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Research article

Spectral stability analysis of the Dirichlet-to-Neumann map for fractional diffusion equations with a reaction coefficient

Ridha Mdimagh[1,3,*] **and Fadhel Jday**[2,3]

[1] Department of Mathematics, College of Science and Arts at Khulis, University of Jeddah, Jeddah, Saudi Arabia

[2] Mathematics Department, Jamoum University College, Umm Al-Qura University, Saudi Arabia

[3] ENIT-LAMSIN, University of Tunis El Manar, Tunisia

* **Correspondence:** Email: rmothman1@uj.edu.sa.

Abstract: This paper focused on the stability analysis of the Dirichlet-to-Neumann (DN) map for the fractional diffusion equation with a reaction coefficient q. The main result provided a Hölder-type stability estimate for the map, which was formulated in terms of the Dirichlet eigenvalues and normal derivatives of eigenfunctions of the operator $A_q := -\Delta + q$.

Keywords: fractional diffusion equation; diffusion potential; Dirichlet-to-Neumann (DN) map; Hölder-type stability; spectral decomposition
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1. Introduction

In this paper, our focus lies on studying the stability of the Dirichlet-to-Neumann (DN) map for the fractional diffusion equation with respect to the reaction coefficient represented by the symbol q. This equation serves as a mathematical model that describes abnormal diffusion in various physical phenomena. Examples include the scattering field data problem in soil [1], material diffusion in heterogeneous media, fluid flow diffusion in inhomogeneous and anisotropic porous media, turbulent plasma behavior, carrier diffusion in amorphous photoconductors, diffusion in turbulent medium flows, percolation models in porous media, biological phenomena, and finance problems (see [2]). More about the applications of fractional derivatives can be found in [3–5].

In mathematical terms, we consider a smooth bounded domain Ω in $\mathbb{R}^d$ ($d \geq 1$) with a smooth boundary denoted by $\partial\Omega$. Our investigation revolves around the initial boundary value problem, formulated as follows:

$$\begin{cases} \partial_t^\alpha u - \Delta u + q(x)u = 0, & \text{in } \Omega_T := \Omega \times (0,T), \\ u(x,0) = 0, & \text{in } \Omega, \\ u = f, & \text{on } \Sigma_T := \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

Here, $T > 0$ is a fixed real number, the diffusion potential q belongs to the space $L^\infty(\bar{\Omega})$, and the Dirichlet data f is taken from the space Ξ defined as

$$\Xi := \{h \in C^1([0,T]; H^{\frac{3}{2}}(\partial\Omega)); h(\cdot,0) = 0, \text{ on } \partial\Omega\}. \tag{1.2}$$

In the above equations, $\partial_t^\alpha u$ denotes the fractional Caputo time derivative, which is defined as

$$\partial_t^\alpha g(t) := \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} g^{(n)}(s)\, ds.$$

Here, $n := [\alpha] + 1$, where $[\cdot]$ represents the integer part function, Γ is the Euler-Gamma function, and $\Delta := \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ stands for the Laplacian operator with respect to the spatial coordinates.

To establish the existence and uniqueness of a solution to the problem described in (1.1), we rely on a proposition given by Kian et al. in [6]:

Proposition 1.1. *[6] Let* $\alpha \in (0,1)$*,* $\rho \in L^\infty(\Omega)$*,* $a \in C^1(\bar{\Omega})$*,* $q \in L^\infty(\Omega)$ *satisfy the conditions*

$$\rho(x) \geq c, \ a(x) \geq c, \ q(x) \geq 0,$$

for some positive constant c, and let $f \in C^1([0,T]; H^{\frac{3}{2}}(\partial\Omega))$ *satisfy* $f(\cdot,0) = 0$ *on* $\partial\Omega$*, then there exists a unique solution* $u \in C([0,T]; L^2(\Omega))$ *to the following boundary value problem:*

$$\begin{cases} \rho\, \partial_t^\alpha u - div(a\nabla u) + q\, u = 0, & in\ \Omega_T, \\ u(x,0) = 0, & in\ \Omega, \\ u = f, & on\ \Sigma_T. \end{cases} \tag{1.3}$$

Moreover, we have $u \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^{2\gamma}(\Omega))$*, for any* $\gamma \in (0,1)$*.*

Let us introduce the DN operator, denoted as $\mathcal{H}_{q,\alpha}$, associated with the problem described in (1.1). This operator is defined as follows:

$$\mathcal{H}_{q,\alpha} : f \in C^1([0,T]; H^{\frac{3}{2}}(\partial\Omega)) \mapsto \frac{\partial u}{\partial \nu} \in C([0,T]; H^{2\gamma-\frac{3}{2}}(\partial\Omega)). \tag{1.4}$$

Here, ν represents the outward unit normal vector to $\partial\Omega$ and u denotes the solution to the problem given in (1.1). According to Proposition 1.1, when $\gamma \in (\frac{3}{4}, 1)$, the operator $\mathcal{H}_{q,\alpha}$ is well-defined in the space $C((0,T]; L^2(\partial\Omega))$.

In this study, our objective is to establish a spectral stability estimate of Hölder type for the DN map $\mathcal{H}_{q,\alpha}$, with respect to the Dirichlet eigenvalues $(\lambda_{k,q})_k$ and the normal derivatives of associated eigenfunctions $(\partial_\nu \varphi_{k,q})_k$ of the operator $A_q := -\Delta + q$. Previous works have explored related stability estimates in different settings. Alessandrini et al. in [7] demonstrated a spectral stability of Hölder type for the DN map associated with a wave diffusion equation using certain approximate spectral data. In [8], the authors established stability estimates for a partial hyperbolic DN map, specifically in cases where measurements are made at the intersection of the domain's boundary with a half-space. They obtained a Hölder type stability estimate in three dimensions and a logarithmic type stability estimate in two dimensions. These results have found applications in various works, such as [9, 10], where log-type stability estimates were proved for the DN map restricted to a specific part of the boundary. These estimates were then used to identify the potential q in the wave equation based on boundary observations. Additionally, in [11], the authors provided Hölder stability results for the DN map and established a stability estimate linked to the multidimensional Borg-Levinson theorem for determining a potential from spectral data.

The structure of this paper is organized as follows: In Section 2, we present fundamental properties related to the spectrum of the operator A_q and state the main result, which can be found in Theorem 2.1. Section 3 is dedicated to the proof of Theorem 2.1.

2. Preliminaries and main result

In this section, we introduce the necessary notations to present our main stability result. We denote by Λ_q the DN map associated with the operator $A_q := -\Delta + q$ defined on the domain $D(A_q) = H_0^1(\Omega) \cap H^2(\Omega)$. The map Λ_q is defined as follows:

$$\Lambda_q : \psi \mapsto \partial_\nu u,$$

where u represents the solution of

$$\begin{cases} A_q u = 0, & \text{in } \Omega, \\ u = \psi, & \text{on } \partial\Omega. \end{cases}$$

Here, $\sigma(A_q) = \lambda_{k,q}$ denotes the spectrum of A_q and $\rho(A_q) = \mathbb{C} \setminus \sigma(A_q)$ represents the resolvent set of A_q.

For any $\lambda \in \rho(A_q)$ and $\psi \in H^{\frac{3}{2}}(\partial\Omega)$, the problem

$$\begin{cases} -\Delta u + qu - \lambda u = 0, & \text{in } \Omega, \\ u = \psi, & \text{on } \partial\Omega \end{cases}$$

has a unique solution $u := u(q, \psi, \lambda) \in H^2(\Omega)$. Furthermore, the operator $\Lambda_q(\lambda) : \psi \mapsto \partial_\nu u$ is bounded from $H^{\frac{3}{2}}(\partial\Omega)$ to $H^{\frac{1}{2}}(\partial\Omega)$ (see [12]).

It is a well-known fact that the spectrum of A_q comprises a sequence of eigenvalues, where each eigenvalue is counted according to its multiplicity. The eigenvalues are ordered as follows:

$$0 \le \lambda_{1,q} \le \lambda_{2,q} \le \ldots \le \lambda_{k,q} \to \infty.$$

The associated sequence of eigenfunctions is denoted by $\varphi_{k,q}$, and we can assume that this sequence forms an orthonormal basis of $L^2(\Omega)$ for the solution $\varphi_{k,q}$ of the problem

$$\begin{cases} (-\Delta + q)\varphi = \lambda_{k,q}\varphi, & \text{in } \Omega, \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases}$$

From the classical $H^2(\Omega)$ a priori estimates, we derive

$$\|\varphi_{k,q}\|_{H^2(\Omega)} \le C\ \lambda_{k,q}\ \|\varphi_{k,q}\|_{L^2(\Omega)} = C\ \lambda_{k,q}. \tag{2.1}$$

For more details, see [9, 13]. Therefore, from trace theorem (see [12]), we obtain

$$\|\partial_\nu \varphi_{k,q}\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\ \lambda_{k,q}. \tag{2.2}$$

Additionally, there exists a positive constant $K \geq 1$ such that

$$K^{-1}\ k^{\frac{2}{d}} \leq \lambda_{k,q} \leq K\ k^{\frac{2}{d}}, \tag{2.3}$$

where the constant K can be chosen uniformly in q within the range $0 \leq q(x) \leq M$ for $x \in \Omega$. Consequently, we obtain the estimates

$$\|\partial_\nu \varphi_{k,q}\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\ k^{\frac{2}{d}}, \tag{2.4}$$

and from [7] (Lemma 2.5), we have

$$\|\partial_\nu \varphi_{k,q}\|_{H^p(\partial\Omega)} \leq C\ (\lambda_{k,q}+1)^{\frac{p}{2}+\frac{3}{4}}, \quad 0 \leq p \leq \frac{1}{2}. \tag{2.5}$$

These estimates provide insights into the behavior of $\varphi_{k,q}$ and its normal derivatives in different function spaces. We can conclude that the sequence $(k^{\frac{-2m}{d}}\|\partial_\nu \varphi_{k,q}\|_{H^{\frac{1}{2}}(\partial\Omega)})_k$ belongs to ℓ^1 if $m > \frac{d}{2}+1$. Here, ℓ^1 is the standard Banach space of real-valued sequences for which the corresponding series are absolutely convergent. It is defined as

$$\ell^1 = \{(x_n)_n \subset \mathbb{R}, \sum_{n=1}^{\infty} |x_n| < \infty\}$$

equipped with its natural norm given by

$$\|(x_n)_n\|_{\ell^1} = \sum_{n=1}^{\infty} |x_n|.$$

Let us fix ζ such that $\frac{d}{2}+1 < \zeta \leq d+1$. It follows that $(k^{\frac{-2\zeta}{d}}\|\partial_\nu \varphi_{k,q}\|_{H^{\frac{1}{2}}(\Gamma)})_k$ belongs to ℓ^1.

Let $r := (r_k)$ be the sequence defined by $r_k = k^{\frac{-2\zeta}{d}}$, $k \geq 1$. We introduce the Banach space

$$\ell^1(H^{\frac{1}{2}}(\Gamma), r) := \{g = (g_k)_{k\geq 1} \subset H^{\frac{1}{2}}(\Gamma),\ \text{and}\ (r_k\|g_k\|_{H^{\frac{1}{2}}(\Gamma)})_k \in \ell^1\}$$

equipped with the norm:

$$\|g\|_{\ell^1(H^{\frac{1}{2}}(\Gamma), r)} = \sum_{k\geq 1} r_k \|g_k\|_{H^{\frac{1}{2}}(\Gamma)}.$$

Furthermore, let $\mu = (\mu_k)$ be the sequence of eigenvalues of A_0 (which corresponds to A_q with $q = 0$). As a consequence of the min-max formula (e.g., [14]), we have

$$|\lambda_{k,q} - \mu_k| \leq \|q\|_{L^\infty(\Omega)},\ k \geq 1$$

which implies that $\|\lambda_q - \mu\|_{\ell^\infty} \leq \|q\|_{L^\infty(\Omega)}$, where $\lambda_q := (\lambda_{k,q})_k$, and ℓ^∞ denotes the standard Banach space of bounded real-valued sequences. In other words, λ_q belongs to the affine space $\tilde{\ell}^\infty = \mu + \ell^\infty$. We equip $\tilde{\ell}^\infty$ with the distance function:

$$d_\infty(\beta_1, \beta_2) = \|(\beta_1 - \mu) - (\beta_2 - \mu)\|_{\ell^\infty} = \|\beta_1 - \beta_2\|_{\ell^\infty},\ \beta_j \in \tilde{\ell}^\infty,\ j = 1, 2. \tag{2.6}$$

Let us define $N_\alpha := [\alpha(d+2)] + 1$ and Ξ_0 as the set

$$\begin{aligned} &\Xi_0 := \Xi \quad \text{if} \;\; N_\alpha = 1, \\ &\Xi_0 = \{g \in H^{N_\alpha}(0,T; H^{\frac{3}{2}}(\partial\Omega)); \partial_t^j g(\cdot, 0) = 0, \;\; j = 0, \dots, N_\alpha - 1\}, \text{ if } N_\alpha \geq 2, \end{aligned} \tag{2.7}$$

where Ξ is the set defined by (1.2).

Instead of considering $\mathcal{H}_{q,\alpha}$ directly as an operator from Ξ to $C([0,T]; L^2(\partial\Omega))$, we will focus on its restriction, denoted by $\mathcal{H}_{q,\alpha}$, to the subspace Ξ_0 of Ξ. Furthermore, we denote $|||\mathcal{H}_{q_1,\alpha} - \mathcal{H}_{q_2,\alpha}|||_p$ as the norm of the operator $\mathcal{H}_{q_1,\alpha} - \mathcal{H}_{q_2,\alpha}$ in the space $\mathcal{L}\left(H^{N_\alpha}(0,T; H^{\frac{3}{2}}(\partial\Omega)); L^2(0,T; H^p(\partial\Omega))\right)$. Here, H^{N_α} represents the L^2-based Sobolev space of order N_α, and $L^2(0,T; H^p(\partial\Omega))$ represents the space of square integrable maps from the interval $(0,T]$ to $H^p(\partial\Omega)$. With these notations established, we are now able to state the main result.

Theorem 2.1. *Let $q_i \in L^\infty(\Omega)$ and let there be a constant M such that $\|q_i\|_{L^\infty(\Omega)} \leq M$, $i = 1,2$, then there exists a constant C such that*

$$|||\mathcal{H}_{q_1,\alpha} - \mathcal{H}_{q_2,\alpha}|||_p \leq C(\delta + \delta^\theta),$$

where $\delta = d_\infty(\lambda_{q_1}, \lambda_{q_2}) + \|\partial_\nu\varphi_{q_1} - \partial_\nu\varphi_{q_2}\|_{\ell^1(H^{\frac{1}{2}}(\partial\Omega),r)}$, $\partial_\nu\varphi_{q_i} = (\partial_\nu\varphi_{k,q_i})_k$, $i = 1,2$, $\theta \in (0,1)$, $0 \leq p < \frac{1}{2}$, and d_∞ is defined in (2.6).

3. Proof of Theorem 2.1

This section aims to establish the stability of the DN map $\mathcal{H}_{q,\alpha}$ with respect to the spectrum $(\lambda_{k,q}, \varphi_{k,q})_k$ of the operator A_q. The proof of Theorem 2.1 relies on two key propositions: Propositions 3.1 and 3.2. Proposition 3.1 extends the results from [7, 9, 13] to the fractional case ($\alpha \in (0,1)$), which originally dealt with the classical case $\alpha = 2$. The extension is made possible by leveraging the properties of the Mittag-Leffler function [15,16]. Proposition 3.2 establishes the stability of the term B_q^α under small perturbations of the potential q in $L^\infty(\Omega)$. The norm of the difference between $B_{q_1}^\alpha$ and $B_{q_2}^\alpha$ is bounded by a constant times the value of δ. This proposition is an essential component in the proof of Theorem 2.1, as it ensures the stability of the additional term B_q^α in the representation of the operator $\mathcal{H}_{q,\alpha}$. The following lemmas, whose proofs can be found in [7, 13], are instrumental in the proof. Alessandrini et al. in [7] used these Lemmas to give an explicit representation of the DN map for the wave equation, which was used in [7, 9, 13] to establish estimates for the DN map and to give stability results for the inverse problem of identifying the potential q according to these estimations. We adapt this technique to establish a spectral stability estimate to the DN map for the fractional diffusion equation.

Lemma 3.1. *([13], Proposition 2.30, p. 64)*

(1) For $\lambda \in \rho(A_q)$, we set $R_q(\lambda) := (A_q - \lambda)^{-1}$. For $h \in L^2(\Omega)$, we have

$$R_q(\lambda)h = \sum_{k\geq 1} \frac{1}{\lambda_{k,q} - \lambda}(h, \varphi_{k,q})\varphi_{k,q},$$

where $(\cdot,\cdot)$ represents the inner product in $L^2(\Omega)$.

(2) *For $\lambda \in \rho(A_q)$, there exists δ_0 and C such that*

$$\|R_q(\lambda + \mu) - R_q(\lambda) - \mu R_q(\lambda)^2\|_{\mathcal{L}(L^2(\Omega), H^2(\Omega))} \leq C|\mu|^2, \ \forall \mu \in \mathbb{C}, \ |\mu| \leq \delta_0.$$

In particular, $\lambda \in \rho(A_q) \mapsto R_q(\lambda) \in \mathcal{L}(L^2(\Omega), H^2(\Omega))$ is a holomorphic function.

Lemma 3.2. *[7, 13] Let $q \in L^\infty(\Omega)$, $m > \frac{d}{2}$, $f \in H^{\frac{3}{2}}(\partial\Omega)$, and $\lambda \in \rho(A_q)$, and we have*

$$\frac{d^m}{d\lambda^m}\Lambda_q(\lambda) f = -m! \sum_{k\geq 1} \frac{1}{(\lambda_{k,q} - \lambda)^{m+1}} \langle f, \partial_\nu \varphi_{k,q} \rangle \partial_\nu \varphi_{k,q|\partial\Omega},$$

where $\langle \cdot, \cdot \rangle$ represents the inner product in $L^2(\partial\Omega)$.

Lemma 3.3. *[7, 13] For nonnegative integer N and $q_1, ; q_2 \in L^\infty(\Omega)$ with $0 \leq q_1, ; q_2 \leq M$. There exists a positive constant C, depending only on Ω and M, such that*

$$\|\frac{d^j}{d\lambda^j}(\Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda))\|_p \leq \frac{C}{|\lambda|^{j+\frac{1-2p}{4}}}, \ \lambda \leq 0 \text{ and } 0 \leq j \leq N,$$

where $\|\cdot\|_p$ denotes the norm in $\mathcal{L}(H^{\frac{3}{2}}(\partial\Omega), H^p(\partial\Omega))$, $0 \leq p < \dfrac{1}{2}$.

Lemma 3.4. *[9, 13] Let $F(\lambda) := \Lambda_{q_1}(\lambda) - \Lambda_{q_2}(\lambda)$. We have*

$$\|F^{(d+1)}(\lambda)\|_p \leq C\delta.$$

In particular, for $0 \leq j \leq d$,

$$\|F^{(j)}(0)\|_p \leq C\delta^\theta,$$

where $\theta \in (0, 1)$.

These lemmas, with the two following propositions, provide the necessary tools to establish the stability result presented in Theorem 2.1.

Proposition 3.1. *For each $f \in \Xi_0$, we have*

$$\mathcal{H}_{q,\alpha} f = \sum_{j=0}^{d+1} \left[\frac{1}{j!} \frac{d^j}{d\lambda^j}(\Lambda_q(\lambda)) \right]_{|\lambda=0} (-\partial_t^\alpha)^j f + B_q^\alpha f, \tag{3.1}$$

where

$$B_q^\alpha f = -\sum_{n=1}^{\infty} \frac{1}{\lambda_{n,q}^{d+2}} \left(\int_0^t \langle (-\partial_s^\alpha)^{d+2} f, \partial_\nu \varphi_{n,q} \rangle (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{n,q}(t-s)^\alpha) \, ds \right) \partial_\nu \varphi_{n,q},$$

and

$$E_{\mu,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\mu k + \beta)}, \ z \in \mathbb{C}, \ \mu > 0, \ \beta \in \mathbb{R} \tag{3.2}$$

is the Mittag-Leffler function.

Proof. We set $u = \sum_{j=0}^{d+1} u^j + h$ as the solution of the problem (1.1), where, for a fixed $t \geq 0$, $u^0 := u^0(\cdot, t)$ is the solution of

$$\begin{cases} -\Delta_x u^0 + q(x)u^0 = 0, & \text{in } \Omega, \\ u^0 = f, & \text{on } \Sigma, \end{cases} \tag{3.3}$$

and for $j = 1, \ldots, d+1$, u^j is the solution of

$$\begin{cases} -\Delta_x u^j + q(x)u^j = -\partial_t^\alpha u^{j-1}, & \text{in } \Omega, \\ u^j = 0, & \text{on } \Sigma, \end{cases} \tag{3.4}$$

$$\begin{cases} \partial_t^\alpha h - \Delta_x h + q(x)h = -\partial_t^\alpha u^{d+1}, & \text{in } \Omega_T, \\ h(x,0) = 0, & \text{in } \Omega, \\ h = 0, & \text{on } \Sigma_T. \end{cases} \tag{3.5}$$

Note that each u^j has zero initial values. In fact, we remark that for any $g \in \Xi$, $\lim_{t\to 0} \partial_t^\alpha g(\cdot, t) = 0$. Indeed, using integration by parts, we obtain

$$\partial_t^\alpha g(\cdot, t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\partial g}{\partial s}(\cdot, s)\, ds, \ t > 0.$$

Since the function $s \mapsto \frac{\partial g}{\partial s}(\cdot, s)$ is continuous on $[0, T]$, then it is bounded on $[0, T]$, and there exists $m, M \in \mathbb{R}$ such that

$$m \int_0^t (t-s)^{-\alpha}\, ds \leq \int_0^t (t-s)^{-\alpha} \frac{\partial g}{\partial s}(\cdot, s)\, ds \leq M \int_0^t (t-s)^{-\alpha}\, ds$$

which implies that

$$\frac{m}{\Gamma(2-\alpha)} t^{1-\alpha} \leq \partial_t^\alpha g(\cdot, t) \leq \frac{M}{\Gamma(1-\alpha)} t^{1-\alpha},$$

and since $1 - \alpha > 0$, $\lim_{t\to 0} \partial_t^\alpha g(\cdot, t) = 0$.

Taking $t = 0$ in Eq (3.3), and since $f(\cdot, 0) = 0$, we deduce that $u^0(\cdot, 0) = 0$; and from the previous remark, $\partial_t^\alpha u^0(\cdot, 0) = 0$. In the same way, we prove that $u^j(\cdot, 0) = 0$ for $j = 1, \ldots, d+1$.

In the following, we show that $\partial_\nu u^j = -\frac{1}{j!} \frac{d^j}{d\lambda^j} (\Lambda_q(\lambda)_{|\lambda=0}[(-\partial_t^\alpha)^j f]$. Indeed, for $j = 0$, we have $\partial_\nu u^0 = \Lambda_q f$. For $j = 1, \ldots, d+1$, let w be the solution of

$$\begin{cases} -\Delta_x w + q(x)w - \lambda w = -\partial_t^\alpha u^{j-1}, & \text{in } \Omega_T, \\ w = 0, & \text{on } \Sigma_T, \end{cases} \tag{3.6}$$

then u^j is the solution of (3.6) for $\lambda = 0$. We have

$$A_q(\lambda)_{|\lambda=0} u^j = -\partial_t^\alpha u^{j-1}$$

which implies that

$$u^j = R_q(\lambda)_{|\lambda=0}(-\partial_t^\alpha u^{j-1}) = \ldots = R_q(\lambda)^j_{|\lambda=0}[(-\partial_t^\alpha)^j u^0].$$

From Lemma 3.1, we have

$$R_q(\lambda) = \sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q} - \lambda} (\cdot, \varphi_{k,q})\, \varphi_{k,q}$$

which implies that

$$R_q(\lambda)^j = \sum_{k=1}^{\infty} \frac{1}{(\lambda_{k,q} - \lambda)^j} (\cdot, \varphi_{k,q})\, \varphi_{k,q}$$

and

$$u^j = \sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q}^j} ((-\partial_t^\alpha)^j u^0, \varphi_{k,q})\, \varphi_{k,q}, \;\; j = 1, \ldots, d+1. \tag{3.7}$$

$(-\partial_t^\alpha)^j u^0$ and $\varphi_{k,q}$ are, respectively, the solutions of

$$\begin{cases} -\Delta_x (-\partial_t^\alpha)^j u^0 + q(x)(-\partial_t^\alpha)^j u^0 = 0, & \text{in } \Omega_T, \\ (-\partial_t^\alpha)^j u^0 = (-\partial_t^\alpha)^j f, & \text{on } \Sigma_T \end{cases} \tag{3.8}$$

and

$$\begin{cases} -\Delta_x \varphi_{k,q} + q(x)\varphi_{k,q} = \lambda_{k,q}\varphi_{k,q}, & \text{in } \Omega, \\ \varphi_{k,q} = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

Applying Green's identity to (3.8) and (3.9),

$$\int_\Omega \Big(\Delta[(-\partial_t^\alpha)^j u^0]\, \varphi_{k,q} - [(-\partial_t^\alpha)^j u^0]\, \Delta\varphi_{k,q} \Big)\, dx = \int_{\partial\Omega} \Big([\partial_\nu (-\partial_t^\alpha)^j u^0]\, \varphi_{k,q} - [(-\partial_t^\alpha)^j u^0]\, \partial_\nu \varphi_{k,q} \Big)\, dx. \tag{3.10}$$

When one obtains

$$((-\partial_t^\alpha)^j u^0, \varphi_{k,q}) = -\frac{1}{\lambda_{k,q}} \langle (-\partial_t^\alpha)^j f, \partial_\nu \varphi_{k,q} \rangle, \tag{3.11}$$

then

$$u^j = -\sum_{k \geq 1} \frac{1}{\lambda_{k,q}^{j+1}} \langle (-\partial_t^\alpha)^j f, \partial_\nu \varphi_{k,q} \rangle\, \varphi_{k,q}$$

and

$$\partial_\nu u^j = -\sum_{k \geq 1} \frac{1}{\lambda_{k,q}^{j+1}} \langle (-\partial_t^\alpha)^j f, \partial_\nu \varphi_{k,q} \rangle\, \partial_\nu \varphi_{k,q}.$$

Then, using Lemma 3.2, we deduce

$$\partial_\nu u^j = \frac{1}{j!} \frac{d^j}{d\lambda^j} (\Lambda_q(\lambda)_{|\lambda=0} [(-\partial_t^\alpha)^j f]. \tag{3.12}$$

Now, we will show that

$$\partial_\nu h = -\sum_{n=1}^{\infty} \frac{1}{\lambda_{n,q}^{d+2}} \left(\int_0^t \langle (-\partial_s^\alpha)^{d+2} f, \partial_\nu \varphi_{n,q} \rangle (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{n,q}(t-s)^\alpha)\, ds \right) \partial_\nu \varphi_{n,q}.$$

From Theorem 2.2 [17], the solution h of (3.5) is given by

$$h = \sum_{n=1}^{\infty} \left(\int_0^t (-\partial_t^\alpha u^{d+1}, \varphi_{n,q})(t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{n,q}(t-s)^\alpha)\, ds \right) \varphi_{n,q}. \tag{3.13}$$

Using (3.7) for $j = d + 1$, we obtain

$$u^{d+1} = \sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q}^{d+1}}((-\partial_t^\alpha)^{d+1}u^0, \varphi_{k,q})\ \varphi_{k,q}$$

which implies that

$$(-\partial_t^\alpha)u^{d+1} = \sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q}^{d+1}}((-\partial_t^\alpha)^{d+2}u^0, \varphi_{k,q})\ \varphi_{k,q}. \tag{3.14}$$

Using the same way for the proof of (3.11), we have

$$((-\partial_t^\alpha)^{d+2}u^0, \varphi_{k,q}) = -\frac{1}{\lambda_{k,q}}\langle(-\partial_t^\alpha)^{d+2}f, \partial_\nu\varphi_{k,q}\rangle$$

and

$$(-\partial_t^\alpha)u^{d+1} = -\sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q}^{d+2}}\langle(-\partial_t^\alpha)^{d+2}f, \partial_\nu\varphi_{k,q}\rangle\ \varphi_{k,q}. \tag{3.15}$$

Substituting (3.15) into (3.13), we obtain

$$h = -\sum_{n=1}^{\infty} \frac{1}{\lambda_{n,q}^{d+2}}\left(\int_0^t \langle(-\partial_s^\alpha)^{d+2}f, \partial_\nu\varphi_{k,q}\rangle(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{k,q}(t-s)^\alpha)\ ds\right)\varphi_{n,q}$$

which implies that

$$B_q^\alpha f = \partial_\nu h = -\sum_{n=1}^{\infty} \frac{1}{\lambda_{n,q}^{d+2}}\left(\int_0^t \langle(-\partial_s^\alpha)^{d+2}f, \partial_\nu\varphi_{k,q}\rangle(t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{k,q}(t-s)^\alpha)\ ds\right)\partial_\nu\varphi_{n,q}. \tag{3.16}$$

Since $u = \sum_{j=1}^{d+1} u^j + h$, $\partial_\nu u = \sum_{j=1}^{d+1} \partial_\nu u^j + \partial_\nu h$ and from Eqs (3.12) and (3.16), we obtain (3.1). □

Proposition 3.2. *For any $q_1,\ q_2 \in L^\infty(\Omega)$ such that $|q_i|_{L^\infty(\Omega)} \le M$ for $i = 1, 2$. There exists a constant C such that*

$$|||B_{q_1}^\alpha - B_{q_2}^\alpha|||_p \le C\delta,$$

where δ is defined in Theorem 2.1 and $|||\cdot|||_p$ represents the norm in the corresponding function space.

Proof.

$$(B_{q_1}^\alpha - B_{q_2}^\alpha)f = J_1 + J_2 + J_3 + J_4,$$

where

$$J_1 = \sum_{k=1}^{\infty}[\frac{1}{\lambda_{k,q_1}^{d+2}} - \frac{1}{\lambda_{k,q_2}^{d+2}}]\partial_\nu\varphi_{k,q_1}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{k,q_1}(t-s)^\alpha)c_{k,q_1}(s)\ ds,$$

$$J_2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q_2}^{d+2}}[\partial_\nu\varphi_{k,q_1} - \partial_\nu\varphi_{k,q_2}]\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_{k,q_1}(t-s)^\alpha)c_{k,q_1}(s)\ ds,$$

$$J_3 = \sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q_2}^{d+2}}\partial_\nu\varphi_{k,q_2}\int_0^t (t-s)^{\alpha-1}\left(E_{\alpha,\alpha}(-\lambda_{k,q_1}(t-s)^\alpha) - E_{\alpha,\alpha}(-\lambda_{k,q_2}(t-s)^\alpha)\right)c_{k,q_1}\ ds,$$

$$J_4 = \sum_{k=1}^{\infty} \frac{1}{\lambda_{k,q_2}^{d+2}} \partial_\nu \varphi_{k,q_2} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_{k,q_2}(t-s)^\alpha)(c_{k,q_1}(s) - c_{k,q_2}(s))\, ds$$

and $c_{k,q_i}(s) = \langle (-\partial_s^\alpha)^{d+2} f, \partial_\nu \varphi_{k,q_i} \rangle,\ i = 1, 2.$

We have the following estimates:

$$\begin{aligned} \|J_1\|_{L^2(0,T;H^p(\partial\Omega))} &\leq d_\infty(\lambda_{q_1}, \lambda_{q_2}) \|f\|_{\Xi_0} \sum_{k\geq 1} \frac{1}{k^{\frac{2(d+2)}{d}}} \\ &\leq C d_\infty(\lambda_{q_1}, \lambda_{q_2}) \|f\|_{\Xi_0}, \end{aligned} \tag{3.17}$$

$$\begin{aligned} \|J_2\|_{L^2(0,T;H^p(\partial\Omega))} + \|J_4\|_{L^2(0,T;H^p(\partial\Omega))} &\leq C \|f\|_{\Xi_0} \sum_{k\geq 1} \frac{1}{k^{\frac{2(d+2)}{d}}} \|\partial_\nu \varphi_{k,q_1} - \partial_\nu \varphi_{k,q_2}\|_{H^{\frac{1}{2}}(\partial\Omega)} \\ &\leq C \|f\|_{\Xi_0} \|\partial_\nu \varphi_{q_1} - \partial_\nu \varphi_{q_2}\|_{\ell^1(H^{\frac{1}{2}}(\partial\Omega),r)}. \end{aligned} \tag{3.18}$$

The estimation of $\|J_3\|_{L^2(0,T;H^p(\partial\Omega))}$:

First, we set $E := E_{\alpha,\alpha}(-\lambda_{k,q_1}(t-s)^\alpha) - E_{\alpha,\alpha}(-\lambda_{k,q_2}(t-s)^\alpha)$, and from (3.2) we have

$$\begin{aligned} E &= \sum_{n=0}^{\infty} \frac{(-1)^n (t-s)^{\alpha n}}{\Gamma(\alpha n + \alpha)} (\lambda_{k,q_1}^n - \lambda_{k,q_2}^n) \\ &= (\lambda_{k,q_1} - \lambda_{k,q_2}) \sum_{n=0}^{\infty} \frac{(-1)^n (t-s)^{\alpha n}}{\Gamma(\alpha n + \alpha)} \sum_{j=0}^{n-1} \lambda_{k,q_1}^j \lambda_{k,q_2}^{n-1-j}. \end{aligned} \tag{3.19}$$

From (2.3), there exists two constants C_1 and C_2 such that

$$C_1 \ k^{\frac{2}{d}(n-1)} \leq \lambda_{k,q_1}^j \lambda_{k,q_2}^{n-1-j} \leq C_2 \ k^{\frac{2}{d}(n-1)}$$

which implies that

$$\tilde{C}_1 \ a_{k,n} \leq \frac{(-1)^n (t-s)^{\alpha n}}{\Gamma(\alpha n + \alpha)} \sum_{j=0}^{n-1} \lambda_{k,q_1}^j \lambda_{k,q_2}^{n-1-j} \leq \tilde{C}_2 \ a_{k,n},$$

where $a_{k,n} := \dfrac{(-1)^n (t-s)^{\alpha n}}{\Gamma(\alpha n + \alpha)} n \ k^{\frac{2}{d}(n-1)}$.

Using the definition of the Mittag-Leffler function, we can prove easily that

$$\sum_{n=1}^{\infty} n \frac{x^n}{\Gamma(\alpha n + \alpha)} = \sum_{n=0}^{\infty} n \frac{x^n}{\Gamma(\alpha n + \alpha)} = \frac{1}{\alpha} E_{\alpha,\alpha-1}(x) - \frac{\alpha - 1}{\alpha} E_{\alpha,\alpha}(x),$$

then

$$\sum_{n=0}^{\infty} a_{k,n} = \frac{1}{k^{\frac{2}{d}}} [\frac{1}{\alpha} E_{\alpha,\alpha-1}(-(t-s)^\alpha k^{\frac{2}{d}}) - \frac{\alpha-1}{\alpha} E_{\alpha,\alpha}(-(t-s)^\alpha k^{\frac{2}{d}})].$$

From [16] (Theorem 1.6, p. 35), there exists a constant $C > 0$ such that

$$|E_{\alpha,\beta}(z)| \leq \frac{C}{1 + |z|},\ z \in \mathbb{C},\ \gamma \leq \arg(z) \leq \pi, \tag{3.20}$$

then

$$|E| \leq \frac{C}{k^{\frac{2}{d}}} |\lambda_{k,q_1} - \lambda_{k,q_2}|.$$

Using (2.3)–(2.5), we obtain

$$\|J_3\|_{H^p(\partial\Omega)} \leq Cd_\infty(\lambda_{q_1}, \lambda_{q_2})\|f\|_{\Xi_0}. \tag{3.21}$$

From the estimations (3.17), (3.18), and (3.21), we conclude that for all $f \in \Xi_0$,

$$\|(B^\alpha_{q_1} - B^\alpha_{q_2})f\|_{H^p(\partial\Omega)} \leq C\delta\|f\|_{\Xi_0},$$

and then

$$|||B^\alpha_{q_1} - B^\alpha_{q_2}|||_p \leq C\delta. \tag{3.22}$$

Proof of Theorem 2.1. We have

$$|||\mathcal{H}_{q_1,\alpha} - \mathcal{H}_{q_2,\alpha}|||_p \leq \sum_{j=0}^{d+1} \|F^{(j)}(0)\|_p + |||B^\alpha_{q_1} - B^\alpha_{q_2}|||_p.$$

From the estimation (3.22) and Lemma 3.4, we obtain

$$|||\mathcal{H}_{q_1,\alpha} - \mathcal{H}_{q_2,\alpha}|||_p \leq C(\delta + \delta^\theta).$$

Therefore,

$$|||\mathcal{H}_{q_1,\alpha} - \mathcal{H}_{q_2,\alpha}|||_p \leq C\delta^\theta$$

for δ sufficiently small.

4. Conclusions

This paper presented a spectral stability estimate for the DN map associated with the fractional diffusion equation. The estimate was formulated in terms of the Dirichlet eigenvalues and normal derivatives of the eigenfunctions of the operator $A_q : -\Delta + q$. The obtained stability result was of Hölder type. The result is novel and interesting, and it has significant implications for the inverse problems of finding the coefficient q in a bounded domain. In future work, the stability estimate will be used as a crucial tool to prove logarithmic stability for this inverse problem, and numerical results will be presented. Overall, this paper contributes to the understanding of the spectral properties and stability analysis of the fractional diffusion equation, paving the way for further investigations in the field of inverse problems and coefficient identification.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

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