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Research article

On integrable and approximate solutions for Hadamard fractional quadratic integral equations

Saud Fahad Aldosary[1,*], Mohamed M. A. Metwali[2], Manochehr Kazemi[3] and Ateq Alsaadi[4]

[1] Department of Mathematics, College of Science and Humanities in Alkharj, Prince Sattam bin Abdulaziz University, Alkharj 11942, Saudi Arabia
[2] Department of Mathematics and Computer Science, Faculty of Science, Damanhour University, Damanhour, Egypt
[3] Department of Mathematics, Ashtian Branch, Islamic Azad University, Ashtian, Iran
[4] Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

* **Correspondence:** Email: Sau.aldosary@psau.edu.sa.

Abstract: This article addressed the integrable and approximate solutions of Hadamard-type fractional Gripenberg's equation in Lebesgue spaces $L_1[1, e]$. It is well known that the Gripenberg's equation has significant applications in mathematical biology. By utilizing the fixed point (FPT) approach and the measure of noncompactness (MNC), we demonstrated the presence of monotonic integrable solutions as well as the uniqueness of the solution for the studied equation in spaces that are not Banach algebras. Moreover, the method of successive approximations was successfully applied and, as a result, we obtained the approximate solutions for these integral equations. To validate the obtained results, we provided several numerical examples.

Keywords: measure of noncompactness (MNC); Hadamard's fractional operator; Gripenberg's equation; approximate solution; fixed point theorem (FPT)
Mathematics Subject Classification: 45G10, 47H10, 47H30, 47N20

1. Introduction

The present article aims to study the integrable and approximate solutions of the following Hadamard-type fractional Gripenberg equations:

$$x(\theta) \;=\; g_3(\theta) + f_3(\theta, x(\theta)) + \left(g_1(\theta) + \frac{1}{\Gamma(\alpha_1)} \int_1^{\theta} \left(\log \frac{\theta}{s}\right)^{\alpha_1 - 1} \frac{f_1(s, x(s))}{s}\, ds\right)$$

$$\times\Big(g_2(\theta) + \frac{1}{\Gamma(\alpha_2)} \int_1^\theta \Big(\log\frac{\theta}{s}\Big)^{\alpha_2-1} \frac{f_2(s,x(s))}{s}\,ds\Big),\ \theta\in[1,e],\ 0<\alpha_1,\alpha_2<1, \tag{1.1}$$

in Lebesgue spaces $L_1[1,e]$, where $e \cong 2.718$.

Many real concrete phenomena are often modeled and described with the aid of integral equations, particularly in physics, economics, engineering, and biology [1–3]. In particular, quadratic integral equations characterize numerous real and concrete issues such as neutron transport, the kinetic theory of gases, radiative transfer theory, and astrophysics (cf. [4,5]). Furthermore, some problems in biology and the queuing theory have led to a quadratic integral equation of the fractional type [1,6]

$$x(\theta) = \lambda\Big(h_1(\theta) + \int_0^\theta b_1(\theta-s)x(s)ds\Big)\Big(h_2(\theta) + \int_0^\theta b_2(\theta-s)x(s)ds\Big),\ \theta\in\mathbb{R}^+,$$

and this equation has various applications in biology and numerous epidemic models, such as the model of the spread of diseases that do not induce permanent immunity [7].

Moreover, various concrete phenomena contain discontinuous data functions. In this paper, we focus on the integrable solutions of the studied problem (cf. [8,9]).

In 1892, the author presented Hadamard fractional operators [10], where the integral kernel contains a logarithmic function of arbitrary order that is not of the convolution type. It is important to study these kinds of operators separately from the well-known Riemann-Liouville and Caputo fractional operators.

Fractional integral equations of the Hadamard type have been analyzed in a variety of function spaces by several researchers (see, e.g., [11–15]).

In [16], the authors examined the quadratic Hadamard fractional equation

$$x(\theta) = \Big(h_1(\theta) + \frac{g_1(\theta,x(\zeta(\theta)))}{\Gamma(q)}\int_1^\theta \frac{u(\theta,x(s))}{s}(\log\frac{\theta}{s})^{q-1}ds\Big)\Big(h_2(\theta) + g_2(\theta,x(\eta(\theta)))\int_0^a v(\theta,s,x(s))ds\Big)$$

in the Banach algebra of continuous functions $C[0,a]$.

In [17], the author discussed the Riemann-Liouville fractional Gripenberg equation

$$x(\theta) = h(\theta,x(\varphi_3(\theta))) + \Big(g_1(\theta) + g_3(\theta)\cdot(Gx)(\varphi_1(\theta))\Big)\Big(g_2(\theta) + \frac{1}{\Gamma(\alpha)}\int_0^\theta \frac{u(s,x(\varphi_2(s)))}{(\theta-s)^{1-\alpha}}ds\Big), \theta\in\mathbb{R}^+$$

in the weighted Lebesgue spaces $L_1^N(\mathbb{R}^+)$, see also [18].

One of our goals is to discuss the monotonicity property of the solution of Eq (1.1), which has been widely studied and is of vital importance in various applications. In [19–21], the authors have studied the monotonicity property of some distinct types of integral equations and have not examined the numerical solutions. As a result, we can numerically and graphically verify that our solutions are nondecreasing.

In general, we cannot find the exact solution of Eq (1.1), so we employ numerical techniques to estimate an approximate solution for that equation. We use the iterative method [22,23] to estimate the solution of (1.1), which shows acceptable accuracy.

The advantages of using the iterative method are as follows:

(1) This method is very effective and has a simple structure for application.

(2) Since most of the numerical methods for solving integral equations, such as interpolation polynomials, quadrature rules, Galerkin methods, finite and divided differences methods, applying Haar wavelets and block pulse functions, and some hybrid methods lead to linear systems, and the singularity of these systems having problems, then using the iterative method based on successive approximations can be very useful to skip these problems [24–26].

For the numerical results of the integral equations, the Nystrom type methods and the iterative methods have been applied in [27,28], and projection methods which contain the well-known Galerkin method and collocation method have been employed in [29].

Many different methods have also been proposed to compute approximate solutions for these equations such as Bernstein's polynomials [30], radial basis functions (RBFs) [31], block-pulse functions [32], degenerate kernel method [33], wavelet method [34], triangular functions method [35], hybrid function method [36], and exponential spline method [37]. In addition, the integral equations have been solved using various analytical-numeric methods, such as the Adomian decomposition approach, the regularization-homotopy method, and the homotopy perturbation method [38, 39].

Here, we inspect the presence of monotonic solutions for the Hadamard-type fractional Gripenberg's equations (1.1), as well as the uniqueness of the solution in $L_1[1, e]$, which is not Banach algebra. We utilize the fixed point theorem (FPT) approach concerning proper measure of noncompactness (MNC) and fractional calculus to obtain our results. We also apply an iterative method to estimate a numerical solution for Eq (1.1) and present an error analysis for that method, which demonstrates that the approximate solution converges to the exact solution. To validate the obtained results, we provide several numerical examples.

2. Notation and auxiliary facts

Let $\mathbb{R} = (-\infty, \infty)$, $J = [1, e] \subset \mathbb{R}$, and MNC refer to the measure of noncompactness. Denoted by $L_p = L_p(J),\ 1 \le p < \infty$ is the Banach space of the measurable functions x under the norm

$$\|x\|_{L_p} = \|x\|_{L_p(J)} = \left(\int_1^e |x(\theta)|^p \, d\theta \right)^{\frac{1}{p}} < \infty.$$

Let $S = S(J)$ allude to the set of all Lebesgue measurable functions on J. The set S concerning the metric

$$d(x, z) = \inf_{\rho > 0} \left[\rho + meas\left(\{ s : |x(\theta) - z(\theta)| \ge \rho \} \right) \right]$$

becomes a complete metric space, where "meas" alludes to the Lebesgue measure on J. Additionally, according to Proposition 2.14 in [3], the convergence in measure on J is similar to the convergence w.r. to the metric d, and we will call the compactness in this space "compactness in measure".

Theorem 2.1. *Suppose that $W \subset L_1$ is a bounded set and $(\Omega_\rho)_{1 \le \rho \le e-1} \subset J$ is a family of measurable sets s.t. $meas(\Omega_\rho) = \rho$ for every $\rho \in J$. Let $w \in W$. We have*

$$w(\theta_1) \ge w(\theta_2); \ \theta_1 \in \Omega_\rho, \ \theta_2 \notin \Omega_\rho,$$

then W forms a compact in measure set in L_1.

Definition 2.2. [40] *Let $\emptyset \neq W \subset L_1$ be a bounded set. The Hausdorff MNC, $\chi(W)$ is defined as*

$$\chi(W) = \inf\{\rho > 0 : \text{ there exists a finite subset } Z \text{ of } L_1 \text{ s.t. } W \subset Z + B_c\},$$

where B_c is the closed ball $B_c = \{z \in L_1 : \|z\|_{L_1} \leq c\},\ c > 0.$

Next, let $\emptyset \neq W \subset L_1$ be a bounded set and $\epsilon > 0$. The measure of equi-integrability c of the set W [41, p. 39] is given by

$$\begin{aligned} c(W) &= \lim_{\epsilon\to 0}\left\{\sup_{w\in W}\left\{\sup\left[\int_D |w(\theta)|\,d\theta :\ D \subset J,\ meas(D) \leq \varepsilon\right]\right\}\right\} \\ &= \lim_{\epsilon\to 0}\ \sup_{\substack{D\subset J,\\ meas(D)\leq\varepsilon}}\ \sup_{w\in W}\|w\|_{L_1(D)}. \end{aligned} \tag{2.1}$$

If the set W is compact in measure, then $c(W)$ is a regular MNC (cf. [41]).

Definition 2.3. *[41] The (Nemytskii) superposition operator is denoted by $F_f(x)(\theta) = f(\theta, x)$, where $f : J \times \mathbb{R} \to \mathbb{R}$ verifies the Carathéodory hypotheses, i.e.,*

(1) It is continuous in x for almost all $\theta \in J$.
(2) It is measurable in θ for any $x \in \mathbb{R}$.

Theorem 2.4. *[41] Assume that f verifies the Carathéodory hypotheses.The operator F_f continuously transforms $L_p \to L_q,\ p, q \geq 1$ if, and only if,*

$$|f(\theta, x)| \leq a(\theta) + b|x|^{\frac{p}{q}}, \tag{2.2}$$

for all $x \in \mathbb{R}$ and $\theta \in J$, where $b \geq 0$ and $a \in L_q$.

Theorem 2.5. *[40] Let $\emptyset \neq U \subset L_1$ be a convex, closed, and bounded set. Let $P : U \to U$ be a continuous mapping and a contraction w.r. to MNC μ, i.e.,*

$$\mu(P(W)) \leq k\mu(W),\ \ k \in [0, 1)$$

for any $\emptyset \neq W \subset U$. Thus, P has a fixed point in U.

Definition 2.6. *[2, 15] The Hadamard-type fractional integral of a function $x \in L_p, 1 \leq p < \infty$ with left hand point 1 takes the structure*

$$I^\alpha x(\theta) = \frac{1}{\Gamma(\alpha)}\int_1^\theta \left(\log\frac{\theta}{s}\right)^{\alpha-1}\frac{x(s)}{s}\,ds,\ \ \theta > 1,\ \ \alpha > 0,$$

where $\Gamma(\alpha) = \int_0^\infty e^{-\nu}\nu^{\alpha-1}\,d\nu$.

Proposition 2.7. *For $\alpha > 0$, we have*

(a) I^α transforms a.e. nondecreasing and nonnegative functions to functions that have similar properties (cf. [11]).
(b) The operator $I^\alpha :\ L_p \to L_p$ is continuous (cf. [2, Lemma 2.32]) with

$$\|I^\alpha x\|_{L_p} \leq \left(M = \frac{1}{\Gamma(\alpha)}\int_0^1 t^{\alpha-1}e^{\frac{t}{p}}\,dt\right)\|x\|_{L_p}.$$

3. Existence of integrable solutions for Eq (1.1)

Let us set (1.1) in operator form as:

$$x = (Hx) = g_3 + F_{f_3}(x) + (A_1x) \cdot (A_2x), \ \ (A_ix) = g_i + I^{\alpha_i} F_{f_i}(x), \ i = 1, 2,$$

where F_{f_i} is as in Definition 2.3 and $I^{\alpha_i}, i = 1, 2$ is as in Definition 2.6.

Let $\frac{1}{p_1} + \frac{1}{p_2} = 1$, and set the hypotheses:

(i) $g_i : J \to [0, \infty), \ i = 1, 2, 3$, are a.e. nondecreasing functions, where $g_1 \in L_{p_1}, \ g_2 \in L_{p_2}$, and $g_3 \in L_1$.

(ii) For $i = 1, 2, 3$, the functions $f_i : J \times \mathbb{R} \to \mathbb{R}$ verify Carathéodory hypotheses, and $(\theta, x) \to f_i(\theta, x) \geq 0$ are nondecreasing w.r.t. to θ and x, separately for $(\theta, x) \in J \times \mathbb{R}$.

(iii) $\exists \, b_i \geq 0, \ i = 1, 2, 3$ and positive functions $a_1 \in L_{p_1}, \ a_2 \in L_{p_2}, \ a_3 \in L_1$ s.t.

$$|f_3(\theta, x)| \leq a_3(\theta) + b_3|x| \ \text{ and } \ |f_j(\theta, x)| \leq a_j(\theta) + b_j|x|^{\frac{1}{p_j}}, \ j = 1, 2,$$

for all $\theta \in [1, e]$ and $x \in \mathbb{R}$.

(iv) Assume that $\exists \, r > 0$ verifies

$$\begin{aligned}
&||g_3||_{L_1} + ||a_3||_{L_1} + \big(||g_1||_{L_{p_1}} + M_1||a_1||_{L_{p_1}}\big)\big(||g_2||_{L_{p_2}} + M_2||a_2||_{L_{p_2}}\big) \\
&+b_2M_2(||g_1||_{L_{p_1}} + M_1||a_1||_{L_{p_1}})r^{\frac{1}{p_2}} + b_1M_1(||g_2||_{L_{p_2}} + M_2||a_2||_{L_{p_2}})r^{\frac{1}{p_1}} \\
&+(b_3 + M_1M_2b_1b_2)r \leq r,
\end{aligned}$$

where $\Big(b_3 + b_1b_2M_1M_2\Big) < 1$ and $M_i = \frac{1}{\Gamma(\alpha_i)} \int_0^1 t^{\alpha_i - 1} e^{\frac{t}{p_i}} \, dt, i = 1, 2.$

Theorem 3.1. *Let (i)–(iv) be fulfilled, then* (1.1) *has at least one a.e. nondecreasing-integrable solution* $x \in L_1$ *on* J.

Proof. **Step 1.** Let $i = 1, 2$. By hypotheses (ii) and (iii) and Theorem 2.4, we indicate that $F_{f_i} : L_1 \to L_{p_i}$ is continuous and $F_{f_3} : L_1 \to L_1$ is continuous. Since the operators $I^{\alpha_i} : L_{p_i} \to L_{p_i}$ are continuous, hypothesis (i) states that $A_i : \ L_1 \ \to \ L_{p_i}$ are continuous. Using the Hölder inequality, we have $(A_1 \cdot A_2) : L_1 \to L_1$ and $H : L_1 \to L_1$, and they are continuous.

Step 2. Recalling our hypotheses and Proposition $2.7_{(b)}$, we have

$$\begin{aligned}
||Hx||_{L_1} &= ||g_3 + F_{f_3}(x) + (A_1x) \cdot (A_2x)||_{L_1} \\
&\leq ||g_3||_{L_1} + ||F_{f_3}(x)||_{L_1} + ||(A_1x) \cdot (A_2x)||_{L_1} \\
&\leq ||g_3||_{L_1} + ||F_{f_3}(x)||_{L_1} + ||(A_1x)||_{L_{p_1}} \cdot ||(A_2x)||_{L_{p_2}} \\
&\leq ||g_3||_{L_1} + ||a_3 + b_3|x|||_{L_1} + ||g_1 + I^{\alpha_1} F_{f_1}(x)||_{L_{p_1}} \cdot ||g_2 + I^{\alpha_2} F_{f_2}(x)||_{L_{p_2}} \\
&\leq ||g_3||_{L_1} + ||a_3||_{L_1} + b_3||x||_{L_1} \\
&\quad + \Big(||g_1||_{L_{p_1}} + ||I^{\alpha_1} F_{f_1}(x)||_{L_{p_1}}\Big) \cdot \Big(||g_2||_{L_{p_2}} + ||I^{\alpha_2} F_{f_2}(x)||_{L_{p_2}}\Big) \\
&\leq ||g_3||_{L_1} + ||a_3||_{L_1} + b_3||x||_{L_1}
\end{aligned}$$

$$
\begin{aligned}
& +\Big(\|g_1\|_{L_{p_1}} + M_1\|F_{f_1}(x)\|_{L_{p_1}}\Big)\cdot\Big(\|g_2\|_{L_{p_2}} + M_2\|F_{f_2}(x)\|_{L_{p_2}}\Big)\\
\leq\ & \|g_3\|_{L_1} + \|a_3\|_{L_1} + b_3\|x\|_{L_1}\\
& +\Big(\|g_1\|_{L_{p_1}} + M_1\Big\|a_1 + b_1|x|^{\frac{1}{p_1}}\Big\|_{L_{p_1}}\Big)\cdot\Big(\|g_2\|_{L_{p_2}} + M_2\Big\|a_2 + b_2|x|^{\frac{1}{p_2}}\Big\|_{L_{p_2}}\Big)\\
\leq\ & \|g_3\|_{L_1} + \|a_3\|_{L_1} + b_3\|x\|_{L_1}\\
& +\Big(\|g_1\|_{L_{p_1}} + M_1\Big(\|a_1\|_{L_{p_1}} + b_1\Big\|x^{\frac{1}{p_1}}\Big\|_{L_{p_1}}\Big)\Big)\Big(\|g_2\|_{L_{p_2}} + M_2\Big(\|a_2\|_{L_{p_2}} + b_2\Big\|x^{\frac{1}{p_2}}\Big\|_{L_{p_2}}\Big)\Big)\\
\leq\ & \|g_3\|_{L_1} + \|a_3\|_{L_1} + b_3\|x\|_{L_1}\\
& +\Big(\|g_1\|_{L_{p_1}} + M_1(\|a_1\|_{L_{p_1}} + b_1\|x\|_{L_1}^{\frac{1}{p_1}})\Big)\Big(\|g_2\|_{L_{p_2}} + M_2(\|a_2\|_{L_{p_2}} + b_2\|x\|_{L_1}^{\frac{1}{p_2}})\Big),
\end{aligned}
$$

where $\Big\|x^{\frac{1}{p_i}}\Big\|_{L_{p_i}} = \|x\|_{L_1}^{\frac{1}{p_i}}$, then $H : L_1 \to L_1$. For $x \in B_r = \{z \in L_1 : \|z\|_{L_1} \leq r\}$, where r is as in assumption (iv),

$$
\begin{aligned}
\|Hx\|_{L_1} \leq\ & \|g_3\|_{L_1} + \|a_3\|_{L_1} + \Big(\|g_1\|_{L_{p_1}} + M_1\|a_1\|_{L_{p_1}}\Big)\Big(\|g_2\|_{L_{p_2}} + M_2\|a_2\|_{L_{p_2}}\Big)\\
& +b_2M_2\Big(\|g_1\|_{L_{p_1}} + M_1\|a_1\|_{L_{p_1}}\Big)r^{\frac{1}{p_2}} + b_1M_1\Big(\|g_2\|_{L_{p_2}} + M_2\|a_2\|_{L_{p_2}}\Big)r^{\frac{1}{p_1}}\\
& +\Big(b_3 + M_1M_2b_1b_2\Big)\cdot r \leq r.
\end{aligned}
$$

Therefore, for $x \in B_r$, the operator H continuously maps the ball B_r into itself.

Step 3. Suppose that $Q_r \subset B_r$ has the functions a.e. nondecreasing on J. The set $\emptyset \neq Q_r$ is closed, convex, bounded in L_1 (cf. [8]), and compact in measure with the aid of Theorem 2.1.

Step 4. Select $x \in Q_r$, then $x(t)$ and, consequently, $f_i, i = 1,2,3$ are a.e. nondecreasing on J (see (ii)). Furthermore, the operators $I^{\alpha_i}, i = 1,2$, are a.e. nondecreasing on J (see Proposition $2.7_{(a)}$). Moreover, each (A_ix), $i = 1,2$, is also of the same type. These properties, along with hypothesis (i), indicate that $H : Q_r \to Q_r$, and it is continuous.

Step 5. Next, let $\emptyset \neq X \subset B_r$ and arbitrary $\varepsilon > 0$. For $x \in X$ and any $D \subset [1,e]$ with $meas(D) \leq \varepsilon$, we derive

$$
\begin{aligned}
& \int_D |(Hx)(\theta)|d\theta \leq \|(Hx)\|_{L_1(D)}\\
\leq\ & \|g_3\|_{L_1(D)} + \|F_{f_3}(x)\|_{L_1(D)} + \|(A_1x)\|_{L_{p_1}(D)}\cdot\|(A_2x)\|_{L_{p_2}(D)}\\
\leq\ & \|g_3\|_{L_1(D)} + \|a_3\|_{L_1(D)} + b_3\|x\|_{L_1(D)} + \Big[\|g_1\|_{L_{p_1}(D)} + M_1\Big(\|a_1\|_{L_{p_1}(D)}\\
& +b_1\|x^{\frac{1}{p_1}}\|_{L_{p_1}(D)}\Big)\Big]\Big[\|g_2\|_{L_{p_2}(D)} + M_2\Big(\|a_2\|_{L_{p_2}(D)} + b_2\|x^{\frac{1}{p_2}}\|_{L_{p_2}(D)}\Big)\Big]\\
\leq\ & \|g_3\|_{L_1(D)} + \|a_3\|_{L_1(D)} + b_3\int_D |x(t)|dt + \Big[\|g_1\|_{L_{p_1}(D)} + M_1\|a_1\|_{L_{p_1}(D)}\\
& +M_1b_1\Big(\int_D |x(t)|dt\Big)^{\frac{1}{p_1}}\Big]\Big[\|g_2\|_{L_{p_2}(D)} + M_2\|a_2\|_{L_{p_2}(D)} + b_2M_2\Big(\int_D |x(t)|dt\Big)^{\frac{1}{p_2}}\Big].
\end{aligned}
$$

Since $g_i, a_i \in L_{p_i}$, $i = 1,2$, we have

$$
\lim_{\varepsilon\to 0}\Big\{\sup_{x\in X}\Big\{\sup\Big[\|g_i\|_{L_{p_i}(D)} + M_i\|a_i\|_{L_{p_i}(D)} : D \subset [1,e],\ meas(D) \leq \varepsilon\Big]\Big\}\Big\} = 0,
$$

and since $g_3, a_3 \in L_1$,

$$\lim_{\varepsilon\to 0}\left\{\sup_{x\in X}\left\{\sup\left[\ \|g_3\|_{L_1(D)}+\|a_3\|_{L_1(D)}:\ D\subset[1,e],\ meas(D)\leq\ \varepsilon\ \right]\right\}\right\}=0.$$

From Eq (2.1), we obtain

$$c(H(X))\leq\Big(b_3+b_1b_2M_1M_2\Big)c(X).$$

Since $\Big(b_3+b_1b_2M_1M_2\Big)<1$, together with the above estimations, we can utilize Theorem 2.5, which completes the proof. □

3.1. Uniqueness of the solution

Presently, we will address and prove the uniqueness of the solutions.

Theorem 3.2. *Suppose that hypotheses of Theorem 3.1 hold, but change hypothesis (iii) with:*

(v) The functions $a_1\in L_{p_1},\ a_2\in L_{p_2},\ a_3\in L_1$, and $b_i\geq 0$ s.t.

$$|f_i(\theta,0)|\leq a_i(\theta),\ i=1,2,3,$$

$$|f_3(\theta,x)-f_3(\theta,y)|\leq b_3|x-y|,\ \ and\ \ |f_j(\theta,x)-f_j(\theta,y)|\leq b_j|x-y|^{\frac{1}{p_j}},\ j=1,2,\ \ x,y\in Q_r,$$

where Q_r is as in Theorem 3.1.

(vi) Assume that

$$C=b_3+\Big(\|g_1\|_{L_{p_1}}+M_1(\|a_1\|_{L_{p_1}}+b_1\cdot r^{\frac{1}{p_1}})\Big)M_2b_2(2r)^{\frac{1}{p_2}}$$
$$+M_1b_1(2r)^{\frac{1}{p_1}}\Big(\|g_2\|_{L_{p_2}}+M_2(\|a_2\|_{L_{p_2}}+b_2\cdot r^{\frac{1}{p_2}})\Big)<1,$$

where M_1, M_2, and r are characterized in hypothesis (iv), then (1.1) has a unique solution $x\in L_1$ in Q_r.

Proof. Using hypothesis (v), we obtain

$$\begin{aligned}\Big||f_i(\theta,x)|-|f_i(\theta,0)|\Big| &\leq |f_i(\theta,x)-f_i(\theta,0)|\leq b_i|x|^{\frac{1}{p_i}}\\ \Rightarrow|f_i(\theta,x)| &\leq |f_i(\theta,0)|+b_i|x|^{\frac{1}{p_i}}\leq a_i(\theta)+b_i|x|^{\frac{1}{p_i}},\ i=1,2.\end{aligned}$$

Similarly, $|f_3(\theta,x)|\leq a_3(\theta)+b_3|x|$. Thus, Theorem 3.1 indicates that (1.1) has at least one solution $x\in L_1$ in Q_r.

Next, let $x,\ y\in Q_r$ be two various solutions of Eq (1.1). We have

$$\begin{aligned}\|x-y\|_{L_1} &\leq \Big\|F_{f_3}(x)-F_{f_3}(y)\Big\|_{L_1}+\Big\|(A_1x)\cdot(A_2x)-(A_1x)\cdot(A_2y)\Big\|_{L_1}\\ &\quad+\Big\|(A_1x)\cdot(A_2y)-(A_1y)\cdot(A_2y)\Big\|_{L_1}\\ &\leq b_3\|x-y\|_{L_1}+\Big\|(A_1x)\Big\|_{L_{p_1}}\Big\|(A_2x)-(A_2y)\Big\|_{L_{p_2}}+\Big\|(A_1x)-(A_1y)\Big\|_{L_{p_1}}\Big\|(A_2y)\Big\|_{L_{p_2}}\end{aligned}$$

$$
\begin{aligned}
&\leq b_3\|x-y\|_{L_1}+\left\|g_1+I^{\alpha_1}F_{f_1}(x)\right\|_{L_{p_1}}\left\|I^{\alpha_2}|F_{f_2}(x)-F_{f_2}(y)|\right\|_{L_{p_2}}\\
&\quad+\left\|I^{\alpha_1}|F_{f_1}(x)-F_{f_1}(y)|\right\|_{L_{p_1}}\left\|g_2+I^{\alpha_2}F_{f_2}(y)\right\|_{L_{p_2}}\\
&\leq b_3\|x-y\|_{L_1}+\Big(\|g_1\|_{L_{p_1}}+M_1(\|a_1\|_{L_{p_1}}+b_1\|x\|_{L_1}^{\frac{1}{p_1}})\Big)M_2\left\|b_2|x-y|^{\frac{1}{p_2}}\right\|_{L_{p_2}}\\
&\quad+M_1\left\|b_1|x-y|^{\frac{1}{p_1}}\right\|_{L_{p_1}}\Big(\|g_2\|_{L_{p_2}}+M_2(\|a_2\|_{L_{p_2}}+b_2\|y\|_{L_1}^{\frac{1}{p_2}})\Big)\\
&\leq b_3\|x-y\|_{L_1}+\Big(\|g_1\|_{L_{p_1}}+M_1(\|a_1\|_{L_{p_1}}+b_1\cdot r^{\frac{1}{p_1}})\Big)M_2b_2\|x-y\|_{L_1}^{\frac{1}{p_2}}\\
&\quad+M_1b_1\Big(\|g_2\|_{L_{p_2}}+M_2(\|a_2\|_{L_{p_2}}+b_2\cdot r^{\frac{1}{p_2}})\Big)\|x-y\|_{L_1}^{\frac{1}{p_1}}\\
&\leq \Bigg(b_3+\Big(\|g_1\|_{L_{p_1}}+M_1(\|a_1\|_{L_{p_1}}+b_1\cdot r^{\frac{1}{p_1}})\Big)M_2b_2(2r)^{\frac{1}{p_2}}\\
&\quad+M_1b_1(2r)^{\frac{1}{p_1}}\Big(\|g_2\|_{L_{p_2}}+M_2(\|a_2\|_{L_{p_2}}+b_2\cdot r^{\frac{1}{p_2}})\Big)\Bigg)\|x-y\|_{L_1},
\end{aligned}
$$

where $\|x-y\|_{L_1}^{\frac{1}{p_i}}=\|x-y\|_{L_1}^{\frac{1}{p_i}-1}\|x-y\|_{L_1}\leq(2r)^{\frac{1}{p_i}}\|x-y\|_{L_1},\ i=1,2.$ The previous estimation with the hypothesis (vi) wraps up the proof. □

Next, we introduce a concrete example that illustrates and fulfills the outcomes presented in Theorems 3.1 and 3.2.

Example 3.3. *Take into consideration the next equation,*

$$
\begin{aligned}
x(\theta) &= \frac{\ln^9\theta}{\theta}+\left(\frac{\ln^9\theta}{\theta}+\frac{1}{50}|x(\theta)|\right)+\left(\frac{\ln^5\theta}{\sqrt{\theta}}+\frac{1}{\Gamma(\frac{1}{2})}\cdot\int_1^\theta\frac{\frac{\ln^5 s}{\sqrt{s}}+\frac{1}{50}|x(s)|^{\frac{1}{2}}}{\sqrt{\left(\log\frac{\theta}{s}\right)}}\frac{ds}{s}\right)\left(\frac{\ln^5\theta}{\sqrt{\theta}}\right.\\
&\quad\left.+\frac{1}{\Gamma(\frac{1}{2})}\cdot\int_1^\theta\frac{\sqrt{\ln\left(1+\frac{|x|}{36}\right)}}{\sqrt{\left(\log\frac{\theta}{s}\right)}}\frac{ds}{s}\right),\quad\theta\in[1,e].
\end{aligned}
\tag{3.1}
$$

Let $p_1=p_2=2$. We have that:

(1) $g_1(\theta)=g_2(\theta)=\frac{\ln^5\theta}{\sqrt{\theta}}\in L_2$ *and* $g_3(\theta)=\frac{\ln^9\theta}{\theta}\in L_1$ *with* $\|g_1\|_{L_2}=\|g_2\|_{L_2}=\frac{1}{\sqrt{11}}$ *and* $\|g_3\|_{L_1}=\frac{1}{10}$.
(2) $|f_1(\theta,x)|\leq\frac{\ln^5\theta}{\sqrt{\theta}}+\frac{1}{50}|x|^{\frac{1}{2}}$*, then* $a_1=\frac{\ln^5\theta}{\sqrt{\theta}},\ b_1=\frac{1}{50}$ *with* $\|a_1\|_{L_2}=\frac{1}{\sqrt{11}}$.
(3) $f_2(\theta,x)=\sqrt{\ln\left(1+\frac{|x|}{36}\right)}$ *and* $|f_2(\theta,x)|\leq\frac{|x|^{\frac{1}{2}}}{6}$*, then* $a_2(\theta)=0,b_2=\frac{1}{6}$.
(4) $|f_3(\theta,x)|\leq\frac{\ln^9\theta}{\theta}+\frac{1}{50}|x|$*, then* $a_3=\frac{\ln^9\theta}{\theta},\ b_3=\frac{1}{50}$ *with* $\|a_3\|_{L_1}=\frac{1}{10}$.
(5) $(b_3+b_1b_2M_1M_2)\leq\frac{1}{50}+\frac{4}{(50)(6)}<1$*, where* $M_1=M_2=1.3483<2$.
(6) Let $\|x\|_{L_1}=r,\ r=0.7720$*, where*

$$
\|g_3\|_{L_1}+\|a_3\|_{L_1}+(\|g_1\|_{L_2}+M_1\|a_1\|_{L_2})(\|g_2\|_{L_2}+M_2\|a_2\|_{L_2})
$$

$$+b_2M_2(\|g_1\|_{L_2}+M_1\|a_1\|_{L_2})\sqrt{r}+b_1M_1(\|g_2\|_{L_2}+M_2\|a_2\|_{L_2})\sqrt{r}$$
$$+(b_3+M_1M_2b_1b_2)r\leq 0.471818+0.094224\sqrt{11}\sqrt{r}+0.033201\, r\leq r,$$

then hypothesis (iv) *holds for* $r = 0.7720$.

Therefore, by utilizing Theorem 3.1, Eq (3.1) has at least one solution $x \in L_1$ *a.e. nondecreasing in* $[1, e]$. *Moreover, we have*

(1) $|f_1(\theta,0)| = \frac{\ln^5\theta}{\sqrt{\theta}}$ *and* $|f_1(\theta,x) - f_1(\theta,y)| \leq \frac{1}{50}|x-y|^{\frac{1}{2}}$.
(2) $|f_2(\theta,0)| = 0$ *and* $|f_2(\theta,x) - f_2(\theta,y)| \leq \frac{1}{6}|x-y|^{\frac{1}{2}}$.
(3) $|f_3(\theta,0)| = \frac{\ln^9\theta}{\theta}$ *and* $|f_3(\theta,x) - f_3(\theta,y)| \leq \frac{1}{50}|x-y|$.
(4) Hypothesis (vi) is fulfilled for $r = 0.7720$, *where*

$$\begin{aligned} C &= b_3+\Big(\|g_1\|_{L_2}+M_1(\|a_1\|_{L_2}+b_1\cdot\sqrt{r})\Big)M_2b_2\sqrt{(2r)} \\ &\quad +\Big(\|g_2\|_{L_2}+M_2(\|a_2\|_{L_2}+b_2\cdot\sqrt{r})\Big)M_1b_1\sqrt{(2r)}\leq 0.4180<1. \end{aligned}$$

Hence, by utilizing Theorem 3.2, Eq (3.1) has a unique solution $x \in L_1$.

4. Numerical successive approximations method

In this section, we apply a numerical method to solve Eq (1.1), which is based on the successive approximations method [22, 23]. This is a well known and applicable classical method for solving initial value problems and various types of integral equations.

The successive approximations method (Picard sequence) for Eq (1.1) is defined by

$$\begin{aligned} x_n(\theta) &= g_3(\theta)+f_3(\theta,x_{n-1}(\theta))+\Big(g_1(\theta)+\frac{1}{\Gamma(\alpha_1)}\int_1^\theta\Big(\log\frac{\theta}{s}\Big)^{\alpha_1-1}\frac{f_1(s,x_{n-1}(s))}{s}\,ds\Big) \\ &\quad\times\Big(g_2(\theta)+\frac{1}{\Gamma(\alpha_2)}\int_1^\theta\Big(\log\frac{\theta}{s}\Big)^{\alpha_2-1}\frac{f_2(s,x_{n-1}(s))}{s}\,ds\Big),\quad n\geq 1, \end{aligned}\tag{4.1}$$

where the zeroth approximation $x_0(\theta)$ is an arbitrary real function. For the zeroth approximation, we select $x_0(\theta) = g_3(\theta)$. By using (4.1), the solution of Eq (1.1) can be computed as:

$$x(\theta) = \lim_{n\to\infty} x_n(\theta).$$

Theorem 4.1. *Under the hypotheses of Theorem 3.2, Eq* (1.1) *has a unique solution* $x^* \in L_1[1, e]$. *Moreover, for any* $x_0 \in L_1[1, e]$, *the Picard sequence is defined as*

$$x_n := (Hx_{n-1}) = g_3 + F_{f_3}(x_{n-1}) + (A_1x_{n-1})\cdot(A_2x_{n-1}),\tag{4.2}$$

where

$$(A_ix_{n-1}) = g_i + I^{\alpha_i}F_{f_i}(x_{n-1}), i = 1, 2,$$

with the initial value $x_0 := g_3$ *converging to* x^* *with respect to the norm* $\|\cdot\|_{L_1}$.

4.1. Error estimation

Here, we derive and estimate the error between the approximate solution $(x_n)_{n\in\mathbb{N}}$ and the exact solution x^* of (1.1), regarding the sequence of the successive approximations method.

Theorem 4.2. *Let the hypotheses of Theorem 3.2 hold, then the error estimation between the approximate solutions $(x_n)_{n\in\mathbb{N}}$ and the exact x^* of (1.1) is given by*

$$\|x^* - x_n\|_{L_1} \leq \frac{C^n}{1-C}\Big(\big\|a_3\big\|_{L_1} + M\Big), \tag{4.3}$$

where C is defined in hypothesis (vi) and

$$M = b_3 r + \left(\|g_1\|_{L_{p_1}} + M_1(\|a_1\|_{L_{p_1}} + b_1 \cdot r^{\frac{1}{p_1}})\right)\left(\|g_2\|_{L_{p_2}} + M_2(\|a_2\|_{L_{p_2}} + b_2 \cdot r^{\frac{1}{p_2}})\right).$$

Proof. We have

$$\begin{aligned}
\|x^* - x_n\|_{L_1} &\leq \Big\|F_{f_3}(x^*) - F_{f_3}(x_{n-1})\Big\|_{L_1} + \Big\|(A_1x^*)\cdot(A_2x^*)(t) - (A_1x^*)\cdot(A_2x_{n-1})\Big\|_{L_1} \\
&\quad + \Big\|(A_1x^*)\cdot(A_2x_{n-1}) - (A_1x_{n-1})\cdot(A_2x_{n-1})\Big\|_{L_1} \\
&\leq b_3\|x^* - x_{n-1}\|_{L_1} + \Big\|(A_1x^*)\Big\|_{L_{p_1}}\Big\|(A_2x^*) - (A_2x_{n-1})\Big\|_{L_{p_2}} \\
&\quad + \Big\|(A_1x^*) - (A_1x_{n-1})\Big\|_{L_{p_1}}\Big\|(A_2x_{n-1})\Big\|_{L_{p_2}} \\
&\leq \Bigg(b_3 + \left(\|g_1\|_{L_{p_1}} + M_1(\|a_1\|_{L_{p_1}} + b_1 \cdot r^{\frac{1}{p_1}})\right)M_2 b_2 (2r)^{\frac{1}{p_2}} \\
&\quad + M_1 b_1 (2r)^{\frac{1}{p_1}}\left(\|g_2\|_{L_{p_2}} + M_2(\|a_2\|_{L_{p_2}} + b_2 \cdot r^{\frac{1}{p_2}})\right)\Bigg)\|x^* - x_{n-1}\|_{L_1} \\
&= C\|x^* - x_{n-1}\|_{L_1}.
\end{aligned} \tag{4.4}$$

Also, we have

$$\|x^* - x_{n-1}\|_{L_1} \leq \|x^* - x_n\|_{L_1} + \|x_n - x_{n-1}\|_{L_1}. \tag{4.5}$$

Combining (4.4) and (4.5), we have

$$\|x^* - x_n\|_{L_1} \leq \frac{C}{1-C}\|x_n - x_{n-1}\|_{L_1}. \tag{4.6}$$

Moreover,

$$\begin{aligned}
\|x_n - x_{n-1}\|_{L_1} &\leq \Big\|F_{f_3}(x_{n-1}) - F_{f_3}(x_{n-2})\Big\|_{L_1} \\
&\quad + \Big\|(A_1x_{n-1})\cdot(A_2x_{n-1}) - (A_1x_{n-1})\cdot(A_2x_{n-2})\Big\|_{L_1} \\
&\quad + \Big\|(A_1x_{n-1})\cdot(A_2x_{n-2}) - (A_1x_{n-2})\cdot(A_2x_{n-2})\Big\|_{L_1} \\
&\leq \Bigg(b_3 + \left(\|g_1\|_{L_{p_1}} + M_1(\|a_1\|_{L_{p_1}} + b_1 \cdot r^{\frac{1}{p_1}})\right)M_2 b_2 (2r)^{\frac{1}{p_2}} \\
&\quad + M_1 b_1 (2r)^{\frac{1}{p_1}}\left(\|g_2\|_{L_{p_2}} + M_2(\|a_2\|_{L_{p_2}} + b_2 \cdot r^{\frac{1}{p_2}})\right)\Bigg)\|x_{n-1} - x_{n-2}\|_{L_1}
\end{aligned}$$

$$= \quad C\|x_{n-1} - x_{n-2}\|_{L_1}.$$

By induction, we get

$$\|x_n - x_{n-1}\|_{L_1} \quad \leq \quad C\|x_{n-1} - x_{n-2}\|_{L_1} \leq ... \leq C^{n-1}\|x_1 - x_0\|_{L_1}. \tag{4.7}$$

By repeating the above procedure for x_1 and x_0, we have

$$\begin{aligned}
\|x_1 - x_0\|_{L_1} &\leq \Big\|g_3 + F_{f_3}(x_0) + (A_1x_0)\cdot(A_2x_0) - g_3\Big\|_{L_1} \\
&\leq \Big\|F_{f_3}(x_0)\Big\|_{L_1} + \Big\|(A_1x_0)\cdot(A_2x_0)\Big\|_{L_1} \\
&\leq \Big\|F_{f_3}(x_0)\Big\|_{L_1} + \Big\|(A_1x_0)\Big\|_{L_{p_1}} \cdot \Big\|(A_2x_0)\Big\|_{L_{p_2}} \\
&\leq \Big\|a_3\Big\|_{L_1} + b_3\Big\|x_0\Big\|_{L_1} + \Big\|g_1 + I^{\alpha_1}F_{f_1}(x_0)\Big\|_{L_{p_1}} \cdot \Big\|g_2 + I^{\alpha_2}F_{f_2}(x_0)\Big\|_{L_{p_2}} \\
&\leq \Big\|a_3\Big\|_{L_1} + b_3 r \\
&\quad + \Big(\|g_1\|_{L_{p_1}} + M_1(\|a_1\|_{L_{p_1}} + b_1 \cdot r^{\frac{1}{p_1}})\Big)\Big(\|g_2\|_{L_{p_2}} + M_2(\|a_2\|_{L_{p_2}} + b_2 \cdot r^{\frac{1}{p_2}})\Big) \\
&= \Big\|a_3\Big\|_{L_1} + M.
\end{aligned} \tag{4.8}$$

From (4.6), (4.7), and (4.8), we have

$$\|x^* - x_n\|_{L_1} \quad \leq \quad \frac{C^n}{1-C}\Big(\Big\|a_3\Big\|_{L_1} + M\Big). \tag{4.9}$$

□

4.2. *Numerical experiments*

Next, we apply our method to a numerical example that demonstrates the accuracy and efficiency of the applied method in solving Eq (1.1).

The absolute errors in the solutions are given by

$$e_n = |x^*(\theta) - x_n(\theta)|, \quad \theta \in [1, e], \tag{4.10}$$

where $x^*(\theta)$ is the exact solution and $x_n(\theta)$ is the approximate solution of (1.1), which is obtained from Picard sequence (4.1). All numerical results are computed using Maple 17.

Example 4.3. *Consider the following equation*

$$\begin{aligned}
x(\theta) &= g_3(\theta) + \frac{1}{10}x(\theta) + \Big(\frac{\sqrt{\theta}}{10} + \frac{1}{\Gamma(\frac{1}{2})}\int_1^\theta \frac{\sqrt{\frac{16-\ln^2(s)}{4} + x(s)}}{\sqrt{\log\frac{\theta}{s}}}\frac{ds}{10s}\Big) \\
&\quad \times\Big(1 - \frac{1}{12}\sqrt{\frac{(\ln\theta)^3}{\pi}} + \frac{1}{\Gamma(\frac{1}{2})}\int_1^\theta \frac{\frac{1}{4}\sqrt{|x(s)|}}{\sqrt{\log\frac{\theta}{s}}}\frac{ds}{2s}\Big), \quad \theta \in [1, e],
\end{aligned} \tag{4.11}$$

where $g_3(\theta) = \frac{9(\ln\theta)^2}{40} - \frac{\sqrt{\theta}}{10} - \frac{2}{5}\sqrt{\frac{\ln\theta}{\pi}}$.

Equation (4.11) has the exact solution $x(\theta) = \frac{(\ln\theta)^2}{4}$.

Let $p_1 = p_2 = 2$. *We have that*

(1) $g_1(\theta) = \frac{\sqrt{\theta}}{10} \in L_2$, $g_2(\theta) = 1 - \frac{1}{12}\sqrt{\frac{(\ln\theta)^3}{\pi}} \in L_2$ and $g_3(\theta) = \frac{9(\ln\theta)^2}{40} - \frac{\sqrt{\theta}}{10} - \frac{2}{5}\sqrt{\frac{\ln\theta}{\pi}} \in L_1$ with $\|g_1\|_{L_2} = \frac{1}{20}\sqrt{2(e^2-1)}$, $\|g_2\|_{L_2} = 1.28103$, and $\|g_3\|_{L_1} = 0.35385$.
(2) $|f_1(\theta,x)| \leq \frac{1}{10}\sqrt{4 - \frac{\ln^2(\theta)}{4}} + \frac{1}{10}\sqrt{x}$, then $a_1(\theta) = \frac{1}{10}\sqrt{4 - \frac{\ln^2(\theta)}{4}}$, $b_1 = \frac{1}{10}$ with $\|a_1\|_{L_2} = \frac{1}{20}\sqrt{15e-14}$.
(3) $f_2(\theta,x) \leq \frac{1}{8}\sqrt{|x|}$, then $b_2 = \frac{1}{8}$ with $\|a_2\|_{L_2} = 0$.
(4) $|f_3(\theta,x)| = \frac{1}{10}|x|$, then $a_3 = 0$, $b_3 = \frac{1}{10}$ with $\|a_3\|_{L_1} = 0$.
(5) $(b_3 + b_1b_2M_1M_2) \leq \frac{1}{10} + \frac{4}{(10)(8)} \leq 1$, where $M_1 = M_2 = 1.3483 \leq 2$.
(6) Let $\|x\|_{L_1} \leq r$, $r = 1.56189$, where

$$\begin{aligned}&\|g_3\|_{L_1} + \|a_3\|_{L_1} + (\|g_1\|_{L_2} + M_1\|a_1\|_{L_2})(\|g_2\|_{L_2} + M_2\|a_2\|_{L_2})\\&+b_2M_2(\|g_1\|_{L_2} + M_1\|a_1\|_{L_2})\sqrt{r} + b_1M_1(\|g_2\|_{L_2} + M_2\|a_2\|_{L_2})\sqrt{r}\\&+(b_3 + M_1M_2b_1b_2)r \leq 1.041124130 + .2632505402\sqrt{r} + .1227812500r \leq r.\end{aligned}$$

Thus, (iv) holds for $r = 1.56189$.

Therefore, by utilizing Theorem 3.1, Eq (4.11) has at least one solution $x^* \in L_1$ *a.e. nondecreasing in* $[1,e]$.

Moreover, we have

(1) $|f_1(\theta,0)| = \frac{1}{10}\sqrt{4 - \frac{\ln^2(\theta)}{4}}$ and $|f_1(\theta,x) - f_1(\theta,y)| \leq \frac{1}{10}|x-y|^{\frac{1}{2}}$.
(2) $|f_2(\theta,0)| = 0$ and $|f_2(\theta,x) - f_2(\theta,y)| \leq \frac{1}{8}|x-y|^{\frac{1}{2}}$.
(3) $|f_3(\theta,0)| = 0$ and $|f_3(\theta,x) - f_3(\theta,y)| \leq \frac{1}{10}|x-y|$.
(4) Hypothesis (vi) is fulfilled with $r = 1.56189$, where

$$\begin{aligned}C &= b_3 + \Big(\|g_1\|_{L_2} + M_1(\|a_1\|_{L_2} + b_1\cdot\sqrt{r})\Big)\cdot M_2b_2\sqrt{2r} + M_1b_1\sqrt{2r}\Big(\|g_2\|_{L_2} + M_2(\|a_2\|_{L_2}\\&\quad +b_2\cdot\sqrt{r})\Big) \leq \frac{1}{10} + \Big(\frac{1}{20}\sqrt{2(e^2-1)} + 2\Big(\frac{1}{20}\sqrt{15e-14} + \frac{1}{10}\sqrt{r}\Big)\Big)\Big(\frac{2}{8}\Big)\sqrt{2r}\\&\quad +\Big(\frac{2}{10}\Big)\sqrt{2r}\Big(1.281026 + 2\Big(0 + \frac{1}{8}\sqrt{r}\Big)\Big) = 0.89871 < 1.\end{aligned}$$

Hence, by utilizing Theorem 3.2, Eq (4.11) has a unique solution $x^* \in L_1$.

We choose $x_0(\theta) = \frac{9(\ln\theta)^2}{40} - \frac{\sqrt{\theta}}{10} - \frac{2}{5}\sqrt{\frac{\ln\theta}{\pi}}$. *The solution is approximated by Picard sequence (4.1) given in Section 4. After 5 and 12 iterations, the absolute errors are obtained in some arbitrary points* $\theta_j = a + \frac{2j-1}{10}$, *for* $j = 1,2,...,10$ *and* $a = 1$. *To compare the exact solution* x^* *of Eq (4.11) and the iterative solutions* x_n, *for* $n = 5, 12$ *iterations, see Table 1.*

In Figure 1, the curves of the exact solution and the approximate solutions with $n = 5$ *and* $n = 12$ *for the proposed methods are plotted. It can be seen that the approximate solution (circle symbol) for* $n = 12$ *is closer to the exact solution. Also, decreasing of the absolute errors by increasing the number of iteration n is shown in Figure 2.*

Table 1. Compare the accuracy of the exact solution and iterative method.

θ_j	$x^*(\theta_j)$	$x_5(t_\theta)$	$x_{12}(\theta_j)$	$e_n, n=5$	$e_n, n=12$
1.1	0.0022710075	0.0022710075	0.0023967892	4.7832602×10^{-4}	1.2578166×10^{-4}
1.3	0.0172087518	0.0172087518	0.0175469736	1.3199705×10^{-3}	3.3822184×10^{-4}
1.5	0.0411004884	0.0411004884	0.0416155839	2.0430314×10^{-3}	5.1509544×10^{-4}
1.7	0.0703915852	0.0703915852	0.0710583528	2.6767101×10^{-3}	6.6676760×10^{-4}
1.9	0.1029941028	0.1029941028	0.1037935905	3.2406831×10^{-3}	7.9948771×10^{-4}
2.1	0.1376177559	0.1376177559	0.1385351622	3.7487879×10^{-3}	9.1740630×10^{-4}
2.3	0.1734344018	0.1734344018	0.1744578406	4.2111160×10^{-3}	1.0234388×10^{-3}
2.5	0.2098971764	0.2098971764	0.2110168927	4.6352514×10^{-3}	1.1197163×10^{-3}
2.7	0.2466372712	0.2466372712	0.2478451158	5.0270369×10^{-3}	1.2078446×10^{-3}
2.9	0.2834022382	0.2834022382	0.2846913036	5.3910690×10^{-3}	1.2890654×10^{-3}

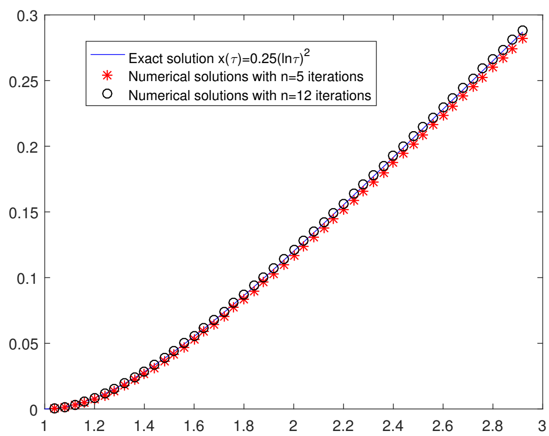

Figure 1. Exact and approximate solutions of Eq (4.11) ($n = 5$ and $n = 12$).

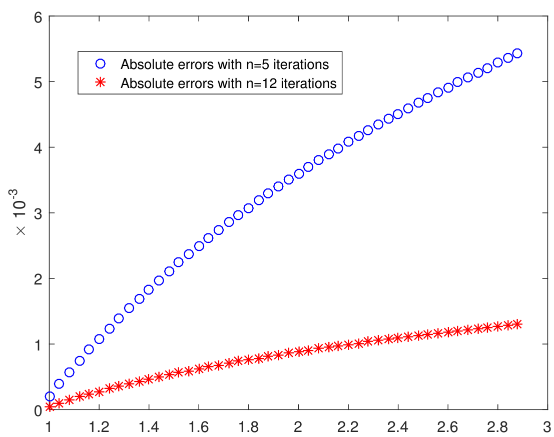

Figure 2. Absolute errors of Eq (4.11) (with $n = 5$ and $n = 12$).

Remark 4.4. *The obtained results from Table 1 and Figure 1 demonstrate the acceptable accuracy of the proposed method. The numerical results show that the accuracy improves with increasing the n.*

Remark 4.5. *According to Figure 1, we can see that the solution of Eq (4.11) is monotonic "nondecreasing," which verifies our assumptions.*

5. Conclusions and perspective

This manuscript addresses the analytical and numerical solutions of the Hadamard-type fractional Gripenberg's equations (1.1) in Lebesgue space $L_1[1, e]$. With the assistance of appropriate MNC and FPT hypotheses, we demonstrated our obtained results of the studied problem in spaces that are not Banach algebras. Two outcomes, namely, Theorems 3.1 and 3.2, are established s.t. the studied problem has at least one monotonic solution and a unique monotonic solution in the mentioned space, respectively. Also, we introduced the numerical iterative method to give approximate solutions of the studied equation with high accuracy. Finally, we estimated the error between the exact solution and the approximate solution of the studied problem via the proposed iterative method. It is worth mentioning that the proposed methods used in this paper are effective and powerful and will be used in future work for other integral equations arising in nonlinear science (see e.g., [42]).

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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