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Theory article

The distribution of ideals whose norm divides n in the Gaussian ring

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Abstract: Let $O_K = \mathbb{Z}[i]$. For each positive integer n, denote $\xi_K(n)$ as the number of integral ideals whose norm divides n in O_K. In this paper, we studied the distribution of ideals whose norm divides n in O_K by using the Selberg-Delange method. This is a natural variant of a result studied by Deshouillers, Dress, and Tenenbaum (often called the DDT Theorem), and we found that the distribution function was subject to beta distribution with density $\sqrt{3}/(2\pi\sqrt[3]{u^2(1-u)})$.

Keywords: Selberg-Delange method; the distribution; Gaussian ring; beta distribution
Mathematics Subject Classification: 11M06, 11N99, 11R04

1. Introduction

For each positive integer n, denote by $\tau(n)$ the number of divisors of n and let $\Omega_n = \{d_1, d_2, \cdots, d_{\tau(n)}\}$ be the set of divisors of n. Let $\mathfrak{S}_n$ be the set of all subsets of Ω_n and let μ_n be the uniform probability measure on Ω_n:

$$\mu_n(d) = \frac{1}{\tau(n)}, \quad d \in \Omega_n.$$

It is easily verified that $(\Omega_n, \mathfrak{S}_n, \mu_n)$ is a probability space. Consider the random variable D_n:

$$D_n : \Omega_n \to \mathbb{R}$$
$$d \mapsto \frac{\log d}{\log n}.$$

The distribution function F_n of D_n is given by

$$F_n(t) = P(D_n \le t) = \frac{1}{\tau(n)} \sum_{d|n,\, d\le n^t} 1 \quad (0 \le t \le 1).$$

It is clear that the sequence $\{F_n\}_{n=1}^{\infty}$ does not converge pointwise on $[0, 1]$ since

$$F_p(t) = \begin{cases} 1/2, & 0 \le t < 1; \\ 1, & t = 1, \end{cases} \qquad F_{p^2}(t) = \begin{cases} 1/3, & 0 \le t < 1/2; \\ 2/3, & 1/2 \le t < 1; \\ 1, & t = 1. \end{cases}$$

However, Deshouillers, Dress, and Tenenbaum [3] proved that its Cesàro means is uniformly convergent on $[0, 1]$. No less remarkable, this limit is the distribution function of a probability law well known to specialists: the arcsine law, with density $1/\left(\pi\sqrt{u(1-u)}\right)$. More precisely,

$$\frac{1}{x}\sum_{n\le x} F_n(t) = \frac{2}{\pi}\arcsin\sqrt{t} + O\left(\frac{1}{\sqrt{\log x}}\right) \tag{1.1}$$

holds uniformly for $x \ge 2$ and $0 \le t \le 1$, and the error term in (1.1) is optimal.

Subsequently, Cui and Wu [1], Feng [6], and Feng and Wu [4] studied the related issues of the Deshouillers-Dress-Tenenbaum (DDT) theorem. Recently, Leung [9] proved that factorization of integers into k parts follows the Dirichlet distribution $\mathrm{Dir}(\frac{1}{k}, \cdots, \frac{1}{k})$ by multidimensional contour integration, thereby generalizing the DDT arcsine law on divisors where $k = 2$. Their results were obtained in $\mathbb{Z}$.

In this paper, we consider a similar problem in the Gaussian ring, unless otherwise stated, and throughout this paper K, O_K, s, and $\sigma_0(\tau)$ will be the Gaussian field, the Gaussian ring(of the form $a + bi$, where $a, b \in \mathbb{Z}$ and $i^2 = -1$), $\sigma + i\tau$, and $c_0/\log(q(|\tau| + 1))$. For each positive integer n, let $\Xi_n = \{\mathfrak{a} \in O_K : N(\mathfrak{a}) \text{ divides } n\}$. Denoting by $\xi_K(n)$ the number of ideals in Ξ_n, then

$$\xi_K(n) = \sum_{N(\mathfrak{a})|n} 1 = \sum_{d|n} a_K(d), \tag{1.2}$$

where $a_K(n)$ is the number of integral ideals with norm n in O_K. Since $a_K(n)$ is multiplicative, so is $\xi_K(n)$.

Let $\mathfrak{S}_n$ be the set of all subsets of Ξ_n and let μ_n be the uniform probability:

$$\mu_n(\mathfrak{a}) = \frac{1}{\xi_K(n)}, \quad \mathfrak{a} \in \Xi_n.$$

It is easily verified that $(\Xi_n, \mathfrak{S}_n, \mu_n)$ is a probability space. Consider the random variable $\mathfrak{D}_n$:

$$\mathfrak{D}_n : \Xi_n \to \mathbb{R}$$
$$\mathfrak{a} \mapsto \frac{\log N(\mathfrak{a})}{\log n}.$$

The distribution function of $\mathfrak{D}_n$ is given by

$$F_{K,n}(t) = P\{\mathfrak{D}_n(\mathfrak{a}) \le t\} = \sum_{N(\mathfrak{a})|n,\ \frac{\log N(\mathfrak{a})}{\log n} \le t} \frac{1}{\xi_K(n)} = \frac{1}{\xi_K(n)} \sum_{N(\mathfrak{a})|n,\ N(\mathfrak{a}) \le n^t} 1.$$

It is clear that the sequence $\{F_{K,n}\}_{n=1}^{\infty}$ does not converge pointwise on $[0, +\infty)$, since

(1) If $p \equiv 1(\mathrm{mod}4)$, then $\Xi_p = \{\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2 \in \mathbb{Z}[i] : N(\mathfrak{a}_0) = 1, N(\mathfrak{a}_1) = N(\mathfrak{a}_2) = p\}$, and so we have

$$F_{K,p}(t) = \begin{cases} 1/3, & 0 \le t < 1; \\ 1, & t \ge 1. \end{cases}$$

(2) If $p \equiv 3(\mathrm{mod}4)$, then $\Xi_p = \{\mathfrak{a}_0 \in \mathbb{Z}[i] : N(\mathfrak{a}_0) = 1\}$, and so we have $F_{K,p}(t) = 1$.
(3) If $p = 2$, $\Xi_p = \{\mathfrak{a}_0, (1 + i) : N(\mathfrak{a}_0) = 1, N(1 + i) = 2\}$, then

$$F_{K,p}(t) = \begin{cases} 1/2, & p = 2 \ \ and \ \ 0 \le t < 1; \\ 1, & p = 2 \ \ and \ \ t \ge 1. \end{cases}$$

However, we shall see that

$$G_N(t) := \frac{1}{N} \sum_{n \le N} F_{K,n}(t)$$

is uniformly convergent on $[0, 1]$, and we get the following result.

Theorem 1.1. *Uniformly for $x \ge 2$ and $0 \le t \le 1$,*

$$\frac{1}{x} \sum_{n \le x} F_{K,n}(t) = B(1/3, 2/3)^{-1} \int_0^t u^{-\frac{2}{3}}(1 - u)^{-\frac{1}{3}} du + O\left(\frac{1}{\sqrt[3]{\log x}} \right)$$

holds, where

$$B(a, b) := \int_0^1 \omega^{a-1}(1 - \omega)^{b-1} d\omega, \quad a, b > 0 \tag{1.3}$$

is beta function. So, as $x \to +\infty$, $x^{-1} \sum_{n \le x} F_{K,n}(t)$ is subject to beta distribution with density $\sqrt{3}/\left(2\pi \sqrt[3]{u^2(1 - u)}\right)$, since $B(1/3, 2/3) = \Gamma(1/3)\Gamma(2/3)/\Gamma(1) = 2\pi/\sqrt{3}$.

This distribution is not arcsince law. Feng and Wu [5] also gave a special case that satisfies the beta distribution.

2. Preliminaries

In order to study $x^{-1} \sum_{n \le x} F_{K,n}(t)$, we need to consider $\sum_{n \le x} 1/\xi_K(nN(\mathfrak{a}))$. Let's start with the properties of $\xi_K(n)$. Dedekind defined the Dedekind zeta function of K as follows:

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \sum_{n=1}^{\infty} \frac{a_K(n)}{n^s}, \tag{2.1}$$

where $\mathfrak{a}$ runs over all nonzero integral ideals in O_K. According to [7, Theorem 2.8], we have $\zeta_K(s) = \zeta(s)L(s, \chi)$, where χ is the primitive character modulo 4. Easily, we get $a_K(n) = \sum_{d|n} \chi(d)$, so formula (1.2) can be converted to

$$\xi_K(n) = \sum_{d|n} \sum_{q|d} \chi(q).$$

When $n = p^m$, p is prime. We have

$$\xi_K(p^m) = \sum_{d|p^m} a_K(d) = a_K(1) + a_K(p) + \cdots + a_K(p^m) = \begin{cases} \frac{(m+1)(m+2)}{2}, & p \equiv 1(\text{mod}4); \\ m+1, & p = 2; \\ \frac{m+2}{2}, & p \equiv 3(\text{mod}4) \text{ and m is even}; \\ \frac{m+1}{2}, & p \equiv 3(\text{mod}4) \text{ and m is odd}. \end{cases}$$

Second, we will get the mean value of $1/\xi_K(nN(\mathfrak{a}))$ based on the Selberg-Delange method. The method was developed by Selberg [10] and Delange [2, 3]. For more details, the reader is referred to the book of Tenenbaum [11].

It's necessary to use Hankel contour when applying the method. For each value of the positive parameter r, we designate the Hankel contour as the path consisting of the circle $|s| = r$ excluding the point $s = -r$ and of the half-line $(-\infty, -r]$ covered twice, with respective arguments $+\pi$ and $-\pi$. The brief introduction of Hankel's formula follows.

Lemma 2.1 (Hankel's formula). *For each $X > 1$, let $H(X)$ denote the part of the Hankel contour situated in the half-plane $\sigma > -X$, then we have, uniformly for $z \in \mathbb{C}$,*

$$\frac{1}{2\pi i}\int_{H(X)} s^{-z}e^s ds = \frac{1}{\Gamma(Z)} + O\left(47^{|z|}\Gamma(1+|z|)e^{-\frac{1}{2}X}\right).$$

Proof. For a detailed description of this lemma, see [11, p.179, Theorem 0.17, Corollary 0.18]. □

The proof of Theorem 1.1 depends on the following two lemmas.

Lemma 2.2. *For any integral ideal $\mathfrak{a} \in O_K$,*

$$\sum_{n\le x}\frac{1}{\xi_K(nN(\mathfrak{a}))} = \frac{hx}{\sqrt[3]{\pi}\log x}\left\{\frac{g(N(\mathfrak{a}))}{\Gamma(2/3)} + O\left(\frac{C\left(\frac{3}{4}\right)^{\omega(N(\mathfrak{a}))}}{\log x}\right)\right\}$$

holds uniformly for $x \ge 2$, where

$$h = 2\log 2\prod_{p\equiv 1(\text{ mod } 4)}\left(1-\frac{1}{p}\right)^{\frac{1}{3}} 2p\left[(p-1)\log\left(1-\frac{1}{p}\right)+1\right]\prod_{p\equiv 3(\text{ mod } 4)} p^2\log\left(1-\frac{1}{p^2}\right)^{-1}\left(1-\frac{1}{p^2}\right)^{\frac{2}{3}},$$

$$g(n) = \prod_{p^{\upsilon}\|n}\sum_{j=0}^{+\infty}\frac{p^{-j}}{\xi_K(p^{j+\upsilon})}\left[\sum_{j=0}^{+\infty}\frac{p^{-j}}{\xi_K(p^j)}\right]^{-1}.$$

Proof. In order to get the mean value of $1/\xi_K(nN(\mathfrak{a}))$, we first consider its Dirichlet series $\sum_{n=1}^{+\infty}\xi_K(nN(\mathfrak{a}))^{-1}n^{-s}$. Let $\upsilon_p(n)$ denote the p-adic valuation of n. By using the formula

$$\xi_K(nN(\mathfrak{a})) = \prod_p \xi_K(p^{\upsilon_p(n)+\upsilon_p(N(\mathfrak{a}))}),$$

we write for $\Re s > 1$:

$$\mathcal{F}_{\mathfrak{a}}(s) = \sum_{n=1}^{+\infty}\frac{1}{\xi_K(nN(\mathfrak{a}))n^s} = \prod_p\sum_{j=0}^{+\infty}\frac{p^{-js}}{\xi_K(p^{j+\upsilon_p(N(\mathfrak{a}))})}$$

$$= \prod_{p\nmid N(\mathfrak{a})} \sum_{j=0}^{+\infty} \frac{p^{-js}}{\xi_K(p^j)} \times \prod_{p|N(\mathfrak{a})} \sum_{j=0}^{+\infty} \frac{p^{-js}}{\xi_K(p^{j+\upsilon_p(N(\mathfrak{a}))})}$$
$$= \prod_{p} \sum_{j=0}^{+\infty} \frac{p^{-js}}{\xi_K(p^j)} \times \prod_{p^{\upsilon}\|N(\mathfrak{a})} \sum_{j=0}^{+\infty} \frac{p^{-js}}{\xi_K(p^{j+\upsilon})} \left[\sum_{j=0}^{+\infty} \frac{p^{-js}}{\xi_K(p^j)}\right]^{-1}$$
$$= L(s,\chi_0)^{\frac{2}{3}} L(s,\chi)^{-\frac{1}{3}} \mathcal{G}_{\mathfrak{a}}\left(s; \frac{2}{3}, -\frac{1}{3}\right),$$

where χ_0 is the principal character mod 4, χ is the primitive character mod 4, and

$$\mathcal{G}_{\mathfrak{a}}\left(s; \frac{2}{3}, -\frac{1}{3}\right) = 2^s \log\left(1 - \frac{1}{2^s}\right)^{-1} \prod_{p\equiv 1(\bmod 4)} \sum_{j=0}^{+\infty} \frac{2p^{-js}}{(j+1)(j+2)} \left(1 - \frac{1}{p^s}\right)^{\frac{1}{3}}$$
$$\times \prod_{p\equiv 3(\bmod 4)} \sum_{j=0}^{+\infty} \frac{p^{-2js}}{j+1} \left(1 - \frac{1}{p^{2s}}\right)^{\frac{2}{3}} \prod_{p^{\upsilon}\|N(\mathfrak{a})} \sum_{j=0}^{+\infty} \frac{p^{-js}}{\xi_K(p^{j+\upsilon})} \left[\sum_{j=0}^{+\infty} \frac{p^{-js}}{\xi_K(p^j)}\right]^{-1}$$

converges absolutely for $\Re s > 1/2$.

Let $\mathcal{G}_{\mathfrak{a}}\left(s; \frac{2}{3}, -\frac{1}{3}\right) = \mathcal{G}_1\left(s; \frac{2}{3}, -\frac{1}{3}\right) \mathcal{G}_2\left(s; \frac{2}{3}, -\frac{1}{3}\right) \mathcal{G}_3\left(s; \frac{2}{3}, -\frac{1}{3}\right) \mathcal{G}_4\left(s; \frac{2}{3}, -\frac{1}{3}\right) \mathcal{G}_5\left(s; \frac{2}{3}, -\frac{1}{3}\right)$, where

$$\mathcal{G}_1\left(s; \frac{2}{3}, -\frac{1}{3}\right) = \sum_{j\geq 0} \frac{1}{(j+1)2^{js}} \prod_{p\equiv 1(\bmod 4)} \left[1 + \sum_{\upsilon\geq 1} \frac{2}{(\upsilon+1)(\upsilon+2)p^{\upsilon s}}\right] \left(1 - \frac{1}{p^s}\right)^{\frac{1}{3}},$$

$$\mathcal{G}_2\left(s; \frac{2}{3}, -\frac{1}{3}\right) = \prod_{p\equiv 3(\bmod 4)} \left(1 - \frac{1}{p^{2s}}\right)^{\frac{2}{3}} \left[1 + \sum_{j\geq 1} \frac{1}{(j+1)p^{2js}}\right],$$

$$\mathcal{G}_3\left(s; \frac{2}{3}, -\frac{1}{3}\right) = \prod_{p^{\upsilon}\|N(\mathfrak{a}), p\equiv 1(\bmod 4)} \sum_{j\geq 0} \frac{2p^{-js}}{(j+\upsilon+1)(j+\upsilon+2)} \left[1 + \sum_{j\geq 1} \frac{2p^{-js}}{(j+1)(j+2)}\right]^{-1},$$

$$\mathcal{G}_4\left(s; \frac{2}{3}, -\frac{1}{3}\right) = \prod_{p^{2\nu}\|N(\mathfrak{a}), p\equiv 3(\bmod 4)} \left[\sum_{j\geq 0} \frac{1}{(\nu+j+1)p^{2js}}\right] \left[1 + \sum_{j\geq 1} \frac{1}{(j+1)p^{2js}}\right]^{-1},$$

$$\mathcal{G}_5\left(s; \frac{2}{3}, -\frac{1}{3}\right) = \left[\sum_{j\geq 0} \frac{2^{-js}}{j+t+1}\right] \left[\sum_{j\geq 0} \frac{2^{-js}}{j+1}\right]^{-1},$$

where $\tau(N(\mathfrak{a})) = (t+1) \prod_{p^{\upsilon}\|N(\mathfrak{a}), p\equiv 1(\bmod 4)} (\upsilon+1) \prod_{p^{2\nu}\|N(\mathfrak{a}), p\equiv 3(\bmod 4)} (2\nu+1)$.

When $\Re s = \sigma > 1/2 + \varepsilon$,

$$\mathcal{G}_{\mathfrak{a}}\left(s; \frac{2}{3}, -\frac{1}{3}\right) = \mathcal{G}_1\left(s; \frac{2}{3}, -\frac{1}{3}\right) \mathcal{G}_2\left(s; \frac{2}{3}, -\frac{1}{3}\right) \mathcal{G}_3\left(s; \frac{2}{3}, -\frac{1}{3}\right) \mathcal{G}_4\left(s; \frac{2}{3}, -\frac{1}{3}\right) \mathcal{G}_5\left(s; \frac{2}{3}, -\frac{1}{3}\right)$$
$$\ll \frac{1}{t+1} \prod_{p^{\upsilon}\|N(\mathfrak{a}), p\equiv 1(\bmod 4)} \frac{1}{(\upsilon+1)(\upsilon+2)} \prod_{p^{2\nu}\|N(\mathfrak{a}), p\equiv 3(\bmod 4)} \frac{1}{\nu+1}$$
$$\leq C\left(\frac{3}{4}\right)^{\omega(N(\mathfrak{a}))}.$$

To deal with the estimation of $\mathcal{F}_{\mathfrak{a}}(s)$ near 1, we introduce the function $Z(s; z_1)$. The function

$$Z(s; z_1) = \{(s-1)L(s, \chi_0)\}^{z_1}/s \tag{2.2}$$

is holomorphic in the $|s-1| < 1$, and admits the Taylor series expansion

$$Z(s; z_1) = \sum_{j=0}^{\infty} \frac{\gamma_j(z_1)}{j!}(s-1)^j$$

where $\gamma_j(z_1)$ is an entire function, for all $\varepsilon > 0$,

$$\frac{\gamma_j(z_1)}{j!} \ll_\varepsilon (1+\varepsilon)^j \quad (j \geq 0).$$

Now, let $z_1 = 2/3$. The function $Z(s; 2/3)\mathcal{G}_{\mathfrak{a}}(s; 2/3, -1/3)L(s, \chi)^{-1/3}$ is holomorphic in the disc $|s-1| < (1-\hat{\beta})/2$, where $\hat{\beta} = \beta_0$ when $L(s, \chi)$ has a real zero β_0, $\hat{\beta} = 1 - \sigma_0(\tau)$ when $L(s, \chi)$ has no real zero β_0, and

$$Z(s; 2/3)\mathcal{G}_{\mathfrak{a}}(s; 2/3, -1/3)L(s, \chi)^{-1/3} \ll_\varepsilon M$$

for $|s-1| < (1-\hat{\beta})/2$. Thus, for $|s-1| < (1-\hat{\beta})/2$, we can write

$$Z(s; 2/3)\mathcal{G}_{\mathfrak{a}}(s; 2/3, -1/3)L(s, \chi)^{-1/3} = \sum_{l=0}^{\infty} g_l(2/3)(s-1)^l,$$

where

$$g_l(2/3) := \frac{1}{l!}\sum_{j=0}^{l}\binom{l}{j}\frac{\partial^{l-j}(\mathcal{G}_{\mathfrak{a}}(s; 2/3, -1/3)L(s, \chi)^{-1/3})}{\partial s^{l-j}}\Big|_{s=1}\gamma_j(2/3). \tag{2.3}$$

We can apply Perron's formula with the choice of parameters $\sigma_a = 1, A(n) = n^\varepsilon, \alpha = 0$ to write

$$\sum_{n \leq x} \frac{1}{\xi_K(nN(\mathfrak{a}))} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} \mathcal{F}_{\mathfrak{a}}(s)\frac{x^s}{s}ds + O\left(\frac{x^{1+\varepsilon}}{T}\right),$$

where $b = 1 + 2/\log x$ and $100 \leq T \leq x$, such that $L(\sigma + iT, \chi) \neq 0$ for $0 < \sigma < 1$.

Let $\mathcal{L}_T$ be the boundary of the modified rectangle with vertices $1/2 + \varepsilon \pm iT$ and $b \pm iT$, where

- $\varepsilon > 0$ is a small constant chosen such that $L(1/2 + \varepsilon + i\gamma, \chi) \neq 0$ for $|\gamma| < T$. Let l_1 be the horizontal line segment with the imaginary part T and the real part $1/2 + \varepsilon$ to b, and let l_2 be the horizontal line segment with the imaginary part $-T$ and the real part b to $1/2 + \varepsilon$. Let l_3 be the vertical line segment with the real part $1/2 + \varepsilon$ and the imaginary part 0^+ to T, and let l_4 be the vertical line segment with the real part $1/2 + \varepsilon$ and the imaginary part $-T$ to 0^-.
- The zeros of $L(s, \chi)$ of the form $\rho = \beta + i\gamma$ with $\beta > 1/2 + \varepsilon$ and $|\gamma| < T$ are avoided by Γ_ρ that horizontal cut drawn from the critical line inside this rectangle to $\rho = \beta + i\gamma$.
- $L(s, \chi)$ has a possible Siegel zero. The possible Siegel zero β_0 of $L(s, \chi)$ is avoided by contour Γ_0 (its upper part is made up of an arc surrounding the point $s = \beta_0$ with radius $r = 1/\log x$ and a line segment joining $\beta_0 - r$ to $1/2 + \varepsilon$).
- The pole of $L(s, \chi_0)$ at the points $s = 1$ is avoided by the truncated Hankel contour Γ (its upper part is

made up of an arc surrounding the point $s = 1$ with radius $r = 1/\log x$ and a line segment joining $1 - r$ to $\tilde{\beta}$), where

$$\tilde{\beta} = \begin{cases} \beta_0 + \frac{1}{\log x}, & L(\beta_0, \chi) = 0; \\ \frac{1}{2} + \varepsilon, & L(\beta_0, \chi) \neq 0. \end{cases}$$

Clearly the function $\mathcal{F}_{\mathfrak{a}}(s)$ is analytic inside $\mathcal{L}_T$. By the Cauchy residue theorem, we can write

$$\sum_{n \le x} \frac{1}{\xi_K(nN(\mathfrak{a}))} = I + I_0 + I_1 + I_2 + I_3 + I_4 + \sum_{\beta > 1/2+\varepsilon, |\gamma| < T} I_\rho + O\left(\frac{x^{1+\varepsilon}}{T}\right) \tag{2.4}$$

where

$$I := \frac{1}{2\pi i} \int_\Gamma \mathcal{F}_{\mathfrak{a}}(s) \frac{x^s}{s} ds, \quad I_\rho := \frac{1}{2\pi i} \int_{\Gamma_\rho} \mathcal{F}_{\mathfrak{a}}(s) \frac{x^s}{s} ds,$$

and

$$I_j := \frac{1}{2\pi i} \int_{l_j} \mathcal{F}_{\mathfrak{a}}(s) \frac{x^s}{s} ds, \quad I_0 := \frac{1}{2\pi i} \int_{\Gamma_0} \mathcal{F}_{\mathfrak{a}}(s) \frac{x^s}{s} ds.$$

A. Evaluation of I.

Let $0 < c < (1-\hat{\beta})/10$ be a small constant. Since $Z(s; 2/3)\mathcal{G}_{\mathfrak{a}}(s; 2/3, -1/3)L(s, \chi)^{-\frac{1}{3}}$ is holomorphic and $O(M)$ in the disc $|s - 1| \le c$, the Cauchy formula implies that

$$g_l(2/3) \ll Mc^{-l} \quad (l \ge 0),$$

where $g_l(2/3)$ is defined as in (2.3). From this and (2.3), it is easy to deduce that for $|s - 1| \le c/2$,

$$\begin{aligned} Z(s; 2/3)\mathcal{G}_{\mathfrak{a}}(s; 2/3, -1/3)L(s, \chi)^{-1/3} &= Z(1; 2/3)\mathcal{G}_{\mathfrak{a}}(1; 2/3, -1/3)L(1, \chi)^{-1/3} + O(|s - 1|) \\ &= \frac{h}{\sqrt[3]{\pi}} g\left(N(\mathfrak{a})\right) + O(|s - 1|), \end{aligned}$$

where

$$g(N(\mathfrak{a})) = \prod_{p^\upsilon \| N(\mathfrak{a})} \sum_{j=0}^{+\infty} \frac{p^{-j}}{\xi_K(p^{j+\upsilon})} \left[\sum_{j=0}^{+\infty} \frac{p^{-j}}{\xi_K(p^j)}\right]^{-1}.$$

So, we have

$$\begin{aligned} I &= \frac{1}{2\pi i} \int_\Gamma \mathcal{F}_{\mathfrak{a}}(s) \frac{x^s}{s} ds \\ &= \frac{1}{2\pi i} \int_\Gamma Z(s; 2/3)\mathcal{G}_{\mathfrak{a}}(s; 2/3, -1/3)L(s, \chi)^{-1/3}(s - 1)^{-\frac{2}{3}} x^s ds \\ &= \frac{h}{\sqrt[3]{\pi}} g\left(N(\mathfrak{a})\right) \frac{1}{2\pi i} \int_\Gamma (s - 1)^{-\frac{2}{3}} x^s ds + O\left(\left|\int_\Gamma (s - 1)^{\frac{1}{3}} x^s ds\right|\right). \end{aligned}$$

Let $s - 1 = \omega/\log x$. According to Lemma 2.1, we have

$$\frac{1}{2\pi i} \int_\Gamma (s - 1)^{-\frac{2}{3}} x^s ds = \frac{x}{\sqrt[3]{\log x}} \frac{1}{2\pi i} \int_{H((1-\tilde{\beta})\log x)} \omega^{-\frac{2}{3}} e^\omega d\omega = \frac{x}{\sqrt[3]{\log x}} \left\{\frac{1}{\Gamma\left(\frac{2}{3}\right)} + O\left(\frac{1}{x^{\frac{1-\tilde{\beta}}{2}}}\right)\right\}. \tag{2.5}$$

On the other hand,

$$\left|\int_{\Gamma}(s-1)^{\frac{1}{3}}x^s ds\right| \ll \int_{|s-1|=\frac{1}{\log x}} |(s-1)^{\frac{1}{3}}x^s||ds| + \int_{\tilde{\beta}}^{1-\frac{1}{\log x}} (1-\sigma)^{\frac{1}{3}}x^{\sigma}d\sigma$$
$$\ll \frac{x}{\sqrt[3]{\log x}}\cdot\frac{1}{\log x} + \frac{x}{\sqrt[3]{\log x}}\cdot\frac{1}{\log x}\int_{\log x(1-\tilde{\beta})}^{1} t^{\frac{1}{3}}e^{-t}dt \tag{2.6}$$
$$\ll \frac{x}{\sqrt[3]{\log x}}\cdot\frac{1}{\log x}.$$

According to (2.5) and (2.6), we have

$$I = \frac{hx}{\sqrt[3]{\pi\log x}}\left\{\frac{g\left(N(\mathfrak{a})\right)}{\Gamma\left(2/3\right)} + O\left(\frac{1}{\log x}\right)\right\}. \tag{2.7}$$

B. Evaluation of I_1 and I_2.

It is well known that

$$|\zeta(\sigma+i\tau)| \ll (|\tau|+1)^{(1-\sigma)/3}\log(|\tau|+1) \quad (1/2 \le \sigma \le 1+\log^{-1}|\tau|, |\tau| \ge 3). \tag{2.8}$$

From (2.8) and [12, Lemma 2.1], we deduce that

$$L(s,\chi_0) = \zeta(s)\left(1-\frac{1}{2^s}\right) \ll (|\tau|+1)^{(1-\sigma)/3}\log(|\tau|+1) \tag{2.9}$$

for $1/2 \le \sigma \le 1+\log^{-1}|\tau|$ and $|\tau| \ge 3$, and

$$L(s,\chi)^{-1} = L(2s,\chi_0)^{-1}\prod_{p\equiv 1(\bmod 4)}\left(1+\frac{1}{p^s}\right)^{-1}\prod_{p\equiv 3(\bmod 4)}\left(1-\frac{1}{p^s}\right)^{-1}$$
$$\ll (\log|\tau|)^{2/3}(\log_2|\tau|)^{1/3} \tag{2.10}$$

for $\sigma > 1/2$. In view of (2.9) and (2.10), we have

$$|I_1|+|I_2| \ll \int_{1/2+\varepsilon}^{1+2/\log x} |L(\sigma\pm iT,\chi_0)|^{\frac{2}{3}}|L(\sigma\pm iT,\chi)|^{-\frac{1}{3}}|\mathcal{G}_{\mathfrak{a}}(s,2/3,-1/3)|\frac{x^{\sigma}}{|s|}d\sigma$$
$$\ll \int_{1/2+\varepsilon}^{1+2/\log x} T^{\frac{2}{9}(1-\sigma)}(\log T)^{\frac{2}{3}}(\log T)^{\frac{2}{9}}(\log_2 T)^{\frac{1}{9}}\frac{x^{\sigma}}{T}d\sigma \tag{2.11}$$
$$\ll \frac{x}{T}\log T\int_{1/2+\varepsilon}^{1+2/\log x}\left(\frac{T^{\frac{2}{9}}}{x}\right)^{1-\sigma}d\sigma \ll \frac{x}{T}\log T.$$

C. Evaluation of I_3 and I_4.

Let $\sigma_0 = 1/2+\varepsilon$, $\tau_0 = |\tau|+3$, for $s = \sigma_0+i\tau$ with $0 \le |\tau| \le T$, In view of (2.9) and (2.10), we have

$$|I_3|+|I_4| \ll \int_0^T |L(\sigma_0+i\tau,\chi_0)|^{\frac{2}{3}}|L(\sigma_0+i\tau,\chi)|^{-\frac{1}{3}}|\mathcal{G}_{\mathfrak{a}}(\sigma_0+i\tau,2/3,-1/3)|\frac{x^{\sigma_0}}{\tau+1}d\tau$$
$$\ll \int_0^T \tau_0^{\frac{2}{9}(1-\sigma_0)}(\log\tau_0)^{\frac{2}{3}}(\log\tau_0)^{\frac{2}{9}}(\log_2\tau_0)^{\frac{1}{9}}\frac{x^{\sigma_0}}{\tau+1}d\tau \tag{2.12}$$
$$\ll x^{\frac{1}{2}+\varepsilon}T^{\frac{1}{9}}.$$

D. Evaluation of I_ρ.

For $s=\sigma+i\gamma$ with $1/2+\varepsilon\le\sigma\le\beta\le 1-\sigma_0(\gamma)$, we have

$$\mathcal{F}_{\mathfrak{a}}(s)\ll|\gamma|^{\frac{2}{9}(1-\sigma)}\log|\gamma|,$$

then we deduce that

$$I_\rho\ll\int_{1/2+\varepsilon}^{\beta}|\gamma|^{\frac{2}{9}(1-\sigma)}\log|\gamma|\frac{x^\sigma}{|\gamma|}d\sigma.$$

Denote by $N(\sigma,T)$ the number of $L(s,\chi_0)$ in the region $\Re s\ge\sigma$ and $|\Im s|\le T$. We have

$$\sum_{\beta>1/2+\varepsilon,|\gamma|<T}|I_\rho|\ll\log T\max_{T_0\le T}\sum_{\beta>1/2+\varepsilon,T_0/2<|\gamma|<T_0}|I_\rho|\ll\log T\max_{T_0\le T}\int_{1/2+\varepsilon}^{1-\sigma_0(T_0)}T_0^{\frac{2}{9}(1-\sigma)}\log T_0\frac{x^\sigma}{T_0}N(\sigma,T_0)d\sigma.$$

According to Huxley [8],

$$N(\sigma,T)\ll T^{\frac{12}{5}(1-\sigma)}(\log T)^9$$

for $1/2+\varepsilon\le\sigma\le 1$, and $T\ge 2$. Thus,

$$\begin{aligned}\sum_{\beta>1/2+\varepsilon,|\gamma|<T}|I_\rho|&\ll\log T\max_{T_0\le T}\int_{1/2+\varepsilon}^{1-\sigma_0(T_0)}T_0^{\frac{2}{9}(1-\sigma)}\log T_0\frac{x^\sigma}{T_0}T_0^{\frac{12}{5}(1-\sigma)}(\log T_0)^9d\sigma\\&\ll\log T\max_{T_0\le T}(\log T_0)^{10}\int_{1/2+\varepsilon}^{1-\sigma_0(T_0)}T_0^{\frac{2}{9}(1-\sigma)}\frac{x\cdot x^{\sigma-1}}{T_0^{2(1-\sigma)}}T_0^{\frac{12}{5}(1-\sigma)}d\sigma\\&\ll x\log T\max_{T_0\le T}(\log T_0)^{10}\int_{1/2+\varepsilon}^{1-\sigma_0(T_0)}\left(\frac{T_0^{28/45}}{x}\right)^{1-\sigma}d\sigma\\&\ll x\log T\left(\frac{T^{28/45}}{x}\right)^{\sigma_0(T)}.\end{aligned}\tag{2.13}$$

E. Evaluation of I_0.

If $L(s,\chi)$ has no Siegel zero, then $I_0=0$. If it has Siegel zero β_0, then $L(s,\chi)=(s-\beta_0)V(s)$, $V(\beta_0)\ne 0$. For $|s-\beta_0|\le 1/\log x$, we can write

$$V(s)^{-1/3}L(s,\chi_0)^{2/3}\mathcal{G}_{\mathfrak{a}}(s;2/3,-1/3)/s=C(\beta_0)+O(|s-\beta_0|),$$

where $C(\beta_0)$ is a constant depending on β_0, then

$$|I_0|=\frac{C(\beta_0)}{2\pi i}\int_{\Gamma_0}(s-\beta_0)^{-1/3}x^sds+O\left(\left|\int_{\Gamma_0}(s-\beta_0)^{2/3}x^sds\right|\right)\ll x^{\beta_0}(\log x)^{1/3}.\tag{2.14}$$

Taking $T=e^{\sqrt{\log x}}$ and inserting (2.11)–(2.14) and (2.7) into (2.4), we have

$$\sum_{n\le x}\frac{1}{\xi_K(nN(\mathfrak{a}))}=\frac{hx}{\sqrt[3]{\pi\log x}}\left\{\frac{g(N(\mathfrak{a}))}{\Gamma(2/3)}+O\left(\frac{C\left(\frac{3}{4}\right)^{\omega(N(\mathfrak{a}))}}{\log x}\right)\right\}.$$

Lemma 2.3. *For any $n \in \mathbb{Z}^+$, we have that*

$$\sum_{n\le x} g(n)a_K(n) = \frac{\sqrt[3]{\pi}x}{h(\log x)^{2/3}}\left\{\frac{1}{\Gamma(1/3)} + O\left(\frac{1}{\log x}\right)\right\}$$

holds uniformly for $x \ge 2$, where $g(n)$ and h are defined in Lemma 2.2.

□

Proof. In order to get the mean value of $g(n)a_K(n)$, we first consider its Dirichlet series $\sum_{n=1}^{+\infty} g(n)a_K(n)n^{-s}$. Since $g(n)$, $a_K(n)$ is multiplicative, the Dirichlet series has Euler expansion. When $\Re s > 1$,

$$\mathcal{F}(s) = \sum_{n=1}^{\infty} \frac{g(n)a_K(n)}{n^s} = L(s,\chi_0)^{1/3}L(s,\chi)^{1/3}\mathcal{P}(s;1/3,1/3),$$

where

$$\begin{aligned}\mathcal{P}(s;1/3,1/3) &= \prod_p \left(1-\frac{\chi_0(p)}{p^s}\right)^{1/3}\left(1-\frac{\chi(p)}{p^s}\right)^{1/3}\sum_{\upsilon\ge 0} g(p^\upsilon)a_K(p^\upsilon)p^{-\upsilon s}\\ &= \sum_{\upsilon\ge 0} 2^{-\upsilon s}\sum_{j=0}^{\infty}\frac{2^{-j}}{j+\upsilon+1}\left[\sum_{j=0}^{\infty}\frac{2^{-j}}{j+1}\right]^{-1}\\ &\times \prod_{p\equiv 1(\bmod 4)}\left(1-\frac{1}{p^s}\right)^{2/3}\sum_{\upsilon\ge 0}\frac{\upsilon+1}{p^{\upsilon s}}\sum_{j\ge 0}\frac{2p^{-j}}{(j+\upsilon+1)(j+\upsilon+2)}\left[\sum_{j\ge 0}\frac{2p^{-j}}{(j+1)(j+2)}\right]^{-1}\\ &\times \prod_{p\equiv 3(\bmod 4)}\left(1-\frac{1}{p^{2s}}\right)^{1/3}\sum_{\upsilon\ge 0} p^{-2\upsilon s}\sum_{j\ge 0}\frac{p^{-2j}}{\upsilon+j+1}\left[\sum_{j\ge 0}\frac{p^{-2j}}{j+1}\right]^{-1}\end{aligned}$$

converges absolutely and is $O(M)$ for $\Re s > 1/2$. Since

$$\sum_{\upsilon\ge 0}\left(1-\frac{1}{p}\right)\sum_{j\ge 0}\frac{p^{(-\upsilon-j)}}{j+\upsilon+1} = 1, \quad \sum_{\upsilon\ge 0}\left(1-\frac{1}{p^2}\right)\sum_{j\ge 0}\frac{p^{-2(j+\upsilon)}}{\upsilon+j+1} = 1,$$

$$\left(1-\frac{1}{p}\right)\sum_{\upsilon\ge 0}\sum_{j\ge 0}(\upsilon+1)\frac{2p^{(-j-\upsilon)}}{(j+\upsilon+1)(j+\upsilon+2)} = 1,$$

then

$$\begin{aligned}\mathcal{P}(1;1/3,1/3) = 2&\left[\sum_{j=0}^{\infty}\frac{2^{-j}}{j+1}\right]^{-1}\prod_{p\equiv 1(\bmod 4)}\left(1-\frac{1}{p}\right)^{-1/3}\left[\sum_{j\ge 0}\frac{2p^{-j}}{(j+1)(j+2)}\right]^{-1}\\ &\times \prod_{p\equiv 3(\bmod 4)}\left(1-\frac{1}{p^2}\right)^{-2/3}\left[\sum_{j\ge 0}\frac{p^{-2j}}{j+1}\right]^{-1} = 2/h.\end{aligned}$$

Applying the Selberg-Delange theorem [11, p.281, Theorem 5.2], we have the formula

$$\sum_{n\le x} g(n)a_K(n) = \frac{\sqrt[3]{\pi}x}{h(\log x)^{2/3}}\left\{\frac{1}{\Gamma(1/3)} + O\left(\frac{1}{\log x}\right)\right\}.$$

□

3. Proof of Theorem 1.1

We only need to consider $0 \le t \le 1$. Now, we have

$$\begin{aligned} S(x,t) &= \frac{1}{x}\sum_{n\le x} F_{K,n}(t) = \frac{1}{x}\sum_{n\le x}\frac{1}{\xi_K(n)}\sum_{N(\mathfrak{a})|n,\, N(\mathfrak{a})\le n^t} 1 \\ &= \frac{1}{x}\sum_{n\le x}\frac{1}{\xi_K(n)}\sum_{N(\mathfrak{a})|n,\, N(\mathfrak{a})\le x^t} 1 - \frac{1}{x}\sum_{n\le x}\frac{1}{\xi_K(n)}\sum_{N(\mathfrak{a})|n,\, n^t< N(\mathfrak{a})\le x^t} 1 \\ &=: S - R. \end{aligned}$$

When $0 \le t \le 1/2$, we will first calculate S. According to Lemma 2.2,

$$\begin{aligned} S &= \frac{1}{x}\sum_{n\le x}\frac{1}{\xi_K(n)}\sum_{N(\mathfrak{a})|n,\, N(\mathfrak{a})\le x^t} 1 = \frac{1}{x}\sum_{N(\mathfrak{a})\le x^t}\sum_{d\le \frac{x}{N(\mathfrak{a})}}\frac{1}{\xi_K(dN(\mathfrak{a}))} \\ &= \frac{1}{x}\sum_{N(\mathfrak{a})\le x^t}\left\{\frac{h\left(\frac{x}{N(\mathfrak{a})}\right)}{\sqrt[3]{\pi\log\left(\frac{x}{N(\mathfrak{a})}\right)}}\left[\frac{g(N(\mathfrak{a}))}{\Gamma(2/3)} + O\left(\frac{C\left(\frac{3}{4}\right)^{\omega(N(\mathfrak{a}))}}{\log\frac{x}{N(\mathfrak{a})}}\right)\right]\right\}. \end{aligned}$$

Since $\log(x/N(\mathfrak{a})) = \log x - \log N(\mathfrak{a}) \ge \log x - \log x^t = (1-t)\log x \ge 1/2\log x$, we have

$$S = \frac{h}{\sqrt[3]{\pi}}\sum_{N(\mathfrak{a})\le x^t}\frac{1}{N(\mathfrak{a})\sqrt[3]{\log\left(\frac{x}{N(\mathfrak{a})}\right)}}\left\{\frac{g(N(\mathfrak{a}))}{\Gamma(2/3)} + O\left(\frac{C\left(\frac{3}{4}\right)^{\omega(N(\mathfrak{a}))}}{\log x}\right)\right\}.$$

Next, we calculate R. According to Lemma 2.2,

$$\begin{aligned} R &= \frac{1}{x}\sum_{n\le x}\frac{1}{\xi_K(n)}\sum_{N(\mathfrak{a})|n,\, n^t<N(\mathfrak{a})\le x^t} 1 = \frac{1}{x}\sum_{N(\mathfrak{a})\le x^t}\sum_{\substack{d\le \frac{x}{N(\mathfrak{a})} \\ (dN(\mathfrak{a}))^t<N(\mathfrak{a})}}\frac{1}{\xi_K(dN(\mathfrak{a}))} \\ &\ll \frac{1}{x}\sum_{N(\mathfrak{a})\le x^t}\sum_{\substack{d\le \frac{x}{N(\mathfrak{a})} \\ d<N(\mathfrak{a})^{\frac{1-t}{t}}}}\frac{1}{\xi_K(d)} = \frac{1}{x}\sum_{N(\mathfrak{a})\le x^t}\sum_{d<N(\mathfrak{a})^{\frac{1-t}{t}}}\frac{1}{\xi_K(d)} \\ &= \frac{1}{x}\sum_{N(\mathfrak{a})\le x^t}\left\{\frac{hN(\mathfrak{a})^{\frac{1-t}{t}}}{\sqrt[3]{\pi\log(N(\mathfrak{a})^{\frac{1-t}{t}})}}\left[\frac{g(1)}{\Gamma(2/3)} + O\left(\frac{1}{\log(N(\mathfrak{a})^{\frac{1-t}{t}})}\right)\right]\right\}. \end{aligned}$$

When $0 \le t \le 1/2$, $(1-t)/t \ge 1$, and since $N(\mathfrak{a}) \ge 2$, $u = \log\left(N(\mathfrak{a})^{\frac{1-t}{t}}\right) \ge \log N(\mathfrak{a}) \ge \log 2 \approx 0.693$. Let $y = (\pi-1)u-1$. Since $(\pi-1)\log\left(N(\mathfrak{a})^{\frac{1-t}{t}}\right)-1 \ge (\pi-1)\log 2-1 > 0$, $\pi\log\left(N(\mathfrak{a})^{\frac{1-t}{t}}\right) \ge \log\left(N(\mathfrak{a})^{\frac{1-t}{t}}\right)+1$, R has the following estimates,

$$R \ll \frac{1}{x}\sum_{N(\mathfrak{a})\le x^t}\frac{N(\mathfrak{a})^{\frac{1-t}{t}}}{\sqrt{1+\log(N(\mathfrak{a})^{\frac{1-t}{t}})}} \ll \frac{1}{x}\sum_{N(\mathfrak{a})\le x^t}\frac{(x^t)^{\frac{1-t}{t}}}{\sqrt[3]{1+\log(x^{t\times\frac{1-t}{t}})}}$$

$$= \frac{1}{x} \times x^{1-t} \sum_{N(\mathfrak{a}) \le x^t} \frac{1}{\sqrt[3]{1 + \log x^{1-t}}} = \frac{1}{x} \times x^{1-t} \frac{1}{\sqrt[3]{1 + \log x^{1-t}}} \sum_{N(\mathfrak{a}) \le x^t} 1$$
$$\le \frac{1}{\sqrt[3]{1 + \log x^{1-t}}} \ll \frac{1}{\sqrt[3]{\log x}}.$$

Therefore,

$$S(x,t) = \frac{h}{\sqrt[3]{\pi}} \sum_{N(\mathfrak{a}) \le x^t} \frac{1}{N(\mathfrak{a}) \sqrt[3]{\log \frac{x}{N(\mathfrak{a})}}} \left[\frac{g(N(\mathfrak{a}))}{\Gamma(2/3)} + O\left(\frac{C\left(\frac{3}{4}\right)^{\omega(N(\mathfrak{a}))}}{\log x} \right) \right] + O\left(\frac{1}{\sqrt[3]{\log x}} \right)$$
$$= \frac{h}{\sqrt[3]{\pi}\Gamma(2/3)} \sum_{N(\mathfrak{a}) \le x^t} \frac{g(N(\mathfrak{a}))}{N(\mathfrak{a}) \sqrt[3]{\log \frac{x}{N(\mathfrak{a})}}} + O\left(\left(\frac{1}{\log x} \right)^{4/3} \sum_{N(\mathfrak{a}) \le x^t} \frac{1}{N(\mathfrak{a})} \right) + O\left(\frac{1}{\sqrt[3]{\log x}} \right).$$

Since

$$\left(\frac{3}{4}\right)^{\omega(N(\mathfrak{a}))} = O(1), \quad \sum_{N(\mathfrak{a}) \le x^t} \frac{1}{N(\mathfrak{a})} = \sum_{n \le x^t} \frac{a_K(n)}{n} = \frac{\pi}{4} \log x^t + \frac{\pi}{4} + O\left(\frac{1}{\sqrt{x}} \right),$$
$$O\left(\left(\frac{1}{\log x} \right)^{4/3} \sum_{N(\mathfrak{a}) \le x^t} \frac{1}{N(\mathfrak{a})} \right) = O\left(\left(\frac{1}{\log x} \right)^{4/3} \times \frac{t\pi}{4} \log x \right) = O\left(\frac{1}{\sqrt[3]{\log x}} \right),$$

we have

$$S(x,t) = \frac{h}{\sqrt[3]{\pi}\Gamma(2/3)} \sum_{n \le x^t} \frac{g(n) a_K(n)}{n \sqrt[3]{\log \frac{x}{n}}} + O\left(\frac{1}{\sqrt[3]{\log x}} \right).$$

Let $G(x) = \sum_{n \le x} g(n) a_K(n)$. According to Lemma 2.3 and using the Abelian Summation formula, we have

$$\frac{h}{\sqrt[3]{\pi}\Gamma(2/3)} \sum_{n \le x^t} \frac{g(n) a_K(n)}{n \sqrt[3]{\log \frac{x}{n}}} = \frac{h}{\sqrt[3]{\pi}\Gamma(2/3)} \times \frac{1}{x^t \sqrt[3]{\log(x/x^t)}} \times G(x^t)$$
$$+ \frac{h}{\sqrt[3]{\pi}\Gamma(2/3)} \int_1^{x^t} \frac{G(u)}{u^2 \sqrt[3]{\log x - \log u}} \left(1 - \frac{1}{3(\log x - \log u)} \right) du$$
$$= \frac{1}{\Gamma(2/3)} \int_1^{x^t} \frac{\frac{1}{\Gamma(1/3)} + O\left(\frac{1}{\log(u+1)} \right)}{u \sqrt[3]{(\log u)^2 (\log x - \log u)}} \left(1 - \frac{1}{3(\log x - \log u)} \right) du$$
$$+ O\left(\frac{1}{\sqrt[3]{\log x}} \right)$$
$$= \frac{1}{\Gamma(2/3)} \int_1^{x^t} \frac{\frac{1}{\Gamma(1/3)} + O(\frac{1}{\log(u+1)})}{u \sqrt[3]{(\log u)^2 (\log x - \log u)}} du + O\left(\frac{1}{\sqrt[3]{\log x}} \right)$$
$$= \frac{1}{\Gamma(2/3)\Gamma(1/3)} \int_1^{x^t} \frac{1}{u \sqrt[3]{(\log u)^2 (\log x - \log u)}} du + O\left(\frac{1}{\sqrt[3]{\log x}} \right)$$
$$= \frac{1}{\Gamma(2/3)\Gamma(1/3)} \int_0^t \frac{1}{\sqrt[3]{\upsilon^2 (1 - \upsilon)}} d\upsilon + O\left(\frac{1}{\sqrt[3]{\log x}} \right),$$

so we can obtain

$$
\begin{aligned}
S(x,t) &= \frac{1}{\Gamma(2/3)\Gamma(1/3)} \int_0^t \frac{1}{\sqrt[3]{v^2(1-v)}} dv + O\left(\frac{1}{\sqrt[3]{\log x}}\right) \\
&= \frac{\sqrt{3}}{2\pi} \log \frac{\sqrt[3]{1-t} + \sqrt[3]{t}}{\sqrt{(1-t)^{2/3} + t^{2/3} - ((1-t)t)^{1/3}}} \\
&\quad - \frac{3}{2\pi} \arctan\left(\frac{2\sqrt{3}\sqrt[3]{1-t}}{3\sqrt[3]{t}} - \frac{\sqrt{3}}{3}\right) + \frac{3}{4} + O\left(\frac{1}{\sqrt[3]{\log x}}\right).
\end{aligned}
\tag{3.1}
$$

Let

$$
\begin{aligned}
\Delta(t) = \frac{\sqrt{3}}{2\pi} \Bigg\{ & 2\log \frac{\sqrt[3]{1-t} + \sqrt[3]{t}}{\sqrt{(1-t)^{2/3} + t^{2/3} - ((1-t)t)^{1/3}}} - \sqrt{3} \arctan\left(\frac{2\sqrt{3}\sqrt[3]{t}}{3\sqrt[3]{1-t}} - \frac{\sqrt{3}}{3}\right) \\
& - \sqrt{3} \arctan\left(\frac{2\sqrt{3}\sqrt[3]{1-t}}{3\sqrt[3]{t}} - \frac{\sqrt{3}}{3}\right)\Bigg\} + \frac{3}{2}.
\end{aligned}
$$

Clearly, $\Delta(t)$ is symmetric with respect to $t = 1/2$, and

$$
S(x,t) + S(x,1-t) = \Delta(t) + O\left(\frac{1}{\sqrt[3]{\log x}}\right),
$$

then when $1/2 \le t \le 1$, $S(x,t)$ is the same as (3.1). This completes the proof.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare no competing interest.

References

1. Z. Cui, J. Wu, The Selberg-Delange method in short intervals with an application, *Acta Arith.*, **163** (2014), 247–260. http://dx.doi.org/10.4064/aa163-3-4
2. H. Delange, Sur les formules dues Atle Selberg, *Bull. Sci. Math.*, **83** (1959), 101–111.
3. H. Delange, Sur les formules de Atle Selberg, *Acta Arith.*, **19** (1971), 105–146. https://doi.org/10.4064/AA-19-2-105-146
4. B. Feng, J. Wu, The arcsine law on divisors in arithmetic progressions modulo prime powers, *Acta Math. Hungar.*, **163** (2021), 392–406. https://doi.org/10.1007/s10474-020-01105-7
5. B. Feng, J. Wu, β-law on divisors of integers representable as sum of two squares, in Chinese, *Sci. China Math.*, **49** (2019), 1563–1572.
6. B. Feng, On the arcsine law on divisors in arithmetic progressions, *Indagat. Math.*, **27** (2016), 749–763. https://doi.org/10.1016/j.indag.2016.01.008

7. Keqin Feng, *Algebraic Number Theory*, in Chinese, Beijing: Science Press, 2000.
8. M. N. Huxley, The difference between consecutive primes, *Invent. Math.*, **15** (1972), 164–170. https://doi.org/10.1007/BF01418933
9. S. K. Leung, Dirichlet law for factorization of integers, polynomials and permutations, preprint paper, 2022.
10. A. Selberg, Note on the paper by L. G. Sathe, *J. Indian Math. Soc.*, **18** (1954), 83–87. https://doi.org/10.18311/JIMS2F19542F17018
11. G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, 3 Eds, Cambridge: Cambridge University Press, 1995.
12. J. Wu, Q. Wu, Mean values for a class of arithmetic functions in short intervals, *Math. Nachr.*, **293** (2020), 178–202. https://doi.org/10.1002/mana.201800276

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