Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(3): 5294–5329.
DOI:10.3934/math.2024256
Received: 15 November 2023
Revised: 19 January 2024
Accepted: 19 January 2024
Published: 26 January 2024

Research article

Local existence of solutions to the 2D MHD boundary layer equations without monotonicity in Sobolev space

Xiaolei Dong*

School of Mathematics and Statistics, Zhoukou Normal University, Zhoukou 466001, China

* **Correspondence:** Email: xld0908@163.com.

Abstract: In this work, we investigated the local existence of the solutions to the 2D magnetohydrodynamic (MHD) boundary layer equations on the half plane by energy methods in weighted Sobolev space. Compared to the existence of solutions to the classical Prandtl equations where the monotonicity condition of the tangential velocity plays an important role, we used the initial tangential magnetic field with a lower bound $\delta > 0$ instead of the monotonicity condition of the tangential velocity.

Keywords: MHD boundary layer equations; the existence of solutions; the energy method; the weighted sobolev space
Mathematics Subject Classification: 35M33, 35Q35, 76D03, 76D10, 76W05

1. Introduction

The magnetohydrodynamic(MHD) boundary layer system was derived by understanding the high Reynolds number limit to the incompressible viscous MHD system ([3, 5, 23]) in a domain with non-slip boundary when both the Reynolds number and the magnetic Reynolds number have the same order. In this paper, we investigate the local existence of the solutions to the following initial boundary value problem for the 2D MHD system in a periodic domain $\mathbb{R}^2_+ = \{(t, x, y) : t \in [0, T], x \in \mathbb{T}, y \in \mathbb{R}_+\}$, which reads as follows

$$\begin{cases} \partial_t \mathbf{u}^\varepsilon + (\mathbf{u}^\varepsilon \cdot \nabla)\mathbf{u}^\varepsilon - (\mathbf{H}^\varepsilon \cdot \nabla)\mathbf{H}^\varepsilon + \nabla p^\varepsilon = \mu\varepsilon\Delta\mathbf{u}^\varepsilon, \\ \partial_t \mathbf{H}^\varepsilon - \nabla \times (\mathbf{u}^\varepsilon \times \mathbf{H}^\varepsilon) = \kappa\varepsilon\Delta\mathbf{H}^\varepsilon, \\ \nabla \cdot \mathbf{u}^\varepsilon = 0, \quad \nabla \cdot \mathbf{H}^\varepsilon = 0, \end{cases} \tag{1.1}$$

where $\mathbb{T}$ stands for a torus or a periodic domain and $\mathbb{R}_+ = [0, +\infty)$. Here, we suppose the viscosity and resistivity coefficients have the same order of a small parameter ε, $\mathbf{u}^\varepsilon = ({u_1}^\varepsilon, {u_2}^\varepsilon)$ denotes the velocity vector, $\mathbf{H}^\varepsilon = ({h_1}^\varepsilon, {h_2}^\varepsilon)$ stands for the magnetic field, and the total pressure $p^\varepsilon = \tilde{p}^\varepsilon + \frac{|\mathbf{H}^\varepsilon|^2}{2}$

with $\tilde{p}^\varepsilon$ represents the pressure of the fluid. Parameters μ and κ are the viscosity and heat conductivity coefficients, respectively.

The initial data of (1.1) is given by

$$(u_1^\varepsilon, h_1^\varepsilon)|_{t=0} = (u_{10}^\varepsilon, h_{10}^\varepsilon). \tag{1.2}$$

The no-slip boundary conditions are imposed on the velocity field and the magnetic field

$$(\mathbf{u}^\varepsilon, \mathbf{H}^\varepsilon)|_{y=0} = \mathbf{0}. \tag{1.3}$$

The far fields boundary conditions are

$$\lim_{y\to+\infty}(u_1, b_1) = (U, B). \tag{1.4}$$

Formally, system (1.1) yields the incompressible ideal MHD system when $\varepsilon = 0$. However, there is no match for the tangential velocity between the equations (1.1) and the limiting equations on the boundary value $y = 0$. This is why a boundary layer forms in the vanishing viscosity and resistivity limit process. To look for the term of system (1.1) whose contribution is essential for the boundary layer, we use the same transform as the one used in [20],

$$t = t, \quad x = x, \quad \tilde{y} = \varepsilon^{-\frac{1}{2}} y,$$

then we set

$$\begin{cases} u_1(t, x, \tilde{y}) = u_1^\varepsilon(t, x, y), \quad b_1(t, x, \tilde{y}) = h_1^\varepsilon(t, x, y), \\ u_2(t, x, \tilde{y}) = \varepsilon^{-\frac{1}{2}} u_2^\varepsilon(t, x, y), \quad b_2(t, x, \tilde{y}) = \varepsilon^{-\frac{1}{2}} h_2^\varepsilon(t, x, y), \end{cases}$$

and

$$p(t, x) = p^\varepsilon(t, x).$$

Next, by taking the limit $\varepsilon \to 0$, the Eqs (1.1)–(1.4) are transformed into the following 2D MHD boundary layer equations

$$\begin{cases} \partial_t u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 = b_1\partial_x b_1 + b_2\partial_y b_1 + \partial_y^2 u_1 - \partial_x p, \\ \partial_t b_1 + \partial_y(u_2 b_1 - u_1 b_2) = \partial_y^2 b_1, \\ \partial_x u_1 + \partial_y u_2 = 0, \ \partial_x b_1 + \partial_y b_2 = 0, \\ (u_1, u_2, b_1, b_2)|_{y=0} = 0, \ \lim\limits_{y\to+\infty}(u_1, b_1) = (U, B), \\ (u_1, b_1)|_{t=0} = (u_0, b_0)(x, y). \end{cases} \tag{1.5}$$

Functions $(U(t, x), B(t, x))$, and $p(t, x)$ are the values on the boundary of the Euler's tangential velocity and Euler's pressure of the outflow, which satisfy the Bernoulli's law,

$$\begin{cases} \partial_t U + U\partial_x U - B\partial_x B + \partial_x p = 0, \\ \partial_t B + U\partial_x B - B\partial_x U = 0. \end{cases}$$

Before exhibiting the main results in this paper, let us briefly review some known results to the problem (1.5). Specifically, when the magnetic field (b_1, b_2) are some constants in (1.5), the system reduces to the classical Prandtl equations, which was first introduced formally by Prandtl [21] in 1904.

This system is the foundation of the boundary layer equations. It describes that the away from the boundary part can be considered as general ideal fluid, but the near a rigid wall part is deeply affected by the viscous force. Formally, the asymptotic limit of the Navier-Stokes equations can be denoted by the Prandtl equations within the boundary layer and by the Euler equations away from boundary. About sixty years later, under the monotonicity condition on the tangential velocity field in the normal variable to the boundary, Oleinik [19] proved the local-in-time well-posedness to the 2D Prandtl equations by using the Crocco transformation, which is the first systematic work in strictly mathematics. This result together with an expanded introduction to the boundary layer theory was showed in Oleinik-Samokhin's classical book [20]. Additionally, under the Oleinik's monotonicity assumption, some authors [2, 17, 31] proved the well-posedness of solution for 2D Prandtl equations by using the energy method and constructing a new unknown function to eliminate the difficult term from the convection term. In addition to Oleinik′s monotonicity assumption on the tangential velocity field, Xin et al. [30] obtained the existence of global weak solutions to the Prandtl equation when the pressure is favourable ($\partial_x p \leq 0$).

When the velocity field equation is coupled with the magnetic field equation, the phenomenon of the boundary layer is different since the boundary layers of the magnetic field may exist and they are more complicated than the classical Prandtl equations. It should be highlighted that the MHD boundary layer equations are an important problem in the investigation of plasma with many known results; See [9,22,26]. There are some results in [3,6] on the so-called Hartmann boundary layer when the magnetic field is transversal to the boundary.

However, we are concerned with the case that the magnetic field is tangent to the boundary in this paper, that is, the Eq (1.5). There are some results in an analytic framework for the 2D MHD boundary layer equations. For example, Xie et al. [28] considered the global existence of solutions to the 2D MHD boundary layer equations in the mixed Prandtl and Hartmann regime when initial data is a small perturbation of the Hartmann profile, and they found that the solution in analytic norm is exponential decay in time. Recently, Liu et al. [16] established the global existence and asymptotic decay estimate of solutions to the 2D MHD boundary layer equations with small initial data. Xie et al. [29] investigated the lifespan of solution to the 2D MHD boundary layer system by using the cancellation mechanism and obtained that the lifespan of solution has a lower bound. Liu et al. [13] studied the well-posedness of solutions to the 2D MHD in an analytic framework. Moreover, inspired by [8] on the classical Prandtl equations, they proved that if the tangential magnetic field is degenerate sufficiently, then the nondegenerate critical point in the tangential velocity does not prevent the formation of singularity. Chen et al. [4] investigated the well-posedness of the MHD boundary layer equation without resistivity by using the paralinearization methods in Sobolev space. Under the assumption that the tangential component of magnetic fields dominates, Li et al. [11] proved the existence and uniqueness of solutions to the MHD boundary layer equations without viscosity in Sobolev spaces. So far, in addition to the well-posedness of solutions in the Sobolev and analystic frameworks, there are some results on the vanishing limits for the incompressible MHD systems; We refer to [25–27] and the references therein for the recent progress.

Under the assumption that the initial tangential magnetic field has a lower bound $\delta_0 > 0$, there are some results to the 2D MHD boundary layer equations. Liu et al. [14] investigated the local existence and uniqueness of solutions in weighed Sobolev space $H_l^m (m \geq 5)$ for the two-dimensional nonlinear MHD boundary layer equations by using the energy method. As a continuation of [14], the same authors [15] proved the validity of the Prandtl boundary layer expansion and gave an L^∞ estimate on the error by multi-scale analysis under the assumptions that both the viscosity and resistivity coefficients with the same order and the initial tangential magnetic field on the boundary are not degenerate. Liu et al. [12] proved the local well-posedness to the 2D MHD boundary layer equations in Sobolev spaces and found the linear instability of the 2D MHD boundary layer when the tangential magnetic field is degenerate at one point. Gao et al. [7] investigated the local well-posedness of solution in weighted conormal Sobolev spaces to the 2D MHD boundary layer equations with any large initial data by energy methods. Huang et al. [10] attained the local well-posedness of solutions to the 2D MHD system in weighted Sobolev spaces by applying the classical iteration scheme.

The main differences between our results and those in [14] are as follows: Liu et al. [14] investigated the local existence and uniqueness of solutions in weighed Sobolev space $H_l^m (m \geq 5)$ for the 2D nonlinear MHD boundary layer equations. However, in this paper, we investigate the local existence of solutions to the 2D MHD equations in weighted Sobolev space H_{k+l}^4 by the energy method, which is a complement for the previous results [14]. The monotonicity condition on the velocity field is not needed for the well-posedness of the 2D MHD boundary layer equations in this work. We use the tangential magnetic field, which has a lower positive bound instead of the monotonicity assumption, on the tangential velocity in the normal direction to the boundary. We first get the boundedness of the approximate solutions to the regularized MHD boundary layer equations in H_{k+l}^4 by calculating the lower order derivative boundary values of variable y for the Eq (4.1) and combine it with Corollary 5.1 in Subsection 5.1. Second, we get the estimates of $D_\tau^\beta(u, h)$ with $|\beta| = 4$ by constructing two new unknown functions in Subsection 5.2. We finally obtain the existence of solution to problem (3.1) in H_{k+l}^4.

To investigate the existence of solution to problem (1.5), we encounter some difficulties. Similar to the Prandtl equation, the difficulty of solving problem (1.5) in the Sobolev framework is the loss of x-derivative in the terms $u_2\partial_y u_1 - b_2\partial_y b_1$ and $u_2\partial_y b_1 - b_2\partial_y u_1$ in the first and second equations of (1.5), respectively. In other words, $u_2 = -\partial_y^{-1}\partial_x u_1$ and $b_2 = -\partial_y^{-1}\partial_x b_1$ by the divergence-free conditions and the boundary conditions. Thus, it creates a loss of the x-derivative and a y-integration to the y-variable, then the standard energy estimates do not work. To overcome this essential difficulty, inspired by recent results in [7, 14], we only need the following two new observations, which can remove the difficult terms in the convection terms. The first one observation is that $\psi := \partial_y^{-1} b_1$ satisfies

$$\partial_t\psi + u_2 b_1 - u_1 b_2 = \partial_y^2\psi.$$

Another observation is that under the assumption on the nondegeneracy of h_1, we use the following unknown functions to lead the cancellation

$$u_m := \partial_x^m u_1 - \frac{\partial_y u_1}{b_1}\partial_x^m\psi, \quad b_m := \partial_x^m b_1 - \frac{\partial_y b_1}{b_1}\partial_x^m\psi.$$

With the help of (u_m, b_m), the difficulties in the analysis on $\partial_x^m u_2\partial_y u_1 - \partial_x^m b_2\partial_y b_1$ and $\partial_x^m u_2\partial_y b_1 - \partial_x^m b_2\partial_y u_1$ mentioned above can be overcame. The detail of the equivalent for (u_m, b_m) and $(\partial_x^m u_1, \partial_x^m b_1)$ in the weighted Sobolev framework will be showed in Subsection 5.2.

The paper is arranged as follows. In Section 2, we introduce some notations and main results in this paper. In Section 3, we give the compatibility condition of the MHD boundary layer equations. In Section 4, we prove some propositions of the initial boundary value to the nonlinear regularized MHD boundary layer equations. In Section 5, we derive the existence of the approximate solutions to the MHD boundary layer equations and prove Theorem 2.1.

Hereafter, let letter C be a general positive constant, which may vary from line to line at each step.

2. Preliminaries and main results

As a preparation, we give some notations. We use the tangential derivative operator

$$\partial_\tau^\beta = \partial_t^{\beta_1}\partial_x^{\beta_2},\ \beta = (\beta_1, \beta_2) \in \mathbb{N}^2,\ |\beta| = \beta_1 + \beta_2.$$

and then denote the derivative operator (in both time and space) by

$$D^\alpha = \partial_\tau^\beta \partial_y^k,\ for\ (\beta_1, \beta_2, k) \in \mathbb{N}^3,\ |\alpha| = \beta_1 + \beta_2 + k.$$

Next, we introduce the weighted Sobolev spaces H^4_{k+l} and Sobolev norms as follows

$$\|f(t)\|^2_{H^4_{k+l}} := \sum_{\alpha\leq 4} \|\langle y\rangle^{k+l} D^\alpha f(t,\cdot)\|^2_{L^2},$$

where $\langle y\rangle = 1 + y$.

We now state our main result as following.

Theorem 2.1. *Let $k \geq \frac{1}{2}$, $l \geq 0$ be real numbers. Assume the initial data $(u_0, b_0) \in H^4_{k+l}$ satisfies the compatibility conditions up to 6 order. Additionally, there exists a small enough $\delta \in (0, 1)$ such that*

$$\begin{cases} \|\langle y\rangle^{k+l+1}\partial_y^i(u_0, b_0)\|_{L^\infty} \leq \delta^{-1},\ \text{for } i = 1, 2 \\ b_0(x, y) \geq \delta, \end{cases} \tag{2.1}$$

then there exists a $T := T(\delta, k, l, \|(u_0, b_0)\|^2_{H^4_{k+l}})$ such that the initial boundary value problem (1.5) has a classical solution (u, b) satisfying $(u, b) \in H^4_{k+l}$.

Before proving the theorem, we introduce some important inequalities. The following inequalities will be used frequently in this paper, whose proofs are given in [14, 31].

Lemma 2.2. *For the proper functions f, g, h, the following inequalities hold:*

(i) For any $l \in \mathbb{R}$, $\lambda > \frac{1}{2}$, an integer $m \geq 3$, and any $\alpha = (\beta, k) \in \mathbb{N}^3, \tilde{\beta} = (\tilde{\beta}_1, \tilde{\beta}_2) \in \mathbb{N}^2$ with $\alpha + \tilde{\beta} \leq m$,

$$\|(D^\alpha g \cdot \partial_\tau^{\tilde{\beta}}\partial_y^{-1}h)(t,\cdot)\|_{L^2_{k+l}} \leq C\|g\|_{H^m_{1+\lambda}}\|h\|_{H^m_{1-\lambda}}. \tag{2.2}$$

(ii) For any $l \in \mathbb{R}$, an integer $m \geq 3$, and any $\alpha = (\beta, k) \in \mathbb{N}^3$, $\tilde{\alpha} = (\tilde{\beta}, \tilde{k}) \in \mathbb{N}^3$ with $\alpha + \tilde{\alpha} \leq m$,

$$\|(D^\alpha f \cdot D^{\tilde{\alpha}} g)(t,\cdot)\|_{L^2_{\tilde{k}+k+l}} \leq C\|f\|_{H^m_{l_1}}\|g\|_{H^m_{l_2}},\ \forall\ l_1 + l_2 = l. \tag{2.3}$$

(iii) If $\lambda > -\frac{1}{2}$ and $\lim\limits_{y\to\infty} f(x, y) = 0$, then,

$$\|\langle y\rangle^\lambda f\|_{L^2(\mathbb{R}^2_+)} \leq C\|\langle y\rangle^{\lambda+1}\partial_y f\|_{L^2(\mathbb{R}^2_+)}. \tag{2.4}$$

3. The compatibility condition of the MHD boundary layer equations

For simplicity, we consider the case of a uniform outflow $(U, B) = (1, 1)$ in this work, which implies that the pressure p is a constant. Thus, the MHD boundary layer Eq (1.5) is reduced to

$$\begin{cases} \partial_t u_1 + u_1\partial_x u_1 + u_2\partial_y u_1 = b_1\partial_x b_1 + b_2\partial_y b_1 + \partial_y^2 u_1, \\ \partial_t b_1 + \partial_y(u_2 b_1 - u_1 b_2) = \partial_y^2 b_1, \\ \partial_x u_1 + \partial_y u_2 = 0,\ \partial_x b_1 + \partial_y b_2 = 0, \\ (u_1, u_2, b_1, b_2)|_{y=0} = 0,\ \lim\limits_{y\to+\infty}(u_1, b_1) = (1, 1), \\ (u_1, b_1)|_{t=0} = (u_0, b_0)(x, y). \end{cases} \tag{3.1}$$

We assume the shear flow u^s is the solution of the following heat equation

$$\begin{cases} \partial_t u^s - \partial_y^2 u^s = 0,\ (t, y) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ u^s|_{y=0} = 0,\ \text{and}\ \lim\limits_{y\to+\infty} u^s = 1, \\ u^s|_{t=0} = u_0^s(y). \end{cases} \tag{3.2}$$

At the moment, we also suppose that

$$\begin{cases} u_1 = u^s + u,\ b_1 = 1 + b, \\ u_2 = v,\ \ b_2 = g, \end{cases} \tag{3.3}$$

then, the MHD boundary layer Eq (3.1) becomes

$$\begin{cases} \partial_t u - \partial_y^2 u + (u^s + u)\partial_x u + v\partial_y u - (1 + b)\partial_x b - g\partial_y b + v\partial_y u^s = 0, \\ \partial_t b - \partial_y^2 b + (u^s + u)\partial_x b + v\partial_y b - (1 + b)\partial_x u - g\partial_y u - g\partial_y u^s = 0, \\ u|_{t=0} = u_0 - u_0^s,\ \ b|_{t=0} = b_0 - 1, \\ (u, v, b, g)|_{y=0} = 0,\ \lim\limits_{y\to+\infty}(u, b) = (0, 0). \end{cases} \tag{3.4}$$

Integrating Eq $(3.4)_2$ over $[0, y]$ yields that

$$\partial_t \int_0^y b d\tilde{y} + v(1 + b) - (u^s + u)g = \partial_y^2 \int_0^y b d\tilde{y}, \tag{3.5}$$

where we have used the boundary conditions $b|_{y=0} = v|_{y=0} = g|_{y=0} = 0$.

Define

$$\psi(t, y) = \int_0^y b d\tilde{y}$$

and ones yields

$$\partial_t \psi + v(1 + b) - (u^s + u)g = \partial_y^2 \psi. \tag{3.6}$$

Next, we give the following basic estimates of a shear flow u^s for the heat Eq (3.2); See [31].

Lemma 3.1. *Let $u^s(t, y)$ be the solution of* (3.2)*, then for any $T_1 > 0$, it holds that for $1 \le p \le 6$,*

$$\left|\partial_y^p u^s(t,y)\right| \le c_1\langle y\rangle^{-k-p+1},\ \forall (t,y)\in[0,T]\times\mathbb{R}_+, \tag{3.7}$$

where $c_1>0$ depends on T_1.

At this moment, let us state the precise version of the compatibility condition for the nonlinear MHD boundary layer Eq (3.4) and give the boundary values.

Proposition 3.2. *Assume that* (u,b) *is a smooth solution of the system* (3.4)*, then the initial data* (u_0,b_0) *has to satisfy the following compatibility conditions up to 6 order:*

$$\begin{cases} u_0(x,0)=0,\ b_0(x,0)=0,\\ \partial_y^2u_0(x,0)=0,\ \ \partial_y^2b_0(x,0)=0,\\ \partial_y^4u_0(x,0)=\partial_x\partial_yu\partial_yu(0,x,0)+\partial_x\partial_yu\partial_yu^s(0,0)-\partial_x\partial_yb\partial_yb(0,x,0),\\ \partial_y^4b_0(x,0)=\partial_yb\partial_x\partial_yu(0,x,0)+\partial_x\partial_yb(0,x,0)\partial_yu(0,x,0)+3\partial_yu^s(0,0)\partial_x\partial_yb(0,x,0). \end{cases} \tag{3.8}$$

Moreover,

$$\begin{aligned} \partial_y^6u_0(x,0)=&-2\partial_x^2\partial_yb\partial_yu(0,x,0)+\partial_x\partial_yu\partial_y^3u(0,x,0)\\ &-2\partial_x^2\partial_yb(0,x,0)\partial_yu^s(0,0)+\partial_x\partial_yu(0,x,0)\partial_y^3u^s(0,0)\\ &-\partial_y^3b\partial_x\partial_yb(0,x,0)+2\partial_yb\partial_x^2\partial_yu(0,x,0)\\ &+\sum_{1\le j\le 3}C_j^4\Big(\partial_y^j(u^s+u)\partial_x\partial_y^{4-j}u+\partial_y^jv\partial_y^{5-j}u-\partial_y^j(1+b)\partial_x\partial_y^{4-j}b\\ &-\partial_y^jg\partial_y^{5-j}b+\partial_y^jv\partial_y^{5-j}u^s\Big)(0,x,0), \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} \partial_y^6b_0(x,0)=&-(3\partial_y^3b+4\partial_x\partial_yu)\partial_x\partial_yu(0,x,0)-\partial_yb(4\partial_x\partial_y^3u+2\partial_x^2\partial_yb)(0,x,0)\\ &+(4\partial_x\partial_y^3b+2\partial_x^2\partial_yu)\partial_yu(0,x,0)+\partial_x\partial_yb(3\partial_y^3u+4\partial_x\partial_yb)(0,x,0)\\ &+3\partial_y^3u^s\partial_x\partial_yb(0,x,0)+\partial_yu^s(4\partial_x\partial_y^3b+2\partial_x^2\partial_yu)(0,x,0)\\ &+\sum_{1\le j\le 3}C_j^4\Big(\partial_y^j(u^s+u)\partial_x\partial_y^{4-j}b-\partial_y^jg\partial_y^{5-j}u\\ &-\partial_y^j(1+b)\partial_x\partial_y^{4-j}u+\partial_y^jv\partial_y^{5-j}b-\partial_y^jg\partial_y^{5-j}u^s\Big)(0,x,0). \end{aligned} \tag{3.10}$$

Proof: By virtue of the Eqs $(3.4)_{1,2}$ and the boundary condition $(3.4)_4$, we get

$$\partial_y^2u(x,0)=0,\ \partial_y^2b(x,0)=0. \tag{3.11}$$

Applying the operator ∂_y on $(3.4)_{1,2}$, respectively, we can derive that

$$\begin{cases} \partial_t\partial_yu-\partial_y^3u+\partial_y\Big((u^s+u)\partial_xu+v\partial_yu\Big)-\partial_y\Big((1+b)\partial_xb+g\partial_yb\Big)+\partial_y\Big(v\partial_yu^s\Big)=0,\\ \partial_t\partial_yb-\partial_y^3b+\partial_y\Big((u^s+u)\partial_xb+v\partial_yb\Big)-\partial_y\Big((1+b)\partial_xu+g\partial_yu\Big)-\partial_y\Big(g\partial_yu^s\Big)=0. \end{cases}$$

Hence, using Eqs $(3.2)_{1,2}$ and Eq $(3.4)_4$, from the above equations, we infer that

$$\begin{cases} \partial_t\partial_y u(t,x,0) = \partial_y^3 u(t,x,0) + \partial_x\partial_y b(t,x,0), \\ \partial_t\partial_y b(t,x,0) = \partial_y^3 b(t,x,0) + \partial_x\partial_y u(t,x,0). \end{cases} \tag{3.12}$$

Differentiating Eq $(3.4)_1$ with respect to y twice, it follows that

$$\partial_t\partial_y^2 u - \partial_y^4 u + \partial_y^2((u^s+u)\partial_x u + v\partial_y u + v\partial_y u^s) - \partial_y((1+b)\partial_x\partial_y b + g\partial_y^2 b) = 0. \tag{3.13}$$

Invoking the Leibniz formula, we can deduce that

$$\begin{aligned} &\partial_y^2((u^s+u)\partial_x u + v\partial_y u + v\partial_y u^s) \\ &= \partial_y^2(u^s+u)\partial_x u + \partial_y^2 v\partial_y u + \partial_y^2 v\partial_y u^s \\ &\qquad +(u^s+u)\partial_x\partial_y^2 u + v\partial_y^3 u + v\partial_y^3 u^s \\ &\qquad +2\partial_y(u^s+u)\partial_x\partial_y u + 2\partial_y v\partial_y^2 u + 2\partial_y v\partial_y^2 u^s. \end{aligned}$$

Therefore,

$$\partial_y^4 u(t,x,0) = \partial_x\partial_y u\partial_y u(t,x,0) + \partial_x\partial_y u\partial_y u^s(t,0) - \partial_x\partial_y b\partial_y b(t,x,0), \tag{3.14}$$

where we used the facts $\partial_y^{2i}u^s(x,0) = 0,\ 0 \le 2i \le 4$.

Differentiating (3.14) with respect to t and using the equality $(3.12)_1$, it follows that

$$\begin{aligned} &\partial_t\partial_y^4 u(t,x,0) \\ = \ &(\partial_x\partial_y^3 u + \partial_x^2\partial_y b)\partial_y u(t,x,0) + \partial_x\partial_y u(\partial_y^3 u + \partial_x\partial_y b)(t,x,0) \\ &\quad +(\partial_x\partial_y^3 u + \partial_x^2\partial_y b)(t,x,0)\partial_y u^s(t,0) + \partial_x\partial_y u(t,x,0)\partial_y^3 u^s(t,0) \\ &\quad -(\partial_x\partial_y^3 b + \partial_x^2\partial_y u)(t,x,0)\partial_y b(t,x,0) - (\partial_y^3 b + \partial_x\partial_y u)(t,x,0)\partial_x\partial_y b(t,x,0) \\ = \ &(\partial_x\partial_y^3 u + \partial_x^2\partial_y b)\partial_y u(t,x,0) + \partial_x\partial_y u\partial_y^3 u(t,x,0) \\ &\quad +(\partial_x\partial_y^3 u + \partial_x^2\partial_y b)(t,x,0)\partial_y u^s(t,0) + \partial_x\partial_y u(t,x,0)\partial_y^3 u^s(t,0) \\ &\quad -(\partial_x\partial_y^3 b + \partial_x^2\partial_y u)(t,x,0)\partial_y b(t,x,0) - \partial_y^3 b(t,x,0)\partial_x\partial_y b(t,x,0). \end{aligned} \tag{3.15}$$

Similar to (3.13), we have the following the results about the magnetic velocity b

$$\partial_t\partial_y^2 b - \partial_y^4 b + \partial_y^2\Big((u^s+u)\partial_x b + v\partial_y b - (1+b)\partial_x u - g\partial_y u - g\partial_y u^s\Big) = 0. \tag{3.16}$$

However, by a direct calculation, we infer that

$$\begin{aligned} &\partial_y^2((u^s+u)\partial_x b + v\partial_y b - g\partial_y u^s) \\ &= \partial_y^2(u^s+u)\partial_x b + \partial_y^2 v\partial_y b - \partial_y^2 g\partial_y u^s \\ &\qquad +(u^s+u)\partial_x\partial_y^2 b + v\partial_y^3 b - g\partial_y^3 u^s \\ &\qquad +2\partial_y(u^s+u)\partial_x\partial_y b + 2\partial_y v\partial_y^2 b - 2\partial_y g\partial_y^2 u^s, \end{aligned}$$

and

$$\partial_y^2((1+b)\partial_x u + g\partial_y u) = \partial_y^2(1+b)\partial_x u + \partial_y^2 g\partial_y u + (1+b)\partial_y^2\partial_x u + g\partial_y^3 u$$

$$+2\partial_y(1+b)\partial_y\partial_x u + 2\partial_y g\partial_y^2 u.$$

Therefore, we can arrive at

$$\begin{aligned}\partial_y^4 b(t,x,0) = &-3\partial_y b(t,x,0)\partial_x\partial_y u(t,x,0) + 3\partial_x\partial_y b(t,x,0)\partial_y u(t,0)\\ &+3\partial_y u^s(t,0)\partial_x\partial_y b(t,x,0).\end{aligned} \tag{3.17}$$

Differentiating (3.17) with respect to t and using equality $(3.12)_2$, it follows that

$$\begin{aligned}\partial_t\partial_y^4 b(t,x,0) = &-3(\partial_y^3 b + \partial_x\partial_y u)\partial_x\partial_y u(t,x,0) - 3\partial_y b(\partial_x\partial_y^3 u + \partial_x^2\partial_y b)(t,x,0)\\ &+3(\partial_x\partial_y^3 b + \partial_x^2\partial_y u)\partial_y u(t,x,0) + 3\partial_x\partial_y b(\partial_y^3 u + \partial_x\partial_y b)(t,x,0)\\ &+3\partial_y^3 u^s\partial_x\partial_y b(t,x,0) + 3\partial_y u^s(\partial_x\partial_y^3 b + \partial_x^2\partial_y u)(t,x,0).\end{aligned} \tag{3.18}$$

Differentiating the Eq $(3.4)_1$ with respect to y four times, it follows that

$$\partial_t\partial_y^4 u + \partial_y^4\Big((u^s+u)\partial_x u + v\partial_y u - (1+b)\partial_x b - g\partial_y b + v\partial_y u^s\Big) = \partial_y^6 u \tag{3.19}$$

and using the Leibniz formula again brings

$$\begin{aligned}&\partial_y^4\Big((u^s+u)\partial_x u + v\partial_y u - (1+b)\partial_x b - g\partial_y b + v\partial_y u^s\Big)\\ &= \partial_y^4(u^s+u)\partial_x u + \partial_y^4 v\partial_y u - \partial_y^4(1+b)\partial_x b - \partial_y^4 g\partial_y b + \partial_y^4 v\partial_y u^s\\ &\quad +(u^s+u)\partial_x\partial_y^4 u + v\partial_y^5 u - (1+b)\partial_x\partial_y^4 b - g\partial_y^5 b + v\partial_y^5 u^s\\ &\quad + \sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s+u)\partial_x\partial_y^{4-j} u + \partial_y^j v\partial_y^{5-j} u - \partial_y^j(1+b)\partial_x\partial_y^{4-j} b\\ &\quad -\partial_y^j g\partial_y^{5-j} b + \partial_y^j v\partial_y^{5-j} u^s\Big).\end{aligned} \tag{3.20}$$

Thus, it follows from (3.15) and (3.19)–(3.20) that

$$\begin{aligned}&\partial_y^6 u(t,x,0)\\ = \;&\partial_t\partial_y^4 u(t,x,0) - \partial_x\partial_y^3 u\partial_y u(t,x,0) + \partial_x\partial_y^3 b\partial_y b(t,x,0) - \partial_x\partial_y^3 u(t,x,0)\partial_y u^s(t,0)\\ &-\partial_x\partial_y^4 b(t,x,0) + \sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s+u)\partial_x\partial_y^{4-j} u + \partial_y^j v\partial_y^{5-j} u - \partial_y^j(1+b)\partial_x\partial_y^{4-j} b\\ &-\partial_y^j g\partial_y^{5-j} b + \partial_y^j v\partial_y^{5-j} u^s\Big)(t,x,0)\\ = \;&-2\partial_x^2\partial_y b\partial_y u(t,x,0) + \partial_x\partial_y u\partial_y^3 u(t,x,0)\\ &-2\partial_x^2\partial_y b(t,x,0)\partial_y u^s(t,0) + \partial_x\partial_y u(t,x,0)\partial_y^3 u^s(t,0)\\ &-\partial_y^3 b\partial_x\partial_y b(t,x,0) + 2\partial_y b\partial_x^2\partial_y u(t,x,0)\\ &+ \sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s+u)\partial_x\partial_y^{4-j} u + \partial_y^j v\partial_y^{5-j} u - \partial_y^j(1+b)\partial_x\partial_y^{4-j} b\\ &-\partial_y^j g\partial_y^{5-j} b + \partial_y^j v\partial_y^{5-j} u^s\Big)(t,x,0).\end{aligned} \tag{3.21}$$

Analogously, derivating $(3.4)_2$ with respect to y four times, it follows that

$$\partial_t\partial_y^4 b + \partial_y^4\Big((u^s+u)\partial_x b - g\partial_y u - (1+b)\partial_x u + v\partial_y b - g\partial_y u^s\Big) = \partial_y^6 b, \tag{3.22}$$

and using the Leibniz formula again,

$$
\begin{aligned}
&\partial_y^4\Big((u^s+u)\partial_x b - g\partial_y u - (1+b)\partial_x u + v\partial_y b - g\partial_y u^s\Big)\\
&= \partial_y^4(u^s+u)\partial_x b - \partial_y^4 g\partial_y u - \partial_y^4(1+b)\partial_x u + \partial_y^4 v\partial_y b - \partial_y^4 g\partial_y u^s\\
&\quad +(u^s+u)\partial_x\partial_y^4 b - g\partial_y^5 u - (1+b)\partial_x\partial_y^4 u + v\partial_y^5 b - g\partial_y^6 u^s\\
&\quad + \sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s+u)\partial_x\partial_y^{4-j} b - \partial_y^j g\partial_y^{5-j} u - \partial_y^j(1+b)\partial_x\partial_y^{4-j} u\\
&\quad +\partial_y^j v\partial_y^{5-j} b - \partial_y^j g\partial_y^{5-j} u^s\Big).
\end{aligned}
\tag{3.23}
$$

Therefore, it follows from (3.18) and (3.22)–(3.23) that

$$
\begin{aligned}
&\partial_y^6 b(t,x,0)\\
= &-(3\partial_y^3 b + 4\partial_x\partial_y u)\partial_x\partial_y u(t,x,0) - \partial_y b(4\partial_x\partial_y^3 u + 2\partial_x^2\partial_y b)(t,x,0)\\
&\quad +(4\partial_x\partial_y^3 b + 2\partial_x^2\partial_y u)\partial_y u(t,x,0) + \partial_x\partial_y b(3\partial_y^3 u + 4\partial_x\partial_y b)(t,x,0)\\
&\quad +3\partial_y^3 u^s\partial_x\partial_y b(t,x,0) + \partial_y u^s(4\partial_x\partial_y^3 b + 2\partial_x^2\partial_y u)(t,x,0)\\
&\quad + \sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s+u)\partial_x\partial_y^{4-j} b - \partial_y^j g\partial_y^{5-j} u - \partial_y^j(1+b)\partial_x\partial_y^{4-j} u\\
&\quad +\partial_y^j v\partial_y^{5-j} b - \partial_y^j g\partial_y^{5-j} u^s\Big)(t,x,0),
\end{aligned}
\tag{3.24}
$$

then we take that the value at $t = 0$ for (3.11) (3.14), (3.17), (3.21) and (3.24) can obtain the desired results.

4. Nonlinear regularized MHD boundary layer equations

To investigate the existence of solution of the MHD boundary layer, we consider a parabolic regularized system for problem (3.4), which we can attain the local existence of the solution by using classical energy methods. More specifically, we discuss the following nonlinear MHD systems, for $0 < \varepsilon < 1$,

$$
\begin{cases}
\partial_t u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon - \partial_y^2 u^\varepsilon + (u^s+u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon - (1+b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s = 0,\\
\partial_t b^\varepsilon - \varepsilon\partial_x^2 b^\varepsilon - \partial_y^2 b^\varepsilon + (u^s+u^\varepsilon)\partial_x b^\varepsilon + v^\varepsilon\partial_y b^\varepsilon - (1+b^\varepsilon)\partial_x u^\varepsilon - g^\varepsilon\partial_y u^\varepsilon - g^\varepsilon\partial_y u^s = 0,\\
(u^\varepsilon, b^\varepsilon)|_{t=0} = (u_0^\varepsilon, b_0^\varepsilon) = (u_0, b_0) + \varepsilon(\mu_1^\varepsilon, \mu_2^\varepsilon),\\
(u^\varepsilon, v^\varepsilon, b^\varepsilon, g^\varepsilon)|_{y=0} = \mathbf{0}, \ \lim_{y\to+\infty}(u^\epsilon, b^\epsilon) = \mathbf{0},
\end{cases}
\tag{4.1}
$$

where we can use the system (4.1) to construct the corrector terms $\varepsilon(\mu_1^\varepsilon, \mu_2^\varepsilon)$, such that the initial data $(u_0, b_0) + \varepsilon(\mu_1^\varepsilon, \mu_2^\varepsilon)$ satisfies the compatibility conditions up to 6 order for the regularized systems (4.1). We show the boundary data of the solution for the regularized system (4.1), which also gives the accurate edition of the compatibility conditions for the system (4.1).

Proposition 4.1. *Let $k \ge \frac{1}{2}$, $l \ge 0$ be real numbers. Assume that (u_0, b_0) satisfies the compatibility conditions* (3.8)–(3.10) *for the equations* (3.4) *and $(\mu_1^\varepsilon, \mu_2^\varepsilon) \in H_{k+l}^6$ such that $(u_0, b_0) + \varepsilon(\mu_1^\varepsilon, \mu_2^\varepsilon)$ satisfies the compatibility conditions up to 6 order for the regularized system* (4.1). *If $(u^\varepsilon, b^\varepsilon)$ is a solution to* *the problem* (4.1) *in* $[0,T]$ *and satisfies* $(u^\varepsilon,b^\varepsilon)\in L^\infty([0,T];H^4_{k+l})$*, then we have*

$$
\begin{cases}
u^\varepsilon(0,x,0)=0,\ b^\varepsilon(0,x,0)=0,\ x\in\times\mathbb{R},\\
\partial_y^2u^\varepsilon(0,x,0)=0,\ \ \partial_y^2b^\varepsilon(0,x,0)=0,\\
\partial_y^4u^\varepsilon(0,x,0)=\partial_x\partial_yu^\varepsilon\partial_yu^\varepsilon(0,x,0)+\partial_x\partial_yu^\varepsilon\partial_yu^s(0,0)-\partial_x\partial_yb^\varepsilon\partial_yb^\varepsilon(0,x,0),\\
\partial_y^4b^\varepsilon(0,x,0)=-3\partial_yb^\varepsilon\partial_x\partial_yu^\varepsilon(0,x,0)+3\partial_x\partial_yb^\varepsilon(0,x,0)\partial_yu^\varepsilon(0,x,0)+3\partial_yu^s(0,0)\partial_x\partial_yb^\varepsilon(0,x,0),\\
\partial_y^6u^\varepsilon(0,x,0)=-2\partial_x^2\partial_yb^\varepsilon\partial_yu^\varepsilon(0,x,0)+\partial_x\partial_yu^\varepsilon\partial_y^3u^\varepsilon(0,x,0)\\
-2\partial_x^2\partial_yb^\varepsilon(0,x,0)\partial_yu^s(0,0)+\partial_x\partial_yu^\varepsilon(0,x,0)\partial_y^3u^s(0,0)\\
-\partial_y^3b^\varepsilon\partial_x\partial_yb^\varepsilon(0,x,0)+2\partial_yb^\varepsilon\partial_x^2\partial_yu^\varepsilon(0,x,0)\\
\underline{+2\varepsilon\partial_x^2\partial_yb^\varepsilon\partial_x\partial_yb^\varepsilon(0,x,0)-2\varepsilon\partial_x^2\partial_yu^\varepsilon\partial_x\partial_yu^\varepsilon(0,x,0)}\\
+\sum_{1\le j\le3}C_j^4\Big(\partial_y^j(u^s+u^\varepsilon)\partial_x\partial_y^{4-j}u^\varepsilon+\partial_y^jv^\varepsilon\partial_y^{5-j}u^\varepsilon-\partial_y^j(1+b^\varepsilon)\partial_x\partial_y^{4-j}b^\varepsilon\\
-\partial_y^jg^\varepsilon\partial_y^{5-j}b^\varepsilon+\partial_y^jv^\varepsilon\partial_y^{5-j}u^s\Big)(0,x,0),\\
\partial_y^6b^\varepsilon(0,x,0)=-(3\partial_y^3b^\varepsilon+4\partial_x\partial_yu^\varepsilon)\partial_x\partial_yu^\varepsilon(0,x,0)-\partial_yb(4\partial_x\partial_y^3u^\varepsilon+2\partial_x^2\partial_yb^\varepsilon)(0,x,0)\\
+(4\partial_x\partial_y^3b^\varepsilon+2\partial_x^2\partial_yu^\varepsilon)\partial_yu^\varepsilon(0,x,0)+\partial_x\partial_yb(3\partial_y^3u^\varepsilon+4\partial_x\partial_yb^\varepsilon)(0,x,0)\\
+3\partial_y^3u^s\partial_x\partial_yb^\varepsilon(0,x,0)+\partial_yu^s(4\partial_x\partial_y^3b^\varepsilon+2\partial_x^2\partial_yu^\varepsilon)(0,x,0)\\
+\underline{6\varepsilon(\partial_x^2\partial_yu^\varepsilon\partial_x\partial_yb^\varepsilon-\partial_x^2\partial_yb^\varepsilon\partial_x\partial_yu^\varepsilon)}(0,x,0)\\
+\sum_{1\le j\le3}C_j^4\Big(\partial_y^j(u^s+u^\varepsilon)\partial_x\partial_y^{4-j}b^\varepsilon-\partial_y^jg^\varepsilon\partial_y^{5-j}u^\varepsilon-\partial_y^j(1+b^\varepsilon)\partial_x\partial_y^{4-j}u^\varepsilon\\
+\partial_y^jv^\varepsilon\partial_y^{5-j}b^\varepsilon-\partial_y^jg^\varepsilon\partial_y^{5-j}u^s\Big)(0,x,0).
\end{cases}
\tag{4.2}
$$

Proof: Looking back to the boundary condition in (4.1)

$$
\begin{cases}
u^\varepsilon(t,x,0)=0,\ v^\varepsilon(t,x,0)=0,\ b^\varepsilon(t,x,0)=0,\\
g^\varepsilon(t,x,0)=0,\ \ (t,x)\in[0,T]\times\mathbb{R},
\end{cases}
$$

the following results are clear:

$$\partial_t\partial_x^n(u^\varepsilon,b^\varepsilon)(t,x,0)=0,\ \partial_t\partial_x^n(v^\varepsilon,g^\varepsilon)(t,x,0)=0,\ \ (t,x)\in[0,T]\times\mathbb{R},0\le n\le4. \tag{4.3}$$

Applying (4.1) and the boundary conditions (4.3), we have

$$\partial_y^2u^\varepsilon|_{y=0}=0,\ \ \partial_y^2b^\varepsilon|_{y=0}=0. \tag{4.4}$$

In addition, we can also derive

$$\partial_x^n\partial_y^2u^\varepsilon|_{y=0}=0,\ \ \partial_x^n\partial_y^2b^\varepsilon|_{y=0}=0,\ \ (t,x)\in[0,T]\times\mathbb{R},0\le n\le4. \tag{4.5}$$

Derivating the equation of (4.1) with respect to y,

$$
\begin{aligned}
\varepsilon\partial_x^2\partial_yu^\varepsilon=\ &\partial_t\partial_yu^\varepsilon-\partial_y^3u^\varepsilon+\partial_y((u^s+u^\varepsilon)\partial_xu^\varepsilon+v^\varepsilon\partial_yu^\varepsilon+v^\varepsilon\partial_yu^s)\\
&-(1+b^\varepsilon)\partial_x\partial_yb^\varepsilon-g\partial_y^2b^\varepsilon,
\end{aligned}
\tag{4.6}
$$

and using the boundary conditions (4.4), we deduce

$$\partial_t\partial_yu^\varepsilon|_{y=0}=\partial_y^3u^\varepsilon|_{y=0}+\partial_x\partial_yb^\varepsilon|_{y=0}+\varepsilon\partial_x^2\partial_yu^\varepsilon|_{y=0}. \tag{4.7}$$

Similarly, derivating the equation of $(4.1)_2$ with respect to y

$$\begin{aligned}&\partial_t\partial_y b^\varepsilon - \partial_y^3 b^\varepsilon + \partial_y^2\Big((u^s+u^\varepsilon)\partial_x b^\varepsilon + v^\varepsilon\partial_y b^\varepsilon - (1+b^\varepsilon)\partial_x u^\varepsilon - g^\varepsilon\partial_y u^\varepsilon - g\partial_y u^s\Big)\\&= \varepsilon\partial_x^2\partial_y b^\varepsilon,\end{aligned}\tag{4.8}$$

and using the boundary condition (4.4) again, then we get

$$\partial_t\partial_y b^\varepsilon|_{y=0} = \partial_y^3 b^\varepsilon|_{y=0} + \partial_x\partial_y u^\varepsilon|_{y=0} + \varepsilon\partial_x^2\partial_y b^\varepsilon|_{y=0}.\tag{4.9}$$

Differentiating (4.6) with respect to y, it follows that

$$\begin{aligned}\varepsilon\partial_x^2\partial_y^2 u^\varepsilon &= \partial_t\partial_y^2 u^\varepsilon - \partial_y^4 u^\varepsilon + \partial_y^2((u^s+u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon + v^\varepsilon\partial_y u^s)\\&\quad -\partial_y((1+b^\varepsilon)\partial_x\partial_y b^\varepsilon + g^\varepsilon\partial_y^2 b^\varepsilon),\end{aligned}\tag{4.10}$$

and applying the Leibniz formula,

$$\begin{aligned}&\partial_y^2((u^s+u^\varepsilon)\partial_x u^\varepsilon + v\partial_y u^\varepsilon + v^\varepsilon\partial_y u^s)\\&= \partial_y^2(u^s+u^\varepsilon)\partial_x u^\varepsilon + \partial_y^2 v^\varepsilon\partial_y u^\varepsilon + \partial_y^2 v^\varepsilon\partial_y u^s\\&\quad +(u^s+u^\varepsilon)\partial_x\partial_y^2 u^\varepsilon + v^\varepsilon\partial_y^3 u^\varepsilon + v^\varepsilon\partial_y^3 u^s\\&\quad +2\partial_y(u^s+u^\varepsilon)\partial_x\partial_y u^\varepsilon + 2\partial_y v^\varepsilon\partial_y^2 u^\varepsilon + 2\partial_y v^\varepsilon\partial_y^2 u^s.\end{aligned}\tag{4.11}$$

Therefore, we can derive

$$\partial_y^4 u^\varepsilon|_{y=0} = \partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon|_{y=0} + \partial_x\partial_y u^\varepsilon\partial_y u^s(t,0) - \partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon|_{y=0},\tag{4.12}$$

where we used the facts $\partial_y^{2i}u^s(x,0)=0,\ 0\le 2i\le 4$.

Differentiating (4.12) with respect to t and using (4.7) and (4.9), it follows that

$$\begin{aligned}&\partial_t\partial_y^4 u^\varepsilon|_{y=0}\\=\ &(\partial_x\partial_y^3 u^\varepsilon + \partial_x^2\partial_y b^\varepsilon + \varepsilon\partial_x^3\partial_y u^\varepsilon)\partial_y u^\varepsilon|_{y=0} + \partial_x\partial_y u^\varepsilon(\partial_y^3 u^\varepsilon + \varepsilon\partial_x^2\partial_y u^\varepsilon)|_{y=0}\\&\quad +(\partial_x\partial_y^3 u^\varepsilon + \partial_x^2\partial_y b^\varepsilon + \varepsilon\partial_x^3\partial_y u^\varepsilon)|_{y=0}\partial_y u^s(t,0) + \partial_x\partial_y u^\varepsilon|_{y=0}\partial_y^3 u^s(t,0)\\&\quad -\partial_x\partial_y b^\varepsilon(\partial_y^3 b^\varepsilon + \varepsilon\partial_x^2\partial_y b^\varepsilon)|_{y=0} - \partial_y b^\varepsilon(\partial_x\partial_y^3 b^\varepsilon + \partial_x^2\partial_y u + \varepsilon\partial_x^3\partial_y b^\varepsilon)|_{y=0}.\end{aligned}\tag{4.13}$$

Analogously, we can arrive at

$$\partial_y^4 b^\varepsilon|_{y=0} = -3\partial_y b^\varepsilon\partial_x\partial_y u^\varepsilon|_{y=0} + 3\partial_x\partial_y b^\varepsilon|_{y=0}\partial_y u^\varepsilon(t,0) + 3\partial_y u^s(t,0)\partial_x\partial_y b^\varepsilon|_{y=0},\tag{4.14}$$

and

$$\begin{aligned}&\partial_t\partial_y^4 b^\varepsilon|_{y=0}\\=\ &-3(\partial_y^3 b^\varepsilon + \partial_x\partial_y u^\varepsilon + \varepsilon\partial_x^2\partial_y b^\varepsilon)\partial_x\partial_y u^\varepsilon|_{y=0} - 3\partial_y b^\varepsilon(\partial_x\partial_y^3 u^\varepsilon + \partial_x^2\partial_y b^\varepsilon + \varepsilon\partial_x^3\partial_y u^\varepsilon)|_{y=0}\\&\quad +3(\partial_x\partial_y^3 b^\varepsilon + \partial_x^2\partial_y u^\varepsilon + \varepsilon\partial_x^3\partial_y b^\varepsilon)\partial_y u^\varepsilon|_{y=0} + 3\partial_x\partial_y b^\varepsilon(\partial_y^3 u^\varepsilon + \partial_x\partial_y b^\varepsilon + \varepsilon\partial_x^2\partial_y u^\varepsilon)|_{y=0}\\&\quad +3\partial_y^3 u^s\partial_x\partial_y b^\varepsilon|_{y=0} + 3\partial_y u^s(\partial_x\partial_y^3 b^\varepsilon + \partial_x^2\partial_y u^\varepsilon + \varepsilon\partial_x^3\partial_y b^\varepsilon)|_{y=0}.\end{aligned}\tag{4.15}$$

Differentiating $(4.1)_1$ with respect to y four times, it follows

$$\partial_y^6 u^\varepsilon + \varepsilon\partial_x^2\partial_y^4 u^\varepsilon = \partial_t\partial_y^4 u^\varepsilon + \partial_y^4\Big((u^s + u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon$$
$$-(1 + b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s\Big), \tag{4.16}$$

and using the Leibniz formula again,

$$\begin{aligned}
&\partial_y^4\Big((u^s + u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon - (1 + b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s\Big)\\
&= \partial_y^4(u^s + u^\varepsilon)\partial_x u^\varepsilon + \partial_y^4 v^\varepsilon\partial_y u^\varepsilon - \partial_y^4(1 + b^\varepsilon)\partial_x b^\varepsilon - \partial_y^4 g^\varepsilon\partial_y b^\varepsilon + \partial_y^4 v^\varepsilon\partial_y u^s\\
&\quad+(u^s + u^\varepsilon)\partial_x\partial_y^4 u^\varepsilon + v^\varepsilon\partial_y^5 u^\varepsilon - (1 + b^\varepsilon)\partial_x\partial_y^4 b^\varepsilon - g^\varepsilon\partial_y^5 b^\varepsilon + v^\varepsilon\partial_y^5 u^s\\
&\quad+\sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s + u^\varepsilon)\partial_x\partial_y^{4-j}u^\varepsilon + \partial_y^j v^\varepsilon\partial_y^{5-j}u^\varepsilon - \partial_y^j(1 + b^\varepsilon)\partial_x\partial_y^{4-j}b^\varepsilon\\
&\quad-\partial_y^j g^\varepsilon\partial_y^{5-j}b^\varepsilon + \partial_y^j v^\varepsilon\partial_y^{5-j}u^s\Big).
\end{aligned}\tag{4.17}$$

Hence, using (4.12) and (4.14), we have

$$\begin{aligned}
\partial_y^6 u^\varepsilon|_{y=0} \;=\;& -2\partial_x^2\partial_y b^\varepsilon\partial_y u^\varepsilon|_{y=0} + \partial_x\partial_y u^\varepsilon\partial_y^3 u^\varepsilon|_{y=0}\\
&-2\partial_x^2\partial_y b^\varepsilon|_{y=0}\partial_y u^s(t,0) + \partial_x\partial_y u^\varepsilon|_{y=0}\partial_y^3 u^s(t,0)\\
&-\partial_y^3 b^\varepsilon\partial_x\partial_y b^\varepsilon|_{y=0} + 2\partial_y b^\varepsilon\partial_x^2\partial_y u^\varepsilon|_{y=0}\\
&+\varepsilon\partial_x^3\partial_y u^\varepsilon(\partial_y u^\varepsilon + \partial_y u^s) - \varepsilon\partial_x^3\partial_y b^\varepsilon\partial_y b^\varepsilon + \varepsilon\partial_x^2\partial_y u^\varepsilon\partial_x\partial_y u^\varepsilon - \varepsilon\partial_x^2\partial_y b^\varepsilon\partial_x\partial_y b^\varepsilon\\
&\underline{-\varepsilon\partial_x^2\partial_y^4 u^\varepsilon|_{y=0}}\\
&+\sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s + u^\varepsilon)\partial_x\partial_y^{4-j}u^\varepsilon + \partial_y^j v^\varepsilon\partial_y^{5-j}u^\varepsilon - \partial_y^j(1 + b^\varepsilon)\partial_x\partial_y^{4-j}b^\varepsilon\\
&-\partial_y^j g^\varepsilon\partial_y^{5-j}b^\varepsilon + \partial_y^j v^\varepsilon\partial_y^{5-j}u^s\Big)\Big|_{y=0}\\
=\;& -2\partial_x^2\partial_y b^\varepsilon\partial_y u^\varepsilon|_{y=0} + \partial_x\partial_y u^\varepsilon\partial_y^3 u^\varepsilon|_{y=0}\\
&-2\partial_x^2\partial_y b^\varepsilon|_{y=0}\partial_y u^s(t,0) + \partial_x\partial_y u^\varepsilon|_{y=0}\partial_y^3 u^s(t,0)\\
&-\partial_y^3 b^\varepsilon\partial_x\partial_y b^\varepsilon|_{y=0} + 2\partial_y b^\varepsilon\partial_x^2\partial_y u^\varepsilon|_{y=0}\\
&\underline{+2\varepsilon\partial_x^2\partial_y b^\varepsilon\partial_x\partial_y b^\varepsilon_{y=0} - 2\varepsilon\partial_x^2\partial_y u^\varepsilon\partial_x\partial_y u^\varepsilon|_{y=0}}\\
&+\sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s + u^\varepsilon)\partial_x\partial_y^{4-j}u^\varepsilon + \partial_y^j v^\varepsilon\partial_y^{5-j}u^\varepsilon - \partial_y^j(1 + b^\varepsilon)\partial_x\partial_y^{4-j}b^\varepsilon\\
&-\partial_y^j g^\varepsilon\partial_y^{5-j}b^\varepsilon + \partial_y^j v^\varepsilon\partial_y^{5-j}u^s\Big)\Big|_{y=0}.
\end{aligned}\tag{4.18}$$

Similarly, derivating $(4.1)_2$ with respect to y four times, it follows that

$$\partial_y^6 b^\varepsilon + \varepsilon\partial_x^2\partial_y^4 b^\varepsilon = \partial_t\partial_y^4 b^\varepsilon + \partial_y^4\Big((u^s + u^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y u^\varepsilon$$
$$-(1 + b^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y b^\varepsilon - g^\varepsilon\partial_y u^s\Big), \tag{4.19}$$

and using the Leibniz formula again,

$$\partial_y^4\Big((u^s + u^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y u^\varepsilon - (1 + b^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y b^\varepsilon - g^\varepsilon\partial_y u^s\Big)$$

$$
\begin{aligned}
&= \partial_y^4(u^s + u^\varepsilon)\partial_x b^\varepsilon - \partial_y^4 g^\varepsilon \partial_y u^\varepsilon - \partial_y^4(1 + b^\varepsilon)\partial_x u^\varepsilon + \partial_y^4 v^\varepsilon \partial_y b^\varepsilon - \partial_y^4 g^\varepsilon \partial_y u^s \\
&\quad +(u^s + u^\varepsilon)\partial_x\partial_y^4 b^\varepsilon - g^\varepsilon \partial_y^5 u^\varepsilon - (1 + b^\varepsilon)\partial_x\partial_y^4 u^\varepsilon + v^\varepsilon \partial_y^5 b^\varepsilon - g^\varepsilon \partial_y^6 u^s \\
&\quad + \sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s + u^\varepsilon)\partial_x\partial_y^{4-j} b^\varepsilon - \partial_y^j g^\varepsilon \partial_y^{5-j} u^\varepsilon - \partial_y^j(1 + b^\varepsilon)\partial_x\partial_y^{4-j} u^\varepsilon \\
&\quad +\partial_y^j v^\varepsilon \partial_y^{5-j} b^\varepsilon - \partial_y^j g^\varepsilon \partial_y^{5-j} u^s\Big). \qquad (4.20)
\end{aligned}
$$

Thus, using (4.9), (4.15), and (4.20), we derive

$$
\begin{aligned}
&\partial_y^6 b^\varepsilon|_{y=0} \\
=\ & -(3\partial_y^3 b^\varepsilon + 4\partial_x\partial_y u^\varepsilon)\partial_x\partial_y u^\varepsilon|_{y=0} - \partial_y b(4\partial_x\partial_y^3 u^\varepsilon + 2\partial_x^2\partial_y b^\varepsilon)|_{y=0} \\
&+(4\partial_x\partial_y^3 b^\varepsilon + 2\partial_x^2\partial_y u^\varepsilon)\partial_y u^\varepsilon|_{y=0} + \partial_x\partial_y b(3\partial_y^3 u^\varepsilon + 4\partial_x\partial_y b^\varepsilon)|_{y=0} \\
&+3\partial_y^3 u^s \partial_x\partial_y b^\varepsilon|_{y=0} + \partial_y u^s(4\partial_x\partial_y^3 b^\varepsilon + 2\partial_x^2\partial_y u^\varepsilon)|_{y=0} \\
&+\underline{6\varepsilon(\partial_x^2\partial_y u^\varepsilon \partial_x\partial_y b^\varepsilon - \partial_x^2\partial_y b^\varepsilon \partial_x\partial_y u^\varepsilon)}|_{y=0} \\
&+ \sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s + u^\varepsilon)\partial_x\partial_y^{4-j} b^\varepsilon - \partial_y^j g^\varepsilon \partial_y^{5-j} u^\varepsilon - \partial_y^j(1 + b^\varepsilon)\partial_x\partial_y^{4-j} u^\varepsilon \\
&+\partial_y^j v^\varepsilon \partial_y^{5-j} b^\varepsilon - \partial_y^j g^\varepsilon \partial_y^{5-j} u^s\Big). \qquad (4.21)
\end{aligned}
$$

Similar to (3.8) and (3.9), we can deduce the desired results. Additionally, we can see that the equalities (3.21) and (3.24) are different from (4.18) and (4.21), respectively. It is obviously that the underlined terms are new terms. The proof is completed. □.

According to the relational expressions of the compatibility conditions (u_0, b_0) and $(u_0^\varepsilon, b_0^\varepsilon)$, respectively, we can also obtain the expression of the corrector terms $\partial_y^{2i}(\mu_1^\varepsilon, \mu_2^\varepsilon)$, $(0 \le i \le 3)$. Thus, we have the following corollary.

Corollary 4.1. *Assume that $(u_0^\varepsilon, b_0^\varepsilon)$ satisfies the compatibility conditions* (4.2) *for the Eq* (4.1) *and* $(u_0^\varepsilon, b_0^\varepsilon) \in L^\infty([0,T]; H_{k+l}^4)$*, then for any* $0 < \varepsilon < 1$*, there exists* $(\mu_1^\varepsilon, \mu_2^\varepsilon) \in H_{k+l}^6$ *such that* $(u_0, b_0) + \varepsilon(\mu_1^\varepsilon, \mu_2^\varepsilon)$ *satisfies the compatibility conditions up to 6 order for the regularized system* (4.1)*,*

$$
\|u_0^\varepsilon\|_{H_{k+l}^4(\mathbb{R}_+^2)} + \|b_0^\varepsilon\|_{H_{k+l}^4(\mathbb{R}_+^2)} \le \frac{3}{2}(\|u_0\|_{H_{k+l}^4(\mathbb{R}_+^2)} + \|b_0\|_{H_{k+l}^4(\mathbb{R}_+^2)}), \qquad (4.22)
$$

and

$$
\lim_{\varepsilon\to 0}\|(u_0^\varepsilon, b_0^\varepsilon) - (u_0, b_0)\|_{H_{k+l}^6(\mathbb{R}_+^2)} = 0. \qquad (4.23)
$$

Proof: We use the proof of the Proposition 4.1 to prove this corollary. Taking the value at $t = 0$ for (4.5) we have the following functions $(\mu_1^\varepsilon, \mu_2^\varepsilon)$ from (4.5):

$$
(\partial_x^n\partial_y^2\mu_1^\varepsilon, \partial_x^n\partial_y^2\mu_2^\varepsilon) = \mathbf{0}, \quad x \in \mathbb{R}
$$

and

$$
\partial_y^4 u_0(x,0) + \varepsilon\partial_y^4\mu_1^\varepsilon(x,0)
$$

$$= (\partial_x\partial_y u_0 + \varepsilon\partial_x\partial_y\mu_1^\varepsilon)(\partial_y u_0 + \varepsilon\partial_y\mu_1^\varepsilon)(x,0) + \partial_y u^s(\partial_x\partial_y u_0 + \varepsilon\partial_x\partial_y\mu_1^\varepsilon)(x,0)$$
$$-(\partial_y b_0 + \varepsilon\partial_y\mu_2^\varepsilon)(\partial_x\partial_y b_0 + \varepsilon\partial_x\partial_y\mu_2^\varepsilon)(x,0). \tag{4.24}$$

Therefore, we get

$$\partial_y^4\mu_1^\varepsilon(x,0) = \partial_x\partial_y u_0\partial_y\mu_1^\varepsilon(x,0) + \partial_y u_0\partial_x\partial_y\mu_1^\varepsilon(x,0) + \varepsilon\partial_y\mu_1^\varepsilon\partial_x\partial_y\mu_1^\varepsilon(x,0)$$
$$+\partial_y u_0^s(0)\partial_x\partial_y\tilde{\mu}_1^\varepsilon(x,0) - \partial_y b_0\partial_x\partial_y\mu_2^\varepsilon(x,0) - \partial_x\partial_y b_0\partial_y\mu_2^\varepsilon(x,0) - \varepsilon\partial_y\mu_2^\varepsilon\partial_x\partial_y\mu_2^\varepsilon(x,0). \tag{4.25}$$

Likewise, we can also derive that μ_2^ε satisfies

$$\begin{aligned}\partial_y^4\mu_2^\varepsilon \quad = \quad & -3(\partial_y b_0\partial_x\partial_y\mu_1^\varepsilon + \partial_x\partial_y u_0\partial_y\mu_2^\varepsilon + \varepsilon\partial_x\partial_y\mu_1^\varepsilon\partial_y\mu_2^\varepsilon)\\ & +3(\partial_y u_0\partial_x\partial_y\mu_2^\varepsilon + \partial_x\partial_y b_0\partial_y\mu_1^\varepsilon + \varepsilon\partial_x\partial_y\mu_2^\varepsilon\partial_y\mu_1^\varepsilon) + 3\partial_y u_0^s(0)\partial_x\partial_y\mu_2^\varepsilon.\end{aligned} \tag{4.26}$$

Taking the values at $t = 0$ for (4.19), we attain a restraint condition for $(\partial_y^6\mu_1^\varepsilon, \partial_y^6\mu_2^\varepsilon)$,

$$\begin{aligned}&\partial_y^6\mu_1^\varepsilon(x,0) = -2(\partial_x^2\partial_y b_0\partial_y\mu_1^\varepsilon + \partial_x^2\partial_y\mu_2^\varepsilon\partial_y u_0 + \varepsilon\partial_x^2\partial_y\mu_2^\varepsilon\partial_y\mu_1^\varepsilon)(x,0)\\
&+(\partial_x\partial_y u_0\partial_y^3\mu_1^\varepsilon + \partial_x\partial_y\mu_1^\varepsilon\partial_y^3 u_0 + \varepsilon\partial_x\partial_y\mu_1^\varepsilon\partial_y^3\mu_1^\varepsilon)(x,0)\\
&+\partial_y^3 u^s\partial_x\partial_y\mu_1^\varepsilon(x,0) - 2\partial_y u^s\partial_x^2\partial_y\mu_2^\varepsilon(x,0)\\
&-(\partial_y^3 b_0\partial_x\partial_y\mu_2^\varepsilon + \partial_y^3\mu_2^\varepsilon\partial_x\partial_y b_0 + \varepsilon\partial_y^3\mu_2^\varepsilon\partial_x\partial_y\mu_2^\varepsilon)(x,0)\\
&+2(\partial_y b_0\partial_x^2\partial_y\mu_1^\varepsilon + \partial_y\mu_2^\varepsilon\partial_x^2\partial_y u_0 + \varepsilon\partial_y\mu_2^\varepsilon\partial_x^2\partial_y\mu_1^\varepsilon)(x,0)\\
&+2(\underline{\partial_x\partial_y b_0\partial_x^2\partial_y b_0} + \varepsilon\partial_x\partial_y\mu_2^\varepsilon\partial_x^2\partial_y b_0 + \varepsilon\partial_x\partial_y b_0\partial_x^2\partial_y\mu_2^\varepsilon + \varepsilon^2\partial_x\partial_y\mu_2^\varepsilon\partial_x^2\partial_y\mu_2^\varepsilon)(x,0)\\
&-2(\underline{\partial_x^2\partial_y u_0\partial_x\partial_y u_0} + \varepsilon\partial_x^2\partial_y u_0\partial_x\partial_y\mu_1^\varepsilon + \varepsilon\partial_x^2\partial_y\mu_1^\varepsilon\partial_x\partial_y u_0 + \varepsilon^2\partial_x^2\partial_y\mu_1^\varepsilon\partial_x\partial_y\mu_1^\varepsilon)(x,0)\\
&+\sum_{1\le j\le 3} C_j^4\Big(\partial_y^j(u^s+u_0)\partial_x\partial_y^{4-j}\mu_1^\varepsilon + \partial_y^j\mu_1^\varepsilon\partial_x\partial_y^{4-j}u_0 + \varepsilon\partial_y^j\mu_1^\varepsilon\partial_x\partial_y^{4-j}\mu_1^\varepsilon\Big)(x,0)\\
&-\sum_{1\le j\le 3} C_j^4\Big(\partial_x\partial_y^{j-1}u_0\partial_y^{5-j}\mu_1^\varepsilon + \partial_x\partial_y^{j-1}\mu_1^\varepsilon\partial_y^{5-j}u_0 + \varepsilon\partial_x\partial_y^{j-1}\mu_1^\varepsilon\partial_y^{5-j}\mu_1^\varepsilon\Big)(x,0)\\
&-\sum_{1\le j\le 3} C_j^4\Big(\partial_y^j b_0\partial_x\partial_y^{4-j}\mu_2^\varepsilon + \partial_y^j\mu_2^\varepsilon\partial_x\partial_y^{4-j}b_0 + \varepsilon\partial_y^j\mu_2^\varepsilon\partial_x\partial_y^{4-j}\mu_2^\varepsilon\Big)(x,0)\\
&+\sum_{1\le j\le 3} C_j^4\Big(\partial_x\partial_y^{j-1}b_0\partial_y^{5-j}\mu_2^\varepsilon + \partial_x\partial_y^{j-1}\mu_2^\varepsilon\partial_y^{5-j}b_0 + \varepsilon\partial_x\partial_y^{j-1}\mu_2^\varepsilon\partial_y^{5-j}\mu_2^\varepsilon\Big)(x,0)\\
&-\sum_{1\le j\le 3} C_j^4\partial_y^{5-j}u_0^s(0)\partial_x\partial_y^{j-1}\mu_1^\varepsilon(x,0).\end{aligned} \tag{4.27}$$

Similar to above,

$$\begin{aligned}\partial_y^6\mu_2^\varepsilon \quad = \quad & -3(\partial_y^3 b_0\partial_x\partial_y\mu_1^\varepsilon + \partial_y^3\mu_2^\varepsilon\partial_x\partial_y u_0 + \varepsilon\partial_y^3\mu_2^\varepsilon\partial_x\partial_y\mu_1^\varepsilon)|_{y=0}\\
& +4(2\partial_x\partial_y u_0\partial_x\partial_y\mu_1^\varepsilon + \varepsilon\partial_x\partial_y\mu_1^\varepsilon\partial_x\partial_y\mu_1^\varepsilon)|_{y=0}\\
& -4(\partial_y b_0\partial_x\partial_y^3\mu_1^\varepsilon + \partial_y\mu_2^\varepsilon\partial_x\partial_y^3 u_0 + \varepsilon\partial_y\mu_2^\varepsilon\partial_x\partial_y^3\mu_1^\varepsilon)|_{y=0}\\
& -2(\partial_y b_0\partial_x^2\partial_y\mu_2^\varepsilon + \partial_y\mu_2^\varepsilon\partial_x^2\partial_y b_0 + \varepsilon\partial_y\mu_2^\varepsilon\partial_x^2\partial_y\mu_2^\varepsilon)|_{y=0}\\
& +4(\partial_x\partial_y^3 b_0\partial_y\mu_1^\varepsilon + \partial_x\partial_y^3\mu_2^\varepsilon\partial_y u_0 + \varepsilon\partial_x\partial_y^3\mu_2^\varepsilon\partial_y\mu_1^\varepsilon)|_{y=0}\\
& +2(\partial_x^2\partial_y u_0\partial_y\mu_1^\varepsilon + \partial_x^2\partial_y\mu_1^\varepsilon\partial_y u_0 + \varepsilon\partial_x^2\partial_y\mu_1^\varepsilon\partial_y\mu_1^\varepsilon)|_{y=0}\end{aligned}$$

$$
\begin{aligned}
&+3(\partial_x\partial_y b_0\partial_y^3\mu_1^\varepsilon+\partial_x\partial_y\mu_2^\varepsilon\partial_y^3u_0+\varepsilon\partial_x\partial_y\mu_2^\varepsilon\partial_y^3\mu_1^\varepsilon)|_{y=0}\\
&+4(\partial_x\partial_y b_0\partial_x\partial_y\mu_2^\varepsilon+\partial_x\partial_y\mu_2^\varepsilon\partial_x\partial_y b_0+\varepsilon\partial_x\partial_y\mu_2^\varepsilon\partial_x\partial_y\mu_2^\varepsilon)|_{y=0}\\
&+3\partial_y^3u_0^s(0)\partial_x\partial_y\mu_2^\varepsilon|_{y=0}+\partial_y u_0^s(0)(4\partial_x\partial_y^3\mu_2^\varepsilon+2\partial_x^2\partial_y\mu_1^\varepsilon)|_{y=0}\\
&+\underline{6(\partial_x\partial_y b_0\partial_x^2\partial_y u_0}+\varepsilon\partial_x\partial_y b_0\partial_x^2\partial_y\mu_1^\varepsilon+\varepsilon\partial_x\partial_y\mu_2^\varepsilon\partial_x^2\partial_y u_0^\varepsilon+\varepsilon^2\partial_x\partial_y\mu_2^\varepsilon\partial_x^2\partial_y\mu_1^\varepsilon)|_{y=0}\\
&-\underline{6(\partial_x^2\partial_y b_0\partial_x\partial_y u_0}+\varepsilon\partial_x^2\partial_y b_0\partial_x\partial_y\mu_1^\varepsilon+\varepsilon\partial_x^2\partial_y\mu_2^\varepsilon\partial_x\partial_y u_0+\varepsilon\partial_x^2\partial_y\mu_2^\varepsilon\partial_x\partial_y\mu_1^\varepsilon)|_{y=0}\\
&+\sum_{1\le j\le 3}C_j^4\Big(\partial_y^j(u^s+u_0)\partial_x\partial_y^{4-j}\mu_2^\varepsilon+\partial_y^j\mu_1^\varepsilon\partial_x\partial_y^{4-j}b_0+\varepsilon\partial_y^j\mu_1^\varepsilon\partial_x\partial_y^{4-j}\mu_2^\varepsilon\Big)(x,0)\\
&+\sum_{1\le j\le 3}C_j^4\Big(\partial_x\partial_y^{j-1}b_0\partial_y^{5-j}\mu_1^\varepsilon+\partial_x\partial_y^{j-1}\mu_2\partial_y^{5-j}u_0+\varepsilon\partial_x\partial_y^{j-1}\mu_2\partial_y^{5-j}\mu_1^\varepsilon\Big)(x,0)\\
&-\sum_{1\le j\le 3}C_j^4\Big(\partial_y^j b_0\partial_x\partial_y^{4-j}\mu_1^\varepsilon+\partial_y^j\mu_2^\varepsilon\partial_x\partial_y^{4-j}u_0+\varepsilon\partial_y^j\mu_2^\varepsilon\partial_x\partial_y^{4-j}\mu_1^\varepsilon\Big)(x,0)\\
&-\sum_{1\le j\le 3}C_j^4\Big(\partial_x\partial_y^{j-1}u_0\partial_y^{5-j}\mu_2^\varepsilon+\partial_x\partial_y^{j-1}\mu_1^\varepsilon\partial_y^{5-j}b_0+\varepsilon\partial_x\partial_y^{j-1}\mu_1^\varepsilon\partial_y^{5-j}\mu_2^\varepsilon\Big)(x,0)\\
&+\sum_{1\le j\le 3}C_j^4\partial_y^{5-j}u_0^s(0)\partial_x\partial_y^{j-1}\mu_2^\varepsilon(x,0).
\end{aligned}
\tag{4.28}
$$

It is clear that $(\partial_y^6\mu_1^\varepsilon(x,0),\partial_y^6\mu_2^\varepsilon(x,0))$ are determined by the low order derivatives of $(\mu_1^\varepsilon,\mu_2^\varepsilon)$ and those of initial data (u_0,b_0). The underlined terms in (4.27) and (4.28) are deduced from the underlined terms in (4.18) and (4.21), respectively. All these underlined terms are from the added regularizing terms $\varepsilon\partial_x^2u^\varepsilon$ and $\varepsilon\partial_x^2b^\varepsilon$ in the Eqs $(4.1)_{1,2}$, respectively. This means that the regularizing terms $\partial_x^2u^\varepsilon$ and $\partial_x^2b^\varepsilon$ have an affect on the boundary. This also explains why we add corrector terms for the initial data in $(4.1)_3$.

We now construct the polynomial functions $\mu_1^\varepsilon(x,y)$ and $\mu_2^\varepsilon(x,y)$ on y by the following forms

$$
\tilde{\mu}_1^\varepsilon(x,y)=\mu_1^\varepsilon(x)\frac{y^6}{6!}\ \text{and}\ \tilde{\mu}_2^\varepsilon(x,y)=\mu_2^\varepsilon(x)\frac{y^6}{6!},
\tag{4.29}
$$

where

$$
\mu_1^\varepsilon(x)=2(\partial_x\partial_y b_0\partial_x^2\partial_y b_0-\partial_x^2\partial_y u_0\partial_x\partial_y u_0)\ \text{and}\ \mu_2^\varepsilon(x)=6(\partial_x\partial_y b_0\partial_x^2\partial_y u_0-\partial_x^2\partial_y b_0\partial_x\partial_y u_0).
$$

We take $\mu_1^\varepsilon(x,y)=\kappa(y)\tilde{\mu}_1^\varepsilon(x,y)$ and $\mu_2^\varepsilon(x,y)=\kappa(y)\tilde{\mu}_2^\varepsilon(x,y)$ with $\kappa\in C^\infty(\mathbb{R}_+)$; $\kappa(y)=1, y\in[0,1]$; $\kappa(y)=0, y\ge 2$. Thus, the proof is completed. □

Remark 4.1. *Actually, if we take* $(\mu_1^\varepsilon(x,y),\mu_2^\varepsilon(x,y))$ *with*

$$
\partial_y^j\mu_1^\varepsilon(x,0)=0\ \text{and}\ \partial_y^j\mu_2^\varepsilon(x,0)=0,\ 0\le j\le 5,
$$

then (4.27) and (4.28) imply

$$
\begin{cases}
\partial_y^6\mu_1^\varepsilon(x,0)=-2\partial_x^2\partial_y u_0\partial_x\partial_y u_0+2\partial_x\partial_y b_0\partial_x^2\partial_y b_0,\\
\partial_y^6\mu_2^\varepsilon(x,0)=6(\partial_x\partial_y b_0\partial_x^2\partial_y u_0-\partial_x^2\partial_y b_0\partial_x\partial_y u_0),
\end{cases}
$$

which are not equal to zero, respectively. Thus, it is necessary to add the corrector terms μ_1^ε *and* μ_2^ε *for the initial data of the regularized system, respectively.*

5. The approximate solutions of the MHD boundary layer equations

In this section, we prove the existence of approximate solutions by establishing a series of the estimates of solutions for the nonlinear MHD boundary layer problem (4.1). To be more specific, we plan to complete the proof of the solution for problem (4.1) by the following two subsections. In the first subsection, we will attain the weighted estimates for $D^\alpha(u,b)$ for $\alpha=(\beta,k)=(\beta_1,\beta_2,k)$, satisfying $\alpha=|\beta|+k\le 4$ and $|\beta|\le 3$, and the weighted estimates for $D_\tau^\beta(u,h)$ with $|\beta|=4$ in the second subsection.

First, we introduce the following lemma, which is helpful to deal with the boundary value.

Lemma 5.1. *([1]) Let $1<p<\infty$. If $U\in W^{m,p}(\mathbb{R}^{n+1})$, then its trace u belongs to the space $B=B_{p,p}^{m-\frac{1}{p}}(\mathbb{R}^n)$ and*

$$\|u\|_B \le K\|U\|_{W^{m,p}(\mathbb{R}^{n+1})},$$

with the constant $K>0$ independent of U.

Corollary 5.1. *Let $1<p<\infty$. If $U\in W^{m,p}(\mathbb{R}^{n+1})$, then its trace u belongs to the space $W^{m-1,p}(\mathbb{R}^n)$ and*

$$\|u\|_{W^{m-1,p}(\mathbb{R}^n)} \le K\|U\|_{W^{m,p}(\mathbb{R}^{n+1})},$$

with the constant $K>0$ independent of U.
Proof: Since $1<p<\infty$, it follows from the fact $W^{m-1,p}(\mathbb{R}^n)=F_{p,2}^{m-1}(\mathbb{R}^n)$ and the embedding theorem $B_{p,1}^{m-1}(\mathbb{R}^n)\hookrightarrow F_{p,2}^{m-1}(\mathbb{R}^n)$ in [24] that $B_{p,p}^{m-\frac{1}{p}}(\mathbb{R}^n)\hookrightarrow B_{p,1}^{m-1}(\mathbb{R}^n)\hookrightarrow W^{m-1,p}(\mathbb{R}^n)$, which gives

$$\|u\|_{W^{m-1,p}(\mathbb{R}^n)} \le C\|U\|_{B_{p,p}^{m-\frac{1}{p}}(\mathbb{R}^n)}.$$

Together with Lemma 5.1, this completes the proof.

5.1. Weighted H_{k+l}^4 with normal derivatives

We use energy methods to establish the weighted estimates for $D^\alpha(u,b)$ with $\alpha=(\beta,k)=(\beta_1,\beta_2,k)$, $\alpha=|\beta|+k\le 4$, and $|\beta|\le 3$. That is, we have the following lemma.

Lemma 5.2. *Let $k\ge\frac{1}{2}$, $l\ge 0$ be real numbers. Assume that $(u^\varepsilon,v^\varepsilon,g^\varepsilon,b^\varepsilon)$ is a solution to the problem (4.1) in $[0,T]$ and satisfies $(u^\varepsilon,b^\varepsilon)\in L^\infty([0,T];H_{k+l}^4)$, then there exists a positive constant C, which may be dependent on k,l such that*

$$\sum_{|\alpha|\le 4,\ |\beta|\le 3}\Big(\frac{d}{dt}\|D^\alpha(u^\varepsilon,b^\varepsilon)(t)\|_{L_{k+l}^2}^2+\varepsilon\|D^\alpha\partial_x(u^\varepsilon,b^\varepsilon)(t)\|_{L_{k+l}^2}^2+\|D^\alpha\partial_y(u^\varepsilon,b^\varepsilon)(t)\|_{L_{k+l}^2}^2\Big)$$
$$\le C\|(u^\varepsilon,b^\varepsilon)\|_{H_{k+l}^4}^2+C\|(u^\varepsilon,b^\varepsilon)\|_{H_{k+l}^4}^4. \tag{5.1}$$

Proof: Applying the operator $D^\alpha = \partial_\tau^\beta \partial_y^k$ on $(4.1)_{1,2}$ for $\alpha = (\beta, k) = (\beta_1, \beta_2, k)$ satisfying $\alpha = |\beta| + k \le 4$ and $|\beta| \le 3$,

$$\begin{cases} \partial_t D^\alpha u^\varepsilon - \varepsilon\partial_x^2 D^\alpha u^\varepsilon = \partial_y^2 D^\alpha u^\varepsilon - D^\alpha\Big((u^s + u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon \\ \qquad -(1 + b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s\Big), \\ \partial_t D^\alpha b^\varepsilon - \varepsilon\partial_x^2 D^\alpha b^\varepsilon = \partial_y^2 D^\alpha b^\varepsilon - D^\alpha\Big((u^s + u^\varepsilon)\partial_x b^\varepsilon + v^\varepsilon\partial_y b^\varepsilon \\ \qquad -(1 + b^\varepsilon)\partial_x u^\varepsilon - g^\varepsilon\partial_y u^\varepsilon - g^\varepsilon\partial_y u^s\Big). \end{cases} \tag{5.2}$$

Multiplying the resulting Eqs $(5.2)_{1,2}$ by $\langle y\rangle^{2k+2l} D^\alpha u^\varepsilon$ and $\langle y\rangle^{2k+2l} D^\alpha b^\varepsilon$, respectively, and integrating it by parts over $\mathbb{R}_+^2$, we derive that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\langle y\rangle^{k+l} D^\alpha(u^\varepsilon, b^\varepsilon)(t)\|_{L^2}^2 + \varepsilon\|\langle y\rangle^{k+l} D^\alpha\partial_x(u^\varepsilon, b^\varepsilon)(t)\|_{L^2}^2 \\ &= \int_{\mathbb{R}_+^2} r_1 \cdot \langle y\rangle^{2k+2l} D^\alpha u^\varepsilon dxdy + \int_{\mathbb{R}_+^2} r_2 \cdot \langle y\rangle^{2k+2l} D^\alpha b^\varepsilon dxdy \\ &\quad + \int_{\mathbb{R}_+^2} \partial_y^2 D^\alpha u^\varepsilon \cdot \langle y\rangle^{2k+2l} D^\alpha u^\varepsilon dxdy + \int_{\mathbb{R}_+^2} \partial_y^2 D^\alpha b^\varepsilon \cdot \langle y\rangle^{2k+2l} D^\alpha b^\varepsilon dxdy, \end{aligned} \tag{5.3}$$

where

$$\begin{cases} r_1 \triangleq -D^\alpha\Big((u^s + u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon - (1 + b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s\Big), \\ r_2 \triangleq -D^\alpha\Big((u^s + u^\varepsilon)\partial_x b^\varepsilon + v^\varepsilon\partial_y b^\varepsilon - (1 + b^\varepsilon)\partial_x u^\varepsilon - g^\varepsilon\partial_y u^\varepsilon - g^\varepsilon\partial_y u^s\Big). \end{cases}$$

Next, we will establish the estimates of the righthand side of nonlinear terms (5.3). First of all, we deal with the first second terms. By the definitions of r_1 and r_2, we have

$$\begin{cases} r_1 = -\Big((u^s + u^\varepsilon)\partial_x D^\alpha u^\varepsilon + v^\varepsilon\partial_y D^\alpha u^\varepsilon - (1 + b^\varepsilon)\partial_x D^\alpha b^\varepsilon - g^\varepsilon\partial_y D^\alpha b^\varepsilon\Big) \\ \quad -\Big([D^\alpha, (u^s + u^\varepsilon)]\partial_x u^\varepsilon + [D^\alpha, v^\varepsilon]\partial_y u^\varepsilon - [D^\alpha, (1 + b^\varepsilon)]\partial_x b^\varepsilon - [D^\alpha, g^\varepsilon]\partial_y b^\varepsilon\Big) \\ \quad +D^\alpha(v^\varepsilon\partial_y u^s) \\ \quad \triangleq r_1^1 + r_1^2 + r_1^3, \\ r_2 = -\Big((u^s + u^\varepsilon)\partial_x D^\alpha b^\varepsilon + v^\varepsilon\partial_y D^\alpha b^\varepsilon - (1 + b^\varepsilon)\partial_x D^\alpha u^\varepsilon - g^\varepsilon\partial_y D^\alpha u^\varepsilon\Big) \\ \quad -\Big([D^\alpha, (u^s + u^\varepsilon)]\partial_x b^\varepsilon + [D^\alpha, v^\varepsilon]\partial_y b^\varepsilon - [D^\alpha, (1 + b^\varepsilon)]\partial_x u^\varepsilon - [D^\alpha, g^\varepsilon]\partial_y u^\varepsilon\Big) \\ \quad -D^\alpha(g^\varepsilon\partial_y u^s) \\ \quad \triangleq r_2^1 + r_2^2 + r_2^3. \end{cases}$$

Therefore, we can divide the term

$$\int_{\mathbb{R}_+^2} r_1 \cdot \langle y\rangle^{2k+2l} D^\alpha u^\varepsilon dxdy + r_2 \cdot \langle y\rangle^{2k+2l} D^\alpha b^\varepsilon dxdy,$$

into the following three parts:

$$\int_{\mathbb{R}_+^2} r_1 \cdot \langle y\rangle^{2k+2l} D^\alpha u^\varepsilon dxdy + r_2 \cdot \langle y\rangle^{2k+2l} D^\alpha b^\varepsilon dxdy$$

$$= \sum_{i=1}^{3} \int_{\mathbb{R}_+^2} r_1^i \cdot \langle y \rangle^{2k+2l} D^\alpha u^\varepsilon dxdy + r_2^i \cdot \langle y \rangle^{2k+2l} D^\alpha b^\varepsilon dxdy \tag{5.4}$$
$$\triangleq I_1 + I_2 + I_3,$$

and the estimates of each term I_i are as follows.

The estimate for I_1:

$$\int_{\mathbb{R}_+^2} r_1^1 \cdot \langle y \rangle^{2k+2l} D^\alpha u^\varepsilon dxdy + r_2^1 \cdot \langle y \rangle^{2k+2l} D^\alpha b^\varepsilon dxdy$$
$$= 2(k+l) \int_{\mathbb{R}_+^2} \langle y \rangle^{2k+2l-1} (v^\varepsilon, g^\varepsilon)(|D^\alpha u^\varepsilon|^2 dxdy + |D^\alpha b^\varepsilon|^2) dxdy \tag{5.5}$$
$$\leq C\|(v^\varepsilon, g^\varepsilon)\|_{L^\infty} \|\langle y \rangle^{2k+2l-1} D^\alpha (u^\varepsilon, b^\varepsilon)\|_{L^2}^2$$
$$\leq C\|(u^\varepsilon, b^\varepsilon)\|_{H_0^2} \|(u^\varepsilon, b^\varepsilon)\|_{H_{k+l}^m}^2,$$

where we have used the integration by parts.

The estimate for I_2:

Notice that

$$I_2 \leq \|r_1^2\|_{L_{k+l}^2} \|D^\alpha u^\varepsilon\|_{L_{k+l}^2} + \|r_2^2\|_{L_{k+l}^2} \|D^\alpha b^\varepsilon\|_{L_{k+l}^2}. \tag{5.6}$$

Therefore, we need to establish the estimates of the terms $\|r_1^2\|_{L_{k+l}^2}$ and $\|r_2^2\|_{L_{k+l}^2}$. However, we know that the terms in $\|r_2^2\|_{L_{k+l}^2}$ are similar to the terms in $\|r_1^2\|_{L_{k+l}^2}$, so we will estimate only the L_{k+l}^2 of r_1^2.

For the commutator operator, we can rewrite it as

$$[D^\alpha, (u^s + u^\varepsilon)]\partial_x u^\varepsilon = \sum_{\hat{\alpha} \leq \alpha,\ 1 \leq \hat{\alpha}} C_\alpha^{\hat{\alpha}} \partial^{\hat{\alpha}} (u^s + u^\varepsilon) \partial^{\alpha - \hat{\alpha}} \partial_x u^\varepsilon,$$

then for $\alpha \leq 4$, we can obtain

$$\|[D^\alpha, (u^s + u^\varepsilon)]\partial_x u^\varepsilon\|_{L_{k+l}^2} \leq C(\|u^\varepsilon\|_{H_{k+l}^4} + \|u^\varepsilon\|_{H_{k+l}^4}^2).$$

Note that

$$[D^\alpha, v^\varepsilon)]\partial_y u^\varepsilon = \sum_{\hat{\alpha} \leq \alpha,\ 1 \leq \hat{\alpha}} C_\alpha^{\hat{\alpha}} \partial^{\hat{\alpha}} v^\varepsilon \partial^{\alpha - \hat{\alpha}} \partial_y u^\varepsilon.$$

Since $1 \leq |\hat{\alpha}|$, $\hat{\alpha} \leq \alpha$, we have for $|\alpha - \hat{\alpha}| \leq 3$,

$$-\partial^{\hat{\alpha}} v^\varepsilon = \partial_\tau^{\hat{\beta}} \partial_y^{\hat{k}} \int_0^y \partial_x u^\varepsilon d\tilde{y} = \begin{cases} \partial_\tau^{\hat{\beta}_1 + e_2} \partial_y^{\hat{k} - e_3} u^\varepsilon, & \hat{k} \geq 1, \\ \int_0^y \partial_\tau^{\hat{\beta} + e_2} u^\varepsilon d\tilde{y}, & \hat{k} = 0. \end{cases}$$

Thus, using Lemma 2.2, we can arrive at

$$\|\partial_\tau^{\hat{\beta}_1 + e_2} \partial_y^{\hat{k} - e_3} u^\varepsilon \partial^{\alpha - \hat{\alpha}} \partial_y u^\varepsilon(t, \cdot)\|_{L_{k+l}^2} \leq C\|u^\varepsilon\|_{H_{l+1}^4} \|u^\varepsilon\|_{H_0^4} \tag{5.7}$$

and

$$\|\partial_\tau^{\hat{\beta} + e_2} u^\varepsilon \partial^{\alpha - \hat{\alpha}} \partial_y u^\varepsilon(t, \cdot)\|_{L_{k+l}^2} \leq C\|u^\varepsilon\|_{H_{l+1}^4} \|u^\varepsilon\|_{H_0^4}. \tag{5.8}$$

Combining (5.7) with (5.8), we derive

$$\|([D^\alpha, v^\varepsilon]\partial_y u^\varepsilon)\|_{L^2_{k+l}} \leq C\|u^\varepsilon\|^2_{H^4_{k+l}}.$$

Similarly, we also have

$$\|([D^\alpha, (1+b^\varepsilon)]\partial_x b^\varepsilon)\|_{L^2_{k+l}} \leq C(\|b^\varepsilon\|_{H^4_{k+l}} + \|b^\varepsilon\|^2_{H^4_{k+l}}),$$

and

$$\|([D^\alpha, g^\varepsilon]\partial_y g^\varepsilon)\|_{L^2_{k+l}} \leq C\|b^\varepsilon\|^2_{H^4_{k+l}}.$$

Using the above three inequalities, we can attain

$$\|r_1^2\|_{L^2_{k+l}} \leq C(\|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}}). \tag{5.9}$$

Similar to above for L^2_{k+l} of r_2^2, we conclude that

$$\|r_2^2\|_{L^2_{k+l}} \leq C(\|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}}). \tag{5.10}$$

Inserting (5.9) and (5.10) into (5.6), we have

$$I_2 \leq C(\|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|^3_{H^4_{k+l}}). \tag{5.11}$$

The estimate for I_3:

By a direct computation, we can get for $|\beta| \geq 1$, $k + |\beta| \leq 4$,

$$\begin{aligned}
&\sum_{|\alpha|=\beta_1+\beta_2+k\leq 4,\ k\geq 1} \partial_t^{\beta_1}\partial_x^{\beta_2}\partial_y^k(v^\varepsilon u_y^s) \\
&= \sum_{|\alpha|\leq 4,\ k\geq k'\geq 1, \beta_1'\leq\beta_1\leq 3} C_k^{k'} C_{\beta_1}^{\beta_1'} \partial_t^{\beta_1'}\partial_x^{\beta_2}\partial_y^{k'} v^\varepsilon \partial_t^{\beta_1-\beta_1'}\partial_y^{k-k'+1} u^s \\
&= \sum_{|\alpha|\leq 4,\ k\geq k'\geq 1, \beta_1'\leq\beta_1\leq 3} C_k^{k'} C_{\beta_1}^{\beta_1'} \partial_t^{\beta_1'}\partial_x^{\beta_2+1}\partial_y^{k'-1} v^\varepsilon \partial_t^{\beta_1-\beta_1'}\partial_y^{k-k'+1} u^s.
\end{aligned}$$

Thus, for $|\alpha| \leq 4$, we can get

$$\|\partial^\alpha(v^\varepsilon u_y^s)\|_{L^2_{k+l}} \leq C\|u^\varepsilon\|_{H^4_{k+l}}.$$

Analogously, for $|\alpha| \leq 4$, we also can derive

$$\|\partial^\alpha(g^\varepsilon u_y^s)\|_{L^2_{k+l}} \leq C\|b^\varepsilon\|_{H^4_{k+l}}.$$

Combining the above two inequalities with $|\alpha| \leq 4$, we have

$$I_3 \leq C\|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}}. \tag{5.12}$$

Hence, we infer from (5.5), (5.11), and (5.12),

$$\Big|\int_{\mathbb{R}^2_+} r_1\cdot\langle y\rangle^{2k+2l}D^\alpha u^\varepsilon dxdy + r_2\cdot\langle y\rangle^{2k+2l}D^\alpha b^\varepsilon dxdy\Big| \le C(\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}} + \|(u^\varepsilon,b^\varepsilon)\|^3_{H^4_{k+l}}). \qquad (5.13)$$

In the following part, we will estimate the remainder terms. We first deal with the term $\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2l}\partial_y^2 D^\alpha u^\varepsilon\cdot D^\alpha u^\varepsilon dxdy$. Similarly, $\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2l}\partial_y^2 D^\alpha b^\varepsilon\cdot D^\alpha b^\varepsilon dxdy$ can be derived. Integrating it by parts and using the boundary value (4.2), we arrive at

$$\begin{aligned}&\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2l}\partial_y^2 D^\alpha u^\varepsilon\cdot D^\alpha u^\varepsilon dxdy\\ &= -\|\langle y\rangle^{2(k+l)}\partial_y D^\alpha u^\varepsilon\|^2_{L^2} - 2(k+l)\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2l-1}\partial_y D^\alpha u^\varepsilon\cdot D^\alpha u^\varepsilon dxdy\\ &+ \int_{\mathbb{R}}\partial_y D^\alpha u^\varepsilon D^\alpha u^\varepsilon|_{y=0}dx. \end{aligned}\qquad (5.14)$$

Applying the Cauchy-Schwarz inequality, we can get

$$2(k+l)\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2l-1}\partial_y D^\alpha u^\varepsilon\cdot D^\alpha u^\varepsilon dxdy \le \frac{1}{8}\|\partial_y u^\varepsilon\|^2_{H^4_{k+l}} + C\|u^\varepsilon\|^2_{H^4_{k+l}}. \qquad (5.15)$$

Now, we study the last term in (5.14), that is the boundary integral $\int_{\mathbb{R}}\partial_y D^\alpha u^\varepsilon D^\alpha u^\varepsilon|_{y=0}dx$. By a direct calculation, we know the boundary integral $\int_{\mathbb{R}}\partial_y D^\alpha u^\varepsilon D^\alpha u^\varepsilon|_{y=0}dx = 0$ when the cases are $k = 1, 2$. Therefore, we only consider the cases $k = 3, 4$.

Case 1: $|\beta| = 1$, $k = 3$, using Corollary 5.1 and the boundary conditions $y = 0$, we lead to

$$\begin{aligned}&\Big|\int_{\mathbb{R}}\partial_\tau\partial_y^4 u^\varepsilon\cdot\partial_\tau\partial_y^3 u^\varepsilon|_{y=0}dx\Big|\\ &\le \|\partial_\tau(\partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon + \partial_x\partial_y u^\varepsilon\partial_y u^s - \partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon|_{y=0})\|\|\partial_\tau\partial_y^3 u^\varepsilon|_{y=0}\|\\ &\le \|\partial_\tau\partial_y(\partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon + \partial_x\partial_y u^\varepsilon\partial_y u^s - \partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon)\|\|\partial_\tau\partial_y^4 u^\varepsilon\|\\ &\le \frac{1}{8}\|\partial_y u^\varepsilon\|^2_{H^4_{k+l}} + C\|u^\varepsilon\|^2_{H^4_{k+l}} + C\|(u^\varepsilon,b^\varepsilon)\|^4_{H^4_{k+l}}.\end{aligned}\qquad (5.16)$$

Case 2: $\beta_1 = \beta_2 = 0$, $k = 4$, i.e., $\Big|\int_{\mathbb{R}}\partial_\tau\partial_y^5 u^\varepsilon\cdot\partial_\tau\partial_y^4 u^\varepsilon|_{y=0}dx\Big|$.

The estimate of this term is the main obstacle. Since there is a higher order partial derivation in y on the boundary value, we use the Eq $(4.1)_1$, Corollary 5.1 and the boundary conditions to overcome this difficulty. We first get the boundary value of $\partial_y^5 u^\varepsilon|_{y=0}$ by using the Eq $(4.1)_1$,

$$\partial_y^5 u^\varepsilon = \partial_y^3\Big(\partial_t u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon + (u^s+u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon - (1+b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s\Big),$$

then using the boundary value (4.2), we can obtain

$$\begin{aligned}&\Big|\int_{\mathbb{R}}\partial_\tau\partial_y^5 u^\varepsilon\cdot\partial_\tau\partial_y^4 u^\varepsilon|_{y=0}dx\Big|\\ &= \Big|\int_{\mathbb{R}}\partial_y^3\Big(\partial_t u^\varepsilon - \varepsilon\partial_x^2 u^\varepsilon + (u^s+u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon - (1+b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s\Big)\end{aligned}$$

$$
\times\Big(\partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon+\partial_x\partial_y u^\varepsilon\partial_y u^s-\partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon\Big)|_{y=0}dx\Big|
$$

$$
=\Big|\int_{\mathbb{R}}\Big(\partial_t\partial_y^3u^\varepsilon-\varepsilon\partial_x^2\partial_y^3u^\varepsilon+(u^s+u^\varepsilon)\partial_x\partial_y^3u^\varepsilon-\partial_x\partial_y^3b^\varepsilon\Big)
\times\Big(\partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon+\partial_x\partial_y u^\varepsilon\partial_y u^s-\partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon\Big)|_{y=0}dx\Big|. \tag{5.17}
$$

From the above equality, we only establish the estimate of the term, which contains the term $\partial_x^2\partial_y^3u^\varepsilon$ by performing an integration by parts in x and using Corollary 5.1; This is the main difficulty. Its estimate shows as follows

$$
\begin{aligned}
&-\int_{\mathbb{R}}\partial_x^2\partial_y^3u^\varepsilon\Big(\partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon+\partial_x\partial_y u^\varepsilon\partial_y u^s-\partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon\Big)|_{y=0}dx\\
&=\int_{\mathbb{R}}\partial_x\partial_y^3u^\varepsilon\partial_x\Big(\partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon+\partial_x\partial_y u^\varepsilon\partial_y u^s-\partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon\Big)|_{y=0}dx\\
&\le\|\partial_x\partial_y^3u^\varepsilon_{y=0}\|\|\partial_x\Big(\partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon+\partial_x\partial_y u^\varepsilon\partial_y u^s-\partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon\Big)|_{y=0}\|\\
&\le\|\partial_x\partial_y^4u^\varepsilon\|\|\partial_x\partial_y\Big(\partial_x\partial_y u^\varepsilon\partial_y u^\varepsilon+\partial_x\partial_y u^\varepsilon\partial_y u^s-\partial_x\partial_y b^\varepsilon\partial_y b^\varepsilon\Big)\|\\
&\le\frac{1}{32}\|\partial_y u^\varepsilon\|^2_{H^4_{k+l}}+C\|u^\varepsilon\|^2_{H^4_{k+l}}+C\|(u^\varepsilon,b^\varepsilon)\|^4_{H^4_{k+l}}.
\end{aligned}\tag{5.18}
$$

Other terms are direct calculations in (5.17) by using the Hölder and Young's inequalities and Corollary 5.1. Hence, we deduce

$$
\Big|\int_{\mathbb{R}}\partial_\tau\partial_y^5u^\varepsilon\cdot\partial_\tau\partial_y^4u^\varepsilon|_{y=0}dx\Big|\le\frac{1}{8}\|\partial_y u^\varepsilon\|^2_{H^4_{k+l}}+C\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}}+C\|(u^\varepsilon,b^\varepsilon)\|^4_{H^4_{k+l}}. \tag{5.19}
$$

Inserting (5.15), (5.16) and (5.19) into (5.14), we arrive at

$$
\Big|\int_{\mathbb{R}^2_+}\partial_y^2D^\alpha u^\varepsilon\cdot\langle y\rangle^{2k+2l}D^\alpha u^\varepsilon dxdy\Big|\le-\frac{5}{8}\|\partial_y u^\varepsilon\|^2_{H^4_{k+l}}+C\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}}+C\|(u^\varepsilon,b^\varepsilon)\|^4_{H^4_{k+l}}. \tag{5.20}
$$

Similar to (5.20), we easily conclude that

$$
\Big|\int_{\mathbb{R}^2_+}\partial_y^2D^\alpha b^\varepsilon\cdot\langle y\rangle^{2k+2l}D^\alpha b^\varepsilon dxdy\Big|\le-\frac{5}{8}\|\partial_y b^\varepsilon\|^2_{H^4_{k+l}}+C\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}}+C\|(u^\varepsilon,b^\varepsilon)\|^4_{H^4_{k+l}}. \tag{5.21}
$$

Plugging (5.13), (5.20) and (5.21) into (5.3), yields (5.1). The proof is thus completed. □

Lemma 5.3. *Let $k\ge\frac{1}{2}$, $l\ge 0$ be real numbers. Assume that $(u^\varepsilon,v^\varepsilon,g^\varepsilon,b^\varepsilon)$ is a solution to the problem* (4.1) *in $[0,T]$ and satisfies $(u^\varepsilon,b^\varepsilon)\in L^\infty([0,T];H^4_{k+l})$, then there exists a positive constant C, which may be dependent on k,l such that*

$$
\sum_{|\beta|=4}\Big(\frac{d}{dt}\|\partial_\tau^\beta(u^\varepsilon,b^\varepsilon)(t)\|^2_{L^2_{k+l}}+\varepsilon\|\partial_\tau^\beta\partial_x(u^\varepsilon,b^\varepsilon)(t)\|^2_{L^2_{k+l}}+\|\partial_\tau^\beta\partial_y(u^\varepsilon,b^\varepsilon)(t)\|^2_{L^2_{k+l}}\Big)
\le C\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}}+C\|(u^\varepsilon,b^\varepsilon)\|^4_{H^4_{k+l}}+\frac{C}{\varepsilon}\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}}. \tag{5.22}
$$

Proof: Applying the operator $\partial_\tau^\beta = \partial_t^{\beta_1}\partial_x^{\beta_2}$ on $(4.1)_{1,2}$ and $|\beta| = |\beta_1| + |\beta_2| = 4$,

$$\begin{cases} \partial_t\partial_\tau^\beta u^\varepsilon - \varepsilon\partial_x^2\partial_\tau^\beta u^\varepsilon = \partial_y^2\partial_\tau^\beta u^\varepsilon - \partial_\tau^\beta\Big((u^s + u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon \\ \qquad -(1 + b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s\Big), \\ \partial_t\partial_\tau^\beta b^\varepsilon - \varepsilon\partial_x^2\partial_\tau^\beta b^\varepsilon = \partial_y^2\partial_\tau^\beta b^\varepsilon - \partial_\tau^\beta\Big((u^s + u^\varepsilon)\partial_x b^\varepsilon + v^\varepsilon\partial_y b^\varepsilon \\ \qquad -(1 + b^\varepsilon)\partial_x u^\varepsilon - g^\varepsilon\partial_y u^\varepsilon - g^\varepsilon\partial_y u^s\Big). \end{cases} \tag{5.23}$$

Multiplying the resulting Eqs $(5.23)_{1,2}$ by $\langle y\rangle^{2k+2l}\partial_\tau^\beta u^\varepsilon$ and $\langle y\rangle^{2k+2l}\partial_\tau^\beta b^\varepsilon$, respectively, and integrating it by parts over $\mathbb{R}_+^2$, we derive that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\langle y\rangle^{k+l}\partial_\tau^\beta(u^\varepsilon, b^\varepsilon)(t)\|_{L^2}^2 + \varepsilon\|\langle y\rangle^{k+l}\partial_\tau^\beta\partial_x(u^\varepsilon, b^\varepsilon)(t)\|_{L^2}^2 \\ &= \int_{\mathbb{R}_+^2} R_1 \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta u^\varepsilon dxdy + \int_{\mathbb{R}_+^2} R_2 \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta b^\varepsilon dxdy \\ &\quad + \int_{\mathbb{R}_+^2} \partial_y^2\partial_\tau^\beta u^\varepsilon \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta u^\varepsilon dxdy + \int_{\mathbb{R}_+^2} \partial_y^2\partial_\tau^\beta b^\varepsilon \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta b^\varepsilon dxdy, \end{aligned} \tag{5.24}$$

where

$$\begin{cases} R_1 = -\partial_\tau^\beta\Big((u^s + u^\varepsilon)\partial_x u^\varepsilon + v^\varepsilon\partial_y u^\varepsilon - (1 + b^\varepsilon)\partial_x b^\varepsilon - g^\varepsilon\partial_y b^\varepsilon + v^\varepsilon\partial_y u^s\Big), \\ R_2 = -\partial_\tau^\beta\Big((u^s + u^\varepsilon)\partial_x b^\varepsilon + v^\varepsilon\partial_y b^\varepsilon - (1 + b^\varepsilon)\partial_x u^\varepsilon - g^\varepsilon\partial_y u^\varepsilon - g^\varepsilon\partial_y u^s\Big). \end{cases}$$

Next, we will establish the estimates of the righthand side terms (5.24). First of all, we deal with the first second terms. By the definitions of R_1 and R_2, we have

$$\begin{cases} R_1 = -\Big((u^s + u^\varepsilon)\partial_x\partial_\tau^\beta u^\varepsilon + v^\varepsilon\partial_y\partial_\tau^\beta u^\varepsilon - (1 + b^\varepsilon)\partial_x\partial_\tau^\beta b^\varepsilon - g^\varepsilon\partial_y\partial_\tau^\beta b^\varepsilon\Big) \\ \quad -\Big([\partial_\tau^\beta, (u^s + u^\varepsilon)]\partial_x u^\varepsilon + [\partial_\tau^\beta, v^\varepsilon]\partial_y u^\varepsilon - [\partial_\tau^\beta, (1 + b^\varepsilon)]\partial_x b^\varepsilon - [\partial_\tau^\beta, g^\varepsilon]\partial_y b^\varepsilon\Big) \\ \quad +\partial_\tau^\beta(v^\varepsilon\partial_y u^s) \\ \quad \triangleq R_1^1 + R_1^2 + R_1^3, \\ R_2 = -\Big((u^s + u^\varepsilon)\partial_x\partial_\tau^\beta b^\varepsilon + v^\varepsilon\partial_y\partial_\tau^\beta b^\varepsilon - (1 + b^\varepsilon)\partial_x\partial_\tau^\beta u^\varepsilon - g^\varepsilon\partial_y\partial_\tau^\beta u^\varepsilon\Big) \\ \quad -\Big([\partial_\tau^\beta, (u^s + u^\varepsilon)]\partial_x b^\varepsilon + [\partial_\tau^\beta, v^\varepsilon]\partial_y b^\varepsilon - [\partial_\tau^\beta, (1 + b^\varepsilon)]\partial_x u^\varepsilon - [\partial_\tau^\beta, g^\varepsilon]\partial_y u^\varepsilon\Big) \\ \quad -\partial_\tau^\beta(g^\varepsilon\partial_y u^s) \\ \quad \triangleq R_2^1 + R_2^2 + R_2^3. \end{cases}$$

Hence, we can divide the term

$$\int_{\mathbb{R}_+^2} R_1 \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta u^\varepsilon dxdy + R_2 \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta b^\varepsilon dxdy$$

into the following three parts:

$$\begin{aligned} &\int_{\mathbb{R}_+^2} R_1 \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta u^\varepsilon dxdy + R_2 \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta b^\varepsilon dxdy \\ &= \sum_{i=1}^{3}\int_{\mathbb{R}_+^2} R_1^i \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta u^\varepsilon dxdy + R_2^i \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta b^\varepsilon dxdy \end{aligned} \tag{5.25}$$

$$\triangleq J_1 + J_2 + J_3,$$

and the estimates of each term J_i are as follows.
The estimate for J_1:

$$\begin{aligned}
&\int_{\mathbb{R}^2_+} R_1^1 \cdot \langle y\rangle^{2k+2l} \partial_\tau^\beta u^\varepsilon dxdy + R_2^1 \cdot \langle y\rangle^{2k+2l} \partial_\tau^\beta b^\varepsilon dxdy \\
&= 2(k+l) \int_{\mathbb{R}^2_+} \langle y\rangle^{2k+2l-1} (v^\varepsilon, g^\varepsilon)(|\partial_\tau^\beta u^\varepsilon|^2 dxdy + |\partial_\tau^\beta b^\varepsilon|^2) dxdy \qquad (5.26)\\
&\leq C\|(v^\varepsilon, g^\varepsilon)\|_{L^\infty} \|\langle y\rangle^{2k+2l-1} \partial_\tau^\beta (u^\varepsilon, b^\varepsilon)\|_{L^2}^2 \\
&\leq C\|(u^\varepsilon, b^\varepsilon)\|_{H_0^2} \|(u^\varepsilon, b^\varepsilon)\|_{H_{k+l}^m}^2,
\end{aligned}$$

where we have used integration by parts.
The estimate for J_2:

Notice that

$$J_2 \leq \|R_1^2\|_{L_{k+l}^2} \|D^\alpha u^\varepsilon\|_{L_{k+l}^2} + \|R_2^2\|_{L_{k+l}^2} \|\partial_\tau^\beta b^\varepsilon\|_{L_{k+l}^2}. \qquad (5.27)$$

Therefore, we need to establish the estimates of the terms $\|R_1^2\|_{L_{k+l}^2}$ and $\|R_2^2\|_{L_{k+l}^2}$. However, we know that the terms in $\|R_2^2\|_{L_{k+l}^2}$ are similar to the terms in $\|R_1^2\|_{L_{k+l}^2}$, so we will estimate only the L_{k+l}^2 of R_1^2.

For the commutator operator, we can rewrite it as

$$[\partial_\tau^\beta, (u^s + u^\varepsilon)]\partial_x u^\varepsilon = \sum_{\hat\beta\leq\beta,\ 1\leq\hat\beta} C_\beta^{\hat\beta} \partial^{\hat\beta}(u^s + u^\varepsilon)\partial^{\beta-\hat\beta}\partial_x u^\varepsilon,$$

then, for $|\beta| = 4$, we can obtain

$$\|[\partial_\tau^\beta, (u^s + u^\varepsilon)]\partial_x u^\varepsilon\|_{L_{k+l}^2} \leq C(\|u^\varepsilon\|_{H_{k+l}^4} + \|u^\varepsilon\|_{H_{k+l}^4}^2).$$

Note that

$$[\partial_\tau^\beta, v^\varepsilon)]\partial_y u^\varepsilon = \sum_{\hat\beta\leq\beta,\ 1\leq\hat\beta} C_\beta^{\hat\beta} \partial^{\hat\beta} v^\varepsilon \partial^{\beta-\hat\beta}\partial_y u^\varepsilon.$$

Since $1 \leq |\hat\beta|, \hat\beta \leq \beta$, we have for $|\beta - \hat\beta| \leq 3$,

$$-\partial^{\hat\beta} v^\varepsilon = \partial_\tau^{\hat\beta} \int_0^y \partial_x u^\varepsilon d\tilde{y} = \int_0^y \partial_\tau^{\hat\beta+e_2} u^\varepsilon d\tilde{y}.$$

Thus, we have

$$\|\partial_\tau^{\hat\beta+e_2} u^\varepsilon \partial^{\beta-\hat\beta}\partial_y u^\varepsilon(t,\cdot)\|_{L_{k+l}^2} \leq C\|u^\varepsilon\|_{H_{l+1}^4} \|u^\varepsilon\|_{H_0^4},$$

i.e.,

$$\|([\partial_\tau^\beta, v^\varepsilon]\partial_y u^\varepsilon)\|_{L_{k+l}^2} \leq C\|u^\varepsilon\|_{H_{k+l}^4}^2.$$

Similarly,

$$\|([\partial_\tau^\beta, (1 + b^\varepsilon)]\partial_x b^\varepsilon)\|_{L_{k+l}^2} \leq C(\|b^\varepsilon\|_{H_{k+l}^4} + \|b^\varepsilon\|_{H_{k+l}^4}^2)$$

and

$$\|([\partial_\tau^\beta, g^\varepsilon]\partial_y g^\varepsilon)\|_{L^2_{k+l}} \le C\|b^\varepsilon\|^2_{H^4_{k+l}}.$$

Combining the above inequalities, we can attain

$$\|R_1^2\|_{L^2_{k+l}} \le C(\|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}}). \tag{5.28}$$

Similar to the above estimates for L^2_{k+l} of R_1^2, we can conclude that

$$\|R_2^2\|_{L^2_{k+l}} \le C(\|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}}). \tag{5.29}$$

Inserting (5.28) and (5.29) into (5.27), we have

$$J_2 \le C(\|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|^3_{H^4_{k+l}}). \tag{5.30}$$

The estimate for J_3:

By a direct computation, we can attain that $|\beta| = 4$,

$$\begin{aligned}
\partial_\tau^\beta(v^\varepsilon u_y^s) &= \sum_{|\beta|=|\beta_1|+|\beta_2|=4} \partial_t^{\beta_1}\partial_x^{\beta_2}(v^\varepsilon u_y^s) \\
&= \sum_{|\beta_1|+|\beta_2|=4,\ \beta_1'\le\beta_1} C_k^{k'} C_{\beta_1}^{\beta_1'} \partial_t^{\beta_1'}\partial_x^{\beta_2} v^\varepsilon \partial_t^{\beta_1-\beta_1'}\partial_y u^s \\
&= \sum_{|\beta_1|+|\beta_2|=4,\ \beta_1'\le\beta_1} C_k^{k'} C_{\beta_1}^{\beta_1'} \partial_t^{\beta_1'}\partial_x^{\beta_2+1} v^\varepsilon \partial_t^{\beta_1-\beta_1'}\partial_y u^s.
\end{aligned}$$

Therefore, $|\beta| = 4$, and we can get

$$\|\partial_\tau^\beta(v^\varepsilon u_y^s)\|_{L^2_{k+l}} \le C\|\partial_x\partial_\tau^\beta u^\varepsilon\|_{L^2_{k+l}}.$$

Analogously, $|\beta| = 4$, and we can also derive

$$\|\partial_\tau^\beta(g^\varepsilon u_y^s)\|_{L^2_{k+l}} \le C\|\partial_x\partial_\tau^\beta b^\varepsilon\|_{L^2_{k+l}}.$$

Combining the above two inequalities with $|\beta| = 4$, we have

$$J_3 \le \frac{\varepsilon}{2}\|\partial_\tau^\beta(u^\varepsilon, b^\varepsilon)\|^2_{L^2_{k+l}} + \frac{C}{\varepsilon}\|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}}. \tag{5.31}$$

Hence, we infer from (5.26), (5.30) and (5.31),

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^2_+} R_1 \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta u^\varepsilon dxdy + R_2 \cdot \langle y\rangle^{2k+2l}\partial_\tau^\beta b^\varepsilon dxdy\Big| \\
&\le C(\|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|^3_{H^4_{k+l}}).
\end{aligned} \tag{5.32}$$

Now, we deal with the last two terms in (5.24). We first estimate the term $\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2l}\partial_y^2\partial_\tau^\beta u^\varepsilon\cdot\partial_\tau^\beta u^\varepsilon dxdy$. Similarly, the term $\int_{\mathbb{R}^2_+}\langle y\rangle^{2k+2l}\partial_y^2\partial_\tau^\beta b^\varepsilon\cdot\partial_\tau^\beta b^\varepsilon dxdy$ can be derived.

Integrating it by parts and using the boundary value (4.2) and the Cauchy-Schwarz inequality, we arrive at

$$\begin{aligned}
&\int_{\mathbb{R}_+^2} \partial_y^2 \partial_\tau^\beta u^\varepsilon \cdot \langle y\rangle^{2k+2l} \partial_\tau^\beta u^\varepsilon dxdy \\
&= -\|\langle y\rangle^{2(k+l)} \partial_y \partial_\tau^\beta u^\varepsilon\|_{L^2}^2 - 2(k+l) \int_{\mathbb{R}_+^2} \partial_y \partial_\tau^\beta u^\varepsilon \cdot \langle y\rangle^{2k+2l-1} \partial_\tau^\beta u^\varepsilon dxdy \\
&\quad -\|\langle y\rangle^{2(k+l)} \partial_y \partial_\tau^\beta u^\varepsilon\|_{L^2}^2 + \frac{1}{2}\|\partial_y u^\varepsilon\|_{H_{k+l}^4}^2 + C\|u^\varepsilon\|_{H_{k+l}^4}^2 \\
&\leq -\frac{1}{2}\|\partial_\tau^\beta \partial_y u^\varepsilon\|_{L_{k+l}^2}^2 + C\|u^\varepsilon\|_{H_{k+l}^4}^2 .
\end{aligned} \tag{5.33}$$

Similarly, we can deduce that

$$\int_{\mathbb{R}_+^2} \partial_y^2 \partial_\tau^\beta b^\varepsilon \cdot \langle y\rangle^{2k+2l} \partial_\tau^\beta b^\varepsilon dxdy \leq -\frac{1}{2}\|\partial_\tau^\beta \partial_y b^\varepsilon\|_{L_{k+l}^2}^2 + C\|b^\varepsilon\|_{H_{k+l}^4}^2 . \tag{5.34}$$

Plugging (5.32)–(5.34) into (5.24), we can conclude the desired result. The proof is completed.

5.2. *Weighted H_{k+l}^4 only in tangential derivatives*

To investigate the existence of solution to problem (4.1), we encounter some difficulties. Similar to the Prandtl equation, the difficulty of solving problem (4.1) in the Sobolev framework is the loss of the x-derivative in the terms $v^\varepsilon \partial_y u^\varepsilon - g^\varepsilon \partial_y b^\varepsilon$ and $v^\varepsilon \partial_y b^\varepsilon - g^\varepsilon \partial_y u^\varepsilon$ in the first and second equations of (4.1), respectively. In other words, $v^\varepsilon = -\partial_y^{-1} \partial_x u^\varepsilon$ and $g^\varepsilon = -\partial_y^{-1} \partial_x b^\varepsilon$, by the divergence-free conditions and boundary conditions. Thus, it creates a loss of the x-derivative and a y-integration to the y-variable, then the standard energy estimates do not work. To overcome this essential difficulty, inspired by recent results in [7, 14], we only need the background tangential magnetic field $(1+b) \geq \delta,\ \delta > 0$ to have a lower positive bound instead of Oleinik's monotonicity assumption on the tangential velocity.

We now apply the differential operator $\partial_\tau^\beta (|\beta| = 4)$ to the first two equations of (4.1). We have that

$$\begin{aligned}
&(\partial_t - \partial_y^2 - \varepsilon\partial_x^2 + (u^s + u^\varepsilon)\partial_x + v^\varepsilon \partial_y) \partial_\tau^\beta u^\varepsilon + \partial_\tau^\beta v \partial_y u^\varepsilon \\
&\quad -(1+b^\varepsilon)\partial_x \partial_\tau^\beta b^\varepsilon - \partial_\tau^\beta g^\varepsilon \partial_y b^\varepsilon + \partial_\tau^\beta v^\varepsilon \partial_y u^s = r_{u^\varepsilon}
\end{aligned} \tag{5.35}$$

and

$$\begin{aligned}
&(\partial_t - \partial_y^2 - \varepsilon\partial_x^2 + (u^s + u^\varepsilon)\partial_x + v^\varepsilon \partial_y) \partial_\tau^\beta b^\varepsilon + \partial_\tau^\beta v \partial_y b^\varepsilon \\
&\quad -(1+b^\varepsilon)\partial_x \partial_\tau^\beta u^\varepsilon - \partial_\tau^\beta g^\varepsilon \partial_y u^\varepsilon - \partial_\tau^\beta g^\varepsilon \partial_y u^s = r_{b^\varepsilon},
\end{aligned} \tag{5.36}$$

where

$$\begin{aligned}
r_{u^\varepsilon} = &- \sum_{0<\tilde{\beta}<\beta} \partial_\tau^{\tilde{\beta}} v^\varepsilon \partial_\tau^{\beta-\tilde{\beta}} \partial_y u^\varepsilon - [\partial_\tau^\beta, (u^s + u^\varepsilon)\partial_x] u^\varepsilon + [\partial_\tau^\beta, (1+b^\varepsilon)\partial_x] b^\varepsilon \\
&+ [\partial_\tau^\beta, \partial_y b^\varepsilon] g^\varepsilon - [\partial_\tau^\beta, \partial_y u^s] v^\varepsilon
\end{aligned} \tag{5.37}$$

and

$$r_{b^\varepsilon} = - \sum_{0<\tilde{\beta}<\beta} \partial_\tau^{\tilde{\beta}} v^\varepsilon \partial_\tau^{\beta-\tilde{\beta}} \partial_y b^\varepsilon - [\partial_\tau^\beta, (u^s + u^\varepsilon)\partial_x] b^\varepsilon + [\partial_\tau^\beta, (1+b^\varepsilon)\partial_x] u^\varepsilon$$

$$+[\partial_\tau^\beta, \partial_y u^\varepsilon]g^\varepsilon + [\partial_\tau^\beta, \partial_y u^s]v^\varepsilon. \tag{5.38}$$

Exploiting the expression (5.37) and the commutator operator, the L^2_{k+l}-estimates of each terms in (5.37) can be controlled, then we can conclude the estimates of $\|r_{u^\varepsilon}\|_{L^2_{k+l}}$ and $\|r_{b^\varepsilon}\|_{L^2_{k+l}}$. We establish the estimate of the term $\|r_{\tilde{u}^\varepsilon}\|_{L^2_{k+l}}$ by using the inequalities (2.2) and (2.3), and we derive

$$\|r_{u^\varepsilon}\|_{L^2_{k+l}} \leq C(\|u^\varepsilon\|_{H^4_{k+l}}\|b^\varepsilon\|_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}}). \tag{5.39}$$

The term $\|r_{b^\varepsilon}\|_{L^2_{k+l}}$ can be estimated similarly:

$$\|r_{b^\varepsilon}\|_{L^2_{k+l}} \leq C(\|u^\varepsilon\|_{H^4_{k+l}}\|b^\varepsilon\|_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}}). \tag{5.40}$$

Next, we consider the Eqs (5.35) and (5.36). It is obvious that the major difficulty derives from the terms

$$\partial_\tau^\beta v^\varepsilon \partial_y u^\varepsilon + \partial_\tau^\beta v^\varepsilon \partial_y u^s - \partial_\tau^\beta g^\varepsilon \partial_y b^\varepsilon = -(\partial_y u^\varepsilon + \partial_y u^s)(\partial_y^{-1}\partial_\tau^{\beta+e_2}u^\varepsilon) + \partial_y^{-1}\partial_\tau^{\beta+e_2}b^\varepsilon \partial_y b^\varepsilon \tag{5.41}$$

and

$$\partial_\tau^\beta v^\varepsilon \partial_y b^\varepsilon - (\partial_\tau^\beta g^\varepsilon \partial_y u^\varepsilon + \partial_\tau^\beta g^\varepsilon \partial_y u^s) = -(\partial_y^{-1}\partial_\tau^{\beta+e_2}u^\varepsilon)\partial_y b^\varepsilon + (\partial_y u^\varepsilon + \partial_y u^s)(\partial_y^{-1}\partial_\tau^{\beta+e_2}b^\varepsilon), \tag{5.42}$$

which imply the 5^{th}-order tangential derivatives, and they cannot be controlled by the standard energy estimates.

To overcome this difficulty, inspired by recent results of [14], we depend on the following two main observations. One is that from the divergence-free condition $\partial_x b^\varepsilon + \partial_y g^\varepsilon = 0$, we give a stream funtion ψ^ε satisfing

$$\partial_y \psi^\varepsilon = b^\varepsilon, \;\; \partial_x \psi^\varepsilon = -g^\varepsilon, \;\; \psi^\varepsilon|_{y=0} = 0, \tag{5.43}$$

then, using the Eq $(4.1)_2$, we can derive

$$(\partial_t - \partial_y^2 - \varepsilon\partial_x^2 + (u^s + u^\varepsilon)\partial_x + v^\varepsilon\partial_y)\psi^\varepsilon + v^\varepsilon = 0.$$

We apply the differential operator ∂_τ^β to the above equation as follows

$$(\partial_t - \partial_y^2 - \varepsilon\partial_x^2 + (u^s + u^\varepsilon)\partial_x + v^\varepsilon\partial_y)\partial_\tau^\beta\psi^\varepsilon + \partial_\tau^\beta v^\varepsilon(1 + b^\varepsilon) = r_{\psi^\varepsilon}, \tag{5.44}$$

where

$$r_{\psi^\varepsilon} = -[\partial_\tau^\beta, (u^s + u^\varepsilon)\partial_x]\psi^\varepsilon - \sum_{1\leq|\tilde{\beta}|<|\beta|\leq 3} \partial_\tau^{\beta-\tilde{\beta}} v^\varepsilon \partial_\tau^{\tilde{\beta}}\partial_y\psi^\varepsilon.$$

At the moment, by using the inequalities (2.2) and (2.3) and the commutator operator, we can conclude the following estimate that $|\beta| < 4$,

$$\|\langle y\rangle^{k+l} r_{\psi^\varepsilon}\|_{L^2} \leq \|\langle y\rangle^{k+l}[\partial_\tau^\beta, (u^s + u^\varepsilon)\partial_x]\psi^\varepsilon\|_{L^2} + \|\langle y\rangle^{k+l}\sum_{\tilde{\beta}<\beta}\partial_\tau^{\beta-\tilde{\beta}}v^\varepsilon\partial_\tau^{\tilde{\beta}}\partial_y\psi^\varepsilon\|_{L^2}$$

$$
\begin{aligned}
&\le \|\langle y\rangle^{k+l}\sum_{\tilde\beta<\beta}\partial_\tau^{\beta-\tilde\beta}(u^s+u^\varepsilon)\partial_\tau^{\tilde\beta}\partial_x\psi^\varepsilon\|_{L^2}+\|\langle y\rangle^{k+l}\sum_{\tilde\beta<\beta}\partial_\tau^{\beta-\tilde\beta}v^\varepsilon\partial_\tau^{\tilde\beta}\partial_y\psi^\varepsilon\|_{L^2}\\
&\le C(1+\|u^\varepsilon\|_{H_0^2})\|g^\varepsilon\|_{H_{k+l}^3}+\|u^\varepsilon\|_{H_0^2}\|b^\varepsilon\|_{H_{k+l}^3}\\
&\le C(1+\|u^\varepsilon\|_{H_0^2})\|b^\varepsilon\|_{H_{k+l}^4}.
\end{aligned}\tag{5.45}
$$

We set $\xi_{u^\varepsilon}=\frac{\partial_y u^\varepsilon+\partial_y u^s}{1+b^\varepsilon}$ and $\xi_{b^\varepsilon}=\frac{\partial_y b^\varepsilon}{1+b^\varepsilon}$ and introduce the following two new unknown functions

$$
u_\beta^\varepsilon:=\partial_\tau^\beta u^\varepsilon-\frac{\partial_y u^\varepsilon+\partial_y u^s}{1+b^\varepsilon}\partial_\tau^m\psi^\varepsilon,\quad b_\beta^\varepsilon:=\partial_\tau^\beta b^\varepsilon-\frac{\partial_y b^\varepsilon}{1+b^\varepsilon}\partial_\tau^\beta\psi^\varepsilon.\tag{5.46}
$$

On the other hand, we use the above two given unknown functions $(u_\beta^\varepsilon,b_\beta^\varepsilon)$ in (5.46) to deal with the loss regularity of $g^\varepsilon=-\partial_y^{-1}\partial_x b^\varepsilon$ by using the convection terms $-(1+b^\varepsilon)\partial_x b^\varepsilon$ and $-(1+b^\varepsilon)\partial_x u^\varepsilon$. More specifically,

$$
\begin{aligned}
&-(1+b^\varepsilon)\partial_x\partial_\tau^\beta b^\varepsilon-\partial_\tau^\beta g^\varepsilon\partial_y b^\varepsilon\\
&=-(1+b^\varepsilon)\partial_x\Big(b_\beta^\varepsilon+\frac{\partial_y b^\varepsilon}{1+b^\varepsilon}\partial_\tau^\beta\psi^\varepsilon\Big)-\partial_\tau^\beta g^\varepsilon\partial_y b^\varepsilon\\
&=-(1+\tilde b^\varepsilon)\partial_x b_\beta^\varepsilon-(1+\tilde b^\varepsilon)\partial_x\xi_{b^\varepsilon}\partial_\tau^\beta\psi^\varepsilon,
\end{aligned}\tag{5.47}
$$

and

$$
\begin{aligned}
&-(1+b^\varepsilon)\partial_x\partial_\tau^\beta u^\varepsilon-\partial_\tau^\beta g^\varepsilon\partial_y u^\varepsilon-\partial_\tau^\beta g^\varepsilon\partial_y u^s\\
&=-(1+b^\varepsilon)\partial_x\Big(u_\beta^\varepsilon+\frac{\partial_y u^\varepsilon+\partial_y u^s}{1+b^\varepsilon}\partial_\tau^\beta\psi^\varepsilon\Big)-\partial_\tau^\beta g^\varepsilon\partial_y u^\varepsilon-\partial_\tau^\beta g^\varepsilon\partial_y u^s\\
&=-(1+b^\varepsilon)\partial_x u_\beta^\varepsilon-(1+b^\varepsilon)\partial_x\xi_{u^\varepsilon}\partial_\tau^\beta\psi^\varepsilon,
\end{aligned}\tag{5.48}
$$

which combined with (5.35) and (5.36), we can derive the following equations of $(u_\beta^\varepsilon,b_\beta^\varepsilon)$

$$
\begin{cases}
(\partial_t-\partial_y^2-\varepsilon\partial_x^2+(u^s+u^\varepsilon)\partial_x+v^\varepsilon\partial_y)u_\beta^\varepsilon-(1+b^\varepsilon)\partial_x b_\beta^\varepsilon=r_{\psi^\varepsilon,u^\varepsilon},\\
(\partial_t-\partial_y^2-\varepsilon\partial_x^2+(u^s+u^\varepsilon)\partial_x+v^\varepsilon\partial_y)b_\beta^\varepsilon-(1+b^\varepsilon)\partial_x u_\beta^\varepsilon=r_{\psi^\varepsilon,b^\varepsilon},
\end{cases}\tag{5.49}
$$

where

$$
\begin{cases}
r_{\psi^\varepsilon,u^\varepsilon}=r_{u^\varepsilon}-\xi_{u^\varepsilon}r_{\psi^\varepsilon}-\partial_\tau^\beta\psi^\varepsilon(\partial_t-\partial_y^2-\varepsilon\partial_x^2+(u^s+u^\varepsilon)\partial_x+v^\varepsilon\partial_y)\xi_{u^\varepsilon}\\
+2\varepsilon\partial_x\xi_{u^\varepsilon}\partial_x\partial_\tau^\beta\psi^\varepsilon+2\varepsilon\partial_y\xi_{u^\varepsilon}\partial_y\partial_\tau^\beta\psi^\varepsilon+(1+b^\varepsilon)\partial_x\xi_{b^\varepsilon}\partial_\tau^\beta\psi^\varepsilon,\\
r_{\psi^\varepsilon,b^\varepsilon}=r_{b^\varepsilon}-\xi_{b^\varepsilon}r_{\psi^\varepsilon}-\partial_\tau^\beta\psi^\varepsilon(\partial_t-\partial_y^2-\varepsilon\partial_x^2+(u^s+u^\varepsilon)\partial_x+v^\varepsilon\partial_y)\xi_{b^\varepsilon}\\
+2\varepsilon\partial_x\xi_{b^\varepsilon}\partial_x\partial_\tau^\beta\psi^\varepsilon+2\varepsilon\partial_y\xi_{b^\varepsilon}\partial_y\partial_\tau^\beta\psi^\varepsilon+(1+b^\varepsilon)\partial_x\xi_{u^\varepsilon}\partial_\tau^\beta\psi^\varepsilon.
\end{cases}\tag{5.50}
$$

We can also get the following initial and boundary conditions

$$
\begin{cases}
u_\beta^\varepsilon|_{t=0}=\partial_\tau^\beta u^\varepsilon(0,x,y)-\frac{\partial_y u^\varepsilon(0,x,y)+\partial_y u^s(0,x,y)}{1+b^\varepsilon(0,x,y)}\int_0^y\partial_\tau^\beta b^\varepsilon(0,x,\tilde y)d\tilde y\triangleq u_{\beta0}^\varepsilon,\\
b_\beta^\varepsilon|_{t=0}=\partial_\tau^\beta b^\varepsilon(0,x,y)-\frac{\partial_y b^\varepsilon(0,x,y)}{1+b^\varepsilon(0,x,y)}\int_0^y\partial_\tau^\beta b^\varepsilon(0,x,\tilde y)d\tilde y\triangleq b_{\beta0}^\varepsilon,\\
u_\beta^\varepsilon|_{y=0}=0,\quad b_\beta^\varepsilon|_{y=0}=0.
\end{cases}\tag{5.51}
$$

Finally, we derive the initial boundary value problem for $(u_\beta^\varepsilon,b_\beta^\varepsilon)$ as follows

$$
\begin{cases}
(\partial_t-\partial_y^2-\varepsilon\partial_x^2+(u^s+u^\varepsilon)\partial_x+v^\varepsilon\partial_y)u_\beta^\varepsilon=r_{\psi^\varepsilon,u^\varepsilon},\\
(\partial_t-\partial_y^2-\varepsilon\partial_x^2+(u^s+u^\varepsilon)\partial_x+v^\varepsilon\partial_y)b_\beta^\varepsilon=r_{\psi^\varepsilon,b^\varepsilon},\\
(u_\beta^\varepsilon,b_\beta^\varepsilon)|_{y=0}=0,\quad(u_\beta^\varepsilon,b_\beta^\varepsilon)|_{t=0}=(b_{\beta0}^\varepsilon,b_{\beta0}^\varepsilon),
\end{cases}\tag{5.52}
$$

where the initial date $(b^\varepsilon_{\beta 0}, b^\varepsilon_{\beta 0})$ and $(r_{\psi^\varepsilon,u^\varepsilon}, r_{\psi^\varepsilon,b^\varepsilon})$ are given by (5.50) and (5.51), respectively.

Moreover, since $\psi = \partial_y^{-1} b^\varepsilon$, $\psi|_{y=0} = 0$, we deduce

$$\|\partial_\tau^\beta \psi^\varepsilon(t)\|_{L^2(\mathbb{R};L^\infty(\mathbb{R}_+))} \le C\|\partial_\tau^\beta b^\varepsilon(t)\|_{L^2}. \tag{5.53}$$

According to the expressions of ξ_{u^ε} and ξ_{b^ε} and the Sobolev embedding inequality, we derive that for $1 \le \lambda < k$,

$$\begin{aligned}
&\|\langle y\rangle^\lambda \xi_{u^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)} = \|\langle y\rangle^\lambda \frac{\partial_y u^\varepsilon + \partial_y u^s}{1+b^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)}\\
&\le \|\langle y\rangle^\lambda \frac{\partial_y u^\varepsilon}{1+b^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)} + \frac{\|\langle y\rangle^\lambda \partial_y u^s}{1+b^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)}\\
&\le C\delta^{-1}(\|\langle y\rangle^\lambda \partial_y u^\varepsilon\|_{H^2_0} + \|\langle y\rangle^{\lambda-k}\|_{L^\infty(\mathbb{R}^2_+)})\\
&\le C\delta^{-1}(\|u^\varepsilon\|_{H^3_{\lambda-1}} + 1).
\end{aligned} \tag{5.54}$$

Analogously, we also have

$$\begin{cases}
\|\langle y\rangle^\lambda \xi_{b^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)} \le C\delta^{-1}\|b^\varepsilon\|_{H^3_{\lambda-1}},\\
\|\langle y\rangle^\lambda \partial_x \xi_{u^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)} \le C\delta^{-1}\|u^\varepsilon\|_{H^4_{\lambda-1}} + C\delta^{-2}(1+\|u^\varepsilon\|_{H^4_{\lambda-1}})\|b^\varepsilon\|_{H^4_0},\\
\|\langle y\rangle^\lambda \partial_y \xi_{u^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)} \le C\delta^{-1}(1+\|u^\varepsilon\|_{H^4_{\lambda-1}}) + C\delta^{-2}(1+\|u^\varepsilon\|_{H^4_{\lambda-1}})\|b^\varepsilon\|_{H^4_0},\\
\|\partial_t \xi_{u^\varepsilon}\|_{L^\infty(\mathbb{R})} \le C\delta^{-1}(1+\|u^\varepsilon\|_{H^3_0}) + C\delta^{-2}(1+\|u^\varepsilon\|_{H^3_0})\|b^\varepsilon\|_{H^2_0},\\
\|\partial_x^2 \xi_{u^\varepsilon}\|_{L^\infty(\mathbb{R})} \le C\delta^{-1}\|u^\varepsilon\|_{H^4_0} + C\delta^{-2}(1+\|u^\varepsilon\|_{H^3_0})\|b^\varepsilon\|_{H^2_0}\\
\qquad\qquad + C\delta^{-3}(1+\|u^\varepsilon\|_{H^2_0})\|b^\varepsilon\|^2_{H^2_0},\\
\|\partial_y^2 \xi_{u^\varepsilon}\|_{L^\infty(\mathbb{R})} \le C\delta^{-1}\|u^\varepsilon\|_{H^4_0} + C\delta^{-2}(1+\|u^\varepsilon\|_{H^3_{\lambda-1}})\|b^\varepsilon\|_{H^3_0}\\
\qquad\qquad + C\delta^{-3}(1+\|u^\varepsilon\|_{H^2_0})\|b^\varepsilon\|^2_{H^2_0},\\
\|\langle y\rangle^\lambda \partial_x \xi_{b^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)} \le C\delta^{-1}\|b^\varepsilon\|_{H^4_{\lambda-1}} + C\delta^{-2}\|b^\varepsilon\|^2_{H^4_{\lambda-1}},\\
\|\langle y\rangle^\lambda \partial_y \xi_{b^\varepsilon}\|_{L^\infty(\mathbb{R}^2_+)} \le C\delta^{-1}\|b^\varepsilon\|_{H^4_{\lambda-1}} + C\delta^{-2}\|b^\varepsilon\|^2_{H^4_{\lambda-1}},\\
\|\partial_t \xi_{b^\varepsilon}\|_{L^\infty(\mathbb{R})} \le C\delta^{-1}\|b^\varepsilon\|_{H^3_0} + C\delta^{-2}\|b^\varepsilon\|^2_{H^2_0},\\
\|\partial_x^2 \xi_{b^\varepsilon}\|_{L^\infty(\mathbb{R})} \le C\delta^{-1}\|b^\varepsilon\|_{H^4_0} + C\delta^{-2}\|b^\varepsilon\|^2_{H^3_0} + C\delta^{-3}\|b^\varepsilon\|^3_{H^2_0},\\
\|\partial_y^2 \xi_{b^\varepsilon}\|_{L^\infty(\mathbb{R})} \le C\delta^{-1}\|b^\varepsilon\|_{H^4_0} + C\delta^{-2}\|b^\varepsilon\|^2_{H^3_0} + C\delta^{-3}\|b^\varepsilon\|^3_{H^2_0},
\end{cases} \tag{5.55}$$

which combined with (5.39),(5.40), (5.45), (5.53), and (5.54), we can infer that for $|\beta| = 4$,

$$\begin{aligned}
&\|r_{\psi^\varepsilon,u^\varepsilon}\|_{L^2_{k+l}}\\
&\le \|r_{u^\varepsilon}\|_{L^2_{k+l}} + \|\xi_{u^\varepsilon} r_{\psi^\varepsilon}\|_{L^2_{k+l}} + \|\partial_\tau^\beta \psi^\varepsilon(\partial_t - \partial_y^2 - \varepsilon\partial_x^2 + (u^s+u^\varepsilon)\partial_x + v^\varepsilon\partial_y)\xi_{u^\varepsilon}\|_{L^2_{k+l}}\\
&\qquad + 2\varepsilon\|\partial_x \xi_{u^\varepsilon}\partial_x\partial_\tau^\beta \psi^\varepsilon\|_{L^2_{k+l}} + 2\varepsilon\|\partial_y \xi_{u^\varepsilon}\partial_y\partial_\tau^\beta \psi^\varepsilon\|_{L^2_{k+l}} + \|(1+b^\varepsilon)\partial_x \xi_{b^\varepsilon}\partial_\tau^\beta \psi^\varepsilon\|_{L^2_{k+l}}\\
&\le C(\|u^\varepsilon\|_{H^4_{k+l}}\|b^\varepsilon\|_{H^4_{k+l}} + \|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}})\\
&\qquad + C\|\partial_\tau^\beta b^\varepsilon\|_{L^2_{k+l}}\Big(\delta^{-1}\|u^\varepsilon\|_{H^4_{k+l}} + \delta^{-2}(1+\|u^\varepsilon\|_{H^4_{k+l}})\|b^\varepsilon\|_{H^4_{k+l}}\\
&\qquad + \delta^{-3}(1+\|u^\varepsilon\|_{H^4_{k+l}})\|b^\varepsilon\|^2_{H^4_{k+l}} + \delta^{-1}\|u^\varepsilon\|^2_{H^4_{k+l}} + \delta^{-2}(1+\|u^\varepsilon\|_{H^4_{k+l}})^2\|b^\varepsilon\|_{H^4_{k+l}}\Big)\\
&\qquad + C\varepsilon\|\partial_x\partial_\tau^\beta b^\varepsilon\|_{L^2_{k+l}}\Big(\delta^{-1}\|u^\varepsilon\|_{H^4_{k+l}} + \delta^{-2}(1+\|u^\varepsilon\|_{H^4_{k+l}})\|b^\varepsilon\|_{H^4_{k+l}}\Big)
\end{aligned}$$

$$
\begin{aligned}
&\quad +C(1+\|b^\varepsilon\|_{H^4_{k+l}})(\delta^{-1}\|b^\varepsilon\|_{H^3_0}+\delta^{-2}\|b^\varepsilon\|^2_{H^2_0})\|\partial_\tau^\beta b^\varepsilon\|_{L^2_{k+l}}\\
&\quad +(\|u^\varepsilon\|_{H^3_{\lambda-1}}+1)(1+\|u^\varepsilon\|_{H^2_0})\|b^\varepsilon\|_{H^4_{k+l}}\\
&\le C\varepsilon\|\partial_x\partial_\tau^\beta b^\varepsilon\|_{L^2_{k+l}}(1+\|(u^\varepsilon,b^\varepsilon)\|_{H^4_{k+l}})\|(u^\varepsilon,b^\varepsilon)\|_{H^4_{k+l}}\\
&\quad +C\delta^{-3}(1+\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}})\|(u^\varepsilon,b^\varepsilon)\|_{H^4_{k+l}}. \qquad (5.56)
\end{aligned}
$$

Similarly, we can derive that

$$
\begin{aligned}
\|r_{\psi^\varepsilon,b^\varepsilon}\|_{L^2_{k+l}} &\le C\delta^{-3}(1+\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}})\|(u^\varepsilon,b^\varepsilon)\|_{H^4_{k+l}}\\
&\quad +C\varepsilon\|\partial_x\partial_\tau^\beta b^\varepsilon\|_{L^2_{k+l}}(1+\|(u^\varepsilon,b^\varepsilon)\|_{H^4_{k+l}})\|(u^\varepsilon,b^\varepsilon)\|_{H^4_{k+l}}. \qquad (5.57)
\end{aligned}
$$

Now, we will conclude the following L^2_{k+1}-norms of $(u^\varepsilon_\beta, b^\varepsilon_\beta)$.

Lemma 5.4. *We have the following estimate of* $(u^\varepsilon_\beta, b^\varepsilon_\beta)$ *given in* (5.46),

$$
\begin{aligned}
&\sum_{|\beta|=4}\Big(\frac{d}{dt}\|(u^\varepsilon_\beta,b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}+\varepsilon\|\partial_x(u^\varepsilon_\beta,b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}+\|\partial_y(u^\varepsilon_\beta,b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}\Big)\\
&\le C\delta^{-6}(1+\|(u^\varepsilon,b^\varepsilon)\|^2_{H^4_{k+l}})^2+C\delta^{-1}(1+\|(u^\varepsilon,b^\varepsilon)\|_{H^4_{k+l}})^4\sum_{|\beta|=4}\|(u^\varepsilon_\beta,b^\varepsilon_\beta)\|^2_{L^2_{k+l}}\\
&\quad +\frac{\varepsilon\delta}{4}\sum_{|\beta|=4}\|\partial_x\partial_\tau^\beta b^\varepsilon\|^2_{L^2_{k+l}}. \qquad (5.58)
\end{aligned}
$$

Proof: Similar to the proof of Lemma 5.3, multiplying $(5.49)_{1,2}$ by $\langle y\rangle^{2(k+1)}u^\varepsilon_\beta$ and $\langle y\rangle^{2(k+1)}b^\varepsilon_\beta$ respectively, and integrating it by parts over Q_T, we deduce that

$$
\begin{aligned}
&\sum_{|\beta|=4}\Big(\frac{1}{2}\frac{d}{dt}\|(u^\varepsilon_\beta,b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}+\varepsilon\|\partial_x(u^\varepsilon_\beta,b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}+\|\partial_y(u^\varepsilon_\beta,b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}\Big)\\
&=\sum_{|\beta|=4}\Big(2(k+l)\int_{\mathbb{R}^2_+}\langle y\rangle^{2(k+l)-1}v^\varepsilon(|u^\varepsilon_\beta|^2+|b^\varepsilon_\beta|^2)dxdy-\int_{\mathbb{R}^2_+}\langle y\rangle^{2(k+l)}b^\varepsilon(u^\varepsilon_\beta b^\varepsilon_\beta)dxdy\\
&\quad +2(k+l)\int_{\mathbb{R}^2_+}\langle y\rangle^{2(k+l)-1}(\partial_y u^\varepsilon_\beta u^\varepsilon_\beta+\partial_y b^\varepsilon_\beta b^\varepsilon_\beta)dxdy\\
&\quad +\int_{\mathbb{R}^2_+}\langle y\rangle^{2(k+l)}(r_{\psi^\varepsilon,u^\varepsilon}u^\varepsilon_\beta+r_{\psi^\varepsilon,b^\varepsilon}b^\varepsilon_\beta)dxdy\Big)=\sum_{i=1}^4\tilde{J}_i, \qquad (5.59)
\end{aligned}
$$

where we have used the boundary conditions $(5.52)_3$ and $(v^\varepsilon, b^\varepsilon)|_{y=0}=0$.

Next, we deal with the estimates of the righthand side terms of (5.59) as follows. By the Sobolev inequality, we first establish the estimate of term $\tilde{J}_1$,

$$
|\tilde{J}_1|\le\|v^\varepsilon\|\|(u^\varepsilon_\beta,b^\varepsilon_\beta)\|^2_{L^2_{k+l}}\le C\|u^\varepsilon\|_{H^2_0}\|(u^\varepsilon_\beta,b^\varepsilon_\beta)\|^2_{L^2_{k+l}}. \qquad (5.60)
$$

Similar to (5.60), exploiting the Sobolev inequality and Young inequality, we can obtain

$$
|\tilde{J}_2|\le C\|b^\varepsilon\|_{H^2_0}\|(u^\varepsilon_\beta,b^\varepsilon_\beta)\|^2_{L^2_{k+l}} \qquad (5.61)
$$

and

$$|\tilde{J}_3| \le \frac{1}{2}\|\partial_y(u^\varepsilon_\beta, b^\varepsilon_\beta)\|^2_{L^2_{k+l}} + C\|(u^\varepsilon_\beta, b^\varepsilon_\beta)\|^2_{L^2_{k+l}}. \tag{5.62}$$

For $\tilde{J}_4$, using (5.56), (5.57), and Young inequality, we infer

$$\begin{aligned}
|\tilde{J}_4| &\le \|r_{\psi^\varepsilon,u^\varepsilon}\|_{L^2_{k+l}}\|u^\varepsilon_\beta\|^2_{L^2_{k+l}} + \|r_{\psi^\varepsilon,b^\varepsilon}\|_{L^2_{k+l}}\|b^\varepsilon_\beta\|^2_{L^2_{k+l}} \\
&\le C\Big(\delta^{-3}(1 + \|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}})\|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}} \\
&\quad + C\varepsilon\|\partial_x\partial^\beta_\tau b^\varepsilon\|_{L^2_{k+l}}(1 + \|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}})\|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}}\Big)\|(u^\varepsilon_\beta, b^\varepsilon_\beta)\|_{L^2_{k+l}} \\
&\le C\delta^{-6}(1 + \|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}})^2 + \delta^{-1}(1 + \|(u^\varepsilon, b^\varepsilon)\|_{H^4_{k+l}})^4\|(u^\varepsilon_\beta, b^\varepsilon_\beta)\|^2_{L^2_{k+l}} \\
&\quad + \frac{\varepsilon\delta}{4}\|\partial_x\partial^\beta_\tau b^\varepsilon\|^2_{L^2_{k+l}}.
\end{aligned} \tag{5.63}$$

Substituting (5.60)–(5.63) into (5.59) and taking the summation over all $|\beta| = 4$ in (5.59), we can derive the desired result (5.58). The proof is therefore completed.

Up to now, we have completed the main estimates of the solutions $(u^\varepsilon, b^\varepsilon)$ for (4.1). However, Lemma 5.4 gives the estimates of $(u^\varepsilon_\beta, b^\varepsilon_\beta)$; Thus, we need to show the equivalence in the L^2_{k+1}-norm between $(\partial^\beta_\tau u^\varepsilon, \partial^\beta_\tau b^\varepsilon)$ and $(u^\varepsilon_\beta, b^\varepsilon_\beta)$ given by (5.46).

Lemma 5.5. *If the smooth function* $(u^\varepsilon, b^\varepsilon)$ *satisfies the problem* (4.1) *in* $[0, T]$ *and the tangential magnetic field has a lower positive bound , then for* $\forall t \in [0, T]$, $k \ge 1$, $l \ge 0$, *and the equality* $(u^\varepsilon_\beta, b^\varepsilon_\beta)$ *with* $|\beta| = 4$ *defined by* (5.46), *we conclude*

$$\gamma(t)^{-1}\|(\partial^\beta_\tau u^\varepsilon, \partial^\beta_\tau b^\varepsilon)\|_{L^2_{k+l}} \le \|(u^\varepsilon_\beta, b^\varepsilon_\beta)\|_{L^2_{k+l}} \le \gamma(t)\|(\partial^\beta_\tau u^\varepsilon, \partial^\beta_\tau b^\varepsilon)\|_{L^2_{k+l}} \tag{5.64}$$

and

$$\begin{aligned}
\|\partial_x\partial^\beta_\tau b^\varepsilon\|_{L^2_{k+l}} \le{}& \|\partial_x b^\varepsilon_\beta\|_{L^2_{k+l}} + \|\partial_y b^\varepsilon\|_{L^\infty_{k+l+1}}\|\partial_x b^\varepsilon_\beta\|_{L^2_{k+l}} \\
&+ \|b^\varepsilon\|_{H^4_{k+l}}\|b^\varepsilon_\beta\|_{L^2_{k+l}} + \|\partial_y b^\varepsilon\|_{L^\infty_{k+l+1}}\|b^\varepsilon\|_{H^4_{k+l}}\|b^\varepsilon_\beta\|_{L^2_{k+l}},
\end{aligned} \tag{5.65}$$

where

$$\gamma(t) = \delta^{-1}(1 + \|\partial_y(u^\varepsilon, b^\varepsilon)\|_{L^\infty_{k+l+1}(\mathbb{R}^2_+)}). \tag{5.66}$$

Proof: According to the definitions of u^ε_β and b^ε_β in (5.46) and using the equalities (2.2) and (2.3), we can derive from (5.53),

$$\begin{aligned}
\|u^\varepsilon_\beta\|_{L^2_{k+l}} &\le \|\partial^\beta_\tau u^\varepsilon\|_{L^2_{k+l}} + \|\xi_{u^\varepsilon}\|_{L^\infty_{k+l+1}(\mathbb{R}^2_+)}\|\langle y\rangle^{-1}\partial^\beta_\tau\psi^\varepsilon\|_{L^2} \\
&\le \|\partial^\beta_\tau u^\varepsilon\|_{L^2_{k+l}} + \delta^{-1}(1 + \|\partial_y u^\varepsilon\|_{L^\infty_{k+l+1}(\mathbb{R}^2_+)})\|\partial^\beta_\tau b^\varepsilon\|_{L^2}
\end{aligned} \tag{5.67}$$

and

$$\|b^\varepsilon_\beta\|_{L^2_{k+l}} \le \|\partial^\beta_\tau b^\varepsilon\|_{L^2_{k+l}} + \|\xi_{b^\varepsilon}\|_{L^\infty_{k+l+1}(\mathbb{R}^2_+)}\|\langle y\rangle^{-1}\partial^\beta_\tau\psi^\varepsilon\|_{L^2(\mathbb{R})}$$

$$\leq \quad \|\partial_\tau^\beta b^\varepsilon\|_{L^2_{k+l}} + \delta^{-1}\|\partial_y b^\varepsilon\|_{L^\infty_{k+l+1}(\mathbb{R}^2_+)}\|\partial_\tau^\beta b^\varepsilon\|_{L^2}. \tag{5.68}$$

Therefore, we derive

$$\|(u^\varepsilon_\beta, b^\varepsilon_\beta)\|_{L^2_{k+l}} \leq \gamma(t)\|(\partial_\tau^\beta u^\varepsilon, \partial_\tau^\beta b^\varepsilon)\|_{L^2_{k+l}}. \tag{5.69}$$

On the other hand, since the equality $\partial_y\psi^\varepsilon = b^\varepsilon$ and expression of b^ε_β in (5.46),

$$b^\varepsilon_\beta := \partial_\tau^\beta b^\varepsilon - \frac{\partial_y b^\varepsilon}{1+b^\varepsilon}\partial_\tau^\beta\psi^\varepsilon = (1+b^\varepsilon)\partial_y(\frac{\partial_\tau^\beta\psi^\varepsilon}{1+b^\varepsilon}),$$

which gives that by $\partial_\tau^\beta\psi^\varepsilon|_{y=0} = 0$,

$$\partial_\tau^\beta\psi^\varepsilon(t,x,y) = (1+b^\varepsilon(t,x,y))\cdot\int_0^y \frac{b^\varepsilon_\beta(t,x,\tilde{y})}{1+b^\varepsilon(t,x,\tilde{y})}d\tilde{y} \tag{5.70}$$

and combined with (5.46), we attain

$$\begin{cases} \partial_\tau^\beta u^\varepsilon = u^\varepsilon_\beta + (\partial_y u^\varepsilon + \partial_y u^s)\cdot\int_0^y \frac{b^\varepsilon_\beta(t,x,\tilde{y})}{1+b^\varepsilon(t,x,\tilde{y})}d\tilde{y}, \\ \partial_\tau^\beta b^\varepsilon = b^\varepsilon_\beta + \partial_y b^\varepsilon\cdot\int_0^y \frac{b^\varepsilon_\beta(t,x,\tilde{y})}{1+b^\varepsilon(t,x,\tilde{y})}d\tilde{y}. \end{cases} \tag{5.71}$$

Thus, we have for $k \geq 1$,

$$\begin{aligned} \|\partial_\tau^\beta u^\varepsilon\|_{L^2_{k+l}} &\leq \|u^\varepsilon_\beta\|_{L^2_{k+l}} + \|\partial_y u^\varepsilon\|_{L^\infty_{k+l+1}}\|\langle y\rangle^{-1}\int_0^y \frac{b^\varepsilon_\beta(t,x,\tilde{y})}{1+b^\varepsilon(t,x,\tilde{y})}d\tilde{y}\|_{L^2} \\ &\quad +\|\langle y\rangle^l\int_0^y \frac{b^\varepsilon_\beta(t,x,\tilde{y})}{1+b^\varepsilon(t,x,\tilde{y})}d\tilde{y}\|_{L^2} \\ &\leq \|u^\varepsilon_\beta\|_{L^2_{k+l}} + C\delta^{-1}(1+\|\partial_y u^\varepsilon\|_{L^\infty_{k+l+1}(\mathbb{R}^2_+)})\|b^\varepsilon_\beta\|_{L^2}. \end{aligned} \tag{5.72}$$

Also,

$$\|\partial_\tau^\beta b^\varepsilon\|_{L^2_{k+l}} \leq \|b^\varepsilon_\beta\|_{L^2_{k+l}} + C\delta^{-1}\|\partial_y b^\varepsilon\|_{L^\infty_{k+l+1}(\mathbb{R}^2_+)}\|b^\varepsilon_\beta\|_{L^2_{k+l}}, \tag{5.73}$$

which gives

$$\|(\partial_\tau^\beta u^\varepsilon, \partial_\tau^\beta b^\varepsilon)\|_{L^2_{k+l}} \leq \gamma(t)^{-1}\|(u^\varepsilon_\beta, b^\varepsilon_\beta)\|_{L^2_{k+l}}, \tag{5.74}$$

provided that $\gamma(t)$ given (5.66). Hence, combining (5.69) with (5.74) implies (5.64).

Similar to (5.72), we also get the desired result (5.65). The proof is thus completed. □

In this part, we will complete the proof of Theorem 2.1. Similar to the proof of Proposition 3.6 in [14], by (2.1) we have

$$\|\langle y\rangle^{k+l+1}\partial_y^i(u^\varepsilon, b^\varepsilon)\|_{L^\infty} \leq \delta^{-1}, \text{ for } i = 1,2,\ t\in[0,T], \tag{5.75}$$

which, combined with (5.46) and (5.66), it follows that for $\delta \in (0,1)$ small enough,

$$\gamma(t) = \delta^{-1}(1+\|\partial_y(u^\varepsilon, b^\varepsilon)\|_{L^\infty_{k+l+1}(\mathbb{R}^2_+)}) \leq 2\delta^{-2}. \tag{5.76}$$

Since the operator $D^\alpha = \partial_\tau^\beta \partial_y^k$, from (5.64), we discover

$$\begin{aligned}\|(u^\varepsilon, b^\varepsilon)\|^2_{H^2_{k+l}} &= \sum_{|\alpha|\le 4,\ |\beta|\le 3} \|D^\alpha(u^\varepsilon, b^\varepsilon)(t)\|^2_{L^2_{k+l}} + \sum_{|\beta|=4} \|\partial_\tau^\beta(u^\varepsilon, b^\varepsilon)(t)\|^2_{L^2_{k+l}} \\ &\le \sum_{|\alpha|\le 4,\ |\beta|\le 3} \|D^\alpha(u^\varepsilon, b^\varepsilon)(t)\|^2_{L^2_{k+l}} + 4\delta^{-2} \sum_{|\beta|=4} \|(u^\varepsilon_\beta, b^\varepsilon_\beta)\|^2_{L^2_{k+l}}. \end{aligned} \tag{5.77}$$

In this position, we show the energy estimate of the approximate solutions $(u^\varepsilon, b^\varepsilon)$. Collecting some established estimates (5.1), (5.58), (5.65), and (5.76) and adding together, then inserting (5.77) into the resultant and using the Young inequality implies

$$\begin{aligned}&\frac{d}{dt}\Big(\sum_{|\alpha|\le 4,\ |\beta|\le 3} \|D^\alpha(u^\varepsilon, b^\varepsilon)(t)\|^2_{L^2_{k+l}} + 4\delta^{-2} \sum_{|\beta|=4} \|(u^\varepsilon_\beta, b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}\Big) \\ &\qquad +\Big(\sum_{|\alpha|\le 4,\ |\beta|\le 3} \|D^\alpha \partial_y(u^\varepsilon, b^\varepsilon)(t)\|^2_{L^2_{k+l}} + 4\delta^{-2} \sum_{|\beta|=4} \|\partial_y(u^\varepsilon_\beta, b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}\Big) \\ &\le C\delta^{-6}(1 + \|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}})^2 + C\delta^{-2}(1 + \|(u^\varepsilon, b^\varepsilon)\|^2_{H^4_{k+l}})^2 \sum_{|\beta|=4} \|(u^\varepsilon_\beta, b^\varepsilon_\beta)\|^2_{L^2_{k+l}} \\ &\le C\delta^{-6}\Big(\sum_{|\alpha|\le 4,\ |\beta|\le 3} \|D^\alpha(u^\varepsilon, b^\varepsilon(t)\|^2_{L^2_{k+l}} + 4\delta^{-2} \sum_{|\beta|=4} \|(u^\varepsilon_\beta, b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}}\Big)^3 \\ &\qquad + C(\delta^{-2} + \delta^{-6}). \end{aligned} \tag{5.78}$$

Define

$$F_0 := \sum_{|\alpha|\le 4,\ |\beta|\le 3} \|D^\alpha(u^\varepsilon, b^\varepsilon)(0)\|^2_{L^2_{k+l}} + 4\delta^{-2} \sum_{|\beta|=4} \|(u^\varepsilon_\beta, b^\varepsilon_\beta)(0)\|^2_{L^2_{k+l}}. \tag{5.79}$$

Consequently, applying the nonlinear Gronwall inequality (Theorem 2, P362, [18]) in (5.78), we have

$$\begin{aligned}&\sum_{|\alpha|\le 4,\ |\beta|\le 3} \|D^\alpha(u^\varepsilon, b^\varepsilon)(t)\|^2_{L^2_{k+l}} + 4\delta^{-2} \sum_{|\beta|=4} \|(u^\varepsilon_\beta, b^\varepsilon_\beta)(t)\|^2_{L^2_{k+l}} \\ &\le (F_0 + (\delta^{-2} + \delta^{-6})t)\{1 - 2C\delta^{-6}(F_0 + (\delta^{-2} + \delta^{-6})t)^2 t\}^{-\frac{1}{2}}. \end{aligned} \tag{5.80}$$

Up to now, we have the following lemma.

Lemma 5.6. *Under the assumptions of Theorem 2.1, there exists a positive constant C, which may be dependent on k, l, for some $k \ge 1,\ l \ge 0$, but independent of ε and t such that*

$$\|(u^\varepsilon, b^\varepsilon)(t)\|^2_{H^4_{k+l}} \le 16\delta^{-2}\|(u, b)(0)\|^2_{H^4_{k+l}},\ \forall t \in [0, T]. \tag{5.81}$$

Proof: First, invoking (5.77) and (5.80), we can derive that

$$\|(u^\varepsilon, b^\varepsilon)(t)\|^2_{H^4_{k+l}} \le (F_0 + (\delta^{-2} + \delta^{-6})t)\{1 - 2C\delta^{-6}(F_0 + (\delta^{-2} + \delta^{-6})t)^2 t\}^{-\frac{1}{2}}, \tag{5.82}$$

then combining (5.79), (5.64), (5.66) with (4.22), we can lead to

$$F_0 \le C\Big(\sum_{|\alpha|\le 4,\ |\beta|\le 3} \|D^\alpha(u, b)(0)\|^2_{L^2_{k+l}} + 4\delta^{-2} \sum_{|\beta|=4} \|(u_\beta, b_\beta)(0)\|^2_{L^2_{k+l}}\Big)$$

$$\leq \quad C\delta^{-2}\|(u,b)(0)\|^2_{H^4_{k+l}}. \tag{5.83}$$

Hence, we can conclude the uniform estimates with respect to $\varepsilon \in (0,1)$ and $\forall t \in [0,T]$

$$\|(u^\varepsilon, b^\varepsilon)(t)\|^2_{H^4_{k+l}} \leq 16\delta^{-2}\|(u,b)(0)\|^2_{H^4_{k+l}},$$

provided that T be determined by (5.82) and (5.83) such that

$$T = \min\Big\{\frac{3C\|(u,b)(0)\|^2_{H^4_{k+l}}}{(1+\delta^{-4})}, \frac{\delta^{10}}{64C\|(u,b)(0)\|^4_{H^4_{k+l}}}\Big\}. \tag{5.84}$$

The proof is thus completed. □

Convergence and consistency

Using evolution Eq (4.1) and uniform H^4_{k+l} bound in (5.81), we conclude that $(\partial_t u^\varepsilon, \partial_t b^\varepsilon)$ is uniformly (in ε) bounded in $L^\infty([0,T]; H^2_{k+l})$. By the Lions-Aubin Lemma and the compact embedding of H^4_{k+l} in $H^{4-\tilde{\delta}}_{\mu,loc}$, for $0 < \tilde{\delta} < 1$, taking a subsequence as $\varepsilon_k \to 0^+$,

$$(u^{\varepsilon_k}, b^{\varepsilon_k}) \overset{*}{\rightharpoonup} (u,b) \quad \text{in} \quad L^\infty([0,T]; H^4_{k+l}) \quad \text{and } (u^{\varepsilon_k}, b^{\varepsilon_k}) \to (u,b) \quad \text{in} \quad C([0,T]; H^{4-\tilde{\delta}}_{k+l,loc}).$$

Applying the local uniform convergence of $(\partial^4_x u^{\varepsilon_k}, \partial^4_x b^{\varepsilon_k})$, we have the following pointwise convergence of $(v^{\varepsilon_k}, g^{\varepsilon_k})$: as $\varepsilon_k \to 0^+$,

$$(v^{\varepsilon_k}, g^{\varepsilon_k}) = \Big(-\int_0^y \partial_x u^{\varepsilon_k} dy, -\int_0^y \partial_x b^{\varepsilon_k} dy\Big) \to \Big(-\int_0^y \partial_x u dy, -\int_0^y \partial_x b dy\Big) =: (v,g). \tag{5.85}$$

Now, we pass the limit in the problem (4.1) and conclude that (u,v,b,g) solves the original problem (3.4). Hence, we finish the proof of Theorem 2.1.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The authors are grateful to the anonymous referees for helpful comments and suggestions that greatly improved the presentation of a previous version of the manuscript. This paper was in part supported by the Zhoukou Normal University high level talents research start funding project (ZKNUC2022008).

Conflict of interest

The author declares to have no competing interest.

References

1. R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, New York-London: Academic Press, 2003.
2. R. Alexandre, Y. G. Wang, C. J. Xu, T. Yang, Well-posedness of the Prandtl equation in Sobolev spaces, *J. Am. Math. Soc.*, **28** (2015), 745–784. http://doi.org/10.1090/S0894-0347-2014-00813-4
3. T. G. Cowling, *Magnetohydrodynamics*, New York: Interscience Publishers Inc., 1957.
4. D. X. Chen, X. L. Li, Long time well-posedness of two dimensional Magnetohydrodynamic boundary layer equation without resistivity, *Math. Method. Appl. Sci.*, **46** (2023), 10186–10202. http://doi.org/10.1002/mma.9110
5. P. A. Davidson, *An introduction to magnetohydrodynamics*, Cambridge: Cambridge University Press, 2001. https://doi.org/10.1017/CBO9780511626333
6. P. G. Drasin, Stability of parallel flow in a parallel magnetic field at small magnetic Reynolds number, *J. Fluid Mech.*, **8** (1960), 130–142. http://doi.org/10.1017/S0022112060000475
7. J. C. Gao, D. E. Huang, Z. A. Yao, Boundary layer problems for the two-dimensional inhomogeneous incompressible magnetohydrodynamics equations, *Calc. Var.*, **60** (2021), 67. http://doi.org/10.1007/s00526-021-01958-y
8. D. Gérard-Varet, E. Dormy, On the ill-posedness of the Prandtl equations, *J. Am. Math. Soc.*, **23** (2010), 591–609.
9. D. Gérard-Varet, M. Prestipino, Formal derivation and stability analysis of boundary layer models in MHD, *Z. Angew. Math. Phys.*, **68** (2016), 76. http://doi.org/10.1007/s00033-017-0820-x
10. Y. T. Huang, C. J. Liu, T. Yang, Local-in-time well-posedness for compressible MHD boundary layer, *J. Differ. Equations*, **266** (2019), 2978–3013. http://doi.org/10.1016/j.jde.2018.08.052
11. W. X. Li, R. Xu, Well-posedness in sobolev spaces of the two-dimensional MHD boundary layer equations without viscosity, *Electron. Res. Arch.*, **29** (2021), 4243–4255. http://doi.org/10.3934/era.2021082
12. C. J. Liu, D. H. Wang, F. Xie, T. Yang, Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces, *J. Funct. Anal.*, **279** (2020), 108637. http://doi.org/10.1016/j.jfa.2020.108637
13. C. J. Liu, F. Xie, T. Yang, A note on the ill-posedness of shear flow for the MHD boundary layer equations, *Sci. China Math.*, **61** (2018), 2065–2078. http://doi.org/10.1007/s11425-017-9306-0
14. C. J. Liu, F. Xie, T. Yang, MHD Boundary layers theory in Sobolev spaces without monotonicity I: well-posedness theory, *Commun. Pur. Appl. Math.*, **72** (2019), 63–121. http://doi.org/10.1002/cpa.21763
15. C. J. Liu, F. Xie, T. Yang, MHD boundary layers in Sobolev spaces without monotonicity II: convergence theory, 2017, arXiv:1704.00523.
16. N. Liu, P. Zhang, Global small analytic solutions of MHD boundary layer equations, *J. Differ. Equations*, **281** (2021), 199–257. http://doi.org/10.1016/j.jde.2021.02.003
17. N. Masmoudi, T. K. Wong, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, *Commun. Pur. Appl. Math.*, **68** (2015), 1683–1741. http://doi.org/10.1002/cpa.21595

18. D. S. Mitrinović, J. E. Pěcarić, A. M. Fink, *Inequalities involving functions and their integrals and derivatives*, Dordrecht: Springer, 1991. https://doi.org/10.1007/978-94-011-3562-7

19. O. A. Oleinik, The Prandtl system of equations in boundary layer theory, *Dokl. Acad. Nauk. SSSR*, **150** (1963), 28–32.

20. O. A. Oleinik, V. N. Samokhin, *Mathematical models in boundary layer theory*, New York: Routledge, 1999. https://doi.org/10.1201/9780203749364

21. L. Prandtl, *ü*ber Fl*ü*ssigkeitsbewegung bei sehr kleiner reibung, *The International Mathematical Congress*, Heidelberg, Germany, 1904, 484–491.

22. V. J. Rossow, Boundary layer stability diagrams for electrically conducting fluids in the presence of a magnetic field, *NACA Technical Note*, **1958** (1958), 4282.

23. M. Sermange, R. Temam, Some mathematical questions related to the MHD equations, *Commun. Pur. Appl. Math.*, **36** (1983), 635–664. http://doi.org/10.1002/cpa.3160360506

24. H. Triebel, *Theory of function spaces*, Basel: Birkhäuser, 1983. https://doi.org/10.1007/978-3-0346-0416-1

25. S. Wang, N. Wang, Boundary layer problem of MHD system with non-characteristic perfect conducting wall, *Appl. Anal.*, **98** (2017), 1–20. http://doi.org/10.1080/00036811.2017.1395867

26. S. Wang, Z. P. Xin, Boundary layer problems in the viscosity diffusion vanishing limits for the incompressible MHD systems, *Scientia Sinica Mathematica*, **47** (2017), 1303–1326. http://doi.org/10.1360/N012016-00211

27. S. Wang, B. Y. Wang, C. D. Liu, N. Wang, Boundary layer problem and zero viscosity-diffusion limit of the incompressible magnetohydrodynamic system with no-slip boundary conditions, *J. Differ. Equations*, **263** (2017), 4723–4749. http://doi.org/10.1016/j.jde.2017.05.025

28. F. Xie, T. Yang, Global-in-time stability of 2D MHD boundary layer in the Prandtl-Hartmann regime, *SIAM J. Math. Anal.*, **50** (2018), 5749–5760. http://doi.org/10.1137/18M1174969

29. F. Xie, T. Yang, Lifespan of solutions to MHD Boundary layer equations with analytic perturbation of general shear flow, *Acta Math. Appl. Sin. Engl. Ser.*, **35** (2019), 209–229. http://doi.org/10.1007/s10255-019-0805-y

30. Z. P. Xin, L. Q. Zhang, On the global existence of solutions to the Prandtl's system, *Adv. Math.*, **181** (2004), 88–133. https://doi.org/10.1016/S0001-8708(03)00046-X

31. C. J. Xu, X. Zhang, Long time well-posedness of Prandtl equations in Sobolev space, *J. Differ. Equations*, **263** (2017), 8749–8803. http://doi.org/10.1016/j.jde.2017.08.046

AIMS Press