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Research article

Fractional 3/8-Simpson type inequalities for differentiable convex functions

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Abstract: The main objective of this study is to establish error estimates of the new parameterized quadrature rule similar to and covering the second Simpson formula. To do this, we start by introducing a new parameterized identity involving the right and left Riemann-Liouville integral operators. On the basis of this identity, we establish some fractional Simpson-type inequalities for functions whose absolute value of the first derivatives are s-convex in the second sense. Also, we examine the special cases $m = 1/2$ and $m = 3/8$, as well as the two cases $s = 1$ and $\alpha = 1$, which respectively represent the classical convexity and the classical integration. By applying the definition of convexity, we derive larger estimates that only used the extreme points. Finally, we provide applications to quadrature formulas, special means, and random variables.

Keywords: Riemann-Liouville integral operators; 3/8-Simpson inequality; convex functions; Hölder inequality
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1. Introduction

Over the last several decades, the study of error estimation of quadrature rules has grown in interest and become an appealing and active subject of research. Numerous extensions and improvements have been suggested for various categories of functions; for example, [1–5].

The 3/8-Simpson rule for four-times continuously differentiable functions, can be declared as follows:

$$\left|\frac{1}{8}\left(\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)\right)-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right| \le \frac{(\gamma_2-\gamma_1)^4}{6480}\left\|\mathcal{P}^{(4)}\right\|_\infty, \tag{1.1}$$

where $\left\|\mathcal{P}^{(4)}\right\|_{\infty} = \sup_{x\in[\gamma_1,\gamma_2]} \left|\mathcal{P}^{(4)}(x)\right|$.

Numerous scholars have examined different Simpson-type disparities. The majority of research has been done on convex function classes, which are significant in many scientific domains including finance, economics, and optimization. Here, we recall the classical definition of the notion of convexity.

Definition 1.1. *[6] A function $\mathcal{P} : I \to \mathbb{R}$ is said to be convex, if*

$$\mathcal{P}(\kappa\gamma_1 + (1-\kappa)\gamma_2) \leq \kappa\mathcal{P}(\gamma_1) + (1-\kappa)\mathcal{P}(\gamma_2)$$

holds for all $\gamma_1, \gamma_2 \in I$ and all $\kappa \in [0, 1]$.

The idea of inequality and the aforementioned principle are closely related, and readers who are interested in learning more about this topic are referred to a rich and diverse literature, see for instance [7–12].

In [13], Laribi and Meftah proposed the following 3/8-Simpson inequalities for s-convex first derivatives.

$$\begin{aligned}
&\left|\frac{1}{8}\left(\mathcal{P}(\gamma_1) + 3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right) + 3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right) + \mathcal{P}(\gamma_2)\right) - \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9(s+1)(s+2)}\left(\left(2\left(\frac{5}{8}\right)^{s+2} + \frac{3s-2}{8}\right)\left(|\mathcal{P}'(\gamma_1)| + |\mathcal{P}'(\gamma_2)|\right)\right. \\
&\left. + \left(\left(1+\left(\frac{3}{4}\right)^{s+2}\right)\left(\frac{1}{2}\right)^{s+1} + \frac{9s+2}{8}\right)\left(\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right| + \left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|\right)\right),
\end{aligned}$$

where $s \in (0, 1]$. For $s = 1$, the above inequality reduces to:

$$\begin{aligned}
&\left|\frac{1}{8}\left(\mathcal{P}(\gamma_1) + 3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right) + 3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right) + \mathcal{P}(\gamma_2)\right) - \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{13824}\left(157\left(|\mathcal{P}'(\gamma_1)| + |\mathcal{P}'(\gamma_2)|\right) + 443\left(\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right| + \left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|\right)\right).
\end{aligned}$$

For functions whose absolute value of the first derivatives are s-convex, they established the following:

$$\begin{aligned}
&\left|\frac{1}{8}\left(\mathcal{P}(\gamma_1) + 3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right) + 3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right) + \mathcal{P}(\gamma_2)\right) - \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9(p+1)^{\frac{1}{p}}}\left(\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q + \left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right. \\
&\left. + \frac{1}{2}\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q + \left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}} + \left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q + |\mathcal{P}'(\gamma_2)|^q}{s+1}\right)^{\frac{1}{q}}\right)
\end{aligned}$$

and

$$\left|\frac{1}{8}\left(\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)\right)-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9}\left(\frac{2}{(s+1)(s+2)}\right)^{\frac{1}{q}}\left(\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\left(\frac{5}{8}\right)^{s+2}+\frac{3s-2}{16}\right)|\mathcal{P}'(\gamma_1)|^q\right.\right.$$
$$\left.+\left(\left(\frac{3}{8}\right)^{s+2}+\frac{5s+2}{16}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left(\frac{s}{4}+\left(\frac{1}{2}\right)^{s+2}\right)^{\frac{1}{q}}\left(\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\frac{17}{64}\right)^{1-\frac{1}{q}}\left(\left(\left(\frac{3}{8}\right)^{s+2}+\frac{5s+2}{16}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q\right.$$
$$\left.\left.+\left(\left(\frac{5}{8}\right)^{s+2}+\frac{3s-2}{16}\right)|\mathcal{P}'(\gamma_2)|^q\right)^{\frac{1}{q}}\right),$$

where $p, q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $s \in (0,1]$.

Erden et al. [14], discussed the above inequality for absolutely continuous functions whose first derivatives belong, to $L^p[\gamma_1,\gamma_2]$, as well as Lipschitzian mappings with bounded variation.

Mahmoudi and Meftah [15] studied a more general form of Simpson's second rule and developed the following results:

$$\left|\frac{1}{2+2\theta}\left(\mathcal{P}(\gamma_1)+\theta\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+\theta\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)\right)-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9(s+1)(s+2)}\left(\left(\frac{3s+4-2\theta}{2+2\theta}+2\left(\frac{2\theta-1}{2+2\theta}\right)^{s+2}\right)(|\mathcal{P}'(\gamma_1)|+|\mathcal{P}'(\gamma_2)|)\right.$$
$$\left.+\left(\frac{3\theta s+(2\theta-4)}{2+2\theta}+\left(\frac{1}{2}\right)^{s+1}+2\left(\frac{3}{2+2\theta}\right)^{s+2}\right)\left(\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|\right)\right),$$

$$\left|\frac{1}{2+2\theta}\left(\mathcal{P}(\gamma_1)+\theta\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+\theta\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)\right)-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{18(p+1)^{p+1}}\left(\left(\frac{3^{p+1}+(2\theta-1)^{p+1}}{2(1+\theta)^{p+1}}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{s+1}\right)^{\frac{1}{q}}+\left(\frac{3^{p+1}+(2\theta-1)^{p+1}}{2(1+\theta)^{p+1}}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{s+1}\right)^{\frac{1}{q}}\right)$$

and

$$\left|\frac{1}{2+2\theta}\left(\mathcal{P}(\gamma_1)+\theta\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+\theta\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)\right)-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9((s+1)(s+2))^{\frac{1}{q}}}\left(\left(\frac{9+(2\theta-1)^2}{8(1+\theta)^2}\right)^{1-\frac{1}{q}}\left(\left(\frac{3s+4-2\theta}{2+2\theta}+2\left(\frac{2\theta-1}{2+2\theta}\right)^{s+2}\right)|\mathcal{P}'(\gamma_1)|^q\right.\right.$$

$$+\left(\frac{(2\theta-1)s+(2\theta-4)}{2+2\theta}+2\left(\frac{3}{2+2\theta}\right)^{s+2}\right|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\left|^{q}\right)^{\frac{1}{q}}$$
$$+\frac{1}{4}\left(2s+\left(\frac{1}{2}\right)^{s-1}\right)^{\frac{1}{q}}\left(\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^{q}+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{9+(2\theta-1)^2}{8(1+\theta)^2}\right)^{1-\frac{1}{q}}\left(\left(\frac{(2\theta-1)s+(2\theta-4)}{2+2\theta}+2\left(\frac{3}{2+2\theta}\right)^{s+2}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^{q}\right.$$
$$\left.+\left(\frac{3s+4-2\theta}{2+2\theta}+2\left(\frac{2\theta-1}{2+2\theta}\right)^{s+2}\right)|\mathcal{P}'(\gamma_2)|^{q}\right)^{\frac{1}{q}}\Bigg),$$

where θ is a positive number, $s \in (0, 1]$, and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Because of its wide range of applications across several domains and its ability to give a better description for evaluating the dynamics of complex systems, fractional calculus, also known as non-integer calculus, has grown in popularity and appeal. This type of computation is often attributed to Liouville, however there are other fractional operators in the literature. First, we review what the Riemann-Liouville operator is.

Definition 1.2. *[16] Let $\mathcal{P} \in L^1[\gamma_1, \gamma_2]$. The Riemann-Liouville fractional integrals $I^{\alpha}_{\gamma_1^+}\mathcal{P}$ and $I^{\alpha}_{\gamma_2^-}\mathcal{P}$ of order $\alpha > 0$ with $\gamma_1 \geq 0$ are defined by*

$$I^{\alpha}_{\gamma_1^+}\mathcal{P}(x) = \frac{1}{\Gamma(\alpha)}\int_{\gamma_1}^{x}(x-\kappa)^{\alpha-1}\mathcal{P}(\kappa)d\kappa, \quad x > \gamma_1$$
$$I^{\alpha}_{\gamma_2^-}\mathcal{P}(x) = \frac{1}{\Gamma(\alpha)}\int_{x}^{\gamma_2}(\kappa-x)^{\alpha-1}\mathcal{P}(\kappa)d\kappa, \quad \gamma_2 > x$$

respectively, where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$ is the Gamma function and $I^0_{\gamma_1^+}\mathcal{P}(x) = I^0_{b^-}\mathcal{P}(x) = \mathcal{P}(x)$.

For papers dealing with fractional integral inequalities, we refer readers to [17–25].

Recently, Ali et al. [7] established some fractional Newton type inequalities for functions whose absolute value of the first derivative is convex by using the follow identity:

Lemma 1.1. *For a differentiable function $\mathcal{P} : [\gamma_1, \gamma_2] \to \mathbb{R}$ over (γ_1, γ_2) with $\mathcal{P} \in L[\gamma_1, \gamma_2]$, the following equality holds:*

$$(1-\lambda-\nu)(\mathcal{P}(\gamma_1)+\mathcal{P}(\gamma_2))+(\nu-\lambda)\left(\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right)$$
$$-\frac{\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\left(I^{\alpha}_{\gamma_1^+}\mathcal{P}(\gamma_2)+I^{\alpha}_{\gamma_2^-}\mathcal{P}(\gamma_1)\right)$$
$$= (\gamma_2-\gamma_1)\int_0^1\Delta(\kappa)\left[\mathcal{P}'((1-\kappa)\gamma_1+\kappa\gamma_2)-\mathcal{P}'(\kappa\gamma_1+(1-\kappa)\gamma_2)\right]d\kappa,$$

where $\lambda, \mu, \nu \geq 0$ and

$$\Delta(\kappa) = \begin{cases} \kappa^{\alpha}-\lambda, & \kappa \in \left[0, \frac{1}{3}\right) \\ \kappa^{\alpha}-\mu, & \kappa \in \left[\frac{1}{3}, \frac{2}{3}\right) \\ \kappa^{\alpha}-\nu, & \kappa \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

Motivated by the above cited results, in this paper, we first introduce a new parameterized identity. Using this identity, we establish some new parameterized Simpson's like type inequalities for differentiable convex functions via Riemann-Liouville integral operators. The obtained results include most of the existing studies. Several estimates are proposed, some of which are finer, and others are larger. Indeed, our results refine those obtained in [7] for the particular case of 3/8-Simpson. It also recovers the results given in [15] by setting $m = \frac{3}{2+2\theta}$ and $\alpha = 1$, in addition to the case $m = \frac{3}{4}$ and $\alpha = 1$. Some of the results obtained in [13] are recaptured by taking $m = \frac{3}{8}$. Applications to composite quadrature formula, special means, and random variables are provided. Note that, in this study, several estimates are proposed, some of which are finer, and others larger.

2. Main results

In order to prove our results, we need the following definitions and lemmas.

Definition 2.1. *[16] For any complex numbers γ_1, γ_2 such that $\Re(\gamma_1) > 0$ and $\Re(\gamma_2) > 0$, the Beta function is defined by*

$$B(\gamma_1, \gamma_2) = \int_0^1 \kappa^{\gamma_1 - 1} (1-\kappa)^{y-1} d\kappa = \frac{\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(\gamma_1+\gamma_2)},$$

where $\Gamma(.)$ is the Euler Gamma function.

Definition 2.2. *[16] The Hypergeometric function is defined for $\Re c > \Re b > 0$ and $|z| < 1$, as follows:*

$${}_2F_1(a, b, c; z) = \frac{1}{B(b, c-b)} \int_0^1 \kappa^{b-1} (1-\kappa)^{c-b-1} (1-z\kappa)^{-a} d\kappa,$$

where $c > b > 0, |z| < 1$ and $B(.,.)$ is the beta function.

Lemma 2.1. *Let $\alpha > 0, l \in (0,1]$ and $p \geq 1$. Then, we have*

$$\int_0^1 |\kappa^\alpha - l|^p d\kappa = \frac{l^{p+\frac{1}{\alpha}} B\left(\frac{1}{\alpha}, p+1\right)}{\alpha} + \frac{(1-l)^{p+1} \cdot {}_2F_1\left(\frac{\alpha-1}{\alpha}, 1, p+2; 1-l\right)}{\alpha(p+1)}.$$

Proof. We have

$$\begin{aligned}
\int_0^1 |\kappa^\alpha - l|^p d\kappa &= \int_0^{l^{\frac{1}{\alpha}}} (l - \kappa^\alpha)^p d\kappa + \int_{l^{\frac{1}{\alpha}}}^1 (\kappa^\alpha - l)^p d\kappa \\
&= \frac{1}{\alpha} \int_0^l (l-u)^p u^{\frac{1}{\alpha}-1} du + \frac{1}{\alpha} \int_l^1 (u-l)^p u^{\frac{1}{\alpha}-1} du \\
&= \frac{l^{p+\frac{1}{\alpha}}}{\alpha} \int_0^1 (1-u)^p u^{\frac{1}{\alpha}-1} du + \frac{1}{\alpha} \int_0^{1-l} (1-l-u)^p (1-u)^{\frac{1}{\alpha}-1} du
\end{aligned}$$

$$= \frac{l^{p+\frac{1}{\alpha}}}{\alpha} B\left(\frac{1}{\alpha}, p+1\right) + \frac{(1-l)^{p+1}}{\alpha} \int_0^1 (1-u)^p \left(1-(1-l)u\right)^{\frac{1}{\alpha}-1} du$$
$$= \frac{l^{p+\frac{1}{\alpha}}}{\alpha} B\left(\frac{1}{\alpha}, p+1\right) + \frac{(1-l)^{p+1}}{\alpha(p+1)} \cdot {}_2F_1\left(\frac{\alpha-1}{\alpha}, 1, p+2; 1-l\right).$$

The proof is completed. □

Lemma 2.2. *Let $\mathcal{P} : [\gamma_1, \gamma_2] \to \mathbb{R}$ be a differentiable function on $[\gamma_1, \gamma_2]$ with $\gamma_1 < \gamma_2$ and $\mathcal{P}' \in L^1[\gamma_1, \gamma_2]$, then the following equality holds:*

$$\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}} \mathcal{Q}(\gamma_1, \gamma_2, \mathcal{P})$$
$$= \frac{\gamma_2-\gamma_1}{9} \left(\int_0^1 (\kappa^{\alpha} - m) \mathcal{P}'\left((1-\kappa)\gamma_1 + \kappa\frac{2\gamma_1+\gamma_2}{3}\right) d\kappa \right.$$
$$- \int_0^1 \left((1-\kappa)^{\alpha} - \frac{1}{2}\right) \mathcal{P}'\left((1-\kappa)\frac{2\gamma_1+\gamma_2}{3} + \kappa\frac{\gamma_1+2\gamma_2}{3}\right) d\kappa$$
$$\left. + \int_0^1 \left(\kappa^{\alpha} - (1-m)\right) \mathcal{P}'\left((1-\kappa)\frac{\gamma_1+2\gamma_2}{3} + \kappa\gamma_2\right) d\kappa \right),$$

where $m \in [0, 1]$ and

$$\mathcal{Q}(\gamma_1, \gamma_2, \mathcal{P}) = I^{\alpha}_{\left(\frac{2\gamma_1+\gamma_2}{3}\right)^-} \mathcal{P}(\gamma_1) + I^{\alpha}_{\left(\frac{2\gamma_1+\gamma_2}{3}\right)^+} \mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right) + I^{\alpha}_{\gamma_2^-} \mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right).$$

Proof. Let

$$I = I_1 - I_2 + I_3, \tag{2.1}$$

where

$$I_1 = \int_0^1 (\kappa^{\alpha} - m) \mathcal{P}'\left((1-\kappa)\gamma_1 + \kappa\frac{2\gamma_1+\gamma_2}{3}\right) d\kappa,$$
$$I_2 = \int_0^1 \left((1-\kappa)^{\alpha} - \frac{1}{2}\right) \mathcal{P}'\left((1-\kappa)\frac{2\gamma_1+\gamma_2}{3} + \kappa\frac{\gamma_1+2\gamma_2}{3}\right) d\kappa,$$
$$I_3 = \int_0^1 \left(\kappa^{\alpha} - (1-m)\right) \mathcal{P}'\left((1-\kappa)\frac{\gamma_1+2\gamma_2}{3} + \kappa\gamma_2\right) d\kappa.$$

Integrating by parts I_1, we obtain

$$I_1 = \frac{3}{\gamma_2-\gamma_1} (\kappa^{\alpha} - m) \mathcal{P}\left((1-\kappa)\gamma_1 + \kappa\frac{2\gamma_1+\gamma_2}{3}\right)\Big|_{\kappa=0}^{\kappa=1} \tag{2.2}$$
$$- \frac{3\alpha}{\gamma_2-\gamma_1} \int_0^1 \kappa^{\alpha-1} \mathcal{P}\left((1-\kappa)\gamma_1 + \kappa\frac{2\gamma_1+\gamma_2}{3}\right) d\kappa$$

$$
\begin{aligned}
&= \tfrac{3(g-m)}{\gamma_2-\gamma_1}\mathcal{P}\left(\tfrac{2\gamma_1+\gamma_2}{3}\right)+\tfrac{3m}{\gamma_2-\gamma_1}\mathcal{P}(\gamma_1) \\
&\quad -\tfrac{3\alpha}{\gamma_2-\gamma_1}\int_0^1 \kappa^{\alpha-1}\mathcal{P}\left((1-\kappa)\gamma_1+\kappa\tfrac{2\gamma_1+\gamma_2}{3}\right)d\kappa \\
&= \tfrac{3(g-m)}{\gamma_2-\gamma_1}\mathcal{P}\left(\tfrac{2\gamma_1+\gamma_2}{3}\right)+\tfrac{3m}{\gamma_2-\gamma_1}\mathcal{P}(\gamma_1) \\
&\quad -\alpha\left(\tfrac{3}{\gamma_2-\gamma_1}\right)^{\alpha+1}\int_{\gamma_1}^{\frac{2\gamma_1+\gamma_2}{3}}(u-\gamma_1)^{\alpha-1}\mathcal{P}(u)\,du \\
&= \tfrac{3(g-m)}{\gamma_2-\gamma_1}\mathcal{P}\left(\tfrac{2\gamma_1+\gamma_2}{3}\right)+\tfrac{3m}{\gamma_2-\gamma_1}\mathcal{P}(\gamma_1)-\tfrac{3^{\alpha+1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha+1}}I^{\alpha}_{\left(\frac{2\gamma_1+b}{3}\right)^-}\mathcal{P}(\gamma_1).
\end{aligned}
$$

Similarly, we obtain

$$
\begin{aligned}
I_2 &= \tfrac{3}{\gamma_2-\gamma_1}\left((1-\kappa)^{\alpha}-\tfrac{1}{2}\right)\mathcal{P}\left((1-\kappa)\tfrac{2\gamma_1+\gamma_2}{3}+\kappa\tfrac{\gamma_1+2\gamma_2}{3}\right)\Big|_{\kappa=0}^{\kappa=1} \\
&\quad +\tfrac{3\alpha}{\gamma_2-\gamma_1}\int_0^1(1-\kappa)^{\alpha-1}\mathcal{P}\left((1-\kappa)\tfrac{2\gamma_1+\gamma_2}{3}+\kappa\tfrac{\gamma_1+2\gamma_2}{3}\right)d\kappa \\
&= -\tfrac{3}{2(\gamma_2-\gamma_1)}\mathcal{P}\left(\tfrac{\gamma_1+2\gamma_2}{3}\right)-\tfrac{3}{2(\gamma_2-\gamma_1)}\mathcal{P}\left(\tfrac{2\gamma_1+\gamma_2}{3}\right) \\
&\quad +\alpha\left(\tfrac{3}{\gamma_2-\gamma_1}\right)^{\alpha+1}\int_{\frac{2\gamma_1+\gamma_2}{3}}^{\frac{\gamma_1+2\gamma_2}{3}}\left(\tfrac{\gamma_1+2\gamma_2}{3}-u\right)^{\alpha-1}\mathcal{P}(u)\,du \\
&= -\tfrac{3}{2(\gamma_2-\gamma_1)}\mathcal{P}\left(\tfrac{\gamma_1+2\gamma_2}{3}\right)-\tfrac{3}{2(\gamma_2-\gamma_1)}\mathcal{P}\left(\tfrac{2\gamma_1+\gamma_2}{3}\right) \\
&\quad +\tfrac{3^{\alpha+1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha+1}}I^{\alpha}_{\left(\frac{2\gamma_1+\gamma_2}{3}\right)^+}\mathcal{P}\left(\tfrac{\gamma_1+2\gamma_2}{3}\right)
\end{aligned}
\tag{2.3}
$$

and

$$
\begin{aligned}
I_3 &= \tfrac{3}{\gamma_2-\gamma_1}\left(\kappa^{\alpha}-(1-m)\right)\mathcal{P}\left((1-\kappa)\tfrac{\gamma_1+2\gamma_2}{3}+\kappa\gamma_2\right)\Big|_{\kappa=0}^{\kappa=1} \\
&\quad -\tfrac{3\alpha}{\gamma_2-\gamma_1}\int_0^1\kappa^{\alpha-1}\mathcal{P}\left((1-\kappa)\tfrac{\gamma_1+2\gamma_2}{3}+\kappa\gamma_2\right)d\kappa \\
&= \tfrac{3m}{\gamma_2-\gamma_1}\mathcal{P}(\gamma_2)+\tfrac{3(1-m)}{\gamma_2-\gamma_1}\mathcal{P}\left(\tfrac{\gamma_1+2\gamma_2}{3}\right) \\
&\quad -\tfrac{3\alpha}{\gamma_2-\gamma_1}\int_0^1\kappa^{\alpha-1}\mathcal{P}\left((1-\kappa)\tfrac{\gamma_1+2\gamma_2}{3}+\kappa\gamma_2\right)d\kappa \\
&= \tfrac{3m}{\gamma_2-\gamma_1}\mathcal{P}(\gamma_2)+\tfrac{3(1-m)}{\gamma_2-\gamma_1}\mathcal{P}\left(\tfrac{\gamma_1+2\gamma_2}{3}\right) \\
&\quad -\alpha\left(\tfrac{3}{\gamma_2-\gamma_1}\right)^{\alpha+1}\int_{\frac{\gamma_1+2\gamma_2}{3}}^{\gamma_2}\left(u-\tfrac{\gamma_1+2\gamma_2}{3}\right)^{\alpha-1}\mathcal{P}(u)\,du \\
&= \tfrac{3m}{\gamma_2-\gamma_1}\mathcal{P}(\gamma_2)+\tfrac{3(1-m)}{\gamma_2-\gamma_1}\mathcal{P}\left(\tfrac{\gamma_1+2\gamma_2}{3}\right)-\tfrac{3^{\alpha+1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha+1}}.I^{\alpha}_{\gamma_2^-}\mathcal{P}(\tfrac{\gamma_1+2\gamma_2}{3}).
\end{aligned}
\tag{2.4}
$$

Using (2.2)–(2.4) in (2.1), and then multiplying the resulting equality by $\frac{\gamma_2-\gamma_1}{9}$, we get the desired result. □

We are now ready to prove our main results. Note that, at the end of each result, we treat certain particular cases which repeat or generalize certain inequalities already known in the literature.

Theorem 2.1. *Let $\mathcal{P}:[\gamma_1,\gamma_2]\to\mathbb{R}$ be a differentiable function on $[\gamma_1,\gamma_2]$ with $\gamma_1<\gamma_2$ and $\mathcal{P}'\in L^1[\gamma_1,\gamma_2]$. If $|\mathcal{P}'|$ is convex, then we have*

$$\begin{aligned}
&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}Q(\gamma_1,\gamma_2,\mathcal{P})\right|\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{2-m(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)}+m^{1+\frac{1}{\alpha}}\frac{2\alpha}{\alpha+1}-m^{1+\frac{2}{\alpha}}\frac{\alpha}{\alpha+2}\right)|\mathcal{P}'(\gamma_1)|\right.\\
&+\left(\frac{8-(1+2m)(\alpha+2)}{4(\alpha+2)}+\left(\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}+m^{1+\frac{2}{\alpha}}\right)\frac{\alpha}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|\\
&+\left(\frac{8-(3-2m)(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)}+\left(\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}+(1-m)^{1+\frac{1}{\alpha}}\right)\frac{2\alpha}{\alpha+1}\right.\\
&\left.-\left(\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}+(1-m)^{1+\frac{2}{\alpha}}\right)\frac{\alpha}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|\\
&\left.+\left(\frac{2-(1-m)(\alpha+2)}{2(\alpha+2)}+(1-m)^{1+\frac{2}{\alpha}}\frac{\alpha}{\alpha+2}\right)|\mathcal{P}'(\gamma_2)|\right).
\end{aligned}$$

Proof. From Lemma 2.2, properties of the modulus, and the convexity of $|\mathcal{P}'|$, we have

$$\begin{aligned}
&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}Q(\gamma_1,\gamma_2,\mathcal{P})\right|\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\int_0^1|\kappa^{\alpha}-m|\left|\mathcal{P}'\left((1-\kappa)\gamma_1+\kappa\frac{2\gamma_1+\gamma_2}{3}\right)\right|d\kappa\right.\\
&+\int_0^1\left|(1-\kappa)^{\alpha}-\frac{1}{2}\right|\left|\mathcal{P}'\left((1-\kappa)\frac{2\gamma_1+\gamma_2}{3}+\kappa\frac{\gamma_1+2\gamma_2}{3}\right)\right|d\kappa\\
&\left.+\int_0^1|\kappa^{\alpha}-(1-m)|\left|\mathcal{P}'\left((1-\kappa)\frac{\gamma_1+2\gamma_2}{3}+\kappa\gamma_2\right)\right|d\kappa\right)
\end{aligned}$$

$$\begin{aligned}
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\int_0^1|\kappa^{\alpha}-m|\left((1-\kappa)|\mathcal{P}'(\gamma_1)|+\kappa\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|\right)d\kappa\right.\\
&+\int_0^1\left|(1-\kappa)^{\alpha}-\frac{1}{2}\right|\left((1-\kappa)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|+\kappa\left|\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|\right)d\kappa\\
&\left.+\int_0^1|\kappa^{\alpha}-(1-m)|\left((1-\kappa)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|+\kappa|\mathcal{P}'(\gamma_2)|\right)d\kappa\right)
\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{\gamma_2-\gamma_1}{9}\Bigg(|\mathcal{P}'(\gamma_1)| \int_0^1 (1-\kappa)\,|\kappa^\alpha - m|\,d\kappa \\
&+ \left|\mathcal{P}'\left(\tfrac{2\gamma_1+\gamma_2}{3}\right)\right| \left(\int_0^1 \kappa\,|\kappa^\alpha - m|\,d\kappa + \int_0^1 (1-\kappa)\left|(1-\kappa)^\alpha - \tfrac{1}{2}\right| d\kappa \right) \\
&+ \left|\mathcal{P}'\left(\tfrac{\gamma_1+2\gamma_2}{3}\right)\right| \left(\int_0^1 \kappa\left|(1-\kappa)^\alpha - \tfrac{1}{2}\right| d\kappa + \int_0^1 (1-\kappa)\,|\kappa^\alpha - (1-m)|\,d\kappa \right) \\
&+ |\mathcal{P}'(\gamma_2)| \int_0^1 \kappa\,|\kappa^\alpha - (1-m)|\,d\kappa \Bigg) \\
&= \tfrac{\gamma_2-\gamma_1}{9}\Bigg(\left(\tfrac{2-m(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)} + m^{1+\frac{1}{\alpha}} \tfrac{2\alpha}{\alpha+1} - m^{1+\frac{2}{\alpha}} \tfrac{\alpha}{\alpha+2} \right) |\mathcal{P}'(\gamma_1)| \\
&+ \left(\tfrac{8-(1+2m)(\alpha+2)}{4(\alpha+2)} + \left(\left(\tfrac{1}{2}\right)^{1+\frac{2}{\alpha}} + m^{1+\frac{2}{\alpha}} \right) \tfrac{\alpha}{\alpha+2} \right) \left|\mathcal{P}'\left(\tfrac{2\gamma_1+\gamma_2}{3}\right)\right| \\
&+ \Bigg(\tfrac{4-(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)} + \tfrac{2-(1-m)(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)} + \left(\left(\tfrac{1}{2}\right)^{1+\frac{1}{\alpha}} + (1-m)^{1+\frac{1}{\alpha}} \right) \tfrac{2\alpha}{\alpha+1} \\
&- \left(\left(\tfrac{1}{2}\right)^{1+\frac{2}{\alpha}} + (1-m)^{1+\frac{2}{\alpha}} \right) \tfrac{\alpha}{\alpha+2} \Bigg) \left|\mathcal{P}'\left(\tfrac{\gamma_1+2\gamma_2}{3}\right)\right| \\
&+ \left(\tfrac{2-(1-m)(\alpha+2)}{2(\alpha+2)} + (1-m)^{1+\frac{2}{\alpha}} \tfrac{\alpha}{\alpha+2} \right) |\mathcal{P}'(\gamma_2)| \Bigg),
\end{aligned}
$$

where we have used the fact that

$$\int_0^1 (1-\kappa)\,|\kappa^\alpha - m|\,d\kappa = \tfrac{2-m(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)} + \tfrac{2\alpha}{\alpha+1} m^{1+\frac{1}{\alpha}} - \tfrac{\alpha}{\alpha+2} m^{1+\frac{2}{\alpha}}, \tag{2.5}$$

$$\int_0^1 \kappa\,|\kappa^\alpha - m|\,d\kappa = \tfrac{2-m(\alpha+2)}{2(\alpha+2)} + \tfrac{\alpha}{\alpha+2} m^{1+\frac{2}{\alpha}}, \tag{2.6}$$

$$\int_0^1 (1-\kappa)\left|(1-\kappa)^\alpha - \tfrac{1}{2}\right| d\kappa = \tfrac{4-(\alpha+2)}{4(\alpha+2)} + \tfrac{\alpha}{\alpha+2}\left(\tfrac{1}{2}\right)^{1+\frac{2}{\alpha}}, \tag{2.7}$$

$$\int_0^1 \kappa\left|(1-\kappa)^\alpha - \tfrac{1}{2}\right| d\kappa = \tfrac{4-(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)} + \tfrac{2\alpha}{\alpha+1}\left(\tfrac{1}{2}\right)^{1+\frac{1}{\alpha}} - \tfrac{\alpha}{\alpha+2}\left(\tfrac{1}{2}\right)^{1+\frac{2}{\alpha}}, \tag{2.8}$$

$$\int_0^1 (1-\kappa)\,|\kappa^\alpha - (1-m)|\,d\kappa = \tfrac{2-(1-m)(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)} + \tfrac{2\alpha(1-m)^{1+\frac{1}{\alpha}}}{\alpha+1} - \tfrac{\alpha(1-m)^{1+\frac{2}{\alpha}}}{\alpha+2} \tag{2.9}$$

and

$$\int_0^1 \kappa\,|\kappa^\alpha - (1-m)|\,d\kappa = \tfrac{2-(1-m)(\alpha+2)}{2(\alpha+2)} + \tfrac{\alpha}{\alpha+2}(1-m)^{1+\frac{2}{\alpha}}. \tag{2.10}$$

The proof is completed. □

Remark 2.1. *Theorem 2.1 will be reduced to Corollary 2.3 from [15], if we take* $\alpha = 1$ *and* $m = \frac{3}{2+2\theta}$.

Corollary 2.1. *In Theorem 2.1, if we take* $m = \frac{3}{8}$*, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{16-3(\alpha+1)(\alpha+2)}{16(\alpha+1)(\alpha+2)} + \left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}\frac{2\alpha}{\alpha+1} - \left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}}\frac{\alpha}{\alpha+2}\right)|\mathcal{P}'(\gamma_1)|\right. \\
&+ \left(\frac{18-7\alpha}{16(\alpha+2)} + \left(\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}} + \left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}}\right)\frac{\alpha}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right| \\
&+ \left(\frac{32-9(\alpha+1)(\alpha+2)}{16(\alpha+1)(\alpha+2)} + \left(\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}} + \left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}\right)\frac{2\alpha}{\alpha+1}\right. \\
&\left. - \left(\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}} + \left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}}\right)\frac{\alpha}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right| \\
&\left.+ \left(\frac{6-5\alpha}{16(\alpha+2)} + \frac{\alpha}{\alpha+2}\left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|\right).
\end{aligned}$$

Corollary 2.2. *In Theorem 2.1, if we take* $m = \frac{1}{2}$*, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{4-(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)} + \frac{\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_1)|\right. \\
&+ \left(\frac{2-\alpha}{2(\alpha+2)} + \left(\frac{1}{2}\right)^{\frac{2}{\alpha}}\frac{\alpha}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right| \\
&+ \left(\frac{4-(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)} + \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\frac{2\alpha}{\alpha+1} - \left(\frac{1}{2}\right)^{\frac{2}{\alpha}}\frac{\alpha}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right| \\
&\left.+ \left(\frac{2-\alpha}{4(\alpha+2)} + \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|\right).
\end{aligned}$$

Corollary 2.3. *In Theorem 2.1, if we use the convexity of* $|\mathcal{P}'|$*, i.e.* $\left|\mathcal{P}'\left(\frac{n\gamma_1+z\gamma_2}{n+z}\right)\right| \leq \frac{n}{n+z}|\mathcal{P}'(\gamma_1)| + \frac{z}{n+z}|\mathcal{P}'(\gamma_2)|$*, we get*

$$\begin{aligned}
&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\frac{16\alpha+36-(5+8m)(\alpha+1)(\alpha+2)}{12(\alpha+1)(\alpha+2)} + \left(\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}} + 3m^{1+\frac{1}{\alpha}} + (1-m)^{1+\frac{1}{\alpha}}\right)\frac{2\alpha}{3(\alpha+1)}\right. \\
&+ \left(\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}} - m^{1+\frac{2}{\alpha}} - (1-m)^{1+\frac{2}{\alpha}}\right)\frac{\alpha}{3(\alpha+2)}\Bigg)|\mathcal{P}'(\gamma_1)| \\
&+ \left(\frac{36+20\alpha-(13-8m)(\alpha+1)(\alpha+2)}{12(\alpha+1)(\alpha+2)} + \left(\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}} + (1-m)^{1+\frac{1}{\alpha}}\right)\frac{4\alpha}{3(\alpha+1)}\right. \\
&\left.+ \left(m^{1+\frac{2}{\alpha}} + (1-m)^{1+\frac{2}{\alpha}} - \left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\frac{\alpha}{3(\alpha+2)}\right)|\mathcal{P}'(\gamma_2)|\Bigg).
\end{aligned}$$

Corollary 2.4. *In Corollary 2.3, if we take $m = \frac{3}{8}$, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\frac{16\alpha+36-8(\alpha+1)(\alpha+2)}{12(\alpha+1)(\alpha+2)} + \left(\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}} + 3\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}} + \left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}\right)\frac{2\alpha}{3(\alpha+1)}\right. \\
&+ \left(\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}} - \left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}} - \left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}}\right)\frac{\alpha}{3(\alpha+2)}\Bigg)\left|\mathcal{P}'(\gamma_1)\right| \\
&+ \left(\frac{36+20\alpha-10(\alpha+1)(\alpha+2)}{12(\alpha+1)(\alpha+2)} + \left(\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}} + \left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}\right)\frac{4\alpha}{3(\alpha+1)}\right. \\
&\left.\left. + \left(\left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}} + \left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}} - \left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\frac{\alpha}{3(\alpha+2)}\right)\left|\mathcal{P}'(\gamma_2)\right|\right).
\end{aligned}$$

Corollary 2.5. *In Corollary 2.3, if we take $m = \frac{1}{2}$, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{4-(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)} + \frac{\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{\frac{1}{\alpha}} - \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'(\gamma_1)\right|\right. \\
&+ \left(\frac{2-\alpha}{2(\alpha+2)} + \left(\frac{1}{2}\right)^{\frac{2}{\alpha}}\frac{\alpha}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right| \\
&+ \left(\frac{4-(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)} + \left(\frac{1}{2}\right)^{\frac{1}{\alpha}}\frac{2\alpha}{\alpha+1} - \left(\frac{1}{2}\right)^{\frac{2}{\alpha}}\frac{\alpha}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right| \\
&\left. + \left(\frac{2-\alpha}{4(\alpha+2)} + \frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'(\gamma_2)\right|\right).
\end{aligned}$$

Corollary 2.6. *In Corollary 2.3, if we take $\alpha = 1$, then we get*

$$\begin{aligned}
&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6} - \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right| \\
\leq\ & \frac{\left(5-8m+8m^2\right)(\gamma_2-\gamma_1)}{72}\left(\left|\mathcal{P}'(\gamma_1)\right| + \left|\mathcal{P}'(\gamma_2)\right|\right).
\end{aligned}$$

Corollary 2.7. *In Corollary 2.6, if we take $m = \frac{3}{8}$, then we get*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8} - \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right| \\
\leq\ & \frac{25(\gamma_2-\gamma_1)}{576}\left(\left|\mathcal{P}'(\gamma_1)\right| + \left|\mathcal{P}'(\gamma_2)\right|\right).
\end{aligned}$$

Corollary 2.8. *In Corollary 2.6, if we take $m = \frac{1}{2}$, then we get*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6} - \frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{24}\left(\left|\mathcal{P}'(\gamma_1)\right| + \left|\mathcal{P}'(\gamma_2)\right|\right).
\end{aligned}$$

Corollary 2.9. *In Theorem 2.1, if we take $\alpha = 1$, then we get*

$$\begin{aligned}&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\frac{1-3m+6m^2-2m^3}{6}\left|\mathcal{P}'(\gamma_1)\right|+\frac{11-12m+8m^3}{24}\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|\right.\\ &\left.+\frac{11-12m+8m^3}{24}\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|+\frac{1-3m+6m^2-2m^3}{6}\left|\mathcal{P}'(\gamma_2)\right|\right).\end{aligned}$$

Remark 2.2. *Corollary 2.9 recaptures the second inequality of Corollary 2.5 from [15] if we take $m = \frac{3}{4}$.*

Corollary 2.10. *In Corollary 2.9, if we take $m = \frac{3}{8}$, then we get*

$$\begin{aligned}&\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\frac{157|\mathcal{P}'(\gamma_1)|+443\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|+443\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|+157|\mathcal{P}'(\gamma_2)|}{1536}\right).\end{aligned}$$

Remark 2.3. *The same results were obtained in Corollary 2.1 from [13].*

Corollary 2.11. *In Corollary 2.9, if we take $m = \frac{1}{2}$, then we get*

$$\begin{aligned}&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \leq\ & \frac{\gamma_2-\gamma_1}{72}\left(|\mathcal{P}'(\gamma_1)|+2\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|+2\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|+|\mathcal{P}'(\gamma_2)|\right).\end{aligned}$$

Theorem 2.2. *Let $\mathcal{P} : [\gamma_1, \gamma_2] \to \mathbb{R}$ be a differentiable function on $[\gamma_1, \gamma_2]$ with $\gamma_1 < \gamma_2$ and $\mathcal{P}' \in L^1[\gamma_1, \gamma_2]$. If $|\mathcal{P}'|^q$ is convex where $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then we have*

$$\begin{aligned}&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^\alpha}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right|\\ \leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{m^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\frac{(1-m)^{p+1}\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;1-m\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.\\ &+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{p+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\\ &\left.+\left(\frac{(1-m)^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\frac{m^{p+1}\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;m\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right),\end{aligned}$$

where B and ${}_2F_1$ are Beta and Hypergeometric functions, respectively.

Proof. From Lemma 2.2, properties of the modulus, Hölder's inequality, and the convexity of $|\mathcal{P}'|^q$, we have

$$
\begin{aligned}
&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^\alpha}Q(\gamma_1,\gamma_2,\mathcal{P})\right|\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\int_0^1|\kappa^\alpha-m|\left|f'\left((1-\kappa)\gamma_1+\kappa\frac{2\gamma_1+\gamma_2}{3}\right)\right|d\kappa\right.\\
&+\int_0^1\left|(1-\kappa)^\alpha-\tfrac{1}{2}\right|\left|\mathcal{P}'\left((1-\kappa)\frac{2\gamma_1+\gamma_2}{3}+\kappa\frac{\gamma_1+2\gamma_2}{3}\right)\right|d\kappa+\left.\int_0^1|\kappa^\alpha-(1-m)|\left|\mathcal{P}'\left((1-\kappa)\frac{\gamma_1+2\gamma_2}{3}+\kappa\gamma_2\right)\right|d\kappa\right)\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\int_0^1|\kappa^\alpha-m|^p\,d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{P}'\left((1-\kappa)\gamma_1+\kappa\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|(1-\kappa)^\alpha-\tfrac{1}{2}\right|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{P}'\left((1-\kappa)\frac{2\gamma_1+\gamma_2}{3}+\kappa\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\\
&+\left.\left(\int_0^1|\kappa^\alpha-(1-m)|^p\,d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left|\mathcal{P}'\left((1-\kappa)\frac{\gamma_1+2\gamma_2}{3}+\kappa\gamma_2\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right)\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\int_0^1|\kappa^\alpha-3|^p\,d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left((1-\kappa)|\mathcal{P}'(\gamma_1)|^q+\kappa\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\right)d\kappa\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|\kappa^\alpha-\tfrac{1}{2}\right|^p d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left((1-\kappa)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\kappa\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q\right)d\kappa\right)^{\frac{1}{q}}\\
&+\left.\left(\int_0^1|\kappa^\alpha-(1-m)|^p\,d\kappa\right)^{\frac{1}{p}}\left(\int_0^1\left((1-\kappa)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+\kappa|\mathcal{P}'(\gamma_2)|^q\right)d\kappa\right)^{\frac{1}{q}}\right)\\
=\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{m^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\frac{(1-m)^{p+1}\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;1-m\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{p+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\\
&+\left.\left(\frac{(1-m)^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{\alpha}+\frac{m^{p+1}\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;m\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

where we have used Lemma 1.1 with $l=m,\frac{1}{2}$, and $1-m$, respectively. The proof is completed. □

Remark 2.4. *Theorem 2.2 will be reduced to Corollary 2.7 from [15] if we take $\alpha = 1$ and $m = \frac{3}{2+2\theta}$.*

Corollary 2.12. *In Theorem 2.2, if we take $m = \frac{3}{8}$, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{3^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{8^{p+1}\alpha} + \frac{5^{p+1}\cdot {}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{5}{8}\right)}{8^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{p+\frac{1}{\alpha}}\alpha} + \frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}} \\
&\left.+\left(\frac{5^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{8^{p+1}\alpha} + \frac{3^{p+1}\cdot {}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{3}{8}\right)}{8^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 2.13. *In Theorem 2.2, if we take $m = \frac{1}{2}$, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{18}\left(\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{\frac{1}{\alpha}}\alpha} + \frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{\frac{1}{\alpha}}\alpha} + \frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}} \\
&\left.+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{\frac{1}{\alpha}}\alpha} + \frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;m\right)}{2\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 2.14. *In Theorem 2.2, if we use the convexity of $|\mathcal{P}'|^q$, i.e. $\left|\mathcal{P}'\left(\frac{n\gamma_1+z\gamma_2}{n+z}\right)\right|^q \leq \frac{n}{n+z}|\mathcal{P}'(\gamma_1)|^q + \frac{z}{n+z}|\mathcal{P}'(\gamma_2)|^q$, we get*

$$\begin{aligned}
&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6} - \frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right| \\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{m^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{\alpha} + \frac{(1-m)^{p+1}\cdot {}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;1-m\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right. \\
&+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{p+\frac{1}{\alpha}}\alpha} + \frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}} \\
&\left.+\left(\frac{(1-m)^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{\alpha} + \frac{m^{p+1}\cdot {}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;m\right)}{\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+5|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Corollary 2.15. *In Corollary 2.14, if we take $m = \frac{3}{8}$, we obtain*

$$\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}Q(\gamma_1,\gamma_2,\mathcal{P})\right|$$

$$\leq \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{3^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{8^{p+\frac{1}{\alpha}}\alpha}+\frac{5^{p+1}\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{5}{8}\right)}{8^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{p+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{5^{p+\frac{1}{\alpha}}B\left(\frac{1}{\alpha},p+1\right)}{8^{p+\frac{1}{\alpha}}\alpha}+\frac{3^{p+1}\cdot{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;m\right)}{8^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+5|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right).$$

Corollary 2.16. *In Corollary 2.14, if we take $m=\frac{1}{2}$, we obtain*

$$\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}Q(\gamma_1,\gamma_2,\mathcal{P})\right|$$

$$\leq \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{p+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{5|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right.$$

$$+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{p+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}$$

$$\left.+\left(\frac{B\left(\frac{1}{\alpha},p+1\right)}{2^{p+\frac{1}{\alpha}}\alpha}+\frac{{}_2F_1\left(\frac{\alpha-1}{\alpha},1,p+2;\frac{1}{2}\right)}{2^{p+1}\alpha(p+1)}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+5|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right).$$

Corollary 2.17. *In Corollary 2.14, if we take $\alpha=1$, then we get*

$$\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$

$$\leq \frac{\gamma_2-\gamma_1}{9}\left(\frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{m^{p+1}+(1-m)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{5|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{P}'(\gamma_1)|^q+5|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.18. *In Corollary 2.17, if we take $m=\frac{3}{8}$, then we get*

$$\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$

$$\leq \frac{\gamma_2-\gamma_1}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{5|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{P}'(\gamma_1)|^q+5|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right)\right),$$

Remark 2.5. *The same result was obtained in Corollary 3.5 from [12].*

Corollary 2.19. *In Corollary 2.17, if we take $m=\frac{1}{2}$, then we get*

$$\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{18}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{5|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}+\left(\frac{|\mathcal{P}'(\gamma_1)|^q+5|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right).$$

Corollary 2.20. *In Theorem 2.2, if we take $\alpha = 1$, then we get*

$$\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{m^{p+1}+(1-m)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$+\frac{1}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{m^{p+1}+(1-m)^{p+1}}{p+1}\right)^{\frac{1}{p}}\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right).$$

Corollary 2.21. *In Corollary 2.20, if we take $m = \frac{3}{8}$, then we get*

$$\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{1}{2}\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{3^{p+1}+5^{p+1}}{8^{p+1}}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.22. *In Corollary 2.20, if we take $m = \frac{3}{8}$, then we get*

$$\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{18}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right).$$

Corollary 2.23. *In Corollary 2.17, using the discrete power mean inequality we get*

$$\begin{aligned}&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \leq\ &\frac{\gamma_2-\gamma_1}{18}\left(\left(\left(\frac{1}{p+1}\right)^{\frac{1}{p}}+4\left(\frac{m^{p+1}+(1-m)^{p+1}}{p+1}\right)^{\frac{1}{p}}\right)\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Corollary 2.24. *In Corollary 2.23, if we take $m=\frac{3}{8}$, we get*

$$\begin{aligned}&\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \leq\ &\frac{\gamma_2-\gamma_1}{36}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(2+\left(\frac{3^{p+1}+5^{p+1}}{8}\right)^{\frac{1}{p}}\right)\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 2.25. *In Corollary 2.23, if we take $m=\frac{1}{2}$, then we get*

$$\begin{aligned}&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \leq\ &\frac{\gamma_2-\gamma_1}{6}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Theorem 2.3. *Let $\mathcal{P}:[\gamma_1,\gamma_2]\to\mathbb{R}$ be a differentiable function on $[\gamma_1,\gamma_2]$ with $\gamma_1<\gamma_2$ and $\mathcal{P}'\in L^1[\gamma_1,\gamma_2]$. If $|\mathcal{P}'|^q$ is convex where $q\geq 1$, then we have*

$$\begin{aligned}&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}Q(\gamma_1,\gamma_2,\mathcal{P})\right|\\ \leq\ &\frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{1-m(\alpha+1)}{\alpha+1}+\frac{2\alpha}{\alpha+1}m^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{2-m(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}m^{1+\frac{1}{\alpha}}\right.\right.\right.\\ &\left.-\frac{\alpha}{\alpha+2}m^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_1)|^q+\left(\frac{2-m(\alpha+2)}{2(\alpha+2)}+\frac{\alpha}{\alpha+2}m^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\Bigg)^{\frac{1}{q}}\\ &+\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{2-\alpha}{4(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\right.\\ &\left.+\left(\frac{4-(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\\ &+\left(\frac{m(\alpha+1)-\alpha}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-m)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{2-(1-m)(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)}+\frac{2\alpha(1-m)^{1+\frac{1}{\alpha}}}{\alpha+1}\right.\right.\\ &\left.\left.\left.-\frac{\alpha(1-m)^{1+\frac{2}{\alpha}}}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+\left(\frac{2-(1-m)(\alpha+2)}{2(\alpha+2)}+\frac{\alpha}{\alpha+2}(1-m)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|^q\right)^{\frac{1}{q}}\right).\end{aligned}$$

Proof. From Lemma 2.2, properties of the modulus, the power mean inequality, and the convexity of $|\mathcal{P}'|^q$, we have

$$
\begin{aligned}
&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}\left(\gamma_1,\gamma_2,\mathcal{P}\right)\right|\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\int_0^1|\kappa^{\alpha}-m|\left|\mathcal{P}'\left((1-\kappa)\gamma_1+\kappa\frac{2\gamma_1+\gamma_2}{3}\right)\right|d\kappa\right.\\
&+\int_0^1\left|(1-\kappa)^{\alpha}-\frac{1}{2}\right|\left|\mathcal{P}'\left((1-\kappa)\frac{2\gamma_1+\gamma_2}{3}+\kappa\frac{\gamma_1+2\gamma_2}{3}\right)\right|d\kappa\\
&\left.+\int_0^1|\kappa^{\alpha}-(1-m)|\left|\mathcal{P}'\left((1-\kappa)\frac{\gamma_1+2\gamma_2}{3}+\kappa\gamma_2\right)\right|d\kappa\right)\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\int_0^1|\kappa^{\alpha}-m|\,d\kappa\right)^{1-\frac{1}{q}}\left(\int_0^1|\kappa^{\alpha}-m|\left|\mathcal{P}'\left((1-\kappa)\gamma_1+\kappa\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right.\\
&+\left(\int_0^1\left|(1-\kappa)^{\alpha}-\frac{1}{2}\right|d\kappa\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|(1-\kappa)^{\alpha}-\frac{1}{2}\right|\left|\mathcal{P}'\left((1-\kappa)\frac{2\gamma_1+\gamma_2}{3}+\kappa\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q d\kappa\right)^{\frac{1}{q}}\\
&+\left(\int_0^1|\kappa^{\alpha}-(1-m)|\,d\kappa\right)^{1-\frac{1}{q}}\\
&\left.\times\left(\int_0^1|\kappa^{\alpha}-(1-m)|\left|\mathcal{P}'\left((1-\kappa)\frac{\gamma_1+2\gamma_2}{3}+\kappa\gamma_2\right)\right|^q d\kappa\right)^{\frac{1}{q}}\right)\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{1-m(\alpha+1)}{\alpha+1}+\frac{2\alpha}{\alpha+1}m^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\int_0^1|\kappa^{\alpha}-m|\left((1-\kappa)|\mathcal{P}'(\gamma_1)|^q+\kappa\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\right)d\kappa\right)^{\frac{1}{q}}\\
&+\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|(1-\kappa)^{\alpha}-\frac{1}{2}\right|\left((1-\kappa)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\kappa\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q\right)d\kappa\right)^{\frac{1}{q}}\\
&+\left(\frac{m(\alpha+1)-\alpha}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-m)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}
\end{aligned}
$$

$$\times\left(\int_0^1 |\kappa^\alpha-(1-m)|\left((1-\kappa)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+\kappa\left|\mathcal{P}'(\gamma_2)\right|^q\right)d\kappa\right)^{\frac{1}{q}}\Bigg)$$

$$\begin{aligned}
=\ & \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{1-m(\alpha+1)}{\alpha+1}+\frac{2\alpha}{\alpha+1}m^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{2-m(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}m^{1+\frac{1}{\alpha}}\right.\right.\right.\\
&\left.-\frac{\alpha}{\alpha+2}m^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'(\gamma_1)\right|^q+\left(\frac{2-m(\alpha+2)}{2(\alpha+2)}+\frac{\alpha}{\alpha+2}m^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{2-\alpha}{4(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\right.\\
&\left.+\left(\frac{4-(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{m(\alpha+1)-\alpha}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-m)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{2-(1-m)(\alpha+1)(\alpha+2)}{2(\alpha+1)(\alpha+2)}+\frac{2\alpha(1-m)^{1+\frac{1}{\alpha}}}{\alpha+1}\right.\right.\\
&\left.-\frac{\alpha(1-m)^{1+\frac{2}{\alpha}}}{\alpha+2}\right)\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+\left(\frac{2-(1-m)(\alpha+2)}{2(\alpha+2)}+\frac{\alpha}{\alpha+2}(1-m)^{1+\frac{2}{\alpha}}\right)\left|\mathcal{P}'(\gamma_2)\right|^q\Bigg)^{\frac{1}{q}},
\end{aligned}$$

where we have used (2.5)–(2.10). The proof is achieved. □

Remark 2.6. *Theorem 2.3 will be reduced to Corollary 2.11 from [15] if we take* $\alpha=1$ *and* $m=\frac{3}{2+2\theta}$.

Corollary 2.26. *In Theorem 2.3, if we take* $m=\frac{3}{8}$*, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^\alpha}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right|\\
\leq\ & \frac{b-a}{9}\left(\left(\frac{5-3\alpha}{8(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{16-3(\alpha+1)(\alpha+2)}{16(\alpha+1)(\alpha+2)}+\frac{2\alpha}{(\alpha+1)}\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}\right.\right.\right.\\
&\left.-\frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}}\right)\left|f'(a)\right|^q+\left(\frac{16-3(\alpha+2)}{16(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}}\right)\left|f'\left(\frac{2a+b}{3}\right)\right|^q\Bigg)^{\frac{1}{q}}\\
&+\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{2-\alpha}{4(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\left|f'\left(\frac{2a+b}{3}\right)\right|^q\right.\\
&\left.+\left(\frac{4-(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{\alpha+2}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{3-5\alpha}{8(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\left(\left(\frac{16-5(\alpha+1)(\alpha+2)}{16(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}\right.\right.\\
&\left.-\frac{\alpha}{\alpha+2}\left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}}\right)\left|f'\left(\frac{a+2b}{3}\right)\right|^q+\left(\frac{6-5\alpha}{16(\alpha+2)}+\frac{\alpha}{\alpha+2}\left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}}\right)\left|f'(b)\right|^q\Bigg)^{\frac{1}{q}}.
\end{aligned}$$

Corollary 2.27. *In Theorem 2.3, if we take* $m=\frac{1}{2}$*, we obtain*

$$\begin{aligned}
&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^\alpha}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right|\\
\leq\ & \frac{\gamma_2-\gamma_1}{9}\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{\alpha}{2^{\frac{1}{\alpha}}(\alpha+1)}\right)^{1-\frac{1}{q}}\left(\left(\Pi_1(\alpha)\left|\mathcal{P}'(\gamma_1)\right|^q+\Pi_2(\alpha)\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.
\end{aligned}$$

$$+\left(\Pi_2(\alpha)\left|\mathcal{P}'\left(\tfrac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\Pi_1(\alpha)\left|\mathcal{P}'\left(\tfrac{\gamma_1+2\gamma_2}{3}\right)\right|^q\right)^{\frac{1}{q}}$$
$$+\left(\Pi_1(\alpha)\left|\mathcal{P}'\left(\tfrac{\gamma_1+2\gamma_2}{3}\right)\right|^q+\Pi_2(\alpha)|\mathcal{P}'(\gamma_2)|^q\right)^{\frac{1}{q}}\Bigg),$$

where

$$\Pi_1(\alpha)=\frac{4-(\alpha+1)(\alpha+2)}{4(\alpha+1)(\alpha+2)}+\frac{\alpha}{2^{\frac{1}{\alpha}}(\alpha+1)}-\frac{\alpha}{2^{1+\frac{2}{\alpha}}(\alpha+2)}$$

and

$$\Pi_2(\alpha)=\frac{2-\alpha}{4(\alpha+2)}+\frac{\alpha}{2^{1+\frac{2}{\alpha}}(\alpha+2)}.$$

Corollary 2.28. *In Theorem 2.3, if we use the convexity of $|\mathcal{P}'|^q$, we get*

$$\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^\alpha}Q(\gamma_1,\gamma_2,\mathcal{P})\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9}\Bigg(\left(\frac{1-m(\alpha+1)}{\alpha+1}+\frac{2\alpha}{\alpha+1}m^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\Bigg(\Bigg(\frac{10+4\alpha-5m(\alpha+1)(\alpha+2)}{6(\alpha+1)(\alpha+2)}+\frac{2\alpha m^{1+\frac{1}{\alpha}}}{\alpha+1}$$
$$-\frac{\alpha m^{1+\frac{2}{\alpha}}}{3(\alpha+2)}\Bigg)|\mathcal{P}'(\gamma_1)|^q+\left(\frac{2-m(\alpha+2)}{6(\alpha+2)}+\frac{\alpha m^{1+\frac{2}{\alpha}}}{3(\alpha+2)}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$
$$\times\Bigg(\left(\frac{4+(\alpha+1)(2-3\alpha)}{12(\alpha+1)(\alpha+2)}+\frac{2\alpha}{3(\alpha+1)}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}+\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_1)|^q$$
$$+\left(\frac{8-(\alpha+1)(2+3\alpha)}{12(\alpha+1)(\alpha+2)}+\frac{4\alpha}{3(\alpha+1)}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{m(\alpha+1)-\alpha}{\alpha+1}+\frac{2\alpha}{\alpha+1}(1-m)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$
$$\times\Bigg(\left(\frac{2-(1-m)(\alpha+1)(\alpha+2)}{6(\alpha+1)(\alpha+2)}+\frac{2\alpha(1-m)^{1+\frac{1}{\alpha}}}{3(\alpha+1)}-\frac{\alpha(1-m)^{1+\frac{2}{\alpha}}}{3(\alpha+2)}\right)|\mathcal{P}'(\gamma_1)|^q$$
$$+\left(\frac{10+6\alpha-5(1-m)(\alpha+1)(\alpha+2)}{6(\alpha+1)(\alpha+2)}+\frac{4\alpha(1-m)^{1+\frac{1}{\alpha}}}{3(\alpha+1)}+\frac{\alpha(1-m)^{1+\frac{2}{\alpha}}}{3(\alpha+2)}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}.$$

Corollary 2.29. *In Corollary 2.28, if we take $m=\frac{3}{8}$, we obtain*

$$\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^\alpha}Q(\gamma_1,\gamma_2,\mathcal{P})\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9}\Bigg(\left(\frac{5-3\alpha}{8(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\Bigg(\Bigg(\frac{80+32\alpha-15(\alpha+1)(\alpha+2)}{48(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{3}{8}\right)^{1+\frac{1}{\alpha}}$$
$$-\frac{\alpha}{3(\alpha+2)}\left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}}\Bigg)|\mathcal{P}'(\gamma_1)|^q+\left(\frac{10-3\alpha}{48(\alpha+2)}+\frac{\alpha}{3(\alpha+2)}\left(\frac{3}{8}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$
$$\times\Bigg(\left(\frac{4+(\alpha+1)(2-3\alpha)}{12(\alpha+1)(\alpha+2)}+\frac{2\alpha}{3(\alpha+1)}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}+\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_1)|^q$$

$$+\left(\frac{8-(\alpha+1)(2+3\alpha)}{12(\alpha+1)(\alpha+2)}+\frac{4\alpha}{3(\alpha+1)}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{3-5\alpha}{8(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$
$$\times\Bigg(\left(\frac{16-5(\alpha+1)(\alpha+2)}{48(\alpha+1)(\alpha+2)}+\frac{2\alpha}{3(\alpha+1)}\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{3(\alpha+2)}\left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_1)|^q$$
$$+\left(\frac{80+48\alpha-25(\alpha+1)(\alpha+2)}{48(\alpha+1)(\alpha+2)}+\frac{4\alpha}{3(\alpha+1)}\left(\frac{5}{8}\right)^{1+\frac{1}{\alpha}}+\frac{\alpha}{3(\alpha+2)}\left(\frac{5}{8}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}.$$

Corollary 2.30. *In Corollary 2.28, if we take $m=\frac{1}{2}$, we obtain*

$$\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{3^{\alpha-1}\Gamma(\alpha+1)}{(\gamma_2-\gamma_1)^{\alpha}}\mathcal{Q}(\gamma_1,\gamma_2,\mathcal{P})\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9}\Bigg(\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}\Bigg(\left(\frac{20+8\alpha-5(\alpha+1)(\alpha+2)}{12(\alpha+1)(\alpha+2)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}$$
$$-\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_1)|^q+\left(\frac{2-\alpha}{12(\alpha+2)}+\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$
$$\times\Bigg(\left(\frac{4+(\alpha+1)(2-3\alpha)}{12(\alpha+1)(\alpha+2)}+\frac{2\alpha}{3(\alpha+1)}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}+\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_1)|^q$$
$$+\left(\frac{8-(\alpha+1)(2+3\alpha)}{12(\alpha+1)(\alpha+2)}+\frac{4\alpha}{3(\alpha+1)}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}$$
$$+\left(\frac{1-\alpha}{2(\alpha+1)}+\frac{2\alpha}{\alpha+1}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}\right)^{1-\frac{1}{q}}$$
$$\times\Bigg(\left(\frac{4-(\alpha+1)(\alpha+2)}{12(\alpha+1)(\alpha+2)}+\frac{2\alpha}{3(\alpha+1)}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}-\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_1)|^q$$
$$+\left(\frac{20+12\alpha-5(\alpha+1)(\alpha+2)}{12(\alpha+1)(\alpha+2)}+\frac{4\alpha}{3(\alpha+1)}\left(\frac{1}{2}\right)^{1+\frac{1}{\alpha}}+\frac{\alpha}{3(\alpha+2)}\left(\frac{1}{2}\right)^{1+\frac{2}{\alpha}}\right)|\mathcal{P}'(\gamma_2)|^q\Bigg)^{\frac{1}{q}}.$$

Corollary 2.31. *In Corollary 2.28, if we take $\alpha=1$, then we get*

$$\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9}\Bigg(\left(\frac{1-2m+2m^2}{2}\right)^{1-\frac{1}{q}}\Bigg(\left(\frac{(7-15m+18m^2-2m^3)|\mathcal{P}'(\gamma_1)|^q+(2-3m+2m^3)|\mathcal{P}'(\gamma_2)|^q}{18}\right)^{\frac{1}{q}}$$
$$+\left(\frac{(2-3m+2m^3)|\mathcal{P}'(\gamma_1)|^q+(7-15m+18m^2-2m^3)|\mathcal{P}'(\gamma_2)|^q}{18}\right)^{\frac{1}{q}}\Bigg)+\frac{1}{4}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\Bigg).$$

Corollary 2.32. *In Corollary 2.31, if we take $m=\frac{3}{8}$, then we get*

$$\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{576}\left(17\left(\left(\frac{973|\mathcal{P}'(\gamma_1)|^q+251|\mathcal{P}'(\gamma_2)|^q}{1224}\right)^{\frac{1}{q}}+\left(\frac{251|\mathcal{P}'(\gamma_1)|^q+973|\mathcal{P}'(\gamma_2)|^q}{1224}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.\left.+16\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right)\right).$$

Corollary 2.33. *In Corollary 2.31, if we take $m=\frac{1}{2}$, then we get*

$$\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{36}\left(\left(\left(\frac{5|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right.\right.$$
$$\left.\left.+\left(\frac{|\mathcal{P}'(\gamma_1)|^q+5|\mathcal{P}'(\gamma_2)|^q}{6}\right)^{\frac{1}{q}}\right)+\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right).$$

Corollary 2.34. *In Theorem 2.3, if we take $\alpha=1$, then we get*

$$\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{9}\left(\left(\frac{1-2m+2m^2}{2}\right)^{1-\frac{1}{q}}\left(\left(\frac{1-3m+6m^2-2m^3}{6}|\mathcal{P}'(\gamma_1)|^q+\frac{2-3m+2m^3}{6}\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q\right)^{\frac{1}{q}}\right.\right.$$
$$+\frac{1}{4}\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}$$
$$\left.+\left(\frac{1-2m+2m^2}{2}\right)^{1-\frac{1}{q}}\left(\frac{2-3m+2m^3}{6}\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+\frac{1-3m+6m^2-2m^3}{6}|\mathcal{P}'(\gamma_2)|^q\right)^{\frac{1}{q}}\right).$$

Remark 2.7. *Corollary 2.34 recaptures the second inequality of Corollary 2.13 from [15] if we take $m=\frac{3}{4}$.*

Corollary 2.35. *In Corollary 2.34, if we take $m=\frac{3}{8}$, then we get*

$$\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|$$
$$\leq \frac{\gamma_2-\gamma_1}{576}\left(16\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.$$
$$\left.+17\left(\left(\frac{157|\mathcal{P}'(\gamma_1)|^q+251\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{408}\right)^{\frac{1}{q}}+\left(\frac{251\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+157|\mathcal{P}'(\gamma_2)|^q}{408}\right)^{\frac{1}{q}}\right)\right).$$

Remark 2.8. *The same result was obtained in Corollary 2.3 from [15].*

Corollary 2.36. *In Corollary 2.34, if we take $m = \frac{1}{2}$, then we get*

$$\begin{aligned}&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \le\ &\frac{\gamma_2-\gamma_1}{36}\left(\left(\left(\frac{|\mathcal{P}'(\gamma_1)|^q+\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{\left|\mathcal{P}'\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|^q+\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q}{2}\right)^{\frac{1}{q}}\right.\right.\\ &\left.+\left(\frac{\left|\mathcal{P}'\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}\right).\end{aligned}$$

Corollary 2.37. *In Corollary 2.31, using the discrete power mean inequality, we get*

$$\begin{aligned}&\left|\frac{2m\mathcal{P}(\gamma_1)+(3-2m)\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+(3-2m)\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+2m\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \le\ &\frac{\left(5-8m+8m^2\right)(\gamma_2-\gamma_1)}{36}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 2.38. *In Corollary 2.37, if we take $m = \frac{3}{8}$, then we get*

$$\begin{aligned}&\left|\frac{\mathcal{P}(\gamma_1)+3\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+3\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{8}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \le\ &\frac{25(\gamma_2-\gamma_1)}{288}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

Corollary 2.39. *In Corollary 2.37, if we take $m = \frac{1}{2}$, then we get*

$$\begin{aligned}&\left|\frac{\mathcal{P}(\gamma_1)+2\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)+\mathcal{P}(\gamma_2)}{6}-\frac{1}{\gamma_2-\gamma_1}\int_{\gamma_1}^{\gamma_2}\mathcal{P}(u)\,du\right|\\ \le\ &\frac{\gamma_2-\gamma_1}{12}\left(\frac{|\mathcal{P}'(\gamma_1)|^q+|\mathcal{P}'(\gamma_2)|^q}{2}\right)^{\frac{1}{q}}.\end{aligned}$$

3. Applications

3.1. Application to composite quadrature formula

Let Q be the partition of the interval $[\mathcal{L}_1, \mathcal{L}_2]$ such that $\mathcal{L}_1 = u_0 < u_1 < ... < u_n = \mathcal{L}_2$, and take the quadrature formula into consideration.

$$\int_{\mathcal{L}_1}^{\mathcal{L}_2}\mathcal{P}(u)\,du = Q(\mathcal{P}, Q) + \mathcal{E}(\mathcal{P}, Q),$$

where

$$Q(\lambda,\mathcal{P},Q)=\sum_{i=0}^{n-1}(u_{i+1}-u_i)\left(\frac{2m\mathcal{P}(u_i)+(3-2m)\mathcal{P}\left(\frac{2u_i+u_{i+1}}{3}\right)+(3-2m)\mathcal{P}\left(\frac{u_i+2u_{i+1}}{3}\right)+2m\mathcal{P}(u_{i+1})}{6}\right),$$

with $m\in[0,1]$ and where $\mathcal{E}(\mathcal{P},Q)$ denotes the associated approximation error.

Proposition 3.1. *Let $\mathcal{P}$ be as in Theorem 2.1. Then, for $m\in[0,1]$, we have*

$$|\mathcal{E}(\mathcal{P},Q)|\le\frac{5-8m+8m^2}{72}\sum_{i=0}^{n-1}(u_{i+1}-u_i)^2\left(|\mathcal{P}'(u_i)|+|\mathcal{P}'(u_{i+1})|\right).$$

Proof. When we apply Corollary 2.6 to the partition Q of the subintervals $[u_i,u_{i+1}]$ $(i=0,1,...,n-1)$, we obtain

$$\begin{aligned}&\left|\frac{2m\mathcal{P}(u_i)+(3-2m)\mathcal{P}\left(\frac{2u_i+u_{i+1}}{3}\right)+(3-2m)\mathcal{P}\left(\frac{u_i+2u_{i+1}}{3}\right)+2m\mathcal{P}(u_{i+1})}{6}-\frac{1}{u_{i+1}-u_i}\int_{u_i}^{u_{i+1}}\mathcal{P}(k)\,dk\right|\\ \le\ &\frac{(5-8m+8m^2)(u_{i+1}-u_i)}{72}\left(|\mathcal{P}'(u_i)|+|\mathcal{P}'(u_{i+1})|\right).\end{aligned}$$

We reach the necessary result by multiplying both sides of the aforementioned inequality by $(u_{i+1}-u_i)$, summing the generated inequalities for all $i=0,1,...,n-1$ and applying the triangular inequality. □

3.2. Application to special means

For arbitrary real numbers ϱ_1,ϱ_2 we have:
The generalized arithmetic mean: $A(\varrho_1,\varrho_2)=\frac{\varrho_1+\varrho_2}{2}$.
The p-logarithmic mean: $L_p(\varrho_1,\varrho_2)=\left(\frac{\varrho_2^{p+1}-\varrho_1^{p+1}}{(p+1)(\varrho_2-\varrho_1)}\right)^{\frac{1}{p}}$, $\varrho_1,\varrho_2>0,\varrho_1\neq\varrho_2$ and $p\in\mathbb{R}\setminus\{-1,0\}$.

Proposition 3.2. *Let $\varrho_1,\varrho_2\in\mathbb{R}$ with $0<\varrho_2<\varrho_2$. Then, we have*

$$\left|A\left(\varrho_1^3,\varrho_2^3\right)+A^3(\varrho_1,\varrho_1,\varrho_2)+A^3(\varrho_1,\varrho_2,\varrho_2)-3L_3^3(\varrho_1,\varrho_2)\right|\le\frac{3(\varrho_2-\varrho_1)}{8}\left(\varrho_1^2+\varrho_2^2\right).$$

Proof. Applying Corollary 2.8 to the function $\mathcal{P}(u)=u^3$ leads to this conclusion. □

3.3. Application to probability

Proposition 3.3. *Let X be a random variable, and let $\mathcal{P}$ be its probability density function that takes values in the finite interval $[\gamma_1,\gamma_2]$ i.e., $\mathcal{P}:[\gamma_1,\gamma_2]\to[0,1]$ with the cumulative distribution function $F(x)=\Pr(X\le x)=\int_{\gamma_1}^{x}\mathcal{P}(u)\,du$. Then, we have*

$$\begin{aligned}&\left|\frac{1+2F\left(\frac{2\gamma_1+\gamma_2}{3}\right)+2F\left(\frac{\gamma_1+2\gamma_2}{3}\right)}{6}-\frac{\gamma_2-E[X]}{\gamma_2-\gamma_1}\right|\\ \le\ &\frac{\gamma_2-\gamma_1}{72}\left(|\mathcal{P}(\gamma_1)|+2\left|\mathcal{P}\left(\frac{2\gamma_1+\gamma_2}{3}\right)\right|+2\left|\mathcal{P}\left(\frac{\gamma_1+2\gamma_2}{3}\right)\right|+|\mathcal{P}(\gamma_2)|\right).\end{aligned}$$

Proof. Replace $\mathcal{P}=F$ in Corollary 2.11 and take into account that $F(\gamma_1)=0,F(\gamma_2)=1$, and $E[X]=\int_{\gamma_1}^{\gamma_2}k\mathcal{P}(k)\,dk=\gamma_2F(\gamma_2)-\gamma_1F(\gamma_1)-\int_{\gamma_1}^{\gamma_2}F(k)\,dk=\gamma_2-\int_{\gamma_1}^{\gamma_2}F(k)\,dk.$ □

4. Conclusions

In this work, we established new a parameterized identity involving the Riemann-Liouville integral operator, thus leading to the construction some fractional $3/8$-Simpson type integral inequalities for functions whose absolute value of the first derivatives are convex. We succeeded in obtaining refinements as well as generalizations of certain known results. Moreover, we presented some applications in numerical integration, special means, and random variables.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare no conflicts of interest.

References

1. M. Alomari, M. Darus, On some inequalities of Simpson-type via quasi-convex functions and applications, *Transylv. J. Math. Mech.*, **2** (2010), 15–24.
2. A. Kashuri, B. Meftah, P. O. Mohammed, Some weighted Simpson type inequalities for differentiable s-convex functions and their applications, *J. Fractional Calculus Nonlinear Syst.*, **1** (2021), 75–94. https://doi.org/10.48185/jfcns.v1i1.150
3. M. Z. Sarikaya, E. Set, M. E. Özdemir, On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex, *J. Appl. Math. Stat. Inf.*, **9** (2013), 37–45. https://doi.org/10.2478/jamsi-2013-0004
4. E. Set, M. E. Özdemir, M. Z. Sarıkaya, On new inequalities of Simpson's type for quasi-convex functions with applications, *Tamkang J. Math.*, **43** (2012), 357–364.
5. J. E. N. Valdés, B. Bayraktar, S. I. Butt, New integral inequalities of Hermite-Hadamard type in a generalized context, *Punjab Univ. J. Math.*, **53** (2021), 765–777. https://doi.org/10.52280/pujm.2021.531101
6. J. E. Pečarić, F. Proschan, Y. L. Tong, *Convex functions, partial orderings, and statistical applications*, Academic Press, 1992.
7. M. A. Ali, C. S. Goodrich, H. Budak, Some new parameterized Newton-type inequalities for differentiable functions via fractional integrals, *J. Inequal. Appl.*, **2023** (2023), 49. https://doi.org/10.1186/s13660-023-02953-x

8. T. Chiheb, N. Boumaza, B. Meftah, Some new Simpson-like type inequalities via preqausiinvexity, *Transylv. J. Math. Mech.*, **12** (2020), 1–10.

9. B. Meftah, Fractional Hermite-Hadamard type integral inequalities for functions whose modulus of derivatives are co-ordinated log-preinvex, *Punjab Univ. J. Math.*, **51** (2019), 21–37.

10. B. Meftah, A. Souahi, M. Merad, Some local fractional Maclaurin type inequalities for generalized convex functions and their applications, *Chaos Solitons Fractals*, **162** (2022), 112504. https://doi.org/10.1016/j.chaos.2022.112504

11. M. Rostamian Delavar, A. Kashuri, M. De La Sen, On weighted Simpson's $\frac{3}{8}$ rule, *Symmetry*, **13** (2021), 1933. https://doi.org/10.3390/sym13101933

12. M. A. Noor, K. I. Noor, S. Iftikhar, Newton inequalities for p-harmonic convex functions, *Honam Math. J.*, **40** (2018), 239–250. https://doi.org/10.5831/HMJ.2018.40.2.239

13. N. Laribi, B. Meftah, 3/8-Simpson type inequalities for functions whose modulus of first derivatives and its q-th powers are s-convex in the second sense, *Jordan J. Math. Stat.*, **16** (2023), 79–98.

14. S. Erden, S. Iftikhar, M. R. Delavar, P. Kumam, P. Thounthong, W. Kumam, On generalizations of some inequalities for convex functions via quantum integrals, *RACSAM*, **114** (2020), 110. https://doi.org/10.1007/s13398-020-00841-3

15. L. Mahmoudi, B. Meftah, Parameterized Simpson-like inequalities for differential s-convex functions, *Analysis*, **43** (2023), 59–70. https://doi.org/10.1515/anly-2022-1068

16. A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and applications of fractional differential equations*, New York: Elsevier, 2006.

17. T. A. Aljaaidi, D. B. Pachpatte, T. Abdeljawad, M. S. Abdo, M. A. Almalahi, S. S. Redhwan, Generalized proportional fractional integral Hermite-Hadamard's inequalities, *Adv. Differ. Equ.*, **2021** (2021), 493. https://doi.org/10.1186/s13662-021-03651-y

18. Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional integration, *Ann. Funct. Anal.*, **1** (2010), 51–58. https://doi.org/10.15352/afa/1399900993

19. T. Du, J. Liao, L. Chen, M. U. Awan, Properties and Riemann-Liouville fractional Hermite-Hadamard inequalities for the generalized (α, m)-preinvex functions, *J. Inequal. Appl.*, **2016** (2016), 306. https://doi.org/10.1186/s13660-016-1251-5

20. T. Du, M. U. Awan, A. Kashuri, S. Zhao, Some k-fractional extensions of the trapezium inequalities through generalized relative semi-(m, h)-preinvexity, *Appl. Anal.*, **100** (2021), 642–662. https://doi.org/10.1080/00036811.2019.1616083

21. A. Kashuri, B. Meftah, P. O. Mohammed, A. A. Lupaş, B. Abdalla, Y. S. Hamed, et al., Fractional weighted Ostrowski-type inequalities and their applications, *Symmetry*, **13** (2021), 968. https://doi.org/10.3390/sym13060968

22. P. O. Mohammed, I. Brevik, A new version of the Hermite-Hadamard inequality for Riemann-Liouville fractional integrals, *Symmetry*, **12** (2020), 610. https://doi.org/10.3390/sym12040610

23. M. A. Noor, K. I. Noor, M. U. Awan, Fractional Ostrowski inequalities for s-Godunova-Levin functions, *Int. J. Anal. Appl.*, **5** (2014), 167–173.

24. S. Rashid, A. O. Akdemir, F. Jarad, M. A. Noor, K. I. Noor, Simpson's type integral inequalities for k-fractional integrals and their applications, *AIMS Mathematics*, **4** (2019), 1087–1100. https://doi.org/10.3934/math.2019.4.1087

25. S. S. Zhou, S. Rashid, F. Jarad, H. Kalsoom, Y. M. Chu, New estimates considering the generalized proportional Hadamard fractional integral operators, *Adv. Differ. Equ.*, **2020** (2020), 275. https://doi.org/10.1186/s13662-020-02730-w