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Research article

Homogenization of the heat equation with random convolutional potential

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Abstract: This paper derived the homogenization of the heat equation with random convolutional potential. By Tartar's method of oscillating test function, the solution of the heat equation with random convolutional potential was shown to converge in distribution to the solution of the effective equation with determined convolutional potential.

Keywords: homogenization; weak convergence; random convolutional potential; heat equation
Mathematics Subject Classification: 35B27, 35K05, 60J45

1. Introduction

The homogenization theory was first developed for elliptic or parabolic equations with periodic coefficients [1, 5, 24] and generalized to the case of random stationary coefficients [2, 16, 18, 28]. There are many classical results about partial differential equations with rapidly oscillatory random coefficients. Bal [3, 4] showed the homogenization of the heat equation with short-range correlated potential by chaos expansion. Zhang & Bal [27] proved the homogenization of the Schrödinger equation with short-range correlated potential by chaos expansion. The authors [17] derived the homogenization of the heat equation with long-range correlated potential by chaos expansion. Gu & Bal [12] presented the homogenization of the heat equation with time-dependent random potential by the probabilistic method. Hairer, Pardoux, & Piatnitski [14] derived the random homogenization of the heat equation with singular potential by analytic approach. The authors [22] showed the homogenization of a singular random one-dimensional partial differential equation (PDE) with time-varying coefficients by the probabilistic approach. Iftimie, Pardoux, & Piatnitski [15] derived the homogenization of a singular random one-dimensional PDE with short-range correlated potential by the probabilistic method. Bessaih, Efendiev, & Maris [6] proved the homogenization of the reaction-diffusion model by Tartar's oscillation test function approach. The authors [7] showed the homogenization of the convection-diffusion equation by Tartar's oscillation test function approach. Recently, some homogenization results appeared in stochastic partial differential equation with periodic coefficients. Wang & Duan [26] proved the homogenization of the stochastic heat equation with oscillatory boundary conditions by the two scale convergence method. The authors [25] derived the homogenization of the stochastic heat equation by Tartar's oscillation test function approach. Mohammed & Sango [20] presented the homogenization of the linear hyperbolic stochastic partial differential equation with rapidly oscillatory coefficients by the two-scale convergence method. The author [21] proved the homogenization of the nonlinear hyperbolic stochastic partial differential equation by Tartar's oscillation test function approach. As far as we know, there are few results about the homogenization of partial differential equation with rapidly oscillatory random coefficients and random convolutional potential.

Here, we are concerned with the following equation defined on $D = (0,1)^d$, $d > 1$,

$$\partial_t u^\epsilon(x,t) = \mathrm{div}(A^\epsilon(x)\nabla u^\epsilon(x,t)) + (q^\epsilon * u^\epsilon)(x,t) + f(x,t),\ (x,t) \in D \times [0,T], \tag{1.1}$$

$$u^\epsilon(x,0) = u_0(x), \tag{1.2}$$

where $A^\epsilon(x) = (a^\epsilon_{ij}(x))_{1\le i,j\le d} = (a_{ij}(\frac{x}{\epsilon}))_{1\le i,j\le d}$ is the diffusion coefficient, which is periodic and $(q^\epsilon * u^\epsilon)(x,t) = \int_D q^\epsilon(y,\omega)u^\epsilon(x-y,t)dy$. The potential $q^\epsilon(x,\omega)$ is highly oscillatory and is of the form $q(\frac{x}{\epsilon},\omega)$, where $0 < \epsilon < 1$ denotes the scale of oscillations. We assume that $q(x,\omega)$, the potential function before scaling, is a stationary ergodic random field on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and is statistically homogeneous. The function $f(x,t)$ is an external force.

Note that the random convolutional term $q^\epsilon * u^\epsilon$ appears in the Eqs (1.1) and (1.2). The convolutional term is related to the mean field limit equation for multiparticle systems [11]. It is determined by the particles interaction potential. If the number of the particles goes to infinity, then in the mean field time scale we have the mean field limit equation with random convolutional term. Here, we are concerned with the case that the interaction potential is random oscillated, which is described by small parameter $\epsilon > 0$. To obtain the limit of the equation as $\epsilon \to 0$, we first derive the tightness of the solution $\{u^\epsilon(x,t)\}_{0<\epsilon<1}$, and then we apply the Tartar's method of the oscillating test function to obtain the limit of $\{u^\epsilon(x,t)\}_{0<\epsilon<1}$ in $L^2(0,T;H)$.

Next, in Section 2, we present some necessary definitions, notations, and assumptions. The main result is stated in Theorem 2.1. Section 3 introduces the cell problem. Section 4 derives the tightness of solution for the Eqs (1.1) and (1.2). The proof of the main result is presented in the last section.

2. Preliminaries and assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and denote by $\mathbb{E}$ the expectation operator with respect to $\mathbb{P}$. The probability space equipped with an ergodic dynamical system $\mathbf{T}_x, x \in \mathbb{R}^d$, that is, a group of measurable maps $\mathbf{T}_x : \Omega \to \Omega$ such that

(*i*) $\mathbf{T}_{x+y} = \mathbf{T}_x \cdot \mathbf{T}_y,\quad x,y \in \mathbb{R}^d,\quad \mathbf{T}_0 = \mathrm{Id}$;

(*ii*) $\mathbb{P}(\mathbf{T}_x A) = \mathbb{P}(A)$ for all $x \in \mathbb{R}^d,\ A \in \mathcal{F}$;

(*iii*) $\mathbf{T}_x(\omega) : \mathbb{R}^d \times \Omega \mapsto \Omega$ is a measurable map from $(\mathbb{R}^d \times \Omega, \mathcal{B} \times \mathcal{F})$ to $(\Omega, \mathcal{F})$, where $\mathcal{B}$ is the Borel σ-algebra;

(*iv*) If $A \in \mathcal{F}$ is invariant with respect to $\mathbf{T}_x,\ x \in \mathbb{R}^d$, then $\mathbb{P}(A) = 0$ or 1.

Let $H = L^2(D)$ be the space of square integrable function on D with the usual inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$. We also need Sobolev functional space $H^1(D) = \{u \in H : \|u\|_1^2 = \|u\|^2 + \|\nabla u\|^2 < \infty\}$ with norm $\|\cdot\|_1$. For Eqs (1.1) and (1.2), we make the following assumptions.

($\mathbf{H_1}$) (Periodicity). $A(x) = (a_{ij}(x))_{1\le i,j\le d}$ is periodic with period D.

($\mathbf{H_2}$) (Uniform ellipticity). There are $0 < \lambda_1 < \lambda_2$ such that

$$\lambda_1|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \lambda_2|\xi|^2$$

for $\xi = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d$ and $x \in D$.

($\mathbf{H_3}$) $\{q(x,\omega)\,, x \in \mathbb{R}^d\}$ is a stationary ergodic random field on $(\Omega, \mathcal{F}, \mathbb{P})$, that is,

$$q(x,\omega) = q(\mathbf{T}_x\omega)$$

for some random variable $q : \Omega \to \mathbb{R}$, and $q(x,\omega)$ is continuous in x.

($\mathbf{H_4}$) There exists a constant $C > 0$,

$$|q(\omega)| = |q(0,\omega)| \le C, \quad \text{for all} \quad \omega \in \Omega\,.$$

($\mathbf{H_5}$) For $T > 0$,

$$f \in L^2([0,T]; H).$$

Remark 2.1. *Assumptions* ($\mathbf{H_1}$) *and* ($\mathbf{H_2}$) *are classical in periodic homogenization, which is applicable in porous media with periodic structure [10, 13, 23]. Assumptions* ($\mathbf{H_3}$) *and* ($\mathbf{H_4}$) *often appear in stochastic homogenization in some ergodic media [9, e.g.]. Assumption* ($\mathbf{H_5}$) *is just for simplicity to bound the solution in $L^2(0,T;H)$. One can make a weaker assumption but need more detail analysis to derive a bound for solutions, which is not our aim here.*

We present our main result.

Theorem 2.1. *Under assumptions* ($\mathbf{H_1}$)–($\mathbf{H_5}$)*, for all $T > 0$, the solutions $\{u^\epsilon(x,t)\}_{0<\epsilon<1}$ to Eqs (1.1) and (1.2) converge in distribution as $\epsilon \to 0$ in space $L^2(0,T;H)$ to u, which is the solution of the following equation*

$$\partial_t u(x,t) = \operatorname{div}(\hat{A}\nabla u(x,t)) + \bar{q}\int_D u(x-y,t)dy + f(x,t), \quad (x,t) \in D\times[0,T], \tag{2.1}$$

$$u(x,0) = u_0(x), \tag{2.2}$$

where $\hat{A}$ is introduced in Section 3 and $\bar{q} = \mathbb{E}q(0,\omega)$.

In our discussion, we need a result on compact embedding. Let $\mathcal{X} \subset \mathcal{Y} \subset \mathcal{Z}$ be three reflexive Banach spaces and let $\mathcal{X} \subset \mathcal{Y}$ with compact and dense embedding. Define a new Banach space

$$G = \{v : v \in L^2(0,T;\mathcal{X}), \frac{dv}{dt} \in L^2(0,T;\mathcal{Z})\}$$

with norm

$$\|v\|_G^2 = \int_0^T \|v\|_{\mathcal{X}}^2 ds + \int_0^T \left\|\frac{dv}{dt}\right\|_{\mathcal{Z}}^2 ds$$

for all $v \in G$, then we have the following result [19].

Lemma 2.1. *A bounded set $B \subset G$ is precompact in $L^2(0,T;\mathcal{Y})$.*

To prove Lemmas 5.1 and 5.2, we also need the following result [19].

Lemma 2.2. *For all $T > 0$, let Q be a bounded region in $D \times [0, T]$. For all given functions g^ϵ and g in $L^p(Q)(1 < p < \infty)$, if*

$$|g^\epsilon|_{L^p(Q)} \leq C \quad \textit{and} \quad g^\epsilon \to g \text{ in } Q \text{ almost everywhere}$$

for some positive constant C, then g^ϵ converges weakly to g in $L^p(Q)$.

In the following part, the positive constants C and $C(T)$ may change from line to line.

3. The cell problem

In this section, we introduce χ, the solution of the cell problem that corresponds to the systems (1.1) and (1.2)

$$\begin{cases} \operatorname{div}(A(y)(I + \nabla\chi(y))) = 0, & \text{in } D, \\ \chi - D \, \text{periodic}, \end{cases}$$

as well as the solution of the adjoint equation χ^*

$$\begin{cases} \operatorname{div}(A^*(y)(I + \nabla\chi^*(y))) = 0, & \text{in } D, \\ \chi^* - D \, \text{periodic}, \end{cases}$$

where A^* is the adjoint of A, $A^* = (a^*_{ij})_{1\leq i,j\leq d}, a^*_{ij} = a^*_{ji}$ for $1 \leq i, j \leq d$. It follows that $\chi^\epsilon(y) = \chi(\frac{y}{\epsilon})$ is the solution for the equation

$$\begin{cases} \operatorname{div}(A^\epsilon(y)(I + \epsilon\nabla\chi^\epsilon(y))) = 0, & \text{in } \epsilon D, \\ \chi^\epsilon - \epsilon D \, \text{periodic}. \end{cases} \tag{3.1}$$

Now, we define the homogenized operator $\hat{A}$ as

$$\hat{A} = \int_D A(y)(I + \nabla\chi(y))dy.$$

4. Tightness of $\{u^\epsilon(x, t)\}_{0<\epsilon<1}$

In this section, we show the tightness of the solution $\{u^\epsilon(x, t)\}_{0<\epsilon<1}$ of Eqs (1.1) and (1.2). Before persenting the tightness, we write systems (1.1) and (1.2). In the following variational weak formulation

$$\begin{aligned} &\langle \partial_t u^\epsilon(x, t), \varphi(x)\rangle_{H^{-1}(D),H^1(D)} \\ &\quad = -\int_D A^\epsilon(x)\nabla u^\epsilon(x, t)\nabla\varphi(x)dx + \int_D f(x, t)\varphi(x)dx \\ &\qquad + \int_D (q^\epsilon(x, \omega) * u^\epsilon(x, t))\varphi(x)dx, \quad \varphi(x) \in H^1(D), \end{aligned} \tag{4.1}$$

with $u^\epsilon(x, 0) = u_0(x)$.

Lemma 4.1. *For all $T > 0$, assume* ($\mathbf{H_1}$)–($\mathbf{H_5}$) *and* $\mathbb{E}\|u_0(x)\|^2 < \infty$ *hold, then*

$$\mathbb{E}\|u^\epsilon(x,t)\|^2 + 2C_{\lambda_1}\mathbb{E}\int_0^t \|u^\epsilon(x,s)\|_1^2 ds \leq C_T(C + \mathbb{E}\|u_0(x)\|^2), \quad \text{for } t \in [0,T] \tag{4.2}$$

and

$$\mathbb{E}\int_0^t \|\partial_t u^\epsilon(x,s)\|_{H^{-1}(D)}^2 ds \leq (\mathbb{E}\|u_0(x)\|^2 + C)C_T, \quad \text{for } t \in [0,T]. \tag{4.3}$$

Proof. Multiplying both sides of the Eq (1.1) by $u^\epsilon(x,t)$ yields

$$\int_D \partial_t u^\epsilon(x,t)u^\epsilon(x,t)dx + \int_D A^\epsilon(x)\nabla u^\epsilon(x,t)\nabla u^\epsilon(x,t)dx = \int_D (q^\epsilon(x,\omega) * u^\epsilon(x,t))u^\epsilon(x,t)dx + \int_D f(x,t)u^\epsilon(x,t)dx.$$

Note that the term $\int_D A^\epsilon(x)\nabla u^\epsilon(x,t)\nabla u^\epsilon(x,t)dx$, first by ($\mathbf{H_2}$),

$$\lambda_1\|\nabla u^\epsilon(x,t)\|^2 \leq \int_D A^\epsilon(x)\nabla u^\epsilon(x,t)\nabla u^\epsilon(x,t)dx$$

and the norm $\|u^\epsilon(x,t)\|_1$ and $\|\nabla u^\epsilon(x,t)\|$ are equivalent, so we have

$$C_{\lambda_1}\|u^\epsilon(x,t)\|_1^2 \leq \int_D A^\epsilon(x)\nabla u^\epsilon(x,t)\nabla u^\epsilon(x,t)dx.$$

Furthermore, by Hölder inequality, Young inequality, and ($\mathbf{H_3}$)–($\mathbf{H_5}$), we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u^\epsilon(x,t)\|^2 + C_{\lambda_1}\|u^\epsilon(x,t)\|_1^2 &\leq \|f(x,t)\|\|u^\epsilon(x,t)\| + \|q^\epsilon(x,\omega) * u^\epsilon(x,t)\|\|u^\epsilon(x,t)\| \\ &\leq \|f(x,t)\|\|u^\epsilon(x,t)\| + C\|u^\epsilon(x,t)\|^2 \\ &\leq \frac{1}{2}\|f(x,t)\|^2 + C\|u^\epsilon(x,t)\|^2.\end{aligned}$$

Integrating from 0 to t yields

$$\|u^\epsilon(x,t)\|^2 + 2C_{\lambda_1}\int_0^t \|u^\epsilon(x,s)\|_1^2 ds \leq \|u_0(x)\|^2 + \int_0^t \|f(x,s)\|^2 ds + C\int_0^t \|u^\epsilon(x,s)\|^2 ds,$$

then the Gronwall inequality yields

$$\|u^\epsilon(x,t)\|^2 + 2C_{\lambda_1}\int_0^t \|u^\epsilon(x,s)\|_1^2 ds \leq C_T(\|u_0(x)\|^2 + C). \tag{4.4}$$

Taking expectation on both sides of (4.4) yields (4.2). By (4.1) and (4.2), we have (4.3). □

By Lemma 2.1, we have the following result.

Lemma 4.2. *Under assumptions* ($\mathbf{H_1}$)–($\mathbf{H_5}$) *and* $\mathbb{E}\|u_0(x)\|^2 < \infty$, *for all $T > 0$, the family $\{u^\epsilon(x,t)\}_{0<\epsilon<1}$ for the Eqs (1.1) and (1.2) is tight in space $L^2(0,T;H)$.*

5. Proof of Theorem 2.1

In this section, we pass the limit of the Eqs (1.1) and (1.2) as $\epsilon \to 0$. By the tightness of $\{u^\epsilon(x,t)\}_{0<\epsilon<1}$, there is a subsequence converging in distribution to $L^2(0,T;H)$ and the subsequence is written as $\{u^\epsilon(x,t)\}_{0<\epsilon<1}$. By the Skorohod theorem [8], one can construct a new probability space and new variable without changing the distribution, such that $\{u^\epsilon(x,t)\}_{0<\epsilon<1}$ (here, we don't change the notations) converges almost surely to $u(x,t)$ in space $L^2(0,T;H)$. Next, we determine the limit process $u(x,t)$.

Before presenting the proof of Theorem 2.1, we first show the following two lemmata.

Lemma 5.1. *For all $\varphi(x) \in C_0^\infty(D)$, $\phi(t) \in C^\infty(0,T)$, almost sure $\omega \in \Omega$,*

$$\lim_{\epsilon\to 0}\int_0^T\int_D (q(\mathbf{T}_{\frac{x}{\epsilon}}\omega) * u^\epsilon(x,t))\varphi(x)\phi(t)dxdt = \int_0^T\int_D\int_D \bar{q}u(x-y,t)\varphi(x)\phi(t)dydxdt.$$

Proof.

$$\begin{aligned}\int_0^T\int_D\int_D q(\mathbf{T}_{\frac{x}{\epsilon}}\omega)u^\epsilon(x-y,t)\varphi(x)\phi(t)dydxdt - \int_0^T\int_D\int_D \bar{q}u(x-y,t)\varphi(x)\phi(t)dydxdt \\ = \int_0^T\int_D\int_D q(\mathbf{T}_{\frac{y}{\epsilon}}\omega)(u^\epsilon(x-y,t) - u(x-y,t))\varphi(x)\phi(t)dydxdt \\ + \int_0^T\int_D\int_D (q(\mathbf{T}_{\frac{y}{\epsilon}}\omega) - \bar{q})u(x-y,t)\varphi(x)\phi(t)dydxdt. \qquad (5.1)\end{aligned}$$

By assumption ($\mathbf{H_4}$), (4.2), Lemma 2.2, and the Birkhoff ergodic theorem [16, Theorem 7.2], (5.1) vanishes as $\epsilon \to 0$. □

Lemma 5.2. *For all $\varphi(x) \in C_0^\infty(D)$, $\phi(t) \in C^\infty(0,T)$, almost sure $\omega \in \Omega$,*

$$\begin{aligned}\lim_{\epsilon\to 0}\int_0^T\int_D A^\epsilon(x)\nabla u^\epsilon(x,t)\nabla(\varphi(x) + \epsilon\nabla\varphi(x)\chi^\epsilon(x))\phi(t)dxdt \\ = \int_0^T\int_D \hat{A}\nabla u(x,t)\nabla\varphi(x)\phi(t)dxdt.\end{aligned}$$

Proof. This convergence can be followed by the discussion in deterministic case [10, Section 8], as here we consider the convergence for almost sure $\omega \in \Omega$. □

Proof of Theorem 2.1. Multiplying both sides of the Eqs (1.1) and (1.2) by the test function $\varphi(x) + \epsilon\nabla\varphi(x)\chi^\epsilon(x)$ with $\varphi(x) \in C_0^\infty(D)$ and $\phi(t) \in C^\infty(0,T)$ with $\phi(0) = 1, \phi(T) = 0$ yields

$$\begin{aligned}&-\int_D u_0(x)(\varphi(x) + \epsilon\nabla\varphi(x)\chi^\epsilon(x))dx - \int_0^T\int_D u^\epsilon(x,s)(\varphi(x) + \epsilon\nabla\varphi(x)\chi^\epsilon(x))\phi_t(s)dxds \\ &= -\int_0^T\int_D A^\epsilon(x)\nabla u^\epsilon(x,s)\nabla(\varphi(x) + \epsilon\nabla\varphi(x)\chi^\epsilon(x))\phi(s)dxds \\ &\quad + \int_0^T\int_D (q(\mathbf{T}_{\frac{x}{\epsilon}}\omega) * u^\epsilon(x,s))(\varphi(x) + \epsilon\nabla\varphi(x)\chi^\epsilon(x))\phi(s)dxds\end{aligned}$$

$$+\int_0^T\int_D f(x,s)(\varphi(x)+\epsilon\nabla\varphi(x)\chi^\epsilon(x))\phi(s)dxds. \tag{5.2}$$

We intend to pass the limit $\epsilon\to 0$ in Eq (5.2). First, simple calculation yields

$$\begin{aligned}
&\int_0^T\int_D A^\epsilon(x)\nabla u^\epsilon(x,s)\nabla(\varphi(x)+\epsilon\nabla\varphi(x)\chi^\epsilon(x))\phi(s)dxds\\
&=\int_0^T\int_D A^\epsilon(x)\nabla u^\epsilon(x,s)\nabla\varphi(x)\phi(s)dxds\\
&\quad+\int_0^T\int_D A^\epsilon(x)\nabla u^\epsilon(x,s)\epsilon\nabla\nabla\varphi(x)\chi^\epsilon(x)\phi(s)dxds\\
&\quad+\int_0^T\int_D A^\epsilon(x)\nabla u^\epsilon(x,s)\epsilon\nabla\varphi(x)\nabla\chi^\epsilon(x)\phi(s)dxds.
\end{aligned}\tag{5.3}$$

By the Eq (3.1), we obtain

$$\int_D A^\epsilon(x)(I+\epsilon\nabla\chi^\epsilon(x))\nabla(u^\epsilon(x,s)\nabla\varphi(x))dx=0.$$

Furthermore,

$$\begin{aligned}
&\int_D A^\epsilon(x)\nabla u^\epsilon(x,s)\nabla\varphi(x)dx+\int_D A^\epsilon(x)u^\epsilon(x,s)\nabla\nabla\varphi(x)dx\\
&=-\int_D A^\epsilon(x)\epsilon\nabla\chi^\epsilon(x)\nabla u^\epsilon(x,s)\nabla\varphi(x)dx-\int_D A^\epsilon(x)\epsilon\chi^\epsilon(x)u^\epsilon(x,s)\nabla\nabla\varphi(x)dx.
\end{aligned}\tag{5.4}$$

Substituting (5.4) into (5.3) yields

$$\begin{aligned}
&\int_0^T\int_D A^\epsilon(x)\nabla u^\epsilon(x,s)\nabla(\varphi(x)+\epsilon\nabla\varphi(x)\chi^\epsilon(x))\phi(s)dxds\\
&=\int_0^T\int_D \epsilon A^\epsilon(x)\nabla u^\epsilon(x,s)\nabla\nabla\varphi(x)\chi^\epsilon(x)\phi(s)dxds\\
&\quad-\int_0^T\int_D A^\epsilon(x)u^\epsilon(x,s)(I+\epsilon\chi^\epsilon(x))\nabla\nabla\varphi(x)\phi(s)dxds.
\end{aligned}$$

By ($\mathbf{H_1}$), we have

$$\int_0^T\int_D \hat{A}\nabla u(x,s)\nabla\varphi(x)\phi(s)dxds=-\int_0^T\int_D \hat{A}u(x,s)\nabla\nabla\varphi(x)\phi(s)dxds \tag{5.5}$$

as $\epsilon\to 0$.

Next, we consider the limit of the Eq (5.2). Now, by Lemmas 5.1 and 5.2,

$$\begin{aligned}
&-\int_D u_0(x)\varphi(x)dx-\int_0^T\int_D u(x,s)\varphi(x)\phi_t(s)dxds\\
&=\int_0^T\int_D \hat{A}\nabla u(x,s)\nabla\varphi(x)\phi(s)dxds
\end{aligned}$$

$$+\int_0^T\int_D \bar{q}\int_D u(x-y,s)dy\varphi(x)\phi(s)dxds$$
$$+\int_0^T\int_D f(x,s)\varphi(x)\phi(s)dxds, \tag{5.6}$$

which implies

$$\partial_t u(x,t) = \operatorname{div}(\hat{A}\nabla u(x,t)) + \bar{q}\int_D u(x-y,t)dy + f(x,t), \quad (x,t)\in D\times[0,T],$$
$$u(x,0) = u_0(x).$$

The proof of Theorem 2.1 is complete. □

6. Conclusions

In this manuscript, we derived the homogenization of the heat equation with random convolutional potential by Tartar's oscillation test function approach.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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