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Research article

Circular evolutes and involutes of spacelike framed curves and their duality relations in Minkowski 3-space

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Abstract: In the present paper, we defined the circular evolutes and involutes for a given spacelike framed curve with respect to Bishop directions in Minkowski 3-space. Then, we studied the essential duality relations among parallel curves, normal surfaces, and circular evolutes and involutes. Furthermore, we also studied the duality relations of their singularities. Based on these studies, we found that it is crucially important to consider the duality relations among different geometric objects for the research of submanifolds with singularities.

Keywords: spacelike framed curves; circular evolutes; involutes; parallel curves; normal surfaces
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1. Introduction

Differential geometry of curves and surfaces is considered to be the beginning of modern geometry. Starting with the beautiful and essential work of Gauss [10], a perfect theory of regular curves and surfaces with moving frame method was established. Now with the development of singularity theory [1–3, 31, 34], non-regular curves and surfaces have been studied in Euclidean 3-space [22, 26]. On the other hand, due to the necessity of physics, the above methods are used in study of the semi-Riemannnian manifolds [17, 21, 27, 29].

It is well known that evolutes and involutes are important research objects in differential geometry and mathematical physics going back to in 1673. In Huygens' book *Horologium oscillatorium*, the elementary properties of the evolutes and involutes of regular plane curves was studied [11]. From the viewpoint of the differential geometry of curves and surfaces with singularities, it has gradually been established for the evolutes and involutes of singular curves [1, 3, 6, 14, 16], such as fronts and frontals. In recent decades, Fukunaga, Honda and Takahashi introduced the concept of framed curves and framed surfaces in Euclidean 3-space [9, 12]. Thereafter, Fukunaga and Takahashi defined the evolutes and involutes of fronts which are plane curves and allowed to contain singular points in Euclidean 2-space [6–8]. Tunçer, Ünal, and Karacan studied the properties the spherical indicatrices of evolutes and involutes of a space curve [33]. Further, Şekerci and Izumiya considered the evolutoids of frontals in the Minkowski plane [4, 5]. Through the work of [18], we can give the definitions of evolutes and focal surfaces for $(1, k)$-type curves with respect to Bishop frames in Euclidean 3-space and discuss their singularities and the classification theorem.

Recently, through the work of Honda and Takahashi [13], we have known the definitions of circular evolutes and involutes of framed curves with respect to Bishop frames in Euclidean 3-space and the duality relations among parallel curves, normal surfaces, and circular evolutes and involutes. We can also learn the local singularity behavior of the circular evolutes and involutes of framed curves through the work of Honda and Takahashi [13]. Furthermore, we studied the nullcone fronts of spacelike framed curves in Minkowski 3-space. We defined the moving frame of a spacelike framed curve, and gave the appropriate Frenet type formula [19, 20]. So, there is natural question of what the phenomena of a circular evolute will be in Minkowski 3-space. In the present paper, we will give a complete answer to this question for spacelike framed curves in Minkowski 3-space.

In the present paper, we research circular evolutes and involutes with a viewpoint of singularity theory in Minkowski 3-space. First, we define the Bishop frame in Minkowski 3-space, and give the relation of the curvature functions between spacelike frames and spacelike Bishop frames. In Section 3, we give the necessary and sufficient condition that a parallel curve is a spacelike or timelike framed curve under causal character in Minkowski 3-space and give the corresponding curvature functions. In Section 4, we give the necessary and sufficient condition that the normal surface of a given spacelike framed curve is a spacelike or timelike framed surface under causal character in Minkowski 3-space and the corresponding invariant functions, as well as give a necessary and sufficient condition that a singularity of a normal surface is a cross cap. Our main contributions are focused on two aspects in Section 5. On the one hand, we study the duality relations among parallel curves, normal surfaces, and circular evolutes and involutes for a given spacelike framed curve in Minkowski 3-space. On the other hand, we study the relations among the sigularities of spacelike framed curves, normal surfaces, and circular evolutes and involutes. Finally, we give an example to illustrate the duality behavior which has been studied in this paper.

All the maps and manifolds considered here are C^∞.

2. Preliminaries

We briefly review some essential concepts of Minkowski 3-space, which are discussed in detail in [27, 29]. Let $\mathbb{R}^3_1$ be the Minkowski 3-space equipped with the canonical pseudo-scalar product $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_1y_1 + x_2y_2 + x_3y_3$, where $\boldsymbol{x} = (x_1, x_2, x_3)$ and $\boldsymbol{y} = (y_1, y_2, y_3)$. We define the norm of $\boldsymbol{x}$ by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$ and the pseudo-vector product of $\boldsymbol{x}$ and $\boldsymbol{y}$ by

$$\boldsymbol{x} \wedge \boldsymbol{y} = \det \begin{pmatrix} -\boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix},$$

where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the canonical basis of $\mathbb{R}^3_1$.

Definition 2.1. A vector $\boldsymbol{x} \in \mathbb{R}^3_1$ is said to be

1) spacelike, if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$ or $\boldsymbol{x} = \mathbf{0}$,
2) timelike, if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$,
3) lightlike, if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$, $\boldsymbol{x} \neq \mathbf{0}$.

It is similar to the concept of the unit sphere in Euclidean 3-space. We can also discuss the pseudo-sphere in Minkowski 3-space. The de Sitter 2-space is defined by

$$\mathbb{S}^2_1 = \{\boldsymbol{x} \in \mathbb{R}^3_1 \,|\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1\},$$

the hyperbolic 2-space is defined by

$$\mathbb{H}^2_0 = \{\boldsymbol{x} \in \mathbb{R}^3_1 \,|\langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1\}.$$

We briefly review the theory of framed curves and framed surfaces in Minkowski 3-space. A (smooth) curve is a differentiable map $\boldsymbol{\gamma} : I \subset \mathbb{R} \to \mathbb{R}^3_1$ where I is an open interval. We say that a curve $\boldsymbol{\gamma}(t)$ is a *spacelike*, *timelike*, or *lightlike* curve if $\dot{\boldsymbol{\gamma}}(t)$ is *spacelike*, *timelike*, or *lightlike*, where $\dot{\boldsymbol{\gamma}}(t) = \frac{d}{dt}\boldsymbol{\gamma}(t)$.

Definition 2.2. ([19]) Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ be a spacelike curve. Then, the C^∞ map $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is called a *spacelike framed curve* if

$$\langle \dot{\boldsymbol{\gamma}}(t), \boldsymbol{\nu}_1(t) \rangle = 0,\ \langle \dot{\boldsymbol{\gamma}}(t), \boldsymbol{\nu}_2(t) \rangle = 0,\ \forall\, t \in I,$$

where

$$\Delta = \{(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) \in \mathbb{S}^2_1 \times \mathbb{H}^2_0 \mid \langle \boldsymbol{\nu}_1(t), \boldsymbol{\nu}_2(t) \rangle = 0\},$$

or

$$\Delta = \{(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) \in \mathbb{H}^2_0 \times \mathbb{S}^2_1 \mid \langle \boldsymbol{\nu}_1(t), \boldsymbol{\nu}_2(t) \rangle = 0\}.$$

Moreover, $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ is said to be a *spacelike framed base curve* if there is a C^∞ map $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta$ such that $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2)$ is a spacelike framed curve.

Let $(\boldsymbol{\gamma}(t), \boldsymbol{\nu}_1(t), \boldsymbol{\nu}_2(t))$ be a spacelike framed curve. We denote $\delta(t) = \operatorname{sign}(\boldsymbol{\nu}_1(t)) = \langle \boldsymbol{\nu}_1(t), \boldsymbol{\nu}_1(t) \rangle$. We define $\boldsymbol{\mu}(t) = \boldsymbol{\nu}_1(t) \wedge \boldsymbol{\nu}_2(t)$, which means $\boldsymbol{\mu}$(t) is a unit spacelike vector field along $\boldsymbol{\gamma}(t)$. Then, we can have a smooth function $\alpha(t)$ satisfying $\dot{\boldsymbol{\gamma}}(t) = \alpha(t)\boldsymbol{\mu}(t)$. Furthermore, we have the following Frenet type formulae.

$$\begin{pmatrix} \dot{\boldsymbol{\nu}}_1(t) \\ \dot{\boldsymbol{\nu}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ \ell(t) & 0 & n(t) \\ -\delta(t)m(t) & \delta(t)n(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}_1(t) \\ \boldsymbol{\nu}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{2.1}$$

where $\ell(t) = \langle \dot{\boldsymbol{\nu}}_1(t), \boldsymbol{\nu}_2(t) \rangle$, $m(t) = \langle \dot{\boldsymbol{\nu}}_1(t), \boldsymbol{\mu}(t) \rangle$, and $n(t) = \langle \dot{\boldsymbol{\nu}}_2(t), \boldsymbol{\mu}(t) \rangle$. We call the functions $(\alpha(t), \ell(t), m(t), n(t))$ the *curvature* of a spacelike framed curve in Minkowski 3-sapce. Then, we consider the frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ which is obtained by

$$\begin{pmatrix} \boldsymbol{\nu}(t) \\ \boldsymbol{\omega}(t) \end{pmatrix} = \begin{pmatrix} \cosh\theta(t) & \sinh\theta(t) \\ \sinh\theta(t) & \cosh\theta(t) \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}_1(t) \\ \boldsymbol{\nu}_2(t) \end{pmatrix},\ \boldsymbol{\mu}(t) = \boldsymbol{\nu}(t) \wedge \boldsymbol{\omega}(t).$$

Then, we have the Frenet type formulae of the frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ as follows:

$$\begin{pmatrix} \dot{\boldsymbol{\nu}}(t) \\ \dot{\boldsymbol{\omega}}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \bar{\ell}(t) & \overline{m}(t) \\ \bar{\ell}(t) & 0 & \bar{n}(t) \\ -\delta(t)\overline{m}(t) & \delta(t)\bar{n}(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}(t) \\ \boldsymbol{\omega}(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{2.2}$$

where $\bar{\ell}(t) = \ell(t) + \dot{\theta}(t)$, $\overline{m}(t) = m(t)\cosh\theta(t) + n(t)\sinh\theta(t)$, and $\bar{n}(t) = m(t)\sinh\theta(t) + n(t)\cosh\theta(t)$.

Remark 2.3. Let $\gamma : I \to \mathbb{R}^3_1$ be a timelike curve. Then, the C^∞ map $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta_5$ is called a *timelike framed curve* if

$$\langle \dot{\gamma}(t), \boldsymbol{\nu}_1(t) \rangle = 0, \ \langle \dot{\gamma}(t), \boldsymbol{\nu}_2(t) \rangle = 0, \ \forall\ t \in I,$$

where

$$\Delta_5 = \{(\boldsymbol{\nu}_1, \boldsymbol{\nu}_2) \in \mathbb{S}^2_1 \times \mathbb{S}^2_1 \mid \langle \boldsymbol{\nu}_1(t), \boldsymbol{\nu}_2(t) \rangle = 0\}.$$

Similar to the discussion of a spacelike framed curve above, we have the Frenet type formulae of a timelike framed curve in Minkowski 3-space as follows:

$$\begin{pmatrix} \dot{\boldsymbol{\nu}}_1(t) \\ \dot{\boldsymbol{\nu}}_2(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & \ell(t) & m(t) \\ -\ell(t) & 0 & n(t) \\ m(t) & n(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}_1(t) \\ \boldsymbol{\nu}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $\ell(t) = \langle \dot{\boldsymbol{\nu}}_1(t), \boldsymbol{\nu}_2(t) \rangle$, $m(t) = \langle \dot{\boldsymbol{\nu}}_1(t), \boldsymbol{\mu}(t) \rangle$, $n(t) = \langle \dot{\boldsymbol{\nu}}_2(t), \boldsymbol{\mu}(t) \rangle$, and $\dot{\gamma}(t) = \alpha(t)\boldsymbol{\mu}(t)$. We call the functions $(\alpha(t), \ell(t), m(t), n(t))$ the curvature of a timelike framed curve in Minkowski 3-space.

Definition 2.4. For a spacelike framed curve $(\gamma, \boldsymbol{\nu}, \boldsymbol{\omega}) : I \to \mathbb{R}^3_1 \times \Delta$, if there exists a smooth function $\beta : I \to \mathbb{R}^3_1$ such that $\dot{\boldsymbol{\nu}}(t) = \beta(t)\boldsymbol{\mu}(t)$, where $\boldsymbol{\mu}(t) = \boldsymbol{\nu}(t) \wedge \boldsymbol{\omega}(t)$, then we call $\boldsymbol{\nu}(t)$ a *Bishop direction*. If $\boldsymbol{\nu}(t)$ and $\boldsymbol{\omega}(t)$ are Bishop directions, then we call the moving frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ a *Bishop frame*.

By the Frenet type formulae of a spacelike framed curve, we can have that there is a function $\theta(t) : I \to \mathbb{R}$ such that $\bar{\ell}(t) = 0$, that is, $\theta(t) = -\ell(t)$, which means that we can always take a frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ to be a Bishop frame by a suitable $\theta(t)$ for a given moving frame $\{\boldsymbol{\nu}_1, \boldsymbol{\nu}_2, \boldsymbol{\mu}\}$ of a spacelike framed curve in Minkowski 3-space.

In the following, we discuss the appropriate moving frame of a surface in Minkowski 3-space. For more detailed discussion, please refer to [9, 32].

Definition 2.5. Let $\boldsymbol{x} : U \to \mathbb{R}^3_1$ be a spacelike surface. Then, the C^∞ map $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3_1 \times \Delta_1$ is said to be *spacelike framed surface* if $\langle \boldsymbol{x}_u(u,v), \boldsymbol{n}(u,v) \rangle = 0$, $\langle \boldsymbol{x}_v(u,v), \boldsymbol{n}(u,v) \rangle = 0$ for all $(u,v) \in U$, where $\boldsymbol{x}_u(u,v) = (\partial\boldsymbol{x}/\partial u)(u,v)$, $\boldsymbol{x}_v(u,v) = (\partial\boldsymbol{x}/\partial v)(u,v)$ and

$$\Delta_1 = \{(\boldsymbol{n}, \boldsymbol{s}) \in \mathbb{H}^2_0 \times \mathbb{S}^2_1 \mid \langle \boldsymbol{n}(u,v), \boldsymbol{s}(u,v) \rangle = 0\}.$$

Moreover, $\boldsymbol{x} : U \to \mathbb{R}^3_1$ is said to be a *spacelike framed base surface* if there is a C^∞ map $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3_1 \times \Delta_1$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a spacelike framed surface.

Definition 2.6. Let $\boldsymbol{x} : U \to \mathbb{R}^3_1$ be a timelike surface. Then, the C^∞ map $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3_1 \times \tilde{\Delta}$ is said to be a *timelike framed surface* if $\langle \boldsymbol{x}_u(u,v), \boldsymbol{n}(u,v) \rangle = 0$, $\langle \boldsymbol{x}_v(u,v), \boldsymbol{n}(u,v) \rangle = 0$ for all $(u,v) \in U$, where $\boldsymbol{x}_u(u,v) = (\partial\boldsymbol{x}/\partial u)(u,v)$, $\boldsymbol{x}_v(u,v) = (\partial\boldsymbol{x}/\partial v)(u,v)$ and

$$\tilde{\Delta} = \{(\boldsymbol{n}, \boldsymbol{s}) \in \mathbb{S}^2_1 \times \mathbb{S}^2_1 \mid \langle \boldsymbol{n}(u,v), \boldsymbol{s}(u,v) \rangle = 0\},$$

or

$$\tilde{\Delta} = \{(\boldsymbol{n}, \boldsymbol{s}) \in \mathbb{S}^2_1 \times \mathbb{H}^2_0 \mid \langle \boldsymbol{n}(u,v), \boldsymbol{s}(u,v)\rangle = 0\}.$$

Moreover, $\boldsymbol{x} : U \to \mathbb{R}^3_1$ is said to be a *timelike framed base surface* if there is a C^∞ map $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s}) : U \to \mathbb{R}^3_1 \times \tilde{\Delta}$ such that $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$ is a timelike framed surface.

We denote $\{a_1, b_1, a_2, b_2, e_1, f_1, g_1, e_2, f_2, g_2\}$ as the *invariant functions* of a spacelike or timelike framed surface $(\boldsymbol{x}, \boldsymbol{n}, \boldsymbol{s})$, where $\boldsymbol{t}(u,v) = \boldsymbol{n}(u,v) \wedge \boldsymbol{s}(u,v)$ and

$$\begin{aligned}
a_1(u,v) &= \langle \boldsymbol{x}_u(u,v), \boldsymbol{s}(u,v)\rangle, \ b_1(u,v) = \langle \boldsymbol{x}_u(u,v), \boldsymbol{t}(u,v)\rangle,\\
a_2(u,v) &= \langle \boldsymbol{x}_v(u,v), \boldsymbol{s}(u,v)\rangle, \ \ b_2(u,v) = \langle \boldsymbol{x}_v(u,v), \boldsymbol{t}(u,v)\rangle,\\
e_1(u,v) &= \langle \boldsymbol{n}_u(u,v), \boldsymbol{s}(u,v)\rangle, \ f_1(u,v) = \langle \boldsymbol{n}_u(u,v), \boldsymbol{t}(u,v)\rangle,\\
g_1(u,v) &= \langle \boldsymbol{s}_u(u,v), \boldsymbol{t}(u,v)\rangle, \ e_2(u,v) = \langle \boldsymbol{n}_v(u,v), \boldsymbol{s}(u,v)\rangle,\\
f_2(u,v) &= \langle \boldsymbol{n}_v(u,v), \boldsymbol{t}(u,v)\rangle, \ \ g_2(u,v) = \langle \boldsymbol{s}_v(u,v), \boldsymbol{t}(u,v)\rangle.
\end{aligned}$$

3. Parallel curves

Let $(\gamma, \boldsymbol{\nu}, \boldsymbol{\omega})$ be a spacelike framed curve in Minkowski 3-space. We will discuss the parallel curve of $\boldsymbol{\gamma}(t)$ with respect to the direction of $\boldsymbol{\omega}(t)$ in this section.

Definition 3.1. We define a curve called a *parallel curve* of $\boldsymbol{\gamma}(t)$ in Minkowski 3-space as

$$P_\gamma[\boldsymbol{\omega}](t) = \boldsymbol{\gamma}(t) + \lambda \boldsymbol{w}(t), \lambda \in \mathbb{R} \setminus \{0\}.$$

By Definition 3.1, we have

$$\dot{P}_\gamma[\boldsymbol{\omega}](t) = \lambda \bar{\ell}(t)\boldsymbol{v}(t) + (\alpha(t) + \lambda \bar{n}(t))\, \boldsymbol{\mu}(t).$$

So, $t_0 \in \mathbb{R}$ is a singularity of the curve $P_\gamma[\boldsymbol{\omega}](t)$ if and only if $\bar{\ell}(t_0) = 0$ and $\alpha(t_0) + \lambda \bar{n}(t_0) = 0$.

Proposition 3.2. *Let $\boldsymbol{w}(t)$ be a timelike vector. Then, $(P_\gamma[\boldsymbol{\omega}], \boldsymbol{n}, \boldsymbol{\omega}) : I \to \mathbb{R}^3_1 \times \Delta$ is a spacelike framed curve where $\boldsymbol{n} : I \to \mathbb{S}^2_1$ if and only if there is a $\varphi(t) : I \to \mathbb{R}$ such that*

$$\lambda \bar{\ell}(t) \cos\varphi(t) + (\alpha(t) + \lambda \bar{n}(t)) \sin\varphi(t) = 0, \ \forall\, t \in I$$

for a fixed $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. Suppose $(P_\gamma[\boldsymbol{\omega}], \boldsymbol{n}, \boldsymbol{\omega})$ is a framed curve, $\boldsymbol{\omega}(t)$ is timelike, and $\langle \boldsymbol{n}(t), \boldsymbol{\omega}(t)\rangle = 0$. So, the vector $\boldsymbol{n}(t)$ is contained in the spacelike plane $\mathrm{Span}_{\mathbb{R}}\{\boldsymbol{\nu}(t), \boldsymbol{\mu}(t)\}$. Then, we have

$$\boldsymbol{n}(t) = \cos\varphi(t)\boldsymbol{\nu}(t) + \sin\varphi(t)\boldsymbol{\mu}(t).$$

Furthermore,

$$\langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{n}(t)\rangle = \lambda \bar{\ell}(t) \cos\varphi(t) + (\alpha(t) + \lambda \bar{n}(t)) \sin\varphi(t) = 0, \ \forall\, t \in I.$$

Conversely, if we have the above equation, then we can define $\boldsymbol{n} : I \to \mathbb{S}^2_1$ by $\boldsymbol{n}(t) = \cos\varphi(t)\boldsymbol{\nu}(t) + \sin\varphi(t)\boldsymbol{\mu}(t)$. It is clear that $(P_\gamma[\boldsymbol{\omega}], \boldsymbol{n}, \boldsymbol{\omega})$ satisfies the definition of a spacelike framed curve. This concludes the proof. □

Proposition 3.3. *Let $\boldsymbol{w}$ be a spacelike vector. Then, we have the following:*

1) $(P_\gamma[\boldsymbol{\omega}], \boldsymbol{n}, \boldsymbol{\omega}) : I \to \mathbb{R}^3_1 \times \Delta$ is a spacelike framed curve where $\boldsymbol{n} : I \to \mathbb{H}^2_0$ if and only if there is a $\varphi(t) : I \to \mathbb{R}$ such that

$$(-\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t))e^{2\varphi(t)} = -(\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t)),\ \forall\ t \in I \tag{3.1}$$

for a fixed $\lambda \in \mathbb{R} \setminus \{0\}$.

2) $(P_\gamma[\boldsymbol{\omega}], \boldsymbol{n}, \boldsymbol{\omega}) : I \to \mathbb{R}^3_1 \times \Delta_5$ is a timelike framed curve where $\boldsymbol{n} : I \to \mathbb{S}^2_1$ if and only if there is a $\varphi(t) : I \to \mathbb{R}$ such that

$$(-\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t))e^{2\varphi(t)} = \lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t),\ \forall\ t \in I \tag{3.2}$$

for a fixed $\lambda \in \mathbb{R} \setminus \{0\}$.

Proof. Suppose $(P_\gamma[\boldsymbol{\omega}], \boldsymbol{n}, \boldsymbol{\omega})$ is a spacelike or timelike framed curve in Minkowski 3-space, $\boldsymbol{\omega}(t)$ is spacelike, and $\langle \boldsymbol{n}(t), \boldsymbol{\omega}(t)\rangle = 0$. So, the vector $\boldsymbol{n}$ is contained in the timelike plane $\mathrm{Span}_{\mathbb{R}}\{\boldsymbol{\nu}(t), \boldsymbol{\mu}(t)\}$. As is known, there are four connected components of a timelike plane with respect to hyperbolic isometries. Then, the vector $\boldsymbol{n}$ has four cases.
1) If $\boldsymbol{n}(t) \in \mathrm{Span}_{\mathbb{R}}\{\boldsymbol{\nu}(t), \boldsymbol{\mu}(t)\}$ is a spacelike vector, then we have $\boldsymbol{n}(t) = \sinh\varphi(t)\boldsymbol{\nu}(t) + \cosh\varphi(t)\boldsymbol{\mu}(t)$ or $\boldsymbol{n}(t) = -\sinh\varphi(t)\boldsymbol{\nu}(t) - \cosh\varphi(t)\boldsymbol{\mu}(t)$. By Definition 2.2, we have

$$\langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{n}(t)\rangle = -\lambda\bar{\ell}(t)\cosh\varphi(t) + (\alpha(t) + \lambda\bar{n}(t))\sinh\varphi(t) = 0,\ \forall\ t \in I.$$

So,

$$(-\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t))e^{2\varphi(t)} = -(\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t)),\ \forall\ t \in I.$$

Conversely, if $\boldsymbol{n}(t)$ satisfies Eq (3.1), we can define $\boldsymbol{n} : I \to \mathbb{H}^2_0$ by the method in the proof of Proposition 3.2, which satisfies that $(P_\gamma[\boldsymbol{\omega}], \boldsymbol{n}, \boldsymbol{\omega}) : I \to \mathbb{R}^3_1 \times \Delta$ is a spacelike framed curve. Then, we conclude the proof of conclusion 1).
2) If $\boldsymbol{n}(t) \in \mathrm{Span}_{\mathbb{R}}\{\boldsymbol{\nu}(t), \boldsymbol{\mu}(t)\}$ is a timelike vector, then we have $\boldsymbol{n}(t) = \cosh\varphi(t)\boldsymbol{\nu}(t) + \sinh\varphi(t)\boldsymbol{\mu}(t)$ or $\boldsymbol{n}(t) = -\cosh\varphi(t)\boldsymbol{\nu}(t) - \sinh\varphi(t)\boldsymbol{\mu}(t)$. By calculation, we obtain

$$\langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{n}(t)\rangle = -\lambda\bar{\ell}(t)\cosh\varphi(t) + (\alpha(t) + \lambda\bar{n}(t))\sinh\varphi(t) = 0,\ \forall\ t \in I.$$

So,

$$(-\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t))e^{2\varphi(t)} = \lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t),\ \forall\ t \in I.$$

Conversely, if $\boldsymbol{n}(t)$ satisfies Eq (3.2), we can define $\boldsymbol{n} : I \to \mathbb{S}^2_1$ by the method in the proof of Proposition 3.2, which satisfies that $(P_\gamma[\boldsymbol{\omega}], \boldsymbol{n}, \boldsymbol{\omega}) : I \to \mathbb{R}^3_1 \times \Delta_5$ is a timelike framed curve. Then, we conclude the proof of conclusion 2). □

If $\bar{\ell}(t) = 0$, the frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ is a Bishop frame, and we can then obviously see that $(P_\gamma[\boldsymbol{w}], \boldsymbol{\nu}, \boldsymbol{\omega})$ is a spacelike or timelike framed curve by taking $\varphi(t) = 0$ in Propositions 3.2 and 3.3.

For a spacelike or timelike framed base curve $\boldsymbol{\gamma}(t)$, the curvature functions are the fundamental invariants of $\boldsymbol{\gamma}(t)$. Therefore, we will give the relations between the curvature functions of $\boldsymbol{\gamma}(t)$ and the parallel curves in the followings:

Proposition 3.4. *Let $\boldsymbol{w}$ be a timelike vector. If $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$ is a spacelike framed curve, then the curvature functions $(\ell_P, m_P, n_P, \alpha_P)$ of $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$ saitisfy*

$$\begin{aligned}
\alpha_P(t) &= \langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{\mu}_P(t)\rangle = -\lambda\bar{\ell}(t)\sin\varphi(t) + (\alpha(t) + \lambda\overline{n}(t))\cos\varphi(t),\\
\ell_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{n}(t)\rangle = \bar{\ell}(t)\cos\varphi(t) + \overline{n}(t)\sin\varphi(t),\\
m_P(t) &= \langle \dot{\boldsymbol{n}}(t), \boldsymbol{\mu}_P(t)\rangle = \overline{m}(t) + \dot{\varphi}(t),\\
n_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{\mu}_P(t)\rangle = -\bar{\ell}(t)\sin\varphi(t) + \overline{n}(t)\cos\varphi(t).
\end{aligned}$$

Proof. If $\boldsymbol{\omega}$ is a timetike vector and $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$ is a spacelike framed curve, then by Proposition 3.2 we have a $\varphi(t) : I \to \mathbb{R}$ satisfying

$$\lambda\bar{\ell}(t)\cos\varphi(t) + (\alpha(t) + \lambda\overline{m}(t))\sin\varphi(t) = 0,\ \forall\ t \in I.$$

We can define $\boldsymbol{n} : I \to \mathbb{S}^2_1$ by $\boldsymbol{n}(t) = \cos\varphi(t)\boldsymbol{v}(t) + \sin\varphi(t)\boldsymbol{\mu}(t)$. Then, $\boldsymbol{\mu}_P(t) = \boldsymbol{n}(t) \wedge \boldsymbol{\omega}(t) = -\sin\varphi(t)\boldsymbol{\nu}(t) + \cos\varphi(t)\boldsymbol{\mu}(t)$. By Definition 2.2, we have $\alpha_P(t) = \langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{\mu}_P(t)\rangle$, $\ell_P(t) = \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{n}(t)\rangle$, $m_P(t) = \langle \dot{\boldsymbol{n}}(t), \boldsymbol{\mu}_P(t)\rangle$, and $n_P(t) = \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{\mu}_P(t)\rangle$. Then, by calculation, we can conclude the proof. □

Remark 3.5. Let $\boldsymbol{w}$ be a spacelike vector. If $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$ is a spacelike framed curve, by Proposition 3.3, we have a $\varphi(t) : I \to \mathbb{R}$ satisfying

$$(-\lambda\bar{\ell}(t) + \alpha(t) + \lambda\overline{n}(t))e^{2\varphi(t)} = -(\lambda\bar{\ell}(t) + \alpha(t) + \lambda\overline{n}(t)),\ \forall\ t \in I.$$

We can define $\boldsymbol{n} : I \to \mathbb{H}^2_0$ by

$$\boldsymbol{n}(t) = \cosh\varphi(t)\boldsymbol{\nu}(t) + \sinh\varphi(t)\boldsymbol{\mu}(t).$$

Then, we have the following curvature functions $(\ell_P, m_P, n_P, \alpha_P)$ of $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$:

$$\begin{aligned}
\boldsymbol{\mu}_P(t) &= \boldsymbol{n}(t) \wedge \boldsymbol{\omega}(t) = \sinh\varphi(t)\boldsymbol{\nu}(t) + \cosh\varphi(t)\boldsymbol{\mu}(t),\\
\alpha_P(t) &= \langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{\mu}_P(t)\rangle = -\lambda\bar{\ell}(t)\sinh\varphi(t) + (\alpha(t) + \lambda\overline{n}(t))\cosh\varphi(t),\\
\ell_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{n}(t)\rangle = -\bar{\ell}(t)\cosh\varphi(t) + \overline{n}(t)\sinh\varphi(t),\\
m_P(t) &= \langle \dot{\boldsymbol{n}}(t), \boldsymbol{\mu}_P(t)\rangle = \overline{m}(t) + \dot{\varphi}(t),\\
n_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{\mu}_P(t)\rangle = -\bar{\ell}(t)\sinh\varphi(t) + \overline{n}(t)\cosh\varphi(t).
\end{aligned}$$

We can also define $\boldsymbol{n} : I \to \mathbb{H}^2_0$ by

$$\boldsymbol{n}(t) = -\cosh\varphi(t)\boldsymbol{\nu}(t) - \sinh\varphi(t)\boldsymbol{\mu}(t).$$

Then, we have the following curvature functions $(\ell_P, m_P, n_P, \alpha_P)$ of $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$:

$$\begin{aligned}
\boldsymbol{\mu}_P(t) &= \boldsymbol{n}(t) \wedge \boldsymbol{\omega}(t) = -\sinh\varphi(t)\boldsymbol{\nu}(t) - \cosh\varphi(t)\boldsymbol{\mu}(t),\\
\alpha_P(t) &= \langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{\mu}_P(t)\rangle = \lambda\bar{\ell}(t)\sinh\varphi(t) - (\alpha(t) + \lambda\overline{n}(t))\cosh\varphi(t),\\
\ell_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{n}(t)\rangle = \bar{\ell}(t)\cosh\varphi(t) - \overline{n}(t)\sinh\varphi(t),\\
m_P(t) &= \langle \dot{\boldsymbol{n}}(t), \boldsymbol{\mu}_P(t)\rangle = \overline{m}(t) + \dot{\varphi}(t),
\end{aligned}$$

$$n_P(t) = \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{\mu}_P(t)\rangle = \bar{\ell}(t)\sinh\varphi(t) - \bar{n}(t)\cosh\varphi(t).$$

Let $\boldsymbol{w}$ be a timelike vector. If $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$ is a timelike framed curve, by Proposition 3.3 we have a $\varphi(t) : I \to \mathbb{R}$ satisfying

$$(-\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t))e^{2\varphi(t)} = \lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t),\ \forall\, t \in I.$$

We can define $\boldsymbol{n} : I \to \mathbb{S}^2_1$ by

$$\boldsymbol{n}(t) = \sinh\varphi(t)\boldsymbol{\nu}(t) + \cosh\varphi(t)\boldsymbol{\mu}(t).$$

Then, we have the following curvature functions $(\ell_P, m_P, n_P, \alpha_P)$ of $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$:

$$\begin{aligned}
\boldsymbol{\mu}_P(t) &= \boldsymbol{n}(t) \wedge \boldsymbol{\omega}(t) = \cosh\varphi(t)\boldsymbol{\nu}(t) + \sinh\varphi(t)\boldsymbol{\mu}(t),\\
\alpha_P(t) &= \langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{\mu}_P(t)\rangle = -\lambda\bar{\ell}(t)\cosh\varphi(t) + (\alpha(t) + \lambda\bar{n}(t))\sinh\varphi(t),\\
\ell_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{n}(t)\rangle = -\bar{\ell}(t)\sinh\varphi(t) + \bar{n}(t)\cosh\varphi(t),\\
m_P(t) &= \langle \dot{\boldsymbol{n}}(t), \boldsymbol{\mu}_P(t)\rangle = -\bar{m}(t) - \dot{\varphi}(t),\\
n_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{\mu}_P(t)\rangle = -\bar{\ell}(t)\cosh\varphi(t) + \bar{n}(t)\sinh\varphi(t).
\end{aligned}$$

We can also define $\boldsymbol{n} : I \to \mathbb{S}^2_1$ by

$$\boldsymbol{n}(t) = -\sinh\varphi(t)\boldsymbol{\nu}(t) - \cosh\varphi(t)\boldsymbol{\mu}(t).$$

Then, we have the following curvature functions $(\ell_P, m_P, n_P, \alpha_P)$ of $(P_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w})$:

$$\begin{aligned}
\boldsymbol{\mu}_P(t) &= \boldsymbol{n}(t) \wedge \boldsymbol{\omega}(t) = -\cosh\varphi(t)\boldsymbol{\nu}(t) - \sinh\varphi(t)\boldsymbol{\mu}(t),\\
\alpha_P(t) &= \langle \dot{P}_\gamma[\boldsymbol{\omega}](t), \boldsymbol{\mu}_P(t)\rangle = \lambda\bar{\ell}(t)\cosh\varphi(t) - \{\alpha(t) + \lambda\bar{n}(t)\}\sinh\varphi(t),\\
\ell_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{n}(t)\rangle = \bar{\ell}(t)\sinh\varphi(t) - \bar{n}(t)\cosh\varphi(t),\\
m_P(t) &= \langle \dot{\boldsymbol{n}}(t), \boldsymbol{\mu}_P(t)\rangle = -\bar{m}(t) - \dot{\varphi}(t),\\
n_P(t) &= \langle \dot{\boldsymbol{\omega}}(t), \boldsymbol{\mu}_P(t)\rangle = \bar{\ell}(t)\cosh\varphi(t) - \bar{n}(t)\sinh\varphi(t).
\end{aligned}$$

4. Normal surfaces

In this section, we will discuss some surfaces constructed by a given spacelike framed curve $\boldsymbol{\gamma}(t)$. First, we will introduce a special ruled surface referred to as a normal surface. Then, we give some essential arguments of such normal surfaces which we use in the next section.

Definition 4.1. Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta$ be a spacelike framed curve with frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$. Then, we define a surface $NS_\gamma[\boldsymbol{w}] : I \times \mathbb{R} \to \mathbb{R}^3_1$ called a *normal surface* by

$$NS_\gamma[\boldsymbol{w}](t, \lambda) = \boldsymbol{\gamma}(t) + \lambda\boldsymbol{w}(t),\ \forall (t, \lambda) \in I \times \mathbb{R}.$$

Then, $\det(\dot{\boldsymbol{\gamma}}(t), \boldsymbol{\omega}(t), \dot{\boldsymbol{\omega}}(t)) = -\alpha(t)\bar{\ell}(t)$. Therefore, $NS_\gamma[\boldsymbol{\omega}](t, \lambda)$ is developable if and only if $\alpha(t)\bar{\ell}(t) = 0$. We see that if the frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ is a Bishop frame, then the normal surface with repect to the direction of $\boldsymbol{\omega}$ is always a developable surface on the regular part of $NS_\gamma[\boldsymbol{\omega}](t,\lambda)$. We call this situation *Bishop normal developable* on the regular part of $NS_\gamma[\boldsymbol{\omega}](t,\lambda)$.

By Definition 4.1, we have

$$\frac{\partial NS_\gamma[\boldsymbol{\omega}](t,\lambda)}{\partial t} \wedge \frac{\partial NS_\gamma[\boldsymbol{\omega}](t,\lambda)}{\partial \lambda} = \lambda\bar{\ell}(t)\boldsymbol{\mu}(t) - \delta(t)(\alpha(t) + \lambda\bar{n}(t))\boldsymbol{\nu}(t). \tag{4.1}$$

Therefore, by Eq (4.1) we have that $(t_0, \lambda_0) \in I \times \mathbb{R}$ is a singularity of $NS_\gamma[\boldsymbol{\omega}](t,\lambda)$ if and only if $\lambda_0\bar{\ell}(t_0) = 0$ and $\alpha(t_0) + \lambda_0\bar{n}(t_0) = 0$.

With we have done in Propositions 3.2 and 3.3, we can have a similar discussion for the problem of whether a normal surface is a framed surface. Because the proof is similar to before, here we will directly give our propositions and omit the proof.

Proposition 4.2. *Let $\boldsymbol{w}$ be a timelike vector. Then, $(NS_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w}) : I \times \mathbb{R} \to \mathbb{R}^3_1 \times \tilde{\Delta}$ is a timelike framed surface where $\boldsymbol{n} : I \times \mathbb{R} \to \mathbb{S}^2_1$ if and only if there is a $\varphi(t,\lambda) : I \times \mathbb{R} \to \mathbb{R}$ such that*

$$\lambda\bar{\ell}(t)\cos\varphi(t,\lambda) + (\alpha(t) + \lambda\bar{m}(t))\sin\varphi(t,\lambda) = 0,\ \forall (t,\lambda) \in I \times \mathbb{R}.$$

Proposition 4.3. *Let $\boldsymbol{w}$ be a spacelike vector. Then, we have the following:*

1) $(NS_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w}) : I \times \mathbb{R} \to \mathbb{R}^3_1 \times \tilde{\Delta}$ is a timelike framed surface $\boldsymbol{n} : I \times \mathbb{R} \to \mathbb{S}^2_1$ if and only if there is a $\varphi(t,\lambda) : I \times \mathbb{R} \to \mathbb{R}$ such that

$$(-\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t))e^{2\varphi(t,\lambda)} = -(\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t)),\ \forall (t,\lambda) \in I \times \mathbb{R}.$$

2) $(NS_\gamma[\boldsymbol{w}], \boldsymbol{n}, \boldsymbol{w}) : I \times \mathbb{R} \to \mathbb{R}^3_1 \times \Delta_1$ is a spacelike framed surface $\boldsymbol{n} : I \times \mathbb{R} \to \mathbb{H}^2_0$ if and only if there is a $\varphi(t,\lambda) : I \times \mathbb{R} \to \mathbb{R}$ such that

$$(-\lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t))e^{2\varphi(t,\lambda)} = \lambda\bar{\ell}(t) + \alpha(t) + \lambda\bar{n}(t),\ \forall (t,\lambda) \in I \times \mathbb{R}.$$

If $\bar{\ell}(t) = 0$, the frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ is a Bishop frame, and we then obviously see that $NS_\gamma[\boldsymbol{\omega}](t,\lambda)$ is always a spacelike or timelike framed base surface in Minkowski 3-space by taking $\varphi(t,\lambda) = 0$ in Propositions 4.2 and 4.3.

In the following theorem, we will show the local behavior of the singularities of the normal surface for a given spacelike framed curve.

Theorem 4.4. *Suppose that $(t_0, \lambda_0) \in I \times \mathbb{R}$ is a singualarity of $NS_\gamma[\boldsymbol{\omega}](t,\lambda)$. Then, the singularity (t_0, λ_0) of $NS_\gamma[\boldsymbol{\omega}](t,\lambda)$ is a cross cap if and only if*

$$\alpha(t_0)\dot{\bar{\ell}}(t_0) + \dot{\alpha}(t_0)\bar{\ell}(t_0) \neq 0.$$

Proof. By Whitney's theorem [34], the sufficient and necessary condition that a singularity (t_0, λ_0) of $NS_\gamma[\boldsymbol{\omega}](t,\lambda)$ is a cross cap is $\det(NS_\gamma[\boldsymbol{\omega}]_\lambda, NS_\gamma[\boldsymbol{\omega}]_{\lambda t}, NS_\gamma[\boldsymbol{\omega}]_{tt}) \neq 0$. According to the calculation results, we have

$$\begin{aligned} NS_\gamma[\boldsymbol{\omega}]_\lambda &= \boldsymbol{\omega}(t), \\ NS_\gamma[\boldsymbol{\omega}]_{\lambda t} &= \bar{\ell}(t)\boldsymbol{\nu}(t) + \bar{n}(t)\boldsymbol{\mu}(t), \end{aligned}$$

$$
\begin{aligned}
NS_{\gamma}[\boldsymbol{\omega}]_t &= \alpha(t)\boldsymbol{\mu}(t) + \lambda\overline{\ell}\boldsymbol{\nu}(t) + \lambda\overline{n}(t)\boldsymbol{\mu}(t),\\
NS_{\gamma}[\boldsymbol{\omega}]_{tt} &= (\dot{\alpha}(t) + \lambda\overline{\ell}(t)\overline{m}(t) + \lambda\dot{\overline{n}}(t))\boldsymbol{\mu}(t)\\
&\quad + (-\delta(t)\alpha(t)\overline{m}(t) + \lambda\dot{\overline{\ell}}(t)) - \lambda\delta(t)\overline{n}(t)\overline{m}(t))\boldsymbol{\nu}(t)\\
&\quad + (\delta(t)\alpha(t)\overline{n}(t) + \lambda\delta(t)\overline{n}^2(t) + \lambda\overline{\ell}^2(t))\boldsymbol{\omega}(t)).
\end{aligned}
$$

On the other hand, if (t_0, λ_0) is a singularity of $NS_{\gamma}[\boldsymbol{\omega}](t)$, then we have $\lambda_0\overline{\ell}(t_0) = 0$ and $\alpha(t_0)+\lambda_0\overline{n}(t_0) = 0$. So, we can get

$$
NS_{\gamma}[\boldsymbol{\omega}]_{tt}\Big|_{(t_0,\lambda_0)} = (\dot{\alpha}(t_0) + \lambda_0\dot{\overline{n}}(t_0))\boldsymbol{\mu}(t_0) + \lambda_0\dot{\overline{\ell}}(t_0)\boldsymbol{\nu}(t_0).
$$

Therefore,

$$
\begin{aligned}
\det(NS_{\gamma}[\boldsymbol{\omega}]_{\lambda}, NS_{\gamma}[\boldsymbol{\omega}]_{\lambda t}, NS_{\gamma}[\boldsymbol{\omega}]_{tt})\Big|_{(t_0,\lambda_0)} &= \overline{\ell}(t_0)(\dot{\alpha}(t_0) + \lambda_0\overline{n}(t_0)) - \overline{n}(t_0)\lambda_0\dot{\overline{\ell}}(t_0)\\
&= \alpha(t_0)\dot{\overline{\ell}}(t_0) + \dot{\alpha}(t_0)\overline{\ell}(t_0).
\end{aligned}
$$

This completes the proof. □

If the frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ is a Bishop frame, which means $\overline{\ell}(t) \equiv 0$, then $\alpha(t_0)\dot{\overline{\ell}}(t_0) + \dot{\alpha}(t_0)\overline{\ell}(t_0) \equiv 0$. Therefore, by Theorem 4.4, we have the following corollary:

Corollary 4.5. *Let $(\gamma(t), \boldsymbol{\nu}, \boldsymbol{\omega})$ be a spacelike framed curve with a Bishop frame of $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$. Then, the singularity of $NS_{\gamma}[\boldsymbol{\omega}](t, \lambda)$ can not be a cross cap.*

Remark 4.6. We already know that the classification of singularities is well established not only for frontals or fronts in Euclidean 3-space, but also non-lightlike frontals or fronts in Minkowski 3-space [15,30,31]. Roughly speaking, the classification of singularities here consists of two parts. The first part is about *non-degenerate singularities*. For the case of fronts about non-degenerate singularities, we can have the necessary and sufficient conditions for the recognization of the singularities of a *cuspidal edge*, *swallowtail*, and *cuspidal butterfly* [9, 15]. For the case of frontals about non-degenerate singularities, we can have the necessary and sufficient conditions for the recognization of the singularities of a *cuspidal cross cap* [15]. In the case of fronts about degenerate singularities with the *corank one* condition, we can have the necessary and sufficient conditions for the recognization of the singularities of *cuspidal lips* and *cuspidal beaks* [15]. In the case of frontals about degenerate singularities with the corank one condition, we can have the necessary and sufficient conditions for the recognization of the *Chen Matumoto Mond ± singularities* [28, 30]. In further related work, we will give detailed classification results, but it is not the main theme of this article. Thus, we will not go into the details in this article.

5. Circular evolutes and involutes with respect to Bishop frames

In this section, we will discuss the circular evolutes and involutes of a spacelike framed curve in Minkowski 3-space with respect to a Bishop frame. First, we give the definition of circular evolutes.

Definition 5.1. Let $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta$ be a spacelike framed curve with a Bishop frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$, that is, $\bar{\ell}(t) = 0$ for all $t \in I$. We assume that $\bar{n}(t) \neq 0$ for all $t \in I$. Then, we define a curve $E_\gamma[\boldsymbol{\omega}] : I \to \mathbb{R}^3_1$ in Minkowski 3-space called a *circular evolute* by

$$E_\gamma[\boldsymbol{\omega}](t) = \gamma(t) - \frac{\alpha(t)}{\bar{n}(t)}\boldsymbol{\omega}(t).$$

Then, in the following two propositions, we will study the relations between circular evolutes and normal surfaces for a given spacelike framed curve.

Proposition 5.2. *Let $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta$ be a spacelike framed curve with a frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ which is Bishop. Then, the circular evolute of $\gamma(t)$ is the striction curve of the normal surface $NS_\gamma[\boldsymbol{\omega}](t, \lambda)$.*

Proof. Suppose that $\boldsymbol{\sigma}(t) : I \to \mathbb{R}^3_1$ is the striction of $NS_\gamma[\boldsymbol{\omega}]$. Then, we have

$$\boldsymbol{\sigma}(t) = \gamma(t) - \frac{\langle \dot{\gamma}(t), \dot{\boldsymbol{\omega}}(t)\rangle}{\langle \dot{\boldsymbol{\omega}}(t), \dot{\boldsymbol{\omega}}(t)\rangle}\boldsymbol{\omega}(t).$$

Because $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ is a Bishop frame, then we have

$$\boldsymbol{\sigma}(t) = \gamma(t) - \frac{\langle \alpha(t)\boldsymbol{\mu}(t), \bar{\ell}(t)\boldsymbol{\nu}(t) + \bar{n}(t)\boldsymbol{\mu}(t)\rangle}{\langle \bar{\ell}(t)\boldsymbol{\nu}(t) + \bar{n}(t)\boldsymbol{\mu}(t), \bar{\ell}(t)\boldsymbol{\nu}(t) + \bar{n}(t)\boldsymbol{\mu}(t)\rangle}\boldsymbol{\omega}(t) = \gamma(t) - \frac{\alpha(t)}{\bar{n}(t)}\boldsymbol{\omega}(t).$$

This concludes the proof. □

Proposition 5.3. *Let $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta$ be a spacelike framed curve with a Bishop frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$, that is, $\bar{\ell}(t) = 0$ for all $t \in I$. Then, the singular point set of $NS_\gamma[\boldsymbol{\omega}](t, \lambda)$ is the circular evolute of $\gamma(t)$.*

Proof. $(t_0, \lambda_0) \in I \times \mathbb{R}$ is a singularity of $NS_\gamma[\boldsymbol{\omega}]$ if and only if

$$\begin{aligned} \lambda_0\bar{\ell}(t_0) &= 0, \\ \alpha(t_0) + \lambda_0\bar{n}(t_0) &= 0. \end{aligned} \tag{5.1}$$

Because $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ is a Bishop frame, Eq (5.1) is equivalent to $\alpha(t_0) + \lambda_0\bar{n}(t_0) = 0$. If (t_0, λ_0) is a singularity of $NS_\gamma[\boldsymbol{\omega}]$, we have

$$NS_\gamma[\boldsymbol{\omega}](t_0, \lambda_0) = \gamma(t_0) - \frac{\alpha(t_0)}{\bar{n}(t_0)} = E_\gamma[\boldsymbol{\omega}](t_0).$$

This concludes the proof. □

By Definition 5.1, we have

$$\dot{E}_\gamma[\boldsymbol{\omega}](t) = \alpha(t)\boldsymbol{\mu}(t) - \frac{\alpha(t)}{\bar{n}(t)}\bar{n}(t)\boldsymbol{\mu}(t)) - \frac{d}{dt}\left(\frac{\alpha(t)}{\bar{n}(t)}\right)\boldsymbol{\omega}(t) = -\frac{d}{dt}\left(\frac{\alpha(t)}{\bar{n}(t)}\right)\boldsymbol{\omega}(t).$$

Obviously, we can find $\langle \dot{E}_\gamma[\omega](t), \boldsymbol{\nu}(t)\rangle = 0$, $\langle \dot{E}_\gamma[\omega](t), \boldsymbol{\mu}(t)\rangle = 0$, and $\boldsymbol{\nu}(t) \wedge \boldsymbol{\mu}(t) = \delta(t)\boldsymbol{\omega}(t)$. So, $(E_\gamma[\boldsymbol{\omega}](t), \boldsymbol{\nu}, \boldsymbol{\mu}) : I \to \mathbb{R}^3_1 \times \Delta$ is a spacelike or timelike framed curve in Minkowski 3-space with moving frame $\{\boldsymbol{\nu}, \boldsymbol{\mu}, \delta\boldsymbol{\omega}\}$. We can get the corresponding invariant functions as follows:

$$\alpha_E(t) = \langle \dot{\gamma}(t), \delta(t)\boldsymbol{\omega}(t)\rangle = -\frac{d}{dt}\left(\frac{\alpha(t)}{\bar{n}(t)}\right),$$

$$\ell_E(t) = \langle \dot{\boldsymbol{\nu}}(t), \boldsymbol{\mu}(t)\rangle = \overline{m}(t),$$
$$m_E(t) = \langle \dot{\boldsymbol{\nu}}(t), \delta(t)\boldsymbol{\omega}(t)\rangle = 0,$$
$$n_E(t) = \langle \dot{\boldsymbol{\mu}}(t), \delta(t)\boldsymbol{\omega}(t)\rangle = -\delta(t)\overline{n}(t).$$

For the convenience of expression, we denote the $\boldsymbol{\omega}$-evolute of $\boldsymbol{\gamma}(t)$ as $E_\gamma[\boldsymbol{\omega}](t)$, and denote the $\boldsymbol{\omega}$-parallel curve of $\gamma(t)$ as $P_\gamma[\boldsymbol{\omega}](t)$. Then, we have the following duality relation between a spacelike framed curve $\boldsymbol{\gamma}(t)$ and the parallel curve with respect to $\boldsymbol{\omega}$-evolute:

Proposition 5.4. *Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}, \boldsymbol{\omega})$ be a spacelike framed curve. Then, we have*

$$E_{P_\gamma}[\omega](t) = E_\gamma[\omega](t).$$

Proof. Because here $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ is a Bishop frame, we have $\overline{\ell}(t) = 0$. Then, if $\boldsymbol{\omega}$ is a timelike vector, by Proposition 3.2, we can take $\varphi(t) = 0$. By Proposition 3.4, we have the following:

$$\begin{aligned} E_{P_\gamma}[\boldsymbol{\omega}](t) &= P_\gamma[\boldsymbol{\omega}](t) - \frac{\alpha_P(t)}{n_P(t)}\boldsymbol{\omega}(t) \\ &= \boldsymbol{\gamma}(t) + \lambda\boldsymbol{\omega}(t) - \frac{\alpha(t) + \lambda\overline{n}(t)}{\overline{n}(t)}\boldsymbol{\omega}(t) \\ &= E_\gamma[\boldsymbol{\omega}](t). \end{aligned}$$

If $\boldsymbol{\omega}$ is a spacelike vector, we can also substitute the corresponding $\alpha_P(t)$ and $n_P(t)$ into the $\boldsymbol{\omega}$-evolutes of $P_\gamma[\boldsymbol{\omega}](t)$. Then, we have $E_{P_\gamma}[\boldsymbol{\omega}](t) = E_\gamma[\boldsymbol{\omega}](t)$. □

In the following, we consider the singular points of circular evolutes.

Definition 5.5. Let $\boldsymbol{\gamma} : I \to \mathbb{R}^3_1$ be a smooth curve in Lorentz-Minkowski 3-space. $t \in I$ is said to be an $(n, n+1)$-*cusp singularity* of $\boldsymbol{\gamma}(t)$ if $\operatorname{rank}(\gamma^{(n)}(t), \gamma^{(n+1)}(t)) = 2$ and $\dot{\gamma}(t) = \ddot{\gamma}(t) = \gamma^{(3)}(t) = \cdots = \gamma^{(n-1)}(t) = 0$.

Theorem 5.6. *Let $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta$ be a spacelike framed curve, and let the frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ be a Bishop frame. We also assume that $\overline{n}(t) \neq 0$ for all $t \in I$. Let t_0 be a singularity of $\boldsymbol{\gamma}(t)$, which means $\alpha(t_0) = 0$. Then, we have the following conclusions:*

1) t_0 is a $(2, 3)$-cusp singularty of $\boldsymbol{\gamma}(t)$ if and only if t_0 is a regular point of $E_\gamma[\boldsymbol{\omega}](t)$.
2) t_0 is an $(n+1, n+2)$-cusp singularity of $\boldsymbol{\gamma}(t)$ if and only if t_0 is an $(n, n+1)$-cusp of $E_\gamma[\boldsymbol{\omega}](t)$ for any $n \geqslant 2, n \in \mathbb{N}$.

Proof. 1) By Eq (2.2), we have

$$\begin{aligned} \ddot{\gamma}(t_0) &= \dot{\alpha}(t_0)\boldsymbol{\mu}(t_0) + \alpha(t_0)\dot{\boldsymbol{\mu}}(t_0), \\ \dddot{\gamma}(t_0) &= \ddot{\alpha}(t_0)\boldsymbol{\mu}(t_0) + 2\dot{\alpha}(t_0)\delta(t_0)(-\overline{m}(t_0)\boldsymbol{\nu}(t_0) + \overline{n}(t_0)\boldsymbol{\omega}(t_0)) + \alpha(t_0)\ddot{\boldsymbol{\mu}}(t_0). \end{aligned} \tag{5.2}$$

If t_0 is a $(2, 3)$-cusp of $\gamma(t)$, then we have $\dot{\gamma}(t_0) = \mathbf{0}$, $\operatorname{rank}(\ddot{\gamma}(t_0), \dddot{\gamma}(t_0)) = 2$, and $\overline{n}(t_0) \neq 0$. Therefore, the singularity of $\boldsymbol{\gamma}(t)$ is a $(2, 3)$-cusp if and only if $\dot{\alpha}(t_0) \neq 0$.

On the other hand,

$$\dot{E}_\gamma[\boldsymbol{\omega}](t_0) = \left[-\frac{d}{dt}\left(\frac{\alpha(t)}{\overline{n}(t)}\right)\boldsymbol{\omega}(t)\right]_{t=t_0} = -\frac{1}{\overline{n}^2(t_0)}\left(\dot{\alpha}(t_0)\overline{n}(t_0) - \alpha(t_0)\dot{\overline{n}}(t_0)\right).$$

Then, we have that $E_\gamma[\boldsymbol{\omega}](t)$ is regular if and only if $\dot{\alpha}(t_0) \neq 0$. Therefore, we have completed the first part of the proof.

2) According to calculations, we have

$$\begin{aligned}
\boldsymbol{\gamma}'(t) &= \alpha(t)\boldsymbol{\mu}(t),\\
\boldsymbol{\gamma}''(t) &= \alpha(t)\boldsymbol{\mu}'(t) + \alpha'(t)\boldsymbol{\mu}(t),\\
\boldsymbol{\gamma}'''(t) &= \alpha(t)\boldsymbol{\mu}''(t) + 2\alpha'(t)\boldsymbol{\mu}'(t) + \alpha''(t)\boldsymbol{\mu}(t),\\
&\cdots\cdots\\
\boldsymbol{\gamma}^{(n+1)}(t) &= C_n^0\alpha(t)\boldsymbol{\mu}^{(n)}(t) + C_n^1\alpha'(t)\boldsymbol{\mu}^{(n-1)}(t)\\
&+ \cdots + C_n^{n-1}\alpha^{(n-1)}(t)\boldsymbol{\mu}'(t) + C_n^n\alpha^{(n)}(t)\boldsymbol{\mu}(t),\\
\boldsymbol{\gamma}^{(n+2)}(t) &= C_{n+1}^0\alpha(t)\boldsymbol{\mu}^{(n+1)}(t) + C_{n+1}^1\alpha'(t)\boldsymbol{\mu}^{(n)}(t)\\
&+ \cdots + C_{n+1}^n\alpha^{(n)}(t)\boldsymbol{\mu}'(t) + C_{n+1}^{n+1}\alpha^{(n+1)}(t)\boldsymbol{\mu}(t)\\
&= C_{n+1}^0\alpha(t)\boldsymbol{\mu}^{(n+1)}(t) + C_{n+1}^1\alpha'(t)\boldsymbol{\mu}^{(n)}(t)\\
&+ \cdots + C_{n+1}^n\alpha^{(n)}(t)\delta(t)\left(-\overline{m}(t)\boldsymbol{\nu}(t) + \overline{n}(t)\boldsymbol{\omega}(t)\right) + C_{n+1}^{n+1}\alpha^{(n+1)}(t)\boldsymbol{\mu}(t).
\end{aligned}$$

If t_0 is an $(n+1, n+2)$-cusp singularity of $\boldsymbol{\gamma}(t)$, then we have

$$\begin{cases}
\operatorname{rank}\left(\boldsymbol{\gamma}^{(n+1)}(t), \boldsymbol{\gamma}^{(n+2)}(t)\right) = 2,\\
\boldsymbol{\gamma}'(t_0) = \boldsymbol{\gamma}''(t_0) = \cdots = \boldsymbol{\gamma}^{(n)}(t_0) = \mathbf{0}.
\end{cases}$$

By the above equations, we can get that t_0 is an $(n+1, n+2)$-cusp of $\boldsymbol{\gamma}(t)$ if and only if

$$\begin{cases}
\alpha^{(n)}(t_0) \neq 0,\\
\alpha(t_0) = \alpha'(t_0) = \cdots = \alpha^{(n-1)}(t_0) = 0.
\end{cases}$$

On the other hand,

$$\begin{aligned}
E_\gamma[\boldsymbol{\omega}]'(t) &= -\frac{d}{dt}\left(\frac{\alpha(t)}{\overline{n}(t)}\right)\boldsymbol{\omega}(t),\\
E_\gamma[\boldsymbol{\omega}]''(t) &= \left(-\left(\frac{\alpha(t)}{\overline{n}(t)}\right)'\boldsymbol{\omega}(t)\right)' = -\left[\left(\frac{\alpha(t)}{\overline{n}(t)}\right)'\boldsymbol{\omega}'(t) + \left(\frac{\alpha(t)}{\overline{n}(t)}\right)''\boldsymbol{\omega}(t)\right],\\
&\cdots\cdots\\
E_\gamma[\boldsymbol{\omega}]^{(n)}(t) &= -\Bigg[C_{n-1}^0\left(\frac{\alpha(t)}{\overline{n}(t)}\right)'\boldsymbol{\omega}^{(n-1)}(t) + C_{n-1}^1\left(\frac{\alpha(t)}{\overline{n}(t)}\right)''\boldsymbol{\omega}^{(n-2)}(t)\\
&\quad + \cdots + C_{n-1}^{n-2}\left(\frac{\alpha(t)}{\overline{n}(t)}\right)^{(n-1)}\boldsymbol{\omega}'(t) + C_{n-1}^0\left(\frac{\alpha(t)}{\overline{n}(t)}\right)^{(n)}\boldsymbol{\omega}(t)\Bigg],\\
E_\gamma[\boldsymbol{\omega}]^{(n+1)}(t) &= -\Bigg[C_n^0\left(\frac{\alpha(t)}{\overline{n}(t)}\right)'\boldsymbol{\omega}^{(n)}(t) + C_n^1\left(\frac{\alpha(t)}{\overline{n}(t)}\right)''\boldsymbol{\omega}^{n-1}(t)\\
&\quad + \cdots + C_n^{n-1}\left(\frac{\alpha(t)}{\overline{n}(t)}\right)^{(n)}(\bar{\ell}(t)\boldsymbol{\nu}(t) + \overline{n}(t)\boldsymbol{\mu}(t)) + C_n^0\left(\frac{\alpha(t)}{\overline{n}(t)}\right)^{(n+1)}\boldsymbol{\omega}(t)\Bigg].
\end{aligned}$$

If the singular point t_0 of $E_\gamma[\boldsymbol{\omega}]$ is an $(n, n+1)$-cusp, by the above equations it is equivalent to

$$\begin{aligned} -\frac{d}{dt}\left(\frac{\alpha(t)}{\overline{n}(t)}\right)|_{t=t_0} = \left(\frac{\alpha(t_0)}{\overline{n}(t_0)}\right)'' = \cdots = \left(\frac{\alpha(t_0)}{\overline{n}(_0)}\right)^{(n-1)} = 0, \\ \left(\frac{\alpha(t_0)}{\overline{n}(_0)}\right)^{(n)} \neq 0. \end{aligned} \tag{5.3}$$

Furthermore,

$$\begin{aligned} &\left(\frac{\alpha(t)}{\overline{n}(t)}\right)' = \alpha(t)\left(\frac{1}{\overline{n}(t)}\right)' + \alpha'(t)\frac{1}{\overline{n}(t)}, \\ &\left(\frac{\alpha(t)}{\overline{n}(t)}\right)'' = \alpha(t)\left(\frac{1}{\overline{n}(t)}\right)'' + 2\alpha'(t)\left(\frac{1}{\overline{n}(t)}\right)' + \alpha''(t)\frac{1}{\overline{n}(t)}, \\ &\quad\cdots\cdots \\ &\left(\frac{\alpha(t)}{\overline{n}(t)}\right)^{(n-1)} = C_{n-1}^0\alpha\left(\frac{1}{\overline{n}}\right)^{(n-1)} + C_{n-1}^1\alpha'\left(\frac{1}{\overline{n}}\right)^{(n-2)} + \cdots + C_{n-1}^{n-1}\alpha^{(n-1)}\left(\frac{1}{\overline{n}}\right), \\ &\left(\frac{\alpha(t)}{\overline{n}(t)}\right)^{(n)} = C_n^0\alpha\left(\frac{1}{\overline{n}}\right)^{(n)} + C_n^1\alpha'\left(\frac{1}{\overline{n}}\right)^{(n-1)} + \cdots + C_n^n\alpha^{(n)}\left(\frac{1}{\overline{n}}\right). \end{aligned} \tag{5.4}$$

Because $\alpha(t_0) = 0$, $\overline{n}(t_0) \neq 0$, by Eqs (5.3) and (5.4), we can get that the singular point t_0 of $E_\gamma[\boldsymbol{\omega}]$ is an $(n, n+1)$-cusp if and only if

$$\begin{cases} \alpha^{(n)}(t_0) \neq 0, \\ \alpha(t_0) = \alpha'(t_0) = \cdots = \alpha^{(n-1)}(t_0) = 0. \end{cases}$$

Then, we have completed the second part of the proof. □

In the following, we will study the relations between circular evolutes and involutes for a given spacelike framed curve.

Definition 5.7. Let $(\gamma, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}_1^3 \times \Delta$ be a spacelike framed curve with $m^2(t) - n^2(t) > 0$ for all $t \in I$. Then, we define a curve $I_\gamma[t_0](t) : I \to \mathbb{R}_1^3$ in Minkowski 3-space called an *involute* of $\boldsymbol{\gamma}(t)$ with respect to a fixed $t_0 \in I$ by

$$I_\gamma[t_0](t) = \boldsymbol{\gamma}(t) - \left(\int_{t_0}^t \alpha(t)dt\right)\boldsymbol{\mu}(t)$$

for a fixed $t_0 \in I$.

We define $\boldsymbol{\xi}(t)$, $\boldsymbol{\eta}(t)$ by

$$\boldsymbol{\xi}(t) = \frac{n(t)\boldsymbol{\nu}_1(t) - m(t)\boldsymbol{\nu}_2(t)}{\sqrt{m^2(t) - n^2(t)}}, \ \boldsymbol{\eta}(t) = \boldsymbol{\xi}(t) \wedge \boldsymbol{\mu}(t) = \delta(t)\frac{-m(t)\boldsymbol{\nu}_1(t) + n(t)\boldsymbol{\nu}_2(t)}{\sqrt{m^2(t) - n^2(t)}}.$$

Then, we have

$$\dot{\boldsymbol{\xi}}(t) = \Big(\frac{\dot{n}(t)(m^2(t) - n^2(t)) - m\dot{m}(t)n(t) + n^2(t)\dot{n}(t)}{(m^2(t) - n^2(t))^{\frac{3}{2}}} - \frac{m(t)\ell(t)}{\sqrt{m^2(t) - n^2(t)}}\Big)\boldsymbol{\nu}_1(t)$$

$$+\left(\frac{\dot{m}(t)(m^2(t)-n^2(t))-m^2(t)\dot{m}(t)+m(t)n(t)\dot{n}(t)}{(m^2(t)-n^2(t))^{\frac{3}{2}}}+\frac{n(t)\ell(t)}{\sqrt{m^2(t)-n^2(t)}}\right)\boldsymbol{\nu}_2(t),$$

$$\dot{I}_\gamma[t_0](t)=\left(\int_{t_0}^{t}\alpha(t)dt\right)\left(\delta(t)m(t)\boldsymbol{\nu}_1(t)-\delta(t)n(t)\boldsymbol{\nu}_2(t)\right),$$

and $\langle \dot{I}_\gamma[t_0](t),\boldsymbol{\xi}(t)\rangle=\langle \dot{I}_\gamma[t_0](t),\boldsymbol{\mu}(t)\rangle=0$. Therefore, $(I_\gamma[t_0](t),\boldsymbol{\xi}(t),\boldsymbol{\mu}(t)):I\to\mathbb{R}^3_1\times\Delta$ is a spacelike or timelike framed curve with the curvature $(\alpha_I,\ell_I,m_I,n_I)$ as follows:

$$\begin{aligned}
\alpha_I(t)&=\langle \dot{I}_\gamma[t_0](t),\boldsymbol{\eta}(t)\rangle=-\delta(t)\left(\int_{t_0}^{t}\alpha(t)dt\right)\sqrt{m^2(t)-n^2(t)},\\
\ell_I(t)&=\langle\dot{\boldsymbol{\xi}}(t),\boldsymbol{\mu}(t)\rangle=0,\\
m_I(t)&=\langle\dot{\boldsymbol{\xi}}(t),\boldsymbol{\eta}(t)\rangle=\frac{-m(t)\dot{n}(t)-\dot{m}(t)n(t)+(m^2(t)-n^2(t)\ell(t))}{m^2(t)-n^2(t)},\\
n_I(t)&=\langle\dot{\boldsymbol{\mu}}(t),\boldsymbol{\eta}(t)\rangle=\delta(t)\sqrt{m^2(t)-n^2(t)}.
\end{aligned}$$

By Definition 2.4, we can see that $\{\boldsymbol{\xi},\boldsymbol{\mu},\boldsymbol{\eta}\}$ is a Bishop frame along $I_\gamma[t_0](t)$.

Proposition 5.8. *Let $(\gamma,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2):I\to\mathbb{R}^3_1\times\Delta$ be a spacelike framed curve with $m^2(t)-n^2(t)>0$ for all $t\in I$. Then, $E_{I_\gamma[t_0]}[\boldsymbol{\mu}](t)=\boldsymbol{\gamma}(t)$ for any fixed $t_0\in I$.*

Proof. By Definitions 5.1 and 5.7, we have

$$\begin{aligned}
E_{I_\gamma[t_0]}[\boldsymbol{\mu}](t)&=I_\gamma[t_0](t)-\frac{\alpha_I(t)}{n_I(t)}\boldsymbol{\mu}(t)\\
&=I_\gamma[t_0](t)-\left(\int_{t_0}^{t}\alpha(t)dt\right)\boldsymbol{\mu}(t)-\frac{-\delta(t)\left(\int_{t_0}^{t}\alpha(t)dt\right)\sqrt{m^2(t)-n^2(t)}}{\delta(t)\sqrt{m^2(t)-n^2(t)}}\boldsymbol{\mu}(t)\\
&=\boldsymbol{\gamma}(t)-\left(\int_{t_0}^{t}\alpha(t)dt\right)\boldsymbol{\mu}(t)+\left(\int_{t_0}^{t}\alpha(t)dt\right)\boldsymbol{\mu}(t)\\
&=\boldsymbol{\gamma}(t).
\end{aligned}$$

This concludes the proof. □

Proposition 5.9. *Let $(\gamma,\boldsymbol{\nu}_1,\boldsymbol{\nu}_2):I\to\mathbb{R}^3_1\times\Delta$ be a spacelike framed curve, and let $\{\boldsymbol{\nu},\boldsymbol{\omega},\boldsymbol{\mu}\}$ be a Bishop frame of $\boldsymbol{\gamma}(t)$ with $\overline{n}(t)\neq 0$ for all $t\in I$. Then, we have that $I_{E_\gamma[\delta\omega]}[t_0](t)$ is a parallel curve of $\boldsymbol{\gamma}(t)$. In particular, if t_0 is a singular point of $\boldsymbol{\gamma}(t)$, we have $I_{E_\gamma[\delta\omega]}[t_0](t)=\boldsymbol{\gamma}(t)$.*

Proof. By Definitions 5.1 and 5.7, we have

$$\begin{aligned}
I_{E_\gamma[\delta\omega]}[t_0](t)&=E_\gamma[\delta(t)\boldsymbol{\omega}](t)-\left(\int_{t_0}^{t}\alpha_E(t)dt\right)\delta(t)\boldsymbol{\omega}(t)\\
&=\boldsymbol{\gamma}(t)-\frac{\alpha(t)}{\overline{n}(t)}\delta(t)\boldsymbol{\omega}(t)-\left(\int_{t_0}^{t}-\frac{d}{dt}\left(\frac{\alpha(t)}{\overline{n}(t)}\right)dt\right)\delta(t)\boldsymbol{\omega}(t)\\
&=\boldsymbol{\gamma}(t)-\frac{\alpha(t_0)}{\overline{n}(t_0)}\delta(t)\boldsymbol{\omega}(t).
\end{aligned}$$

This concludes the proof. □

We now consider the singular points of involutes in the following:

Theorem 5.10. *Let* $(\boldsymbol{\gamma}, \boldsymbol{\nu}_1, \boldsymbol{\nu}_2) : I \to \mathbb{R}^3_1 \times \Delta$ *be a spacelike framed curve, and let the frame* $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ *be a Bishop frame. We also assume that* $m^2(t) - n^2(t) > 0$ *for all* $t \in I$*, and let* t_1 *be a singularity of* $I_\gamma[t_0](t)$*, which means* $\alpha_I(t_1) = 0$*. Then, we have the following conclusions:*

1) The point t_1 of $\boldsymbol{\gamma}(t)$ is regular if and only if t_1 of $I_\gamma[t_0]$ is a $(2,3)$-cusp.
2) The singular point t_1 of $\boldsymbol{\gamma}(t)$ is an $(n, n+1)$-cusp if and only if t_1 of $I_\gamma[t_0](t)$ is an $(n+1, n+2)$-cusp for any $n \geqslant 2, n \in \mathbb{N}$.

Proof. 1) The point t_1 of $\boldsymbol{\gamma}(t)$ is regular if and only if $\alpha(t_1) \neq 0$. The singularity t_1 of $I_\gamma[t_0](t)$ is a $(2,3)$-cusp if and only if $\operatorname{rank}(\ddot{I}_\gamma[t_0](t_1), \dddot{I}_\gamma[t_0](t_1)) = 2$ and $\dot{I}_\gamma[t_0](t_1) = \mathbf{0}$, which means $\alpha_I(t_1) = 0$ and $\dot{\alpha}_I(t_1) \neq 0$. Since

$$
\begin{aligned}
\dot{I}_\gamma[t_0](t) &= \delta(t)\alpha_I(t)\boldsymbol{\eta}(t),\\
\alpha_I(t) &= -\left(\delta(t)\int_{t_0}^{t}\alpha(t)dt\right)\sqrt{m^2(t) - n^2(t)}\,,\\
\dot{\alpha}_I(t) &= -\delta(t)\alpha(t)\sqrt{m^2(t) - n^2(t)} - \left(\delta(t)\int_{t_0}^{t}\alpha(t)dt\right)\frac{d}{dt}\left(\sqrt{m^2(t) - n^2(t)}\right)
\end{aligned}
$$

and

$$m^2(t) - n^2(t) > 0,$$

we can get the conclusion of 1).

2) By the calculations of Theorem 5.6, we have

$$
\begin{aligned}
I_\gamma^{(n)}(t) &= (\delta\alpha_I\boldsymbol{\eta})^{(n-1)} = C_{n-1}^0\delta\alpha_I\boldsymbol{\eta}^{(n-1)} + \cdots + (\delta\alpha_I)^{(n-1)}\boldsymbol{\eta},\\
I_\gamma^{(n+1)}(t) &= C_n^0\delta\alpha_I\boldsymbol{\eta}^{(n)} + \cdots + C_n^{n-1}(\delta\alpha_I)^{(n-1)}\boldsymbol{\eta}' + C_n^n(\delta\alpha_I)^{(n)}\boldsymbol{\eta},\\
I_\gamma^{(n+2)}(t) &= C_{n+1}^0\delta\alpha_I\boldsymbol{\eta}^{(n+1)} + \cdots + C_{n+1}^n(\delta\alpha_I)^{(n)}\boldsymbol{\eta}' + C_{n+1}^{n+1}(\delta\alpha_I)^{(n+1)}\boldsymbol{\eta}.
\end{aligned}
$$

Thus, t_1 is an $(n+1, n+2)$-cusp of $I_\gamma[t_0](t)$ if and only if

$$
\begin{cases}
\alpha_I(t_1) = \dot{\alpha}_I(t_1) = \cdots = \alpha_I^{(n-1)}(t_1) = 0,\\
\alpha_I^{(n)}(t_1) \neq 0.
\end{cases}
\tag{5.5}
$$

Furthermore,

$$
\begin{aligned}
-\delta(t)\alpha_I^{(n)}(t) = &C_n^0\left(\int_{t_0}^{t}\alpha(t)dt\right)\sqrt{m^2(t) - n^2(t)}^{(n)} + C_n^1\alpha(t)\sqrt{m^2(t) - n^2(t)}^{(n-1)} + \cdots\\
&+ C_n^{n-1}\alpha^{(n-2)}(t)\sqrt{m^2(t) - n^2(t)}' + C_n^n\alpha(t)^{(n-1)}\sqrt{m^2(t) - n^2(t)}
\end{aligned}
$$

and

$$m^2(t) - n^2(t) > 0.$$

We have that Eq (5.5) is equivalent to

$$
\begin{cases}
\alpha(t_1) = \dot{\alpha}(t_1) = \cdots = \alpha^{(n-2)}(t_1) = 0,\\
\alpha^{(n-1)}(t_1) \neq 0.
\end{cases}
\tag{5.6}
$$

Namely, the singular point t_1 of $\boldsymbol{\gamma}(t)$ is an $(n, n+1)$-cusp if and only if t_1 is an $(n+1, n+2)$-cusp of $I_\gamma[t_0](t)$ for any $n \geqslant 2, n \in \mathbb{N}$. This concludes the proof. □

6. Example

In the following example, we give a spacelike framed curve in Minkowski 3-space. In this example, we will discuss its circular evolutes, involutes, normal surfaces, and their singularities. Then, we show the relationships among them by their geometric figure.

Example 6.1. *Let* $\gamma(t) = (\sinh^3 t, \cosh^3 t, 1)$*. We can see that* $(0, 1, 1)$ *is a* $(2, 3)$*-cusp of the curve* $\gamma(t)$*, and* $\gamma(t)$ *is a spacelike framed curve with singularities.*

By $\dot{\gamma}(t) = (3\sinh^2 t\cosh t, 3\cosh^2 t\sinh t, 0)$, naturally we can take the Bishop frame $\{\boldsymbol{\nu}, \boldsymbol{\omega}, \boldsymbol{\mu}\}$ of $\gamma(t)$ as $\boldsymbol{\mu}(t) = (\sinh t, \cosh t, 0)$, $\boldsymbol{\nu}(t) = (\sqrt{2}\cosh t, \sqrt{2}\sinh t, -1)$, $\boldsymbol{\omega}(t) = (\cosh t, \sinh t, -\sqrt{2})$. Then, we have the Frenet formulae

$$\begin{pmatrix} \dot{\boldsymbol{\nu}}(t) \\ \dot{\boldsymbol{\omega}}(t) \\ \dot{\boldsymbol{\mu}}(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \sqrt{2} \\ 0 & 0 & 1 \\ \sqrt{2} & -1 & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\nu}(t) \\ \boldsymbol{\omega}(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$
$$\dot{\gamma}(t) = (3\sinh t\cosh t, 3\cosh 2t\sinh t, 0)\boldsymbol{\mu}(t).$$

By the definitions of normal surfaces and circular evolutes and involutes, we have

$$E_\gamma[\boldsymbol{\omega}](t) = (\sinh^3 t - 3\sinh t\cosh^2 t, \cosh^3 t - 3\sinh^2 t\cosh t, 1 + 3\sqrt{2}\sinh t\cosh t),$$
$$I_\gamma[0](t) = (\frac{3}{2}\sinh^3 t, \cosh^3 t - \frac{3}{2}\sinh^2 t\cosh t, 1),$$
$$NS_\gamma[\boldsymbol{\omega}](t) = (\sinh^3 t + \lambda\cosh t, \cosh^3 t + \lambda\sinh t, 1 - \sqrt{2}\lambda).$$

We show the geometric locus of $\gamma(t)$, $E_\gamma[\boldsymbol{\omega}](t)$, $I_\gamma[0](t)$, $NS_\gamma[\boldsymbol{\omega}](t)$ in Figure 1. $\gamma(t)$ is the blue curve. The purple curve in Figure 1 is $I_\gamma[0](t)$. $E_\gamma[\boldsymbol{\omega}](t)$ is the red curve. The green surface $NS_\gamma[\boldsymbol{\omega}](t)$ is the normal surface of $\gamma(t)$, and this is a singular surface with a singularity type of cuspidal edge. We see that $E_\gamma[\boldsymbol{\omega}](t)$ lies in the singular set of $NS_\gamma[\boldsymbol{\omega}](t)$. We can also see that the black point is a regular point in $E_\gamma[\boldsymbol{\omega}](t)$, but it is a $(2, 3)$-cusp of $\gamma(t)$ and a $(3, 4)$-cusp of $I_\gamma[0](t)$. Moreover, we find that the circular evolute of $\gamma(t)$ can be a regular curve, even if $\gamma(t)$ is a spacelike framed curve with singularities.

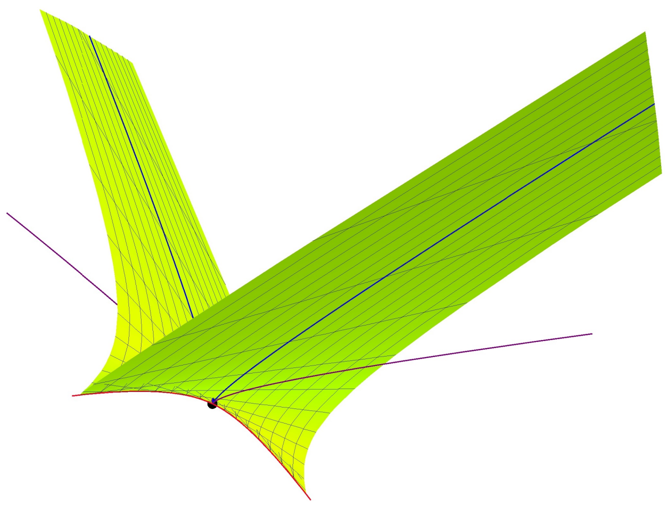

Figure 1. $\gamma(t)$, $E_\gamma[\boldsymbol{\omega}](t)$, $I_\gamma[0](t)$, $NS_\gamma[\boldsymbol{\omega}](t)$.

7. Conclusions

Through our research we have found that there are fancy duality relations not only among parallel curves, normal surfaces, and circular evolutes and involutes, but also for their singularities. Our example also shows more clearly that duality relations are a kind of relation that are very canonical and natural in our geometric imagination. Based on these studies, we can further consider the a family of curves and surfaces and research their related properties, such as the corresponding behaviors of one-parameter families of framed curves, or a family of curves that satisfies certain equations. On the other hand, although the equations are more complex with growth of dimensions, there has already been some related research [23–25]. Thus, it makes sense to further consider circular evolutes and involutes in higher dimensional space. In any case, we find that it is crucially important to consider the duality relations among different geometric objects for the research of submanifolds with singularities.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest that may influence the publication of this work.

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