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Research article

Sandwich theorems involving fractional integrals applied to the q-analogue of the multiplier transformation

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Abstract: In this paper, the research discussed involves fractional calculus applied to a q-operator. Fractional integrals applied to the q-analogue of the multiplier transformation gives a new operator, and the research is conducted applying the differential subordination and superordination theories. The best dominant and the best subordinant are obtained by the theorems and corollaries discussed. Combining the results from the both theories, sandwich-type results are presented as a conclusion of this research.

Keywords: fractional integral of q-analogue of multiplier transformation; differential subordination; differential superordination; best dominant; best subordinant
Mathematics Subject Classification: 30C45

1. Introduction

More and more fields of research have used fractional calculus to develop and find new applications. Similarly, q-calculus is involved in several engineering domains, physics, and in mathematics. The combination of fractional and q-calculus in geometric functions theory and some interesting applications were obtained by Srivastava [1].

Jackson [2, 3] established the q-derivative and the q-integral in the field of mathematical analysis via quantum calculus. The foundations of quantum calculus in the theory of geometric functions were laid by Srivastava [4]. Continued research in this field has led to the obtaining of numerous q-analogue operators, such as the q-analogue of the Sălăgean differential operator [5], giving new applications in [6–8]; the q-analogue of the Ruscheweyh differential operator introduced by Răducanu and Kanas [9] and studied by Mohammed and Darus [10] and Mahmood and Sokół [11]; and the q-analogue of the multiplier transformation [12, 13].

This study involves an operator defined by applying the Riemann-Liouville fractional integral to the q-analogue of the multiplier transformation. Many operators have been defined and studied in recent years by using the Riemann-Liouville or Atagana-Baleanu fractional integrals.

First, we recall the classically used notations and notions from geometric functions theory.

Working on the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$, we establish here the class of analytic functions denoted by $\mathcal{H}(U)$ and its subclasses $\mathcal{H}[a,n]$ containing the functions $f \in \mathcal{H}(U)$ defined by $f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots$, with $z \in U$, $a \in \mathbb{C}$, $n \in \mathbb{N}$, as well as $\mathcal{A}_n$, containing the functions $f \in \mathcal{H}(U)$ of the form $f(z) = z + a_{n+1} z^{n+1} + \ldots$, $z \in U$. When $n = 1$, the notation $\mathcal{A}_1 = \mathcal{A}$ is used.

We also recall the Riemann-Liouville fractional integral definition introduced in [14, 15]:

Definition 1. *([14, 15]) The fractional integral of order λ applied to the analytic function f in a simply-connected region of the z-plane which contains the origin is defined by*

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt,$$

where $\lambda > 0$ and the multiplicity of $(z-t)^{\lambda-1}$ is removed by the condition that $\log(z-t)$ is real when $(z-t) > 0$.

The q-analogue of the multiplier transformation is defined below.

Definition 2. *([13]) The q-analogue of the multiplier transformation, denoted by $\mathcal{I}_q^{m,l}$, has the following form:*

$$\mathcal{I}_q^{m,l} f(z) = z + \sum_{j=2}^{\infty} \left(\frac{[l+j]_q}{[l+1]_q} \right)^m a_j z^j,$$

where $q \in (0,1)$, $m, l \in \mathbb{R}$, $l > -1$, and $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$, $z \in U$.

Remark 1. *We notice that* $\lim\limits_{q\to 1} \mathcal{I}_q^{m,l} f(z) = \lim\limits_{q\to 1} \left(z + \sum_{j=2}^{\infty} \left(\frac{[l+j]_q}{[l+1]_q} \right)^m a_j z^j \right) = z + \sum_{j=2}^{\infty} \left(\frac{l+j}{l+1} \right)^m a_j z^j = I(m,1,l)$. *The operator $I(m,1,l)$ was studied by Cho and Srivastava [16] and Cho and Kim [17]. The operator $I(m,1,1)$ was studied by Uralegaddi and Somanatha [18], and the operator $I(\alpha,\lambda,0)$ was introduced by Acu and Owa [19]. Cătaş [20] studied the operator $I_p(m,\lambda,l)$ which generalizes the operator $I(m,\lambda,l)$. Alb Lupaş studied the operator $I(m,\lambda,l)$ in [21–23].*

Now, we introduce definitions from the differential subordination and differential superordination theories.

Definition 3. *([24]) Between the analytic functions f and g there is a differential subordination, denoted $f(z) \prec g(z)$, if there exists ω, a Schwarz analytic function with the properties $|\omega(z)| < 1$, $z \in U$ and $\omega(0) = 0$, such that $f(z) = g(\omega(z))$, $\forall\, z \in U$. In the special case where g is an univalent function in U, the above differential subordination is equivalent to $f(U) \subset g(U)$ and $f(0) = g(0)$.*

Definition 4. *([24]) Considering a univalent function h in U and $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$, when the analytic function p satisfies the differential subordination*

$$\psi(p(z), zp'(z), z^2 p''(z); z) \prec h(z), \quad z \in U, \tag{1.1}$$

then p is a solution of the differential subordination. When $p \prec g$ for all solutions p, the univalent function g is a dominant of the solutions. A dominant $\widetilde{g}$ with the property $\widetilde{g} \prec g$ for every dominant g is called the best dominant of the differential subordination.

Definition 5. *([25]) Considering an analytic function h in U and $\varphi : \mathbb{C}^3 \times \overline{U} \to \mathbb{C}$, when p and $\varphi\left(p(z), zp'(z), z^2p''(z); z\right)$ are univalent functions in U fulfilling the differential superordination*

$$h(z) \prec \varphi(p(z), zp'(z), z^2p''(z); z), \tag{1.2}$$

then p is a solution of the differential superordination. When $g \prec p$ for all solutions p, the analytic function g is a subordinant of the solutions. A subordinant $\widetilde{g}$ with the property $g \prec \widetilde{g}$ for every subordinant g is called the best subordinant of the differential superordination.

Definition 6. *([24]) Q denotes the class of injective functions f analytic on $\overline{U}\backslash E(f)$, with the property $f'(\zeta) \neq 0$ for $\zeta \in \partial U\backslash E(f)$, when $E(f) = \{\zeta \in \partial U : \lim_{z\to\zeta} f(z) = \infty\}$.*

The obtained results from this paper are constructed based on the following lemmas.

Lemma 1. *([24]) Considering the univalent function g in U and the analytic functions θ, η in a domain $D \supset g(U)$, such that $\eta(w) \neq 0$, $\forall\, w \in g(U)$, define the functions $G(z) = zg'(z)\eta(g(z))$ and $h(z) = \theta(g(z)) + G(z)$. Assuming that G is starlike univalent in U and $Re\left(\frac{zh'(z)}{G(z)}\right) > 0$, $\forall\, z \in U$, when the analytic function p having the properties $p(U) \subseteq D$ and $p(0) = g(0)$, satisfies the differential subordination $\theta(p(z)) + zp'(z)\eta(p(z)) \prec \theta(g(z)) + zg'(z)\eta(g(z))$, for $z \in U$, then $p \prec g$ and g is the best dominant.*

Lemma 2. *([26]) Considering the convex univalent function g in U and the analytic functions θ, η in a domain $D \supset g(U)$, define the function $G(z) = zg'(z)\eta(g(z))$. Assuming that G is starlike univalent in U and $Re\left(\frac{\theta'(g(z))}{\eta(g(z))}\right) > 0$, $\forall\, z \in U$, when $p \in \mathcal{H}[g(0), 1] \cap Q$, with $p(U) \subseteq D$, the function $\theta(p(z)) + zp'(z)\eta(p(z))$ is univalent in U, and the differential superordination $\theta(g(z)) + zg'(z)\eta(g(z)) \prec \theta(p(z)) + zp'(z)\eta(p(z))$ is satisfied, then $g \prec p$ and g is the best subordinant.*

2. Main results

The operator obtained by applying the the Riemann-Liouville fractional integral to the q-analogue of the multiplier transformation is written as follows:

Definition 7. *Let q, m, l be real numbers, $q \in (0, 1)$, $l > -1$, and $\lambda \in \mathbb{N}$. The fractional integral applied to the q-analogue of the multiplier transformation is defined by*

$$D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z) = \frac{1}{\Gamma(\lambda)}\int_0^z \frac{\mathcal{I}_q^{m,l}f(t)}{(z-t)^{1-\lambda}}dt = \tag{2.1}$$

$$\frac{1}{\Gamma(\lambda)}\int_0^z \frac{t}{(z-t)^{1-\lambda}}dt + \sum_{j=2}^{\infty}\left(\frac{[l+j]_q}{[l+1]_q}\right)^m a_j \int_0^z \frac{t^j}{(z-t)^{1-\lambda}}dt.$$

After a laborious computation, we discover that the fractional integral applied to the q-analogue of the multiplier transformation takes the following form:

$$D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z) = \frac{1}{\Gamma(\lambda+2)}z^{\lambda+1} + \sum_{j=2}^{\infty}\left(\frac{[l+j]_q}{[l+1]_q}\right)^m \frac{\Gamma(j+1)}{\Gamma(j+\lambda+1)}a_j z^{j+\lambda}, \tag{2.2}$$

when $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$. We note that $D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z) \in \mathcal{H}[0, \lambda+1]$.

Remark 2. *When $q \to 1$, we obtain the classical case, and the fractional integral applied to the multiplier transformation is defined by*

$$D_z^{-\lambda} I(m,1,l) f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{I(m,1,l) f(t)}{(z-t)^{1-\lambda}} dt = \tag{2.3}$$

$$\frac{1}{\Gamma(\lambda)} \int_0^z \frac{t}{(z-t)^{1-\lambda}} dt + \sum_{j=2}^{\infty} \left(\frac{l+j}{l+1}\right)^m a_j \int_0^z \frac{t^j}{(z-t)^{1-\lambda}} dt,$$

which, after several calculus can be written in the form

$$D_z^{-\lambda} I(m,1,l) f(z) = \frac{1}{\Gamma(\lambda+2)} z^{\lambda+1} + \sum_{j=2}^{\infty} \left(\frac{l+j}{l+1}\right)^m \frac{\Gamma(j+1)}{\Gamma(j+\lambda+1)} a_j z^{j+\lambda}, \tag{2.4}$$

when $f(z) = z + \sum_{j=2}^{\infty} a_j z^j \in \mathcal{A}$. We note that $D_z^{-\lambda} I(m,1,l) f(z) \in \mathcal{H}[0, \lambda+1]$.

The main subordination result product regarding the operator introduced in Definition 7 is exposed in the following theorem:

Theorem 1. *Consider $f \in \mathcal{A}$ and g an analytic function univalent in U with the property that $g(z) \neq 0$, $\forall\, z \in U$, with real numbers q, m, l, $q \in (0,1)$, $l > -1$, and $\lambda, n \in \mathbb{N}$. Assuming that $\frac{zg'(z)}{g(z)}$ is a starlike function univalent in U and*

$$Re\left(1 + \frac{b}{d} g(z) + \frac{2c}{d} (g(z))^2 - \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)}\right) > 0, \tag{2.5}$$

for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, $z \in U$, denote

$$\psi_\lambda^{m,l,q}(n,a,b,c,d;z) := a + b\left[\frac{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)}{z}\right]^n + \tag{2.6}$$

$$c\left[\frac{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)}{z}\right]^{2n} + dn\left[\frac{z\left(D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)\right)'}{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)} - 1\right].$$

If the differential subordination

$$\psi_\lambda^{m,l,q}(n,a,b,c,d;z) \prec a + bg(z) + c(g(z))^2 + d\frac{zg'(z)}{g(z)}, \tag{2.7}$$

is satisfied by the function g, for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, then the differential subordination

$$\left(\frac{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)}{z}\right)^n \prec g(z), \tag{2.8}$$

holds and g is the best dominant for it.

Proof. Setting $p(z) := \left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^n$, $z \in U$, $z \neq 0$, and differentiating with respect to z, we get

$$p'(z) = n\left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^{n-1}\left[\frac{\left(D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)\right)'}{z} - \frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z^2}\right] =$$

$$n\left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^{n-1}\frac{\left(D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)\right)'}{z} - \frac{n}{z}p(z)$$

and

$$\frac{zp'(z)}{p(z)} = n\left[\frac{z\left(D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)\right)'}{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)} - 1\right].$$

Defining the functions θ and η by $\theta(w) := a + bw + cw^2$ and $\eta(w) := \frac{d}{w}$, it can be easily certified that θ is analytic in $\mathbb{C}$, η is analytic in $\mathbb{C}\backslash\{0\}$, and that $\eta(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$.

Considering the functions $G(z) = zg'(z)\eta(g(z)) = d\frac{zg'(z)}{g(z)}$ and

$h(z) = \theta(g(z)) + G(z) = a + bg(z) + c(g(z))^2 + d\frac{zg'(z)}{g(z)}$,

we deduce that $G(z)$ is starlike univalent in U.

Differentiating the function h with respect to z we get

$$h'(z) = bg'(z) + 2cg(z)g'(z) + d\frac{(g'(z) + zg''(z))g(z) - z(g'(z))^2}{(g(z))^2}$$

and

$$\frac{zh'(z)}{G(z)} = \frac{zh'(z)}{d\frac{zg'(z)}{g(z)}} = 1 + \frac{b}{d}g(z) + \frac{2c}{d}(g(z))^2 - \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)}.$$

The condition

$$Re\left(\frac{zh'(z)}{G(z)}\right) = Re\left(1 + \frac{b}{d}g(z) + \frac{2c}{d}(g(z))^2 - \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)}\right) > 0$$

is satisfied by relation (2.5), and we deduce that

$$a + bp(z) + c(p(z))^2 + d\frac{zp'(z)}{p(z)} = a + b\left[\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right]^n +$$

$$c\left[\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right]^{2n} + d\gamma\left[\frac{z\left(D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)\right)'}{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)} - 1\right] = \psi_\lambda^{m,l,q}(n,a,b,c,d;z),$$

which is the function from relation (2.6).

Rewriting the differential subordination (2.7), we obtain

$$a + bp(z) + c(p(z))^2 + d\frac{zp'(z)}{p(z)} \prec a + bg(z) + c(g(z))^2 + d\frac{zg'(z)}{g(z)}.$$

The hypothesis of Lemma 1 being fulfilled, we get the conclusion $p(z) \prec g(z)$, written as

$$\left(\frac{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)}{z}\right)^n \prec g(z)$$

and g is the best dominant. $\square$

Corollary 1. *Suppose that the relation (2.5) is fulfilled for real numbers q, m, l, $q \in (0,1)$, $l > -1$, and $\lambda, n \in \mathbb{N}$. If the differential subordination*

$$\psi_\lambda^{m,l,q}(n,a,b,c,d;z) \prec a + b\frac{\alpha z+1}{\beta z+1} + c\left(\frac{\alpha z+1}{\beta z+1}\right)^2 + d\frac{(\alpha-\beta)z}{(\alpha z+1)(\beta z+1)}$$

is verified for $a,b,c,d \in \mathbb{C}$, $d \neq 0$, $-1 \leq \beta < \alpha \leq 1$, and the function $\psi_\lambda^{m,l,q}$ is given by relation (2.6), then the differential subordination

$$\left(\frac{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)}{z}\right)^n \prec \frac{\alpha z+1}{\beta z+1}$$

is satisfied with the function $g(z) = \dfrac{\alpha z+1}{\beta z+1}$ as the best dominant.

Proof. Considering in Theorem 1 the function $g(z) = \dfrac{\alpha z+1}{\beta z+1}$, with $-1 \leq \beta < \alpha \leq 1$, the corollary is verified. $\square$

Corollary 2. *Assume that relation (2.5) is satisfied for real numbers q, m, l, $q \in (0,1)$, $l > -1$, and $\lambda, n \in \mathbb{N}$. If the differential subordination*

$$\psi_\lambda^{m,l,q}(n,a,b,c,d;z) \prec a + b\left(\frac{z+1}{1-z}\right)^s + c\left(\frac{z+1}{1-z}\right)^{2s} + \frac{2sdz}{1-z^2}$$

holds for $a,b,c,d \in \mathbb{C}$, $0 < s \leq 1$, $d \neq 0$, and the function $\psi_\lambda^{m,l,q}$ is defined by relation (2.6), then the differential subordination

$$\left(\frac{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)}{z}\right)^n \prec \left(\frac{z+1}{1-z}\right)^s$$

is satisfied with the function $g(z) = \left(\dfrac{z+1}{1-z}\right)^s$ as the best dominant.

Proof. Considering in Theorem 1 the function $g(z) = \left(\dfrac{z+1}{1-z}\right)^s$, with $0 < s \leq 1$, the corollary is obtained. $\square$

When $q \to 1$ in Theorem 1, we get the classical case:

Theorem 2. *Consider $f \in \mathcal{A}$ and g an analytic function univalent in U with the property that $g(z) \neq 0$, $\forall\, z \in U$, with real numbers m, l, $l > -1$, and $\lambda, n \in \mathbb{N}$. Assuming that $\frac{zg'(z)}{g(z)}$ is a starlike function univalent in U and*

$$Re\left(1 + \frac{b}{d}g(z) + \frac{2c}{d}(g(z))^2 - \frac{zg'(z)}{g(z)} + \frac{zg''(z)}{g'(z)}\right) > 0, \tag{2.9}$$

for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, $z \in U$, denote

$$\psi_{m,l,\lambda}(n, a, b, c, d; z) := a + b\left[\frac{D_z^{-\lambda} I(m, 1, l) f(z)}{z}\right]^n + \tag{2.10}$$

$$c\left[\frac{D_z^{-\lambda} I(m, 1, l) f(z)}{z}\right]^{2n} + dn\left[\frac{z\left(D_z^{-\lambda} I(m, 1, l) f(z)\right)'}{D_z^{-\lambda} I(m, 1, l) f(z)} - 1\right].$$

If the differential subordination

$$\psi_{m,l,\lambda}(n, a, b, c, d; z) \prec a + bg(z) + c(g(z))^2 + d\frac{zg'(z)}{g(z)}, \tag{2.11}$$

is satisfied by the function g, for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, then the differential subordination

$$\left(\frac{D_z^{-\lambda} I(m, 1, l) f(z)}{z}\right)^n \prec g(z) \tag{2.12}$$

holds and g is the best dominant for it.

Proof. The proof of the theorem follows the same steps as the proof of Theorem 1 and it is omitted. □

The corresponding superordination results regarding the operator introduced in Definition 7 are exposed in the following:

Theorem 3. *Consider $f \in \mathcal{A}$ and g an analytic function univalent in U with the properties $g(z) \neq 0$ and $\frac{zg'(z)}{g(z)}$ is starlike univalent in U, with real numbers q, m, l, $q \in (0, 1)$, $l > -1$, and $\lambda, n \in \mathbb{N}$. Assuming that*

$$Re\left(\frac{2c}{d}(g(z))^2 + \frac{b}{d}g(z)\right) > 0, \text{ for } b, c, d \in \mathbb{C},\ d \neq 0 \tag{2.13}$$

and the function $\psi_\lambda^{m,l,q}(n, a, b, c, d; z)$ is defined in relation (2.6), if the differential superordination

$$a + bg(z) + c(g(z))^2 + d\frac{zg'(z)}{g(z)} \prec \psi_\lambda^{m,l,q}(n, a, b, c, d; z) \tag{2.14}$$

is fulfilled for the function g, for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, then the differential superordination

$$g(z) \prec \left(\frac{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)}{z}\right)^n \tag{2.15}$$

holds and g is the best subordinant for it.

Proof. Set $p(z) := \left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^n$, $z \in U$, $z \neq 0$.

Defining the functions θ and η by $\theta(w) := a + bw + cw^2$ and $\eta(w) := \dfrac{d}{w}$, it is evident that $\eta(w) \neq 0$, $w \in \mathbb{C}\backslash\{0\}$ and we can certify that θ is analytic in $\mathbb{C}$ and η is analytic in $\mathbb{C}\backslash\{0\}$.

With easy computation, we get that

$$\frac{\theta'(g(z))}{\eta(g(z))} = \frac{g'(z)\left[b + 2cg(z)\right]g(z)}{d}$$

and relation (2.13) can be written as

$$Re\left(\frac{\theta'(g(z))}{\eta(g(z))}\right) = Re\left(\frac{2c}{d}(g(z))^2 + \frac{b}{d}g(z)\right) > 0,$$

for $b, c, d \in \mathbb{C}$, $d \neq 0$.

Following the same computations as in the proof of Theorem 1, the differential superordination (2.14) can be written as

$$a + bg(z) + c(g(z))^2 + d\frac{zg'(z)}{g(z)} \prec a + bp(z) + c(p(z))^2 + d\frac{zp'(z)}{p(z)}.$$

The hypothesis of Lemma 2 being fulfilled, we obtain the conclusion

$$g(z) \prec p(z) = \left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^n$$

and g is the best subordinant. □

Corollary 3. *Assume that relation (2.13) is verified for real numbers q, m, l, $q \in (0,1)$, $l > -1$, and $\lambda, n \in \mathbb{N}$. If the differential superordination*

$$a + b\frac{\alpha z + 1}{\beta z + 1} + c\left(\frac{\alpha z + 1}{\beta z + 1}\right)^2 + d\frac{(\alpha - \beta)z}{(\alpha z + 1)(\beta z + 1)} \prec \psi_\lambda^{m,l,q}(n, a, b, c, d; z)$$

is satisfied for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, $-1 \leq \beta < \alpha \leq 1$, and the function $\psi_\lambda^{m,l,q}$ is defined by the relation (2.6), then the differential superordination

$$\frac{\alpha z + 1}{\beta z + 1} \prec \left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^n$$

holds with the function $g(z) = \dfrac{\alpha z + 1}{\beta z + 1}$ as the best subordinant.

Proof. Considering in Theorem 3 the function $g(z) = \dfrac{\alpha z + 1}{\beta z + 1}$, with $-1 \leq \beta < \alpha \leq 1$, the corollary is proved. □

Corollary 4. *Suppose that relation (2.13) is fulfilled for real numbers q, m, l, $q \in (0,1)$, $l > -1$, and $\lambda, n \in \mathbb{N}$. If the differential superordination*

$$a + b\left(\frac{z+1}{1-z}\right)^{s} + c\left(\frac{z+1}{1-z}\right)^{2s} + \frac{2sdz}{1-z^2} \prec \psi_{\lambda}^{m,l,q}(n, a, b, c, d; z)$$

is satisfied for $a, b, c, d \in \mathbb{C}$, $0 < s \leq 1$, $d \neq 0$, and the function $\psi_{\lambda}^{m,l,q}$ is given by the relation (2.6), then the differential superordination

$$\left(\frac{z+1}{1-z}\right)^{s} \prec \left(\frac{D_z^{-\lambda} \mathcal{I}_q^{m,l} f(z)}{z}\right)^{n}$$

is satisfied with the function $g(z) = \left(\frac{z+1}{1-z}\right)^{s}$ as the best subordinant.

Proof. Considering in Theorem 3 the function $g(z) = \left(\frac{z+1}{1-z}\right)^{s}$, with $0 < s \leq 1$, the corollary is obtained. □

When $q \to 1$ in Theorem 3, we get the classical case:

Theorem 4. *Consider $f \in \mathcal{A}$ and g an analytic function univalent in U with the properties $g(z) \neq 0$ and $\frac{zg'(z)}{g(z)}$ is starlike univalent in U, with real numbers m, l, $l > -1$, and $\lambda, n \in \mathbb{N}$. Assuming that*

$$Re\left(\frac{2c}{d}(g(z))^2 + \frac{b}{d}g(z)\right) > 0, \text{ for } b, c, d \in \mathbb{C},\ d \neq 0, \tag{2.16}$$

and the function $\psi_{m,l,\lambda}(n, a, b, c, d; z)$ is defined in relation (2.10), if the differential superordination

$$a + bg(z) + c(g(z))^2 + d\frac{zg'(z)}{g(z)} \prec \psi_{m,l,\lambda}(n, a, b, c, d; z) \tag{2.17}$$

is fulfilled for the function g, for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, then the differential superordination

$$g(z) \prec \left(\frac{D_z^{-\lambda} I(m, 1, l) f(z)}{z}\right)^{n} \tag{2.18}$$

holds and g is the best subordinant for it.

Proof. The proof of the theorem follows the same steps as the proof of Theorem 3 and it is omitted. □

The sandwich-type result is obtained by combining Theorems 1 and 3.

Theorem 5. *Consider $f \in \mathcal{A}$ and g_1, g_2 analytic functions univalent in U with the properties that $g_1(z) \neq 0$, $g_2(z) \neq 0$, $\forall\ z \in U$, and, respectively, $\frac{zg_1'(z)}{g_1(z)}$, $\frac{zg_2'(z)}{g_2(z)}$ are starlike univalent, with real numbers q, m, l, $q \in (0,1)$, $l > -1$, and $\lambda, n \in \mathbb{N}$. Assuming that relation (2.5) is verified by the function g_1 and the relation (2.13) is verified by the function g_2, and the function $\psi_{\lambda}^{m,l,q}(n, a, b, c, d; z)$ defined by relation (2.6) is univalent in U, if the sandwich-type relation*

$$a + bg_1(z) + c(g_1(z))^2 + d\frac{zg_1'(z)}{g_1(z)} \prec \psi_{\lambda}^{m,l,q}(n, a, b, c, d; z) \prec a + bg_2(z) + c(g_2(z))^2 + d\frac{zg_2'(z)}{g_2(z)}$$

is satisfied for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, *then the below sandwich-type relation*

$$g_1(z) \prec \left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^n \prec g_2(z)$$

holds for g_1 *as the best subordinant and* g_2 *the best dominant.*

Considering in Theorem 5 the functions $g_1(z) = \frac{\alpha_1 z + 1}{\beta_1 z + 1}$, $g_2(z) = \frac{\alpha_2 z + 1}{\beta_2 z + 1}$, with $-1 \leq \beta_2 < \beta_1 < \alpha_1 < \alpha_2 \leq 1$, the following corollary holds.

Corollary 5. *Suppose that relations (2.5) and (2.13) are fulfilled for real numbers* q, m, l, $q \in (0,1)$, $l > -1$, *and* $\lambda, n \in \mathbb{N}$. *If the sandwich-type relation*

$$a + b\frac{\alpha_1 z + 1}{\beta_1 z + 1} + c\left(\frac{\alpha_1 z + 1}{\beta_1 z + 1}\right)^2 + d\frac{(\alpha_1 - \beta_1) z}{(\alpha_1 z + 1)(\beta_1 z + 1)} \prec \psi_\lambda^{m,l,q}(n, a, b, c, d; z)$$

$$\prec a + b\frac{\alpha_2 z + 1}{\beta_2 z + 1} + c\chi\left(\frac{\alpha_2 z + 1}{\beta_2 z + 1}\right)^2 + d\frac{(\alpha_2 - \beta_2) z}{(\alpha_2 z + 1)(\beta_2 z + 1)}$$

is satisfied for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, $-1 \leq \beta_2 \leq \beta_1 < \alpha_1 \leq \alpha_2 \leq 1$, *and the function* $\psi_\lambda^{m,l,q}$ *is defined by the relation (2.6), then the following sandwich-type relation*

$$\frac{\alpha_1 z + 1}{\beta_1 z + 1} \prec \left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^n \prec \frac{\alpha_2 z + 1}{\beta_2 z + 1}$$

holds for $g_1(z) = \frac{\alpha_1 z + 1}{\beta_1 z + 1}$ *as the best subordinant and* $g_2(z) = \frac{\alpha_2 z + 1}{\beta_2 z + 1}$ *the best dominant.*

Considering in Theorem 5 the functions $g_1(z) = \left(\frac{z+1}{1-z}\right)^{s_1}$, $g_2(z) = \left(\frac{z+1}{1-z}\right)^{s_2}$, with $0 < s_1, s_2 \leq 1$, the following corollary holds.

Corollary 6. *Assume that the relations (2.5) and (2.13) are satisfied for real numbers* q, m, l, $q \in (0,1)$, $l > -1$, *and* $\lambda, n \in \mathbb{N}$. *If the sandwich-type relation*

$$a + b\left(\frac{z+1}{1-z}\right)^{s_1} + c\left(\frac{z+1}{1-z}\right)^{2s_1} + \frac{2s_1 dz}{1-z^2} \prec \psi_\lambda^{m,l,q}(n, a, b, c, d; z)$$

$$\prec a + b\left(\frac{z+1}{1-z}\right)^{s_2} + v\left(\frac{z+1}{1-z}\right)^{2s_2} + \frac{2s_2 dz}{1-z^2}$$

holds for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, $-1 \leq \beta_2 \leq \beta_1 < \alpha_1 \leq \alpha_2 \leq 1$, *and the function* $\psi_\lambda^{m,l,q}$ *is defined by the relation (2.6), then the following sandwich-type relation*

$$\left(\frac{z+1}{1-z}\right)^{s_1} \prec \left(\frac{D_z^{-\lambda}\mathcal{I}_q^{m,l}f(z)}{z}\right)^n \prec \left(\frac{z+1}{1-z}\right)^{s_2}$$

is satisfied for $g_1(z) = \left(\frac{z+1}{1-z}\right)^{s_1}$ *as the best subordinant and* $g_2(z) = \left(\frac{z+1}{1-z}\right)^{s_2}$ *the best dominant.*

The sandwich-type result is obtained by combining Theorems 2 and 4 for the classical case when $q \to 1$.

Theorem 6. *Consider $f \in \mathcal{A}$ and g_1, g_2 analytic functions univalent in U with the properties that $g_1(z) \neq 0$, $g_2(z) \neq 0$, $\forall\, z \in U$, and, respectively, $\frac{zg_1'(z)}{g_1(z)}$, $\frac{zg_2'(z)}{g_2(z)}$ are starlike univalent, with real numbers m, l, $l > -1$, and $\lambda, n \in \mathbb{N}$. Assuming that relation (2.9) is verified by the function g_1 and relation (2.16) is verified by the function g_2, and the function $\psi_{m,l,\lambda}(n, a, b, c, d; z)$ from relation (2.10) is univalent in U, if the sandwich-type relation*

$$a + bg_1(z) + c(g_1(z))^2 + d\frac{zg_1'(z)}{g_1(z)} \prec \psi_{m,l,\lambda}(n, a, b, c, d; z) \prec a + bg_2(z) + c(g_2(z))^2 + d\frac{zg_2'(z)}{g_2(z)},$$

is satified for $a, b, c, d \in \mathbb{C}$, $d \neq 0$, then the below sandwich-type relation

$$g_1(z) \prec \left(\frac{D_z^{-\lambda} I(m, 1, l) f(z)}{z}\right)^n \prec g_2(z)$$

holds for g_1 as the best subordinant and g_2 the best dominant.

3. Conclusions

The results presented in this paper are determined as applications of fractional calculus combined with q-calculus in geometric functions theory. We obtain a new operator described in Definition 7 by applying a fractional integral to the q-analogue of the multiplier transformation. The new fractional q-analogue of the multiplier transformation operator introduced in this paper yields new subordination and superordination results.

The subordination theory used in Theorem 1 gives the best dominant of the differential subordination and, considering well-known functions in geometric functions theory as the best dominant, some illustrative corollaries are obtained. Using the duality, the superordination theory used in Theorem 3 gives the best subordinant of the differential superordination, and illustrative corollaries are established taking the same well-known functions. Combining Theorem 1 and Theorem 3, we present a sandwich-type theorem involving the two dual theories of differential subordination and superordination. Considering the functions studied in the previous corollaries, we establish the other sandwich-type results. The classical case when $q \to 1$ is also presented.

For future studies, using the fractional integral of the q-analogue of the multiplier transformation introduced in this paper, and following [27] and [28], we can define q- subclasses of univalent functions and study some properties, such as coefficient estimates, closure theorems, distortion theorems, neighborhoods, radii of starlikeness, convexity, and close-to-convexity of functions belonging to the defined subclass.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

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