a partial metric space.

The following topological properties for partial metric spaces were proved by Mathews in [29]. A partial metric on Ω generates a T_0 topology τ_p on Ω, which has a family of open p-balls such that $\{B_p(\xi, \epsilon) : \xi \in \Omega, \epsilon > 0\}$, where $B_p(\xi, \epsilon) = \{\zeta \in \Omega : p(\xi, \zeta) < p(\xi, \xi) + \epsilon\}$ for all $\xi \in \Omega$ and $\epsilon > 0$. If p is a partial metric on Ω, then the metric $d_p : \Omega \times \Omega \to [0, \infty)$ is given by

$$d_p(\xi, \zeta) = 2p(\xi, \zeta) - p(\xi, \xi) - p(\zeta, \zeta).$$

Furthermore, $\lim_{n\to\infty} d_p(\xi_n, \xi) = 0$ if and only if

$$p(\xi, \xi) = \lim_{n\to\infty} p(\xi_n, \xi) = \lim_{n,m\to\infty} p(\xi_n, \xi_m).$$

Let (Ω, p) be a partial metric space. Then,

(i) a sequence (ξ_n) in (Ω, p) converges to a point $\xi \in \Omega$ if and only if $p(\xi, \xi) = \lim_{n\to\infty} p(\xi_n, \xi)$;

(ii) a sequence (ξ_n) in (Ω, p) is called a Cauchy sequence if $\lim_{n,m\to\infty} p(\xi_n, \xi_m)$ exist(and is finite);

(iii) (Ω, p) is said to be complete if every Cauchy sequence (ξ_n) in Ω converges with respect to τ_p to a point $\xi \in \Omega$ such that $p(\xi, \xi) = \lim_{n,m\to\infty} p(\xi_n, \xi_m)$.

Let Δ and Γ be non-empty subsets of a partial metric space $(\Omega, \|.\|)$. Then, $p(\Delta, \Gamma) = \inf\{p(\zeta, \mu) : \zeta \in \Delta, \mu \in \Gamma\}$.

Lemma 1.1. *[29] Let (Ω, p) be a partial metric space and (ξ_n) be a sequence in Ω. Then,*

(i) (ξ_n) is a Cauchy sequence in (Ω, p) if and only if it is a Cauchy sequence in the metric space (Ω, d_p);

(ii) the space (Ω, p) is complete if and only if the metric space (Ω, d_p) is complete.

There are a lot of of studies on the application of fixed point theory in the field of computational mathematics and inverse problems [33]. For work on the uniqueness of inverse problems in parabolic partial differential equations, see [34–38],and, for more works on fixed points, see [39, 40].

Motivated by the works of Berinde and pacurar [3] and Popescu [2], in this article, we introduce a generalization of Górnicki mappings by considering a non-self-map ϕ and prove the existence and uniqueness of best proximity points for proximal Górnicki mappings. Which generalizes many existing results in the literature. As a part of this study, the large class of generalized enriched contractions called Górnicki mappings are introduced to partial metric spaces and we prove the existence of fixed points and best proximity points in partial metric spaces.

2. Main results

In this section, we are going to define a proximal Górnicki mapping. By using the definition we are going to prove the existence and uniqueness of best proximity points. As a continuation, we define the concept of a proximal enriched contraction, proximal enriched Kannan mapping, and proximal enriched Chatterjea mapping; we also prove that a proximal Górnicki mapping is a generalization of all of these results.

Definition 2.1. Let Δ and Γ be a non-empty convex subset of a normed linear space $(\Omega, \|.\|)$. Given any $\lambda \in (0, 1)$, the proximal averaged multi-valued mapping ϕ_λ on Δ is defined by

$$\phi_\lambda \xi = (1 - \lambda)\xi + \lambda\mu,$$

where $\|\mu - \phi\xi\| = d(\Delta, \Gamma)$ for all $\xi, \mu \in \Delta$.

Remark 2.1. Suppose that $\Delta = \Gamma$ in (2.1); then, the proximal averaged multi-valued mapping reduced the averaged mapping in [4]. That is,

$$\phi_\lambda \xi = (1-\lambda)\xi + \lambda\phi\xi.$$

Definition 2.2. Let Δ and Γ be a non-empty subset of a complete metric space (Ω, d). A mapping $\phi : \Delta \to \Gamma$ is said to be a proximal Górnicki mapping if, for every $\xi, \zeta, \mu, \nu \in \Delta$ under the following condition

$$\left.\begin{array}{l} d(\phi\xi,\mu) = d(\Delta,\Gamma) \\ d(\phi\zeta,\nu) = d(\Delta,\Gamma) \end{array}\right\} \implies d(\mu,\nu) \le M[d(\xi,\mu) + d(\xi,\zeta) + d(\zeta,\nu)]$$

with $0 \le M < 1$, and there exist $\alpha, \beta \ge 0$ with $\alpha \le 1$ such that, for this any $\xi \in \Delta$, there exists $\eta \in \Delta$ whenever $d(\phi\eta, \varrho) = d(\Delta, \Gamma)$ this implies that

$$d(\eta,\varrho) \le \alpha d(\xi,\mu), \quad d(\xi,\eta) \le \beta d(\xi,\mu),$$

where $\varrho \in \Delta$.

Theorem 2.1. *Let (Ω, d) be a complete metric space, and let $\Delta and \Gamma$ be closed subsets of Ω. A mapping $\phi : \Delta \to \Gamma$ is a proximal Górnicki mapping; then, ϕ has a unique best proximity point.*

Proof. Suppose that $\xi_0 \in \Delta$; there exists $\mu_0 \in \Delta$ such that $d(\phi\xi_0, \mu_0) = d(\Delta, \Gamma)$.

For $\xi_0, \mu_0 \in \Delta$ there exist $\xi_1, \mu_1 \in \Delta$ such that $d(\xi_1, \mu_1) \le \alpha d(\xi_0, \mu_0)$ and $d(\xi_0, \xi_1) \le \beta d(\xi_0, \mu_0)$.

Continuing this, we have that $\xi_n, \mu_n \in \Delta$, where $d(\phi\xi_n, \mu_n) = d(\Delta, \Gamma)$ and

$$d(\xi_{n+1},\mu_{n+1}) \le \alpha d(\xi_n,\mu_n), \;\; d(\xi_{n+1},\xi_n) \le \beta d(\xi_n,\mu_n).$$

Note that,

$$d(\xi_{n+1},\xi_n) \le \beta d(\xi_n,\mu_n) \le ... \le \beta\alpha^n d(\xi_0,\mu_0).$$

Here $\lim_{n\to\infty} d(\xi_{n+1}, \xi_n) = 0$. To prove that (ξ_n) is a Cauchy sequence, without loss of generality for $n, m \in \mathbb{N}$, consider that $n < m$:

$$\begin{aligned} d(\xi_n,\xi_m) &\le d(\xi_n,\xi_{n+1}) + d(\xi_{n+1},\xi_{n+2}) + ... + d(\xi_{m-1},\xi_m) \\ &\le \beta\alpha^n d(\xi_0,\mu_0) + \beta\alpha^{n+1} d(\xi_0,\mu_0) + ... + \beta\alpha^{m-1} d(\xi_0,\mu_0) \\ &= \beta\alpha^n d(\xi_0,\mu_0)(1 + \alpha + ... + \alpha^{m-n-1}) \\ &\le \beta\alpha^n (1-\alpha)^{-1} d(\xi_0,\phi\xi_0), \end{aligned}$$

where $d(\xi_n, \xi_m) \to 0$ as $n \to \infty$. Since Ω is complete, there exists $\eta \in \Omega$ such that $\xi_n \to \eta$; also, $d(\xi_n, \mu_n) \to 0$ implies that $\mu_n \to \eta$ as $n \to \infty$. For $\eta \in \Delta$ there exists $\nu \in \Delta$, such that $d(\phi\eta, \nu) = d(\Delta, \Gamma)$. Now,

$$d(\mu_n,\nu) \le M[d(\xi_n,\mu_n) + d(\eta,\nu) + d(\xi_n,\eta)].$$

Applying the limit we will get that $d(\eta, \nu) = 0$ implies that $\eta = \nu$. Hence we have that $d(\nu, \phi\eta) = d(\Delta, \Gamma)$. Therefore, η is a best proximity point for ϕ. To prove uniqueness, suppose that $\eta_1, \eta_2 \in \Delta$ are the best proximity points for ϕ. Now, we have

$$d(\mu_1,\mu_2) \le M[d(\eta_1,\mu_1) + d(\eta_2,\mu_2) + d(\eta_1,\eta_2)],$$

where $d(\mu_1, \phi\eta_1) = d(\Delta, \Gamma) = d(\mu_2, \phi\eta_2)$.

$$\begin{aligned} d(\eta_1, \eta_2) &\leq Md(\eta_1, \eta_2), \\ (1 - M)d(\eta_1, \eta_2) &\leq 0, \\ d(\eta_1, \eta_2) &\leq 0. \end{aligned}$$

Therefore we have $\eta_1 = \eta_2$. This implies the uniqueness.

Remark 2.2. Let ξ be a best proximity point for ϕ if and only if ξ is a fixed point for the proximal multi-valued averaged mapping $\phi\lambda$.

Suppose that $d(\phi\xi, \xi) = d(\Delta, \Gamma)$; then, the proximal multi valued averaged mapping becomes

$$\phi_\lambda \xi = (1 - \lambda)\xi + \lambda\xi = \xi.$$

Therefore ξ becomes a fixed point of ϕ_λ, and the converse is also true.

Here we are going to prove that in a normed linear space, the class of proximal enriched contractions are contained in the class of proximal Górnicki mappings.

Definition 2.3. Let $(\Omega, ||.||)$ be a normed linear space and Δ and Γ be non-empty subsets of Ω. A mapping $\phi : \Delta \to \Gamma$ is said to be a proximal enriched contraction if there exist $\beta \in [0, \infty)$ and $\sigma \in [0, \beta + 1)$ such that, for every ξ, ζ there exist $\mu, \nu \in \Omega$ whenever

$$\begin{aligned} ||\phi\xi - \mu|| &= d(\Delta, \Gamma), \\ ||\phi\zeta - \nu|| &= d(\Delta, \Gamma), \end{aligned}$$

implies

$$||\beta(\xi - \zeta) + \mu - \nu|| \leq \sigma||\xi - \zeta||.$$

Theorem 2.2. *Let Δ and Γ be non-empty subsets of Ω. If $\phi : \Delta \to \Gamma$ is a proximal enriched contraction, then ϕ is a proximal Górnicki mapping.*

Proof. Consider a proximal multi-valued averaged mapping ϕ_λ given by

$$\phi_\lambda \xi = (1 - \lambda)\xi + \lambda\mu,$$

where $||\mu - \phi\xi|| = d(\Delta, \Gamma)$. Since

$$||\phi_\lambda \xi - \phi_\lambda \zeta|| \leq c||\xi - \zeta||$$

implies that a proximal multi valued averaged mapping is a contraction mapping with $\lambda = \frac{1}{\beta+1} < 1$ and $c = \frac{\sigma}{\beta+1} < 1$, consider that

$$\begin{aligned} ||\mu - \nu|| &\leq ||\mu - \phi_\lambda \xi|| + ||\phi_\lambda \xi - \phi_\lambda \zeta|| + ||\phi_\lambda \zeta - \nu|| \\ &\leq (1 - \lambda)||\xi - \mu|| + (1 - \lambda)||\zeta - \nu|| + c||\xi - \zeta||. \end{aligned}$$

Let $M = \max\{1 - \lambda, c\}$; this implies that

$$||\mu - \nu|| \leq M[||\xi - \mu|| + ||\zeta - \nu|| + ||\xi - \zeta||].$$

That is,

$$d(\mu, \nu) \leq M[d(\xi, \mu) + d(\xi, \zeta) + d(\zeta, \nu)].$$

For given $\xi \in \Delta$, let $\eta = \phi_\lambda \xi$ with $d(\phi\eta, \varrho) = d(\Delta, \Gamma)$ and $d(\phi\xi, u) = d(A, B)$, where $u, w \in A$; we have

$$\lambda\|\eta - \varrho\| = \|\phi_\lambda \eta - \eta\| = \|\phi_\lambda \eta - \phi_\lambda \xi\| \leq c\|\xi - \eta\|$$

this implies that $\|\eta - \varrho\| \leq c\|\xi - \mu\|$; also, $\|\xi - \eta\| = \|\xi - \rho_\lambda \xi\| = \lambda\|\xi - \mu\|$.

Therefore, ρ is a proximal Górinicki mapping with $\alpha = c$ and $\beta = \lambda$.

Definition 2.4. A mapping $\rho : \Delta \to \Gamma$ is said to be a proximal enriched Kannan mapping, if there exists $\beta \in [0, \infty)$ and $\sigma \in [0, \frac{1}{2})$ such that, for all ξ, ζ there exist $\mu, \nu \in \Delta$, whenever

$$\|\phi\xi - \mu\| = d(\Delta, \Gamma),$$
$$\|\phi\zeta - \nu\| = d(\Delta, \Gamma),$$

implies

$$\|\beta(\xi - \zeta) + \mu - \nu\| \leq \sigma(\|\xi - \mu\| + \|\zeta - \nu\|).$$

Theorem 2.3. *Let Δ and Γ be a nonempty subset of a normed linear space Ω, suppose that $\phi : \Delta \to \Gamma$ is a proximal enriched Kannan mapping then ϕ is a proximal Górnicki mapping.*

Proof. Consider the proximal multi-valued averaged mapping ϕ_λ such that

$$\begin{aligned}
\|\phi_\lambda \xi - \phi_\lambda \zeta\| &= \|(1 - \lambda)(\xi - \zeta) + \lambda(\mu - \nu)\| \\
&= \lambda\|\frac{(1 - \lambda)}{\lambda}(\xi - \zeta) + (\mu - \nu)\| \\
&\leq \lambda c(\|\xi - \mu\| + \|\zeta - \nu\|) \\
&\leq c(\|\phi_\lambda \xi - \xi\| + \|\phi_\lambda \zeta - \zeta\|),
\end{aligned}$$

which implies that the proximal averaged mapping ϕ_λ is a Kannan mapping with $\lambda = \frac{1}{\beta+1} < 1$ and $c = \sigma < \frac{1}{2}$.

Here,

$$\begin{aligned}
\|\mu - \nu\| &\leq \|\mu - \phi_\lambda \xi\| + \|\phi_\lambda \xi - \phi_\lambda \zeta\| + \|\phi_\lambda \zeta - \nu\| \\
&\leq (1 - \lambda)\|\mu - \xi\| + (1 - \lambda)\|\nu - \zeta\| + c(\|\phi_\lambda \xi - \xi\| \\
&\qquad + \|\phi_\lambda \zeta - \zeta\|) \\
&\leq (1 - \lambda + c\lambda)(\|\mu - \xi\| + \|\nu - \zeta\|),
\end{aligned}$$

and $(1 - \lambda + c\lambda) < 1$. Therefore we have

$$d(\mu, \nu) \leq M(d(\mu, \xi) + d(\nu, \zeta) + d(\xi, \zeta)$$

with $M = 1 - \lambda + c\lambda < 1$.

For given $\xi \in \Delta$, let $\eta = \phi_\lambda \xi$ with $d(\phi\eta, \varrho) = d(\Delta, \Gamma)$ and $d(\phi\xi, \mu) = d(\Delta, \Gamma)$, where $\mu, \varrho \in \Gamma$; then, we have

$$\lambda\|\eta - \varrho\| = \|\phi_\lambda \eta - \eta\| = \|\phi_\lambda \eta - \phi_\lambda \xi\|$$

$$
\begin{aligned}
&\leq c(\|\xi - \phi_\lambda \xi\| + \|\eta - T_\lambda \eta\|) \\
&= c\lambda(\|\xi - \mu\| + \|\eta - \varrho\|) \\
\|\eta - \varrho\| &\leq \frac{c}{1-c}\|\xi - \mu\|.
\end{aligned}
$$

Hence, $d(\eta, \varrho) \leq \frac{c}{1-c}\|\xi - \mu\|$.

Also, $d(\xi, \eta) = \|\xi - \eta\| = \lambda\|\xi - \mu\| = \lambda d(\xi, \mu)$.

Therefore, T is a proximal Górnicki mapping with $\alpha = \frac{c}{1-c} < 1$ and $\beta = \lambda$.

Definition 2.5. Let Δ and Γ be nonempty subset of a normed linear space Ω; consider $T : \Delta \to \Gamma$ to be proximal enriched Chatterjea mapping if there exist $\beta \in [0, \infty)$ and $\sigma \in [0, \frac{1}{2})$ such that for all ξ, ζ there exist $\mu, \nu \in \Omega$, whenever

$$
\begin{aligned}
\|\phi\xi - \mu\| &= d(\Delta, \Gamma), \\
\|\phi\zeta - \nu\| &= d(\Delta, \Gamma),
\end{aligned}
$$

implies

$$
\|\beta(\xi - \zeta) + \mu - \nu\| \leq \sigma(\|(\beta + 1)(\xi - \zeta) + \zeta - \nu\| + \|(\beta + 1)(\zeta - \xi) + \xi - \mu\|).
$$

Theorem 2.4. *Let δ, Γ be a non-empty subsets of a normed linear space Ω; suppose that $\phi : \Delta \to \Gamma$ is a proximal enriched Chatterjea mapping then ϕ is a proximal Gornicki mapping.*

Proof. Consider the proximal averaged mapping ϕ_λ; we have

$$
\begin{aligned}
\|\phi_\lambda \xi - \phi_\lambda \zeta\| &= \|(1 - \lambda)\xi + \lambda\mu - (1 - \lambda)\zeta - \lambda\nu\| \\
&= \lambda\|\frac{(1 - \lambda)}{\lambda}(\xi - \zeta) + (\mu - \nu)\|.
\end{aligned}
$$

Let $\beta = \frac{1-\lambda}{\lambda}$; we have

$$
\begin{aligned}
\|\phi_\lambda \xi - \phi_\lambda \zeta\| &\leq \lambda c\left(\|(\beta + 1)(\xi - \zeta) + \zeta - \nu\| + \|(\beta + 1)(\zeta - \xi) + \xi - \mu\|\right) \\
\|\phi_\lambda \xi - \phi_\lambda \zeta\| &\leq c\left(\|\phi_\lambda \zeta - \xi\| + \|\phi_\lambda \zeta - \xi\|\right).
\end{aligned}
$$

Therefore the proximal averaged mapping is a Chatterjea mapping with $c = \sigma < \frac{1}{2}$.

Suppose that, for $\xi, \zeta \in \Delta$, there exist $\mu, \nu \in \Delta$ such that $d(\mu, \phi\xi) = d(\nu, \phi\zeta) = d(\Delta, \Gamma)$. Now,

$$
\begin{aligned}
\|\mu - \nu\| &\leq \|\mu - \phi_\lambda \xi\| + \|\phi_\lambda \xi - \phi_\lambda \zeta\| + \|\phi_\lambda \zeta - \nu\| \\
&\leq (1 - \lambda)\left(\|\mu - \xi\| + \|\nu - \zeta\|\right) + c\left(\|\phi_\lambda \zeta - \xi\| + \|\phi_\lambda \xi - \zeta\|\right) \\
&\leq (1 - \lambda + \lambda c)\left(\|\mu - \xi\| + \|\nu - \zeta\|\right) + 2c\|\xi - \zeta\|
\end{aligned}
$$

implies that $d(\mu, \nu) \leq M\left(d(\mu, \xi) + d(\nu, \zeta) + d(\xi, \zeta)\right)$ with $M = \max\{(1 - \lambda + c\lambda, 2c\} < 1$.

For given $\xi \in \Delta$, let $\eta = \phi_\lambda \xi$ with $d(\phi\eta, \varrho) = d(\Delta, \Gamma)$ and $d(\phi\xi, \mu) = d(\Delta, \Gamma)$, where $\mu, \varrho \in \Delta$; then, we have

$$
\begin{aligned}
\lambda\|\eta - \varrho\| &= \|\phi_\lambda \eta - \eta\| \\
&\leq c\left(\|\phi_\lambda \eta - \xi\| + \|\phi_\lambda \xi - \eta\|\right) \\
&\leq c\lambda\left(\|\xi - \mu\| + \|\eta - \varrho\|\right).
\end{aligned}
$$

Here $\|\eta - \varrho\| \leq \frac{c}{1-c}\|\xi - \mu\|$; also, $\|\eta - \xi\| = \lambda\|\xi - \mu\|$.

Therefore we conclude that ϕ is a proximal Górnicki mapping with $\alpha = \frac{c}{1-c}$ and $\beta = \lambda$.

Remark 2.3. Let $\Delta = \Gamma$ in Theorems 2.1–2.4; then, the results can be reduced to Theorems 4 and 6–8 in [2], respectively. Our work generalizes the work done by Popescu [2].

Example. Let $\Omega = \mathbb{R}^2$, $\|(\xi_1, \xi_2) - (\zeta_1, \zeta_2)\| = \sqrt{(\xi_1 - \zeta_1)^2 + (\xi_2 - \zeta_2)^2}$, $\Delta = \{(\xi, 0) : \xi \in \mathbb{R}\}$ and $\Gamma = \{(\xi, 1) : \xi \in \mathbb{R}\}$. Consider $\phi : \Delta \to \Gamma$ to be the following non-self mapping:

$$\phi\xi = \begin{cases} (\frac{\xi+2}{2}, 1), & \text{if } \xi \leq 3, \\ (\frac{\xi}{2}, 1), & \text{if } \xi > 3. \end{cases}$$

Therefore, ϕ is a proximal Górnicki mapping but ϕ is not a proximal enriched contraction; also,ϕ is not a proximal enriched Kannan mapping or proximal enriched Chatterjea mapping.

Proof. Let $(\xi, 0), (\zeta, 0) \in \Delta$.
Case I. For $\xi \leq 3$, $\zeta \leq 3$, there exist $\mu = (\frac{\xi+2}{2}, 0)$ and $\nu = (\frac{\zeta+2}{2}, 0)$ such that $\|\mu - \nu\| = \frac{|\xi-\zeta|}{2}$, where $d(\phi(\xi, 0), \mu) = d(\phi(\zeta, 0), \nu) = d(\Delta, \Gamma) = 1$.
Case II. For $\xi > 3$, $\zeta > 3$, there exist $\mu = (\frac{\xi}{2}, 0)$ and $\nu = (\frac{\zeta}{2}, 0)$ such that $\|\mu - \nu\| = \frac{\xi-\zeta}{2}$, where $d(\phi(\xi, 0), \mu) = d(\phi(\zeta, 0), \nu) = d(\Delta, \Gamma) = 1$.
Case III. For $\xi \leq 3$, $\zeta > 3$, there exists $\mu = (\frac{\xi+2}{2}, 0)$ and $\nu = (\frac{\zeta}{2}, 0)$; we have

$$\begin{aligned} \|\mu - \nu\| = |\frac{\xi + 2}{2} - \frac{\zeta}{2}| &\leq |\frac{\xi - \zeta}{2}| + 1 \\ &\leq \frac{2}{3}(|\xi - \zeta| + \frac{3}{2}) \\ &\leq \frac{2}{3}(|\xi - \zeta| + \frac{\zeta}{2}) \\ &\leq \frac{2}{3}(|\xi - \zeta| + \frac{|\xi - 2|}{2} + \frac{\zeta}{2}). \end{aligned}$$

For every $(\xi, 0), (\zeta, 0) \in \mathbb{R}^2$ we have,

$$\|\mu - \nu\| \leq \frac{2}{3}(\|(\xi, 0) - (\zeta, 0)\| + \|(\xi, 0) - \mu\| + \|(\zeta, 0) - \nu\|).$$

Let $\eta = \mu$; that is,

$$\eta = \begin{cases} (\frac{\xi+2}{2}, 0), & \xi \leq 3, \\ (\frac{\xi}{2}, 0), & \xi > 3. \end{cases}$$

Choose

$$w\varrho = \begin{cases} (\frac{\xi+6}{4}, 0), & \xi \leq 3, \\ (\frac{\xi+4}{4}, 0), & \xi > 3, \frac{\xi}{2} \leq 3, \\ (\frac{\xi}{4}, 0), & \xi > 3, \frac{\xi}{2} > 3, \end{cases}$$

$$d(\eta, \varrho) = \|(\frac{\xi + 2}{2}, 0) - (\frac{\xi + 6}{4}, 0)\| = |\frac{\xi - 2}{4}| \text{ if } \xi \leq 3,$$

$$d(\eta, \varrho) = \|(\frac{\xi}{2}, 0) - (\frac{\xi + 4}{4}, 0)\| = |\frac{\xi - 4}{4}|, \text{ if } \xi > 3 \text{ and } \frac{\xi}{2} \leq 3,$$

$$d(\eta, \varrho) = \|(\frac{\xi}{2}, 0) - (\frac{\xi}{4}, 0)\| = |\frac{\xi}{2}|, \text{ if } \xi > 3 \text{ and } \frac{\xi}{2} > 3,$$

then, $d(\xi,\mu) = \begin{cases} |\frac{\xi-2}{2}|, & \text{if } \xi \le 3, \\ |\frac{\xi}{2}|, & \text{if } \xi > 3. \end{cases}$

Also note that

$$d(\xi,\eta) = \|(\xi,0) - (\frac{\xi+2}{2},0)\| = |\frac{\xi-2}{2}| = d(\xi,\mu)\ if \xi \le 3,$$

$$d(\xi,\eta) = \|(\xi,0) - (\frac{\xi}{2},0)\| = |\frac{\xi}{2}| = d(\xi,\mu)\ if \xi > 3.$$

Table 1 shows the convergence behavior of initial points $(0,0),(0.5,0),(2,0),(3,0)$ to fixed point (2,0) using iteration process.

Table 1. Iteration of convergence for a proximal Górnicki mapping.

ξ_n	$\xi_0 = (0,0)$	$\xi_0 = (0.5,0)$	$\xi_0 = (2,0)$	$\xi_0 = (3,0)$
ξ_1	(0,0)	(5.0000e-01,0)	(2,0)	(3,0)
ξ_2	(1,0)	(1.2500e+00,0)	(2,0)	(2.5000e+00,0)
ξ_3	(1.5000e+00,0)	(1.6250e+00,0)	(2,0)	(2.2500e+00,0)
ξ_4	(1.7500e+00,0)	(1.8125e+00,0)	(2,0)	(2.1250e+00,0)
ξ_5	(1.8750e+00,0)	(1.9062e+00,0)	(2,0)	(2.0625e+00,0)
ξ_6	(1.9375e+00,0)	(1.9531e+00,0)	(2,0)	(2.0312e+00,0)
ξ_7	(1.9687e+00,0)	(1.9765e+00,0)	(2,0)	(2.0156e+00,0)
ξ_8	(1.9843e+00,0)	(1.9882e+00,0)	(2,0)	(2.0078e+00,0)
ξ_9	(1.9921e+00,0)	(1.9941e+00,0)	(2,0)	(2.0039e+00,0)
ξ_{10}	(1.9960e+00,0)	(1.9970e+00,0)	(2,0)	(2.0019e+00,0)
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
ξ_{53}	(2,0)	(2,0)	(2,0)	(2,0)

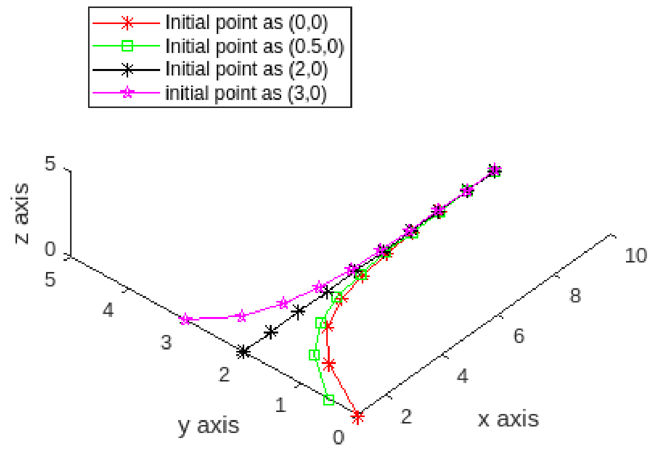

Figure 1. Convergence behavior for proximal Górnicki mapping.

Figure 1 shows convergence behavior graphically to the fixed point $(2,0)$. Therefore, we conclude that ϕ is a proximal Górnicki mapping with $\alpha = \frac{1}{2}$ and $\beta = 1$. Now, let $(\xi,0) = (3,0)$ and $(\zeta,0) = (3+\frac{1}{n},0)$,

$n \geq 1$, and let $\mu = (\frac{5}{2}, 0)$ and $\nu = (\frac{1}{2}(3 + \frac{1}{n}), 0)$. We have

$$\|\beta((\xi, 0) - (\zeta, 0) + \mu - \nu)\| = \|(1 - \frac{\beta}{n} - \frac{1}{2n}, 0)\| = |1 - \frac{\beta}{n} - \frac{1}{2n}| \to 1.$$

As $n \to \infty$, since

$$\sigma\|(\xi, 0) - (\zeta, 0)\| = \sigma\|(\frac{1}{n}, 0)\| = \sigma|\frac{1}{n}| \to 0,$$

as $n \to \infty$, we conclude that ϕ is not a proximal enriched contraction.

$$\sigma(\|(\xi, 0) - \mu\| + \|(\zeta, 0) - \nu\|) = \sigma(\|(\frac{1}{2}, 0)\| + \|\frac{1}{2}(3 + \frac{1}{n}, 0)\|) = \sigma(2 + \frac{1}{2n}) \to 2\sigma < 1,$$

as $n \to \infty$, we conclude that ϕ is not a proximal enriched Kannan mapping. Also, we have

$$\begin{aligned}\sigma(\|(\beta + 1)((\xi, 0) - (\zeta, 0)) + (\zeta, 0) - \nu\| + \|(\beta + 1)((\zeta, 0) - (\xi, 0)) + (\xi, 0) - \mu\|)\\ = \sigma(\|(\frac{3}{2} - \frac{\beta}{n} + \frac{1}{2n}, 0)\| + \|(\frac{\beta}{n} + \frac{1}{n} + \frac{1}{2}, 0)\|)\\ = \sigma(|\frac{3}{2} - \frac{b}{n} + \frac{1}{2n}| + |\frac{1}{2} + \frac{\beta + 1}{n}|) \to 2\sigma < 1,\end{aligned}$$

as $n \to \infty$.

Therefore, we conclude that ϕ is not a proximal enriched Chatterjea mapping

3. Fixed points and best proximity points for Górnicki mappings on a partial metric space

In this section, we extend our results to partial metric spaces and prove the existence of fixed points and best proximity points for Górnicki mappings and proximal Górnicki mappings on partial metric spaces.

Definition 3.1. Let (Ω, p) be a complete partial metric space; a mapping $\phi : \Omega \to \Omega$ is said to be a Górnicki mapping if

$$p(\phi\xi, \phi\zeta) \leq M[p(\xi, \phi\xi) + p(\zeta, \phi\zeta) + p(\xi, \zeta)]$$

with $M < 1$ for all $\xi, \zeta \in \phi$ and there exist $\alpha, \beta \geq 0$ with $\alpha < 1$ such that, for all $\xi \in \Omega$, there exists $\mu \in \Omega$ with $p(\mu, \phi\mu) \leq \alpha p(\xi, \phi\xi)$, $p(\mu, \xi) \leq \beta p(\xi, \phi\xi)$.

Theorem 3.1. *Let (Ω, p) be a complete partial metric space, and let $\phi : \Omega \to \Omega$ be a Górnicki mapping; then, ϕ has a fixed point.*

Proof. Let $\xi_0 \in \Omega$; there exists $\xi_1 \in \Omega$ such that

$$\begin{aligned}p(\xi_1, \phi\xi_1) \leq \alpha p(\xi_0, \phi\xi_0),\\ p(\xi_1, \xi_0) \leq \beta p(\xi_0, \phi\xi_0).\end{aligned}$$

In general,

$$\begin{aligned}p(\xi_{n+1}, \phi\xi_{n+1}) \leq \alpha p(\xi_n, \phi\xi_n),\\ p(\xi_{n+1}, \xi_n) \leq \beta p(\xi_n, \phi\xi_n),\end{aligned}$$

implies that

$$p(\xi_{n+1}, \xi_n) \leq \beta p(\xi_n, \phi\xi_n) \leq ... \leq \beta\alpha^n p(\xi_0, \phi\xi_0).$$

Therefore,

$$\lim_{n\to\infty} p(\xi_{n+1}, \xi_n) = 0.$$

To prove that$\{\xi_n\}$ is Cauchy, consider the following:

$$\begin{aligned} p(\xi_n, \xi_m) &\leq p(\xi_n, \xi_{n+1}) + p(\xi_{n+1}, \xi_m) - p(\xi_{n+1}, \xi_{n+1}) \\ &\leq p(\xi_n, \xi_{n+1}) + p(\xi_{n+1}, \xi_{n+2}) + ... + p(\xi_{m-1}, \xi_m) \\ &\leq \beta\alpha^n p(\xi_0, \phi T\xi_0) + \beta\alpha^{n+1} p(\xi_0, \phi\xi_0) + ... + \beta\alpha^{m-1} p(\xi_0, \phi\xi_0) \\ &= \beta\alpha^n p(\xi_0, \phi\xi_0)(1 + \alpha + ... + \alpha^{m-n-1}) \\ &\leq \beta\alpha^n (1-\alpha)^{-1} p(\xi_0, \phi\xi_0), \end{aligned}$$

$\lim_{n\to\infty} p(\xi_n, \xi_m) = 0$. Since $\{\xi_n\}$ is Cauchy with respect to the metric d_p, as induced by the partial metric p, then there exists ξ such that $\{\xi_n\}$ converges to ξ with respect to the metric such that $\lim_{n\to\infty} d_p(\xi_n, \xi) = 0$; then, we have

$$p(\xi, \xi) = \lim_{n\to\infty} p(\xi_n, \xi) = \lim_{n,m\to\infty} p(\xi_m, \xi_n) = 0.$$

Also, $\lim_{n\to\infty} p(\xi_n, \phi\xi_n) = 0$ since

$$p(\xi_n, \phi\xi_n) \leq \alpha p(\xi_{n-1}, \phi\xi_{n-1}) \leq ... \leq \alpha^n p(\xi_0, \phi\xi_0).$$

Note that $\lim_{n\to\infty} p(\phi\xi_n, \phi\xi) = p(\xi, \phi\xi)$ since

$$\begin{aligned} p(\phi\xi_n, \phi\xi) &\leq p(\phi\xi_n, \xi) + p(\xi, \phi\xi) - p(\xi, \xi), \\ \lim_{n\to\infty} p(\phi\xi_n, \phi\xi) &\leq p(\xi, \phi\xi), \end{aligned}$$

also, we have

$$p(\xi, \phi\xi) \leq p(\xi, \phi\xi_n) + p(\phi\xi_n, \phi\xi - p(\phi\xi_n, \phi\xi_n),$$

applying limits on both sides, we have that, $p(\xi, \phi\xi) \leq \lim_{n\to\infty} p(\phi\xi_n, \phi\xi)$, which implies that $\lim_{n\to\infty} p(\phi\xi_n, \phi\xi) = p(\xi, \phi\xi)$.

Now to prove $\phi(\xi) = \xi$ consider the following

$$p(\phi\xi_n, \phi\xi) \leq M\left(p(\xi_n, \phi\xi_n) + p(\xi, \phi\xi) + p(\xi_n, \xi)\right),$$

applying the limit on both sides implies that $\phi\xi = \xi$.

Theorem 3.2. *In the above Theorem 3.1, if $M < \frac{1}{2}$ then ϕ has a unique fixed point.*

Proof. To prove the uniqueness, suppose that there exists two fixed points $\xi_1, \xi_2 \in \omega$ such that $\phi\xi_1 = \xi_1$ and $\phi\xi_2 = \xi_2$. Then, we have

$$\begin{aligned} p(\phi\xi_1, \phi\xi_2) &\leq M(p(\xi_1, \phi\xi_1) + p(\xi_2, \phi\xi_2) + p(\xi_1, \xi_2)), \\ p(\xi_1, \xi_2) &\leq 2Mp(\xi_1, \xi_2). \end{aligned}$$

Since $M < \frac{1}{2}$, we have that $p(\xi_1, \xi_2) = 0$. This implies that $\xi_1 = \xi_2$.

Example. Consider $\Omega = [0,1]$ with a partial metric $p(\xi,\zeta) = \max\{\xi,\zeta\}$. Let $\phi(\xi) = \frac{\xi}{2}$ and $p(\frac{\xi}{2},\frac{\zeta}{2}) = \frac{p(\xi,\zeta)}{2}$, which implies that $p(\frac{\xi}{2},\frac{\zeta}{2}) \le \frac{1}{2}[p(\xi,\frac{\xi}{2}) + p(\zeta,\frac{\zeta}{2}) + p(\xi,\zeta)]$; also, $p(\frac{\xi}{2},\frac{\xi}{4}) \le \frac{1}{2}p(\xi,\frac{\xi}{2})$ and $p(\frac{\xi}{2},\xi) \le p(\xi,\frac{\xi}{2})$; in this case, ϕ, satisfies the conditions for a Górnicki mapping with $M = \frac{1}{2}$, $\mu = \frac{\xi}{2}$, $\alpha = \frac{1}{2}$ and $\beta = 1$ for all $\xi \in \Omega$. Therefore, ϕ has a unique fixed point.

Example. Consider $\mathcal{F}$ to be the set of all polynomials of degree less than or equal to n with non-negative real coefficients; suppose $f_1, f_2 \in \mathcal{F}$ has a partial metric space such that $p(f_1, f_2) = \max_i\{a_i, b_i\}$, where $f_1 = a_0 + a_1t + ... + a_nt^n$ and $f_2 = b_0 + b_1t + ... + b_nt^n$; suppose that $\phi : \mathcal{F} \to \mathcal{F}$ with $\phi(a_0 + a_1t + ... + a_nt^n) = \left(\frac{a_1}{2}t + \frac{a_2}{2}t^2 + ... + \frac{a_n}{2}t^n\right)$ for every $f_1, f_2 \in \mathcal{F}$; then,

$$p\left(\frac{a_1}{2}t + ... + \frac{a_n}{2}t^n, \frac{b_1}{2}t + ... + \frac{b_n}{2}t^n\right) \le \frac{1}{2}p\left(a_0 + ... + a_nt^n, b_0 + ... + b_nt^n\right),$$

also

$$p\left(\frac{a_1}{2}t + ... + \frac{a_n}{2}t^n, \frac{a_2}{4}t + ... + \frac{a_n}{4}t^n\right) \le \frac{1}{2}p(a_0 + a_1t + ... + a_nt^n, \frac{a_1}{2}t + ... + \frac{a_n}{2}t^n),$$

and $p(\frac{a_1}{2}t + ... + \frac{a_n}{2}t^n, a_0 + a_1t + ... + a_nt^n) \le p(a_0 + a_1t + ... + a_nt^n, \frac{a_1}{2}t + ... + \frac{a_n}{2}t^n)$; in this case, ϕ satisfies the condition of a Górnicki mapping with $M = \frac{1}{2}$ and $\mu = \left(\frac{a_1}{2}t + \frac{a_2}{2}t^2 + ... + \frac{a_n}{2}t^n\right)$ for every $a_0 + a_1t + ... + a_nt^n \in \mathcal{F}$ with $\alpha = \frac{1}{2}$ and $\beta = 1$ this implies that ϕ has a unique fixed point.

The definition of a proximal Górnicki mapping on partial metric spaces is as follows

Definition 3.2. Let (Ω, p) be a complete partial metric space, and let Δ and Γ be non-empty subsets of Ω. A mapping $\Omega : \Delta \to \Gamma$ is said to be a proximal Górinicki mapping if for all ξ, ζ there exist $\mu, \nu \in \Delta$ whenever

$$p(\phi\xi, \mu) = p(\Delta, \Gamma),$$
$$p(\phi\zeta, \nu) = p(\Delta, \Gamma),$$

implies

$$p(\mu, \nu) \le M\left(p(\xi, \mu) + p(\xi, \zeta) + p(\nu, \zeta)\right)$$

with $0 \le M < 1$ and there exist $\alpha, \beta \ge 0$ with $\alpha \le 1$ such that, for any $\xi \in \Delta$, there exists $\eta \in \Delta$ when $p(\phi\eta, \varrho) = p(\Delta, \Gamma)$ this implies that

$$p(\eta, \varrho) \le \alpha p(\xi, \mu), \quad p(\xi, \eta) \le \beta p(\xi, \mu),$$

where $\varrho \in \Delta$.

Theorem 3.3. *Let (Ω, p) be a complete partial metric space, and let Δ and Γ be closed subsets of Ω. A mapping $\phi : \Delta \to \Gamma$ is a proximal Górnicki mapping; then, ϕ has a best proximity point.*

Proof. Suppose that $\xi_0 \in \Delta$; there exists $\mu_0 \in \Delta$ such that $p(\phi\xi_0, \mu_0) = p(\Delta, \Gamma)$.

For $\xi_0, \mu_0 \in \Delta$, there exist $\xi_1, \mu_1 \in \Delta$ such that $p(\xi_1, \mu_1) \le \alpha p(\xi_0, \mu_0)$ and $p(\xi_0, \xi_1) \le \beta d(\xi_0, \mu_0)$. Continuing this, we have that $\xi_n, \mu_n \in \Delta$, where $p(\phi\xi_n, \mu_n) = p(\Delta, \Gamma)$ and

$$p(\xi_{n+1}, \mu_{n+1}) \le \alpha p(\xi_n, \mu_n), \quad p(\xi_{n+1}, \xi_n) \le \beta p(\xi_n, \mu_n).$$

Note that,

$$p(\xi_{n+1},\xi_n) \leq \beta p(\xi_n,\mu_n) \leq ... \leq \beta\alpha^n p(\xi_0,\mu_0).$$

Here $\lim_{n\to\infty} p(\xi_{n+1},\xi_n) = 0$. To prove that (ξ_n) is a Cauchy sequence, without loss of generality for $n, m \in \mathbb{N}$, consider that $n < m$.

$$\begin{aligned}
p(\xi_n,\xi_m) &\leq p(\xi_n,\xi_{n+1}) + p(\xi_{n+1},\xi_m) - p(\xi_{n+1},\xi_{n+1}) \\
&\leq p(\xi_n,\xi_{n+1}) + p(\xi_{n+1},\xi_{n+2}) + ... + p(\xi_{m-1},\xi_m) \\
&\leq \beta\alpha^n p(\xi_0,\mu_0) + \beta\alpha^{n+1} p(\xi_0,\mu_0) + ... + \beta\alpha^{m-1} p(\xi_0,\mu_0) \\
&= \beta\alpha^n p(\xi_0,\mu_0)(1 + \alpha + ... + \alpha^{m-n-1}) \\
&\leq \beta\alpha^n (1-\alpha)^{-1} p(\xi_0,\mu_0),
\end{aligned}$$

$\lim_{n,m\to\infty} p(x_n, x_m) = 0$. Since $\{\xi_n\}$ is Cauchy with respect to the metric d_p, as induced by the partial metric p, then there exists η such that $\{\xi_n\}$ converges to η with respect to the metric such that $\lim_{n\to\infty} d_p(\xi_n,\eta) = 0$, and We have

$$p(\eta,\eta) = \lim_{n\to\infty} p(\xi_n,\eta) = \lim_{n,m\to\infty} p(\xi_m,\xi_n) = 0.$$

Also $\lim_{n\to\infty} p(\xi_n,\mu_n) = 0$ since

$$p(\xi_n,\mu_n) \leq \alpha p(\xi_{n-1},\mu_{n-1}) \leq ... \leq \alpha^n p(\xi_0,\mu_0).$$

For $\eta \in \Delta$, there exists $\mu \in \Delta$ such that $d(\phi\eta,\mu) = d(\Delta,\Gamma)$.

Note that $\lim_{n\to\infty} p(\mu_n,\mu) = p(\eta,\mu)$ since

$$\begin{aligned}
p(\mu_n,\mu) &\leq p(\mu_n,\eta) + p(\eta,\mu) - p(\eta,\eta), \\
\lim_{n\to\infty} p(\mu_n,\mu) &\leq p(\eta,\mu),
\end{aligned}$$

also, we have

$$p(\eta,\mu) \leq p(\eta,\mu_n) + p(\mu_n,\mu) - p(\mu_n,\mu_n),$$

applying the limits on both sides, we have that $p(\eta,\mu) \leq \lim_{n\to\infty} p(\mu_n,\mu)$, which implies that $\lim_{n\to\infty} p(\mu_n,\mu) = p(\eta,\mu)$.

Now,

$$p(\mu_n,\mu) \leq M\left(p(\xi_n,\mu_n) + p(\mu,\eta) + p(\xi_n,\eta)\right),$$

applying the limit on both sides implies that $p(\eta,\mu) = 0$; hence, $\eta = \mu$. Therefore, we have that $p(\eta,\phi\eta) = p(\Delta,\Gamma)$ hence η becomes a best proximity point for ϕ.

Theorem 3.4. *In the above Theorem 3.3, if $M < \frac{1}{2}$, then ϕ has a unique best proximity point.*

Proof. To prove uniqueness, suppose that $\eta_1, \eta_2 \in \Delta$ are best proximity points for ϕ. Now, we have

$$p(\eta_1, \eta_2) \leq M[p(\eta_1, \eta_1) + p(\eta_2, \eta_2) + p(\eta_1, \eta_2)],$$

where $p(\eta_1, \phi\eta_1) = p(\Delta, \Gamma) = p(\eta_2, \phi\eta_2)$.

$$\begin{aligned} p(\eta_1, \eta_2) &\leq 2Mp(\eta_1, \eta_2), \\ (1 - 2M)p(\eta_1, \eta_2) &\leq 0, \\ p(\eta_1, \eta_2) &\leq 0. \end{aligned}$$

Therefore, we have the $\eta_1 = \eta_2$. This implies the uniqueness.

4. Application to Górnicki mappings on variational inequality problems

In the year 1966, Hartman and Stampachia proved that a mapping $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is continuous on a compact, convex subset Ω of $\mathbb{R}^n$; considering this, we can find $\zeta \in \Omega$ such that

$$\langle \phi\zeta \ , \ \zeta - \eta \rangle \geq 0,$$

for every $\eta \in \Omega$.

Now consider Ψ as a real Hilbert space with the inner product $\langle \cdot \ , \ \cdot \rangle$ and the induced norm $||\cdot||$. Let $\aleph$ be a non-empty closed convex subset of Ψ. An element $\zeta_0 \in \aleph$ is known as the best approximation if $||\xi - \zeta_0|| = D(\xi, \aleph)$, where $D(\xi, \aleph) = \inf_{\zeta\in\aleph} ||\xi - \zeta||$. The operator $\Upsilon_\aleph : \Psi \to \aleph$ is called the metric projection of Ψ onto $\aleph$ such that, for all $\xi \in \Psi$,

$$\Upsilon_\aleph(\xi) = \{\zeta \in \aleph : ||\xi - \zeta|| = D(\xi, \aleph)\}.$$

For each point $\xi \in \Psi$, there exists a unique nearest point in $\aleph$, denoted by $\Upsilon_\aleph(\xi)$.

That is,

$$||\xi - \Upsilon_\aleph(\xi)|| \leq ||\xi - \zeta||$$

for all $\zeta \in \aleph$.

The projection operator $\Upsilon_\aleph$ plays an important role in proving the existence of the solution to variational inequality problems.

We consider the following variational inequality problem:

$$\text{Find } \mu \in \aleph \text{ such that } \langle \Pi\mu \ , \ \nu - \mu \rangle \geq 0 \text{ for all } \nu \in \aleph. \tag{4.1}$$

Kinderlehrer and Stampacchia [41]introduced and applied variational inequality problems to solve the deflection of an elastic beam problem, filtration of a liquid through porous media, and free boundary problems of lubrication. These are the references for application-related fixed point theory [38,42–44].

Lemma 4.1. *Let* $\eta \in \Psi$. *Then,* $\mu \in \aleph$ *satisfies the inequality* $\langle \mu - \eta \ , \ \zeta - \mu \rangle \geq 0$ *for all* $\zeta \in \aleph$ *if and only if* $\mu = \Upsilon_\aleph \eta$.

Lemma 4.2. *Let* $\Pi : \Psi \to \Psi$. *Then* $\mu \in \aleph$ *is a solution of* $\langle \Pi\mu \ , \ \nu - \mu \rangle \geq 0$ *for all* $\nu \in \aleph$ *if and only if* $\mu = \Upsilon_\aleph(\mu - \lambda\Pi\mu)$ *with* $\lambda > 0$.

Theorem 4.1. *Suppose that $\lambda > 0$ and $\Upsilon_{\aleph}(I - \lambda\Pi)$ is a proximal Górnicki mapping on $\aleph$. Then, there exists a unique solution μ^* for the variational inequality problem (4.1).*

Proof. Consider the operator $\phi : \aleph \to \aleph$ defined by $\phi\xi = \Upsilon_{\aleph}(\xi - \lambda\Pi\xi)$ for all $\xi \in \aleph$. By Lemma 4.2, there exists $\mu^* \in \aleph$ as a solution of $\langle \Pi\mu^*\ ,\ \nu - \mu^* \rangle \geq 0$ for all $\nu \in \aleph$ if and only if $\mu^* = \phi\mu^*$. In Theorem 2.1 let $\Delta = \Gamma = \aleph$; then, ϕ satisfies the hypothesis of Theorem 2.1. Therefore, there exists a unique fixed point μ^* for ϕ.

5. Conclusions

Through this work the concept of the best proximity points has been are introduced for proximal Górnicki mappings which include fixed points for Górnicki mappings, enriched contractions, enriched Kannan mappings, and enriched Chatterjea mappings. Also, proximal Górnicki mappings generalize proximal enriched contractions, proximal enriched Kannan mappings, and proximal enriched Chatterjea mappings.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that there are no competing interests.

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