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Research article

Toeplitz operators on two poly-Bergman-type spaces of the Siegel domain $D_2 \subset \mathbb{C}^2$ with continuous nilpotent symbols

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Abstract: We studied Toeplitz operators acting on certain poly-Bergman-type spaces of the Siegel domain $D_2 \subset \mathbb{C}^2$. Using continuous nilpotent symbols, we described the C^*-algebras generated by such Toeplitz operators. Bounded measurable functions of the form $\tilde{c}(\zeta) = c(\operatorname{Im}\zeta_1, \operatorname{Im}\zeta_2 - |\zeta_1|^2)$ are called nilpotent symbols. In this work, we considered symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$ and $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$, where both limits $\lim_{s\to 0^+} b(s)$ and $\lim_{s\to+\infty} b(s)$ exist, and a belongs to the set of piece-wise continuous functions on $\overline{\mathbb{R}} = [-\infty, +\infty]$ and with one-sided limits at 0. We described certain C^*-algebras generated by such Toeplitz operators that turned out to be isomorphic to subalgebras of $M_n(\mathbb{C}) \otimes C(\overline{\Pi})$, where $\overline{\Pi} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}_+$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$.

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1. Introduction

In recent years, the theory of Toeplitz operators has been generalized from Bergman spaces of square-integrable holomorphic functions to poly-Bergman spaces of square-integrable polyanalytic functions [1, 2]. Bianalytic functions emerged in the mathematical theory of elasticity, but the mathematical relevance of more general polyanalytic functions was soon realized [3].

Similar to the study of Toeplitz operators on spaces of analytic functions, we select a set of symbols $E \subset L^\infty$ in such a way that the C^*-algebra generated by Toeplitz operators with symbols in E can be explicitly described up to isomorphism, that is, as an algebra of matrix-valued functions. For the unit disk $\mathbb{D}$ and the Siegel domain $D_n \subset \mathbb{C}^n$, in [4–6] the authors considered the set E_G of symbols invariant under the action of a maximal Abelian subgroup G of biholomorphisms, they found that the C^*-algebra $\mathcal{T}_G$ generated by Toeplitz operators acting on the Bergman spaces with symbols in E_G is commutative. The authors also proved that $\mathcal{T}_G$ is isomorphic and isometric to a C^*-subalgebra of continuous functions on a locally compact Hausdorff space X. Of course, $\mathcal{T}_G$ is isomorphic to $C(\sigma(\mathcal{T}_G))$, where $\sigma(\mathcal{T}_G)$ is the spectrum of $\mathcal{T}_G$. Unfortunately, X is far away to be $\sigma(\mathcal{T}_G)$. In this sense, we can note that the full description of a commutative C^*-algebra depends on the spectrum of the algebra. It makes sense to select a smaller class $E \subset E_G$ in order to fully describe the C^*-algebra $\mathcal{T}_E$ generated by the Toeplitz operators with symbols in E. For the case of the Siegel domain D_2, in [7] the authors choose a family E of piece-wise continuous symbols invariant under a nilpotent group and prove that the spectrum of the C^*-algebra $\mathcal{T}_E$ is a compactification of the upper half-plane $\Pi = \{z = x + iy \in \mathbb{C} \mid y > 0\}$. In this work, we extend the results in [7] by studying the noncommutative C^*-algebra generated by Toeplitz operators acting on two type-poly-Bergman spaces of the Siegel domain D_2, where it is possible to identify the space of all irreducible representation of such algebra. In general, the spectrum of a commutative C^*-algebra generated by Toeplitz operators is too large, and it is impossible to have a full description of it. In [8–12], we can find outstanding contributions about the spectrum of commutative algebras and the spectrum of operators acting on Hilbert spaces with reproducing kernel, where the Berezin transform play a significant role.

In the case of the upper half-plane Π, a vertical symbol is a bounded measurable function $a(z)$ depending only on $y = \operatorname{Re} z$. Taking vertical symbols, in [13–15] the authors studied Toeplitz operators acting on the true-poly-Bergman space $\mathcal{A}^2_{(n)}(\Pi)$ from the point of view of wavelet spaces. Toeplitz operators with vertical symbols acting on poly-Bergman-type spaces have also be studied. Taking vertical symbols with limits at $y = 0$ and $y = \infty$, in [13, 16] the authors described the C^*-algebra generated by all Toeplitz operators on the poly-Bergman space $\mathcal{A}^2_n(\Pi)$; this algebra is isomorphic to a subalgebra of $M_n(\mathbb{C}) \otimes C[0, +\infty]$. Similar research has studied Toeplitz operators on poly-Bergman spaces with homogeneous symbols ([17, 18]). Taking horizontal symbols having one-sided limits at $x = \pm\infty$, in [19, 20] the authors studied Toeplitz operators acting on poly-Fock spaces $F^2_k(\mathbb{C})$ and they proved that the C^*-algebra generated by such Toeplitz operators is isomorphic to a subalgebra of $M_n(\mathbb{C}) \otimes C[-\infty, +\infty]$. In [21–24], the authors studied the decomposition of the von Neumann algebra of radial operators acting on the poly-Fock spaces $F^2_k(\mathbb{C})$ and weighted poly-Bergman spaces $\mathcal{A}^2_n(\mathbb{D})$.

In [4, 5], the authors found all classes of bounded measurable symbols associated to commutative algebras generated by Toeplitz operators acting on the Bergman space of the Siegel domain $D_n \subset \mathbb{C}^n$. In particular, they studied the C^*-algebra $\mathcal{T}_{N_n}$ generated by all Toeplitz operators with bounded nilpotent symbols, which are functions of the form $\tilde{c}(\zeta) = c(\operatorname{Im}\zeta_1, ..., \operatorname{Im}\zeta_{n-1}, \operatorname{Im}\zeta_n - |\zeta'|^2)$, where $\zeta' = (\zeta_1, ..., \zeta_{n-1})$. Furthermore, in [7] the authors studied Toeplitz operators on the true-poly-Bergman-type space $\mathcal{A}^2_{(L)}(D_2)$, with nilpotent symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$ and $\tilde{b}(z) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$. In this paper, we consider two poly-Bergman-type spaces of the Siegel domain $D_2 \subset \mathbb{C}^2$, in which Toeplitz operators with continuous nilpotent symbols are studied. The main purpose of this work is to describe the C^*-algebra generated by all the Toeplitz operators acting on the poly-Bergman-type spaces $\mathcal{A}^2_{(1,n)}(D_2)$ and $\mathcal{A}^2_{(n,1)}(D_2)$. We take nilpotent symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$ and $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$.

In Section 2, we introduce poly-Bergman-type spaces for the Siegel domain and discuss how they are identified through a Bargmann-type transform. In Section 3, we define Toeplitz operators acting on $\mathcal{A}^2_{(1,n)}(D_2)$ with nilpotent symbols; such Toeplitz operators are unitarily equivalent to multiplication operators.

In Section 3.1, we take symbols of the form $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$ for which both limits $\lim_{s\to 0^+} b(s)$ and $\lim_{s\to+\infty} b(s)$ exist; the C^*-algebra generated by all Toeplitz operators T_b is isomorphic to $\mathfrak{C} = \{M \in M_n(\mathbb{C}) \otimes C[0,\infty] : M(0), M(+\infty) \in \mathbb{C}I\}$. Let $PC(\overline{\mathbb{R}}, \{0\})$ denote the set of all functions continuous on $\overline{\mathbb{R}} \setminus \{0\}$ and having one-side limit values at 0, where $\overline{\mathbb{R}}$ is the two-point compactification of $\mathbb{R}$. In Section 3.2, we take nilpotent symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$, where $a \in PC(\overline{\mathbb{R}}, \{0\})$; the C^*-algebra generated by all Toeplitz operators T_a is isomorphic to $C(\overline{\Pi})$, where $\overline{\Pi} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}_+$.

In Section 4, we introduce Toeplitz operators acting on $\mathcal{A}^2_{(n,1)}(D_2)$ with nilpotent symbols $\tilde{c}$ and we show that such Toeplitz operators are unitarily equivalent to multiplication operators $\gamma^c I$ acting on $L^2(\mathbb{R}\times\mathbb{R}_+)$, where γ^c is a continuous matrix-valued function on Π. In this work, we take symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$, where $a \in PC(\overline{\mathbb{R}}, \{0\})$. In Section 4.1, we prove that the matrix-valued function γ^a can be continuously extended to $\overline{\Pi}$ under a change of variable, which is one of our main results. In Section 4.3, we prove that the C^*-algebra generated by all Toeplitz operators T_a is isomorphic to a C^*-subalgebra of $M_n(\mathbb{C}) \otimes C(\overline{\Pi})$, thus the spectrum of such algebra is fully described.

2. Poly-Bergman type spaces of the Siegel domain

In this section, we recall some results obtained in [25] that are needed in this paper. We recall how the poly-Bergman-type spaces are defined and how they are identified with a tensor product of L_2-spaces. This allows us to study Toeplitz operators with nilpotent symbols through a Bargmann-type transform. We clarify that if X is any positive-measure subset of a Euclidean space, then $L^2(X)$ refers to $L^2(X, d\mu)$, where $d\mu$ is Lebesgue measure restricted to X. We will study Toeplitz operators acting on certain poly-Bergman-type subspaces of $L^2(D_2)$, where D_2 is the two-dimensional Siegel domain

$$D_2 = \{\zeta = (\zeta_1, \zeta_2) \in \mathbb{C}^2 : \operatorname{Im}\zeta_2 - |\zeta_1|^2 > 0\}.$$

For each pair $L = (j,k) \in \mathbb{N}^2$, the poly-Bergman-type space $\mathcal{A}^2_L(D_2)$ is the closed subspace of $L^2(D_2)$ consisting of all L-analytic functions, that is, all functions $f \in C^\infty(D_2)$ satisfying the equations

$$\left(\frac{\partial}{\partial\overline{\zeta}_1} - 2i\zeta_1\frac{\partial}{\partial\overline{\zeta}_2}\right)^j f = 0, \qquad \left(\frac{\partial}{\partial\overline{\zeta}_2}\right)^k f = 0.$$

Note that $\mathcal{A}^2_L(D_2)$ is just the Bergman space when $L = (1,1)$. The anti-poly-Bergman type space $\widetilde{\mathcal{A}}^2_L(D_2)$ is defined to be the complex conjugate of $\mathcal{A}^2_L(D_2)$. Now, true-poly-Bergman-type spaces are defined as follows:

$$\begin{aligned}
\mathcal{A}^2_{(L)}(D_2) &= \mathcal{A}^2_L(D_2) \ominus \left(\sum_{m=1}^{2} \mathcal{A}^2_{L-e_m}(D_2)\right), \\
\widetilde{\mathcal{A}}^2_{(L)}(D_2) &= \widetilde{\mathcal{A}}^2_L(D_2) \ominus \left(\sum_{m=1}^{2} \widetilde{\mathcal{A}}^2_{L-e_m}(D_2)\right),
\end{aligned}$$

where $e_1 = (1,0)$ and $e_2 = (0,1)$. We assume that $\mathcal{A}^2_L(D_2) = \{0\}$ whenever $L \in \mathbb{Z}^2 \setminus \mathbb{N}^2$. The Hilbert space $L^2(D_2)$ is the direct sum of all the true-poly-Bergman-type spaces and the true-anti-poly-Bergman-type spaces; see [25] for details. Let us briefly recall how the authors also constructed a unitary map from $\mathcal{A}^2_{(L)}(D_n)$ to the tensor product $L^2(\mathbb{R}) \otimes \mathcal{H}_{j-1} \otimes L^2(\mathbb{R}_+) \otimes \mathcal{L}_{k-1}$, where $\mathcal{H}_m$ and $\mathcal{L}_m$ are the one-dimensional spaces generated by the Hermite and Laguerre functions, respectively, which are given by

$$h_m(y) = \frac{1}{(2^m \sqrt{\pi} m!)^{1/2}} H_m(y) e^{-y^2/2}$$

and

$$\ell_m(y) = (-1)^m L_m(y) e^{-y/2} \chi_+(y)$$

for $m = 0, 1, 2, ...$ As usual, $\mathbb{R}_+ = \{x \in \mathbb{R} : \ x > 0\}$, and χ_+ denotes the indicator function of $\mathbb{R}_+$. The Hermite and Laguerre polynomials are defined by the Rodrigues formulae as follows:

$$H_m(y) := (-1)^m e^{y^2} \frac{d^m}{dy^m}(e^{-y^2}), \quad L_m(y) := e^y \frac{1}{m!} \frac{d^m}{dy^m}(e^{-y} y^m).$$

It is well known that $\{h_m\}_{m=0}^{\infty}$ and $\{\ell_m\}_{m=0}^{\infty}$ are orthonormal bases for $L^2(\mathbb{R})$ and $L^2(\mathbb{R}_+)$, respectively. Finally, $\mathcal{H}_m = \text{span}\{h_m\}$ and $\mathcal{L}_m = \text{span}\{\ell_m\}$.

Consider the flat domain $\mathcal{D} = \mathbb{C} \times \Pi$, where $\Pi = \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{C}$, then $\mathcal{D}$ can be identified with D_2 under the mapping $\kappa : \mathcal{D} \to D_2$ given by the rule

$$\zeta = \kappa(w_1, w_2) = (w_1, w_2 + i|w_1|^2).$$

Thus we have the unitary operator $U_0 : L^2(D_2) \longrightarrow L^2(\mathcal{D})$ given by

$$(U_0 h)(w) = h(\kappa(w)).$$

Take $w = (w_1, w_2) \in \mathbb{C} \times \Pi$, with $w_m = u_m + iv_m$ and $m = 1, 2$. We identify $w = (u_1 + iv_1, u_2 + iv_2)$ with (u_1, v_1, u_2, v_2), then

$$L^2(\mathcal{D}) = L^2(\mathbb{R}, du_1) \otimes L^2(\mathbb{R}, dv_1) \otimes L^2(\mathbb{R}, du_2) \otimes L^2(\mathbb{R}_+, dv_2).$$

Introduce the unitary operator $U_1 = F \otimes I \otimes F \otimes I : L^2(\mathcal{D}) \to L^2(\mathcal{D})$, where F is the Fourier transform acting on $L^2(\mathbb{R})$ by the rule

$$(Fg)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-itx} dx.$$

Consider now the following two mappings $\psi_1,\ \psi_2 : \mathcal{D} \to \mathcal{D}$ defined by

$$\psi_1(\xi_1, \xi_2) = (\xi_1, t_2 + i\frac{s_2}{2|t_2|})$$

and

$$\psi_2(z_1, z_2) = \left(\sqrt{|x_2|}(x_1 + y_1) + i\frac{1}{2\sqrt{|x_2|}}(-x_1 + y_1), z_2 \right),$$

where $\xi_m = t_m + is_m$ and $z_m = x_m + iy_m$. Both functions ψ_1 and ψ_2 induce the unitary operators acting on $L^2(\mathcal{D})$ by

$$(V_1 g)(\xi) = \frac{1}{(2|t_2|)^{1/2}} g(\psi_1(\xi)), \quad (V_2 f)(z) = f(\psi_2(z)).$$

Theorem 2.1. *[25] The operator $U = V_2V_1U_1U_0$ is unitary and maps $L^2(D_2)$ onto the space*

$$L^2(\mathcal{D}) = L^2(\mathbb{R}, dx_1) \otimes L^2(\mathbb{R}, dy_1) \otimes L^2(\mathbb{R}, dx_2) \otimes L^2(\mathbb{R}_+, dy_2).$$

For each $L = (j, k) \in \mathbb{N}^2$, the operator U maps the true-Bergman-type space $\mathcal{A}^2_{(L)}(D_2)$ onto the space

$$\mathcal{H}^+_{(L)} = L^2(\mathbb{R}) \otimes \mathbb{C}h_{j-1} \otimes L^2(\mathbb{R}_+) \otimes \mathbb{C}\ell_{k-1}.$$

We will study Toeplitz operators with nilpotent symbols acting on $\mathcal{A}^2_L(D_2)$ in the cases $L = (1, n)$ and $L = (n, 1)$. In both cases, the poly-Bergman-type space can be identified with $(L^2(\mathbb{R}\times\mathbb{R}_+))^n$ through a Bargmann type transform [25]. Since $\mathcal{A}^2_{(1,n)}(D_2) = \bigoplus_{k=1}^n \mathcal{A}^2_{((1,k))}(D_2)$, the operator U isometrically maps $\mathcal{A}^2_{(1,n)}(D_2)$ onto the space

$$\mathcal{H}^+_{(1,n)} = L^2(\mathbb{R}) \otimes \mathbb{C}h_0 \otimes L^2(\mathbb{R}_+) \otimes \operatorname{span}\{\ell_0, ..., \ell_{n-1}\}.$$

Analogously, the operator U restricted to $\mathcal{A}^2_{(n,1)}(D_2)$ is an isometric isomorphism onto the space

$$\mathcal{H}^+_{(n,1)} = L^2(\mathbb{R}) \otimes \operatorname{span}\{h_0, ..., h_{n-1}\} \otimes L^2(\mathbb{R}_+) \otimes \mathbb{C}\ell_0.$$

We introduce the following linear isometric embeddings

$$R_{0(1,n)},\ R_{0(n,1)} : (L^2(\mathbb{R} \times \mathbb{R}_+))^n \longrightarrow L^2(\mathcal{D})$$

defined by

$$\begin{aligned}(R_{0(1,n)}g)(x_1, y_1, x_2, y_2) &= \chi_+(x_2)\ h_0(y_1)[N(y_2)]^T g(x_1, x_2),\\ (R_{0(n,1)}g)(x_1, y_1, x_2, y_2) &= \chi_+(x_2)\, \ell_0(y_2)[H(y_1)]^T g(x_1, x_2),\end{aligned}$$

where $g = (g_1, ..., g_n)^T \in (L^2(\mathbb{R} \times \mathbb{R}_+))^n$ and

$$H(y_1) = (h_0(y_1), ..., h_{n-1}(y_1))^T \text{ and } N(y_2) = (\ell_0(y_2), ..., \ell_{n-1}(y_2)^T.$$

Clearly, $\mathcal{H}^+_{(1,n)}$ and $\mathcal{H}^+_{(n,1)}$ are the images of $R_{0(1,n)}$ and $R_{0(n,1)}$, respectively. Consequently, the operators

$$R_{(1,n)} := R^*_{0(1,n)}U,\ R_{(n,1)} := R^*_{0(n,1)}U : L^2(D_2) \longrightarrow (L^2(\mathbb{R} \times \mathbb{R}_+))^n,$$

isometrically map the poly-Bergman-type spaces $\mathcal{A}^2_{(1,n)}(D_2)$ and $\mathcal{A}^2_{(n,1)}(D_2)$ onto $(L^2(\mathbb{R} \times \mathbb{R}_+))^n$. Therefore,

$$R_{(1,n)}R^*_{(1,n)} = I = R_{(n,1)}R^*_{(n,1)},$$

$$R^*_{(1,n)}R_{(1,n)} = B_{(1,n)} \qquad \text{and} \qquad R^*_{(n,1)}R_{(n,1)} = B_{(n,1)},$$

where $B_{(1,n)}$ and $B_{(n,1)}$ are the orthogonal projections from $L^2(D_2)$ onto $\mathcal{A}^2_{(1,n)}(D_2)$ and $\mathcal{A}^2_{(n,1)}(D_2)$, respectively. Thus, $R^*_{(1,n)}$ and $R^*_{(n,1)}$ play the role of the Segal-Bargmann transform for the poly-Bergman-type spaces $\mathcal{A}^2_{(1,n)}(D_2)$ and $\mathcal{A}^2_{(n,1)}(D_2)$, where the adjoint operators $R^*_{0(1,n)}$ and $R^*_{0(n,1)}$ are given by

$$\begin{aligned}(R^*_{0(1,n)}f)(x_1, x_2) &= \int_{\mathbb{R}}\int_{\mathbb{R}_+} f(x_1, y_1, x_2, y_2)h_0(y_1)N(y_2)dy_2dy_1,\\ (R^*_{0(n,1)}f)(x_1, x_2) &= \int_{\mathbb{R}}\int_{\mathbb{R}_+} f(x_1, y_1, x_2, y_2)\ell_0(y_2)H(y_1)dy_2dy_1,\end{aligned}$$

where $(x_1, x_2) \in \mathbb{R} \times \mathbb{R}_+$.

3. Toeplitz operators on the poly-Bergman space $\mathcal{A}^2_{(1,n)}(D_2)$

Toeplitz operators with nilpotent symbols acting on the poly-Bergman-type space $\mathcal{A}^2_{(1,n)}(D_2)$ are studied in this section. The Vasilevski's techniques, as in [6], allow us to identify Toeplitz operators with multiplication operators. Recall that $\tilde{c} \in L^\infty(D_2, d\mu)$ is said to be a nilpotent symbol if it has the form $\tilde{c}(\zeta_1, \zeta_2) = c(\operatorname{Im}\zeta_1, \operatorname{Im}\zeta_2 - |\zeta_1|^2)$, where $c : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{C}$, then the Toeplitz operator acting on $\mathcal{A}^2_{(1,n)}(D_2)$ with nilpotent symbol $\tilde{c}$ is defined by

$$(T_c f)(\zeta) = (B_{(1,n)}(\tilde{c} f))(\zeta),$$

where $B_{(1,n)}$ is the orthogonal projection from $L^2(D_2)$ onto $\mathcal{A}^2_{(1,n)}(D_2)$. If we define

$$\begin{aligned} M_f : L^2(D_2) &\to L^2(D_2) \\ g &\mapsto fg, \end{aligned}$$

then T_c is equal to $B_{(1,n)} M_{\tilde{c}}$ restricted to $\mathcal{A}^2_{(1,n)}(D_2)$.

The Bargmann-type operator $R_{(1,n)}$ identifies the space $\mathcal{A}^2_{(1,n)}(D_2)$ with $(L^2(\mathbb{R} \times \mathbb{R}_+))^n$, and it fits properly in the study of the Toeplitz operators $T_{\tilde{c}}$.

Theorem 3.1. *Let $\tilde{c}$ be a nilpotent symbol, then the Toeplitz operator T_c is unitary equivalent to the multiplication operator M_{γ^c} and, in fact, $M_{\gamma^c} = R_{(1,n)} T_c R^*_{(1,n)}$, where the matrix-valued function $\gamma^c : \mathbb{R} \times \mathbb{R}_+ \to M_n(\mathbb{C})$ is given by*

$$\gamma^c(x_1, x_2) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} c\left(\frac{-x_1 + y_1}{2\sqrt{x_2}}, \frac{y_2}{2x_2} \right) (h_0(y_1))^2 N(y_2) [N(y_2)]^T dy_2 dy_1. \tag{3.1}$$

Proof. We have

$$\begin{aligned} R_{(1,n)} T_c R^*_{(1,n)} &= R_{(1,n)} B_{(1,n)} M_{\tilde{c}} R^*_{(1,n)} \\ &= R_{(1,n)} R^*_{(1,n)} R_{(1,n)} M_{\tilde{c}} R^*_{(1,n)} \\ &= R_{(1,n)} M_{\tilde{c}} R^*_{(1,n)} \\ &= R^*_{0(1,n)} V_2 V_1 U_1 U_0 (M_{\tilde{c}}) U_0^{-1} U_1^{-1} V_1^{-1} V_2^{-1} R_{0(1,n)}. \end{aligned}$$

Recall that $\zeta = \kappa(w) = (w_1, w_2 + i|w_1|^2)$, where $w = (w_1, w_2) \in \mathcal{D}$ and $w_m = u_m + iv_m$. For $f \in L^2(\mathcal{D})$,

$$(U_0 M_{\tilde{c}} U_0^{-1} f)(w) = \tilde{c}(\kappa(w))(U_0^{-1} f)(\kappa(w)) = \tilde{c}(\kappa(w)) f(w).$$

That is, $U_0 M_{\tilde{c}} U_0^{-1} = M_{\tilde{c}\circ\kappa}$. It is easy to see that $U_1 M_{\tilde{c}\circ\kappa} U_1^{-1} = M_{\tilde{c}\circ\kappa}$. Furthermore,

$$V_1 M_{\tilde{c}\circ\kappa} V_1^{-1} = M_{\tilde{c}\circ\kappa\circ\psi_1}$$

and

$$V_2 V_1 M_{\tilde{c}\circ\kappa} V_1^{-1} V_2^{-1} = M_{\tilde{c}\circ\kappa\circ\psi_1\circ\psi_2}.$$

Denote $\tilde{c} \circ \kappa \circ \psi_1 \circ \psi_2$ by C. It is easy see that

$$C(z) = c\left(\frac{-x_1 + y_1}{2\sqrt{|x_2|}}, \frac{y_2}{2|x_2|} \right) \tag{3.2}$$

where $z = (x_1 + iy_1, x_2 + iy_2) \in \mathcal{D}$. Finally, let $g = (g_1, ..., g_n)^T \in (L^2(\mathbb{R} \times \mathbb{R}_+))^n$ and $A = (R_{(1,n)} T_c R^*_{(1,n)} g)(x_1, x_2)$, then

$$\begin{aligned} A &= (R^*_{0(1,n)} M_C R_{0(1,n)} g)(x_1, x_2) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} C(z)(R_{0(1,n)} g)(x_1, y_1, x_2, y_2) h_0(y_1) N(y_2) dy_2 dy_1 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} C(z) h_0(y_1) [N(y_2)]^T g(x_1, x_2) h_0(y_1) N(y_2) dy_2 dy_1 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} C(z)(h_0(y_1))^2 N(y_2)[N(y_2)]^T g(x_1, x_2) dy_2 dy_1. \end{aligned}$$

Thus, $R_{(1,n)} T_c R^*_{(1,n)} = M_{\gamma^c}$, where $\gamma^c(x_1, x_2)$ is given in (3.1). □

Studying the full C^*-algebra generated by all Toeplitz operators with nilpotent symbols is a difficult task due to the fact that its spectrum is too large. For this reason, we study Toeplitz operators in special cases. In particular, we consider two specific cases of nilpotent symbols. First, we study the Toeplitz operators with symbols of the form $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$, for which

$$\gamma^b(x_1, x_2) = \gamma^b(x_2) = \int_{\mathbb{R}_+} b\left(\frac{y_2}{2x_2}\right) N(y_2)[N(y_2)]^T dy_2. \tag{3.3}$$

Second, we analyze Toeplitz operators with symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$, for which

$$\gamma^a(x_1, x_2) = \int_{\mathbb{R}} a\left(\frac{-x_1 + y_1}{2\sqrt{x_2}}\right)(h_0(y_1))^2 dy_1 I_{n\times n}. \tag{3.4}$$

3.1. *Toeplitz operators with symbols $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$*

In this section, we study the C^*-algebra generated by all Toeplitz operators T_b with symbols of the form $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$, where $b : \mathbb{R}_+ \to \mathbb{C}$ has limits at 0 and $+\infty$. Under this continuity condition, we will see that γ^b is continuous on $\overline{\Pi} := \overline{\mathbb{R}} \times \overline{\mathbb{R}}_+$, where $\overline{\mathbb{R}} = [-\infty, +\infty]$ and $\overline{\mathbb{R}}_+ = [0, +\infty]$ are the two-point compactification of $\mathbb{R} = (-\infty, +\infty)$ and $\mathbb{R}_+ = (0, +\infty)$, respectively. The spectral function $\gamma^b = (\gamma^b_{jk}) : \mathbb{R} \times \mathbb{R}_+ \to M_n(\mathbb{C})$ is continuous if all of its matrix entries are continuous. These matrix entries are given by

$$\gamma^b_{jk}(x_1, x_2) = \int_{\mathbb{R}_+} b\left(\frac{y_2}{2x_2}\right) \ell_{j-1}(y_2)\ell_{k-1}(y_2) dy_2, \quad j, k = 1, ..., n. \tag{3.5}$$

Let $L^\infty_{\{0,+\infty\}}(\mathbb{R}_+)$ denote the subspace of $L^\infty(\mathbb{R}_+)$ consisting of all functions having limit values at 0 and $+\infty$. For $b \in L^\infty_{\{0,+\infty\}}(\mathbb{R}_+)$, define

$$b_0 := \lim_{y\to 0^+} b(y), \quad b_\infty := \lim_{y\to+\infty} b(y).$$

We sometimes think of γ^b_{jk} as a function from $\mathbb{R}_+$ to $\mathbb{C}$, as it depends only on the variable x_2. The form of the matrix-valued function γ^b was obtained in [16] as the spectral function of a Toeplitz operator acting on poly-Bergman spaces of the upper half-plane with vertical symbols, i.e., symbols depending only on the imaginary part of z. Thus, we have at least two scenarios in which γ^b appears as a spectral matrix-valued function.

Lemma 3.2. *[16] Let $b \in L^{\infty}_{\{0,+\infty\}}(\mathbb{R}_+)$, then the spectral matrix-valued function $\gamma^b : \mathbb{R}_+ \to M_n(\mathbb{C})$ is continuous and satisfies*

$$b_\infty I = \lim_{x_2 \to 0^+} \gamma^b(x_2), \quad b_0 I = \lim_{x_2 \to +\infty} \gamma^b(x_2).$$

Obviously, in this context, the spectral matrix-valued function γ^b is defined and continuous on $\overline{\Pi}$, but it is constant along each horizontal straight line. Thus, γ^b is identified with a continuous function on $\overline{\mathbb{R}}_+$.

Let $M_n(C([0,\infty])) = M_n(\mathbb{C}) \otimes C([0,\infty])$, where $M_n(\mathbb{C})$ is the algebra of all $n \times n$ matrices with complex entries. Let $\mathfrak{C}$ be the C^*-subalgebra of $M_n(C([0,\infty]))$ given by

$$\mathfrak{C} = \{M \in M_n(C([0,\infty])) : M(0), M(+\infty) \in \mathbb{C}I\}.$$

By Lemma 3.2 and Theorem 4.8 in [16], we have the following

Theorem 3.3. *For all $b \in L^{\infty}_{\{0,+\infty\}}(\mathbb{R}_+)$, the spectral matrix-valued function γ^b belongs to the C^*-algebra $\mathfrak{C}$. Moreover, the C^*-algebra generated by all the matrix-valued functions $\gamma^b : \overline{\Pi} \to M_n(\mathbb{C})$, with $b \in L^{\infty}_{\{0,+\infty\}}(\mathbb{R}_+)$, is equal to $\mathfrak{C}$. That is, the C^*-algebra generated by all the Toeplitz operators T_b, with $b \in L^{\infty}_{\{0,+\infty\}}(\mathbb{R}_+)$, is isomorphic to $\mathfrak{C}$, where the isomorphism is defined on the generators by*

$$T_b \longmapsto \gamma^b.$$

3.2. Toeplitz operators with continuous symbols $a(\operatorname{Im}\zeta_1)$

Our next stage is the study of the C^*-algebra generated by all Toeplitz operators T_a acting on the poly-Bergman space $\mathcal{A}^2_{(1,n)}(D_2)$, with symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$. Recall that γ^a is given by

$$\gamma^a(x_1, x_2) = \int_{\mathbb{R}} a\left(\frac{-x_1 + y_1}{2\sqrt{x_2}}\right)(h_0(y_1))^2 dy_1 I_{n\times n}$$

for all $(x_1, x_2) \in \Pi = \mathbb{R} \times \mathbb{R}_+$. It is easy to see that γ^a is continuous on Π.

Based on the results obtained in [7], we have the following theorem.

Theorem 3.4. *The C^*-algebra generated by all Toeplitz operators of the form T_a, where $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$ with $a \in C(\overline{\mathbb{R}})$, is isomorphic and isometric to the C^*-algebra $C(\triangle)$, where the quotient space $\triangle = \overline{\Pi}/(\overline{\mathbb{R}} \times \{+\infty\})$ is defined by the identification of $\overline{\mathbb{R}} \times \{\infty\}$ to a point. Furthermore, the C^*-algebra generated by all Toeplitz operators T_a with $a \in PC(\overline{\mathbb{R}}, \{0\})$ is isomorphic to the C^*-algebra $C(\overline{\Pi})$, where $PC(\overline{\mathbb{R}}, \{0\})$ consists of all functions continuous on $\overline{\mathbb{R}} \setminus \{0\}$ and have one-sided limits at 0.*

Proof. Note that γ^a can be identified with the scalar function

$$\begin{aligned} \Pi &\to \mathbb{C} \\ (x_1, x_2) &\mapsto \int_{\mathbb{R}} a\left(\frac{-x_1 + y_1}{2\sqrt{x_2}}\right)(h_0(y_1))^2 dy_1. \end{aligned}$$

This function was obtained in [7] as the spectral function of the Toeplitz operator acting on the Bergman space $\mathcal{A}^2(D_2)$ with symbol $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$. Theorems 10 and 14 of [7] complete the proof. □

4. Toeplitz operators on the poly-Bergman space $\mathcal{A}^2_{(n,1)}(D_2)$

In this section, we study certain C^*-algebras generated by Toeplitz operators with nilpotent symbols acting on the poly-Bergman-type space $\mathcal{A}^2_{(n,1)}(D_2)$. We apply techniques as in Section 3. A Toeplitz operator acting on $\mathcal{A}^2_{(n,1)}(D_2)$ with nilpotent symbol $\tilde{c}(\zeta_1, \zeta_2) = c(\operatorname{Im}\zeta_1, \operatorname{Im}\zeta_2 - |\zeta_1|^2)$ is defined by

$$(T_c f)(\zeta) = (B_{(n,1)}(\tilde{c} f))(\zeta),$$

where $B_{(n,1)}$ is the orthogonal projection from $L^2(D_2)$ onto $\mathcal{A}^2_{(n,1)}(D_2)$. The Bargmann-type operator $R_{(n,1)}$ identifies the space $\mathcal{A}^2_{(n,1)}(D_2)$ with $(L^2(\mathbb{R} \times \mathbb{R}_+))^n$.

Theorem 4.1. *Let $\tilde{c}$ be a nilpotent symbol, then the Toeplitz operator T_c is unitary equivalent to the multiplication operator $\gamma^c I = R_{(n,1)} T_c R^*_{(n,1)}$, where the matrix-valued function $\gamma^c : \mathbb{R} \times \mathbb{R}_+ \to M_n(\mathbb{C})$ is given by*

$$\gamma^c(x_1, x_2) = \int_{\mathbb{R}} \int_{\mathbb{R}_+} c\left(\frac{-x_1 + y_1}{2\sqrt{x_2}}, \frac{y_2}{2x_2}\right) (\ell_0(y_2))^2 H(y_1)[H(y_1)]^T dy_2 dy_1. \tag{4.1}$$

Proof. We have

$$\begin{aligned} R_{(n,1)} T_c R^*_{(n,1)} &= R_{(n,1)} B_{(n,1)}(M_{\tilde{c}}) R^*_{(n,1)} \\ &= R_{(n,1)} R^*_{(n,1)} R_{(n,1)} M_{\tilde{c}} R^*_{(n,1)} \\ &= R_{(n,1)} M_{\tilde{c}} R^*_{(n,1)} \\ &= R^*_{0(n,1)} V_2 V_1 U_1 U_0 M_{\tilde{c}} U_0^{-1} U_1^{-1} V_1^{-1} V_2^{-1} R_{0(n,1)}, \end{aligned}$$

where

$$V_2 V_1 U_1 U_0 (M_{\tilde{c}}) U_0^* U_1^* V_1^{-1} V_2^{-1} = M_C,$$

$z = (x_1 + iy_1, x_2 + iy_2) \in \mathcal{D}$, and C is given in (3.2).

Finally, let $g = (g_1, ..., g_n)^T \in (L^2(\mathbb{R} \times \mathbb{R}_+))^n$ and $B = (R_{(n,1)} T_c R^*_{(n,1)} g)(x_1, x_2)$, then

$$\begin{aligned} B &= (R^*_{0(n,1)} M_C R_{0(n,1)} g)(x_1, x_2) \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} C(z)(R_{0(n,1)} g)(x_1, y_1, x_2, y_2) \ell_0(y_2) H(y_1) dy_2 dy_1 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} C(z) \ell_0(y_2)[H(y_1)]^T g(x_1, x_2) \ell_0(y_2) H(y_1) dy_2 dy_1 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}_+} C(z)(\ell_0(y_2))^2 H(y_1)[H(y_1)]^T g(x_1, x_2) dy_2 dy_1. \end{aligned}$$

Thus, $R_{(n,1)} T_c R^*_{(n,1)} = \gamma^c I$, where $\gamma^c(x_1, x_2)$ is given in (4.1). □

As in Section 3, we consider two specific cases of nilpotent symbols. First, we will take Toeplitz operators with symbols of the form $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$, for which

$$\gamma^b(x_1, x_2) = \gamma^b(x_2) = \int_{\mathbb{R}_+} b\left(\frac{y_2}{2x_2}\right) (\ell_0(y_2))^2 dy_2 I_{n\times n}. \tag{4.2}$$

This spectral function can be identified with the scalar function

$$\begin{aligned}\mathbb{R}_+ &\to \mathbb{C}\\ x_2 &\mapsto \int_{\mathbb{R}_+} b\left(\frac{y_2}{2x_2}\right)(\ell_0(y_2))^2 dy_2,\end{aligned}$$

which was studied in [16]. Thus, the algebra generated by Toeplitz operators of the form T_b, where $b \in L^\infty_{\{0,+\infty\}}(\mathbb{R}_+)$, has been completely described.

Second, we analyze Toeplitz operators with symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$; in this case, we have

$$\gamma^a(x_1, x_2) = \int_{\mathbb{R}} a\left(\frac{-x_1 + y_1}{2\sqrt{x_2}}\right) H(y_1)[H(y_1)]^T dy_1. \tag{4.3}$$

From this point on, we focus on describing the C^*-algebra generated by matrix-valued functions of this type.

4.1. Continuity of the spectral function γ^a

In order to describe the C^*-algebra generated by Toeplitz operators acting on $\mathcal{A}^2_{(n,1)}(D_2)$ with nilpotent symbols of the form $\tilde{a}(\zeta) = a(\operatorname{Im}\zeta_1)$, first we will analyze the continuous extension of $\gamma^a = (\gamma^a_{jk})$ to the compactification $\overline{\Pi} := \overline{\mathbb{R}} \times \overline{\mathbb{R}}_+$. Make the change of variable $y_1 \mapsto 2\sqrt{x_2}y_1 + x_1$ in the integral representation of γ^a_{jk}, then

$$\gamma^a_{jk}(x_1, x_2) = 2\sqrt{x_2}\int_{\mathbb{R}} a(y_1) h_{j-1}(2\sqrt{x_2}y_1 + x_1) h_{k-1}(2\sqrt{x_2}y_1 + x_1) dy_1.$$

The function γ^a_{jk} is continuous at each point $(x_1, x_2) \in \Pi$ by the continuity of $h_{j-1}h_{k-1}$ and the Lebesgue dominated convergence theorem. Next, we will prove that γ^a_{jk} has a one-sided limit at each point of $\mathbb{R} \times \{0\}$. For $a \in L^\infty(\mathbb{R})$, we introduce the notation

$$a_- = \lim_{y\to-\infty} a(y) \quad \text{and} \quad a_+ = \lim_{y\to+\infty} a(y) \tag{4.4}$$

if such limits exist.

Lemma 4.2. *Let $a \in L^\infty(\mathbb{R})$ and suppose that a has limits at $\pm\infty$, then for each $x_0 \in \mathbb{R}$, the spectral matrix-valued function $\gamma^a : \Pi \to M_n(C)$ satisfies*

$$\begin{aligned}\lim_{(x_1,x_2)\to(x_0,0)} \gamma^a(x_1, x_2) = a_- &\int_{-\infty}^{x_0} H(y_1)[H(y_1)]^T dy_1\\ &+ a_+ \int_{x_0}^{+\infty} H(y_1)[H(y_1)]^T dy_1.\end{aligned} \tag{4.5}$$

Proof. Let A denote the (j, k)-entry of the righthand side of (4.5). Take $\epsilon > 0$. We will prove that there exists $\delta > 0$ such that $|\gamma^a(x_1, x_2) - A| < \epsilon$ whenever $|x_1 - x_0| < \delta$ and $0 < x_2 < \delta$. Note that $|a_-|, |a_+| \leq \|a\|_\infty$. Since $C_{jk} = \int_{-\infty}^{\infty} |h_{j-1}(y_1)h_{k-1}(y_1)| dy_1 > 0$, there exists $\delta_1 > 0$ such that

$$\|a\|_\infty \int_{-\delta_1+x_0}^{\delta_1+x_0} |h_{j-1}(y_1)h_{k-1}(y_1)| dy_1 < \frac{\epsilon}{5}.$$

Then

$$
\begin{aligned}
I &:= |\gamma^a_{jk}(x_1,x_2) - A| \\
&= \bigg| \int_{-\infty}^{\infty} a\left(\frac{-x_1+y_1}{2\sqrt{x_2}}\right) h_{j-1}(y_1)h_{k-1}(y_1)dy_1 \\
&\quad -a_- \int_{-\infty}^{x_0} h_{j-1}(y_1)h_{k-1}(y_1)dy_1 - a_+ \int_{x_0}^{\infty} h_{j-1}(y_1)h_{k-1}(y_1)dy_1 \bigg| \\
&\leq \int_{-\infty}^{-\delta_1+x_0} \left| a\left(\frac{-x_1+y_1}{2\sqrt{x_2}}\right) - a_- \right| |h_{j-1}(y_1)h_{k-1}(y_1)| dy_1 \\
&\quad +|a_-| \int_{-\delta_1+x_0}^{x_0} |h_{j-1}(y_1)h_{k-1}(y_1)| dy_1 \\
&\quad +|a_+| \int_{x_0}^{\delta_1+x_0} |h_{j-1}(y_1)h_{k-1}(y_1)| dy_1 \\
&\quad + \int_{-\delta_1+x_0}^{\delta_1+x_0} \left| a\left(\frac{-x_1+y_1}{2\sqrt{x_2}}\right) h_{j-1}(y_1)h_{k-1}(y_1) \right| dy_1 \\
&\quad + \int_{\delta_1+x_0}^{\infty} \left| a\left(\frac{-x_1+y_1}{2\sqrt{x_2}}\right) - a_+ \right| |h_{j-1}(y_1)h_{k-1}(y_1)| dy_1 \\
&\leq C_{jk} \max_{-\infty<y_1<-\delta_1+x_0} \left| a\left(\frac{-x_1+y_1}{2\sqrt{x_2}}\right) - a_- \right| + \frac{3\epsilon}{5} \\
&\quad +C_{jk} \max_{\delta_1+x_0<y_1<\infty} \left| a\left(\frac{-x_1+y_1}{2\sqrt{x_2}}\right) - a_+ \right|.
\end{aligned}
$$

We have assumed that a converges at $\pm\infty$. Thus, there exists $N > 0$ such that $|a(y) - a_-| < \epsilon/(5C_{jk})$ and $|a(y) - a_+| < \epsilon/(5C_{jk})$ for $|y| > N$. Let $\delta = \min\{\delta_1/2, \delta_1^2/(16N^2)\}$, then we have $\frac{1}{2\sqrt{x_2}}|-x_1+y_1| > N$ if $|x_1 - x_0| < \delta$, $0 < x_2 < \delta$ and $|y_1 - x_0| \geq \delta_1$. Thus,

$$
\max_{-\infty<y_1<-\delta_1+x_0} \left| a\left(\frac{-x_1+y_1}{2\sqrt{x_2}}\right) - a_- \right| < \frac{\epsilon}{5C_{jk}}
$$

and

$$
\max_{\delta_1+x_0<y_1<\infty} \left| a\left(\frac{-x_1+y_1}{2\sqrt{x_2}}\right) - a_+ \right| < \frac{\epsilon}{5C_{jk}}.
$$

Finally, we conclude that $|\gamma^a_{jk}(x_1,x_2) - A| < \epsilon$ whenever $|x_1 - x_0| < \delta$ and $0 < x_2 < \delta$. □

In general, the matrix-valued function γ^a does not converge at the points $(\pm\infty, +\infty) \in \overline{\Pi}$; however, γ^a has limit values along the parabolas $x_2 = \alpha(x_1^2 + 1)$, with $\alpha > 0$. For this reason, we introduce the mapping $\Phi : \Pi \longrightarrow \Pi$ given by

$$
\Phi(x_1, x_2) = \left(x_1, \frac{x_2}{x_1^2+1}\right).
$$

We will prove that $\phi^a = \gamma^a \circ \Phi^{-1} : \Pi \to M_n(\mathbb{C})$ has a continuous extension to $\overline{\Pi} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}_+$ with the usual topology. It is easy to see that $\Phi^{-1}(t_1, t_2) = (t_1, (t_1^2 + 1)t_2)$. Concerning the spectral properties of T_a, the matrix-valued function ϕ^a contains the same information as γ^a, but ϕ^a behaves much better than

γ^a, at least for a continuous on $\overline{\mathbb{R}}$. From now on, we take ϕ^a as the spectral matrix-valued function for the Toeplitz operator T_a. A direct computation shows that

$$\phi^a(t_1, t_2) = \int_{-\infty}^{\infty} a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2+1)}}\right) H(s_1)[H(s_1)]^T ds_1.$$

Note that both Φ and Φ^{-1} are continuous on $\mathbb{R} \times [0, +\infty)$. In addition, the spectral function $\phi^a = \gamma^a \circ \Phi^{-1}$ is continuous on $\mathbb{R} \times [0, +\infty)$ because γ^a is. Since $\Phi^{-1}(t_1, 0) = (t_1, 0)$, we have that $\phi^a(t_1, 0) = \gamma^a(t_1, 0)$ for all $t_1 \in \mathbb{C}$.

Theorem 4.3. *For $a \in C(\overline{\mathbb{R}})$, the spectral matrix-valued function $\phi^a : \Pi \to M_n(\mathbb{C})$ can be extended continuously to $\overline{\Pi} = \overline{\mathbb{R}} \times \overline{\mathbb{R}}_+$. Furthermore, ϕ^a is constant along $\overline{\mathbb{R}} \times \{+\infty\}$.*

Proof. The result follows from Lemmas 4.2 and 4.4–4.6 below. □

Lemma 4.4. *Let $a \in L^\infty(\mathbb{R})$ and suppose that a converges at $\pm\infty$, then $\phi^a = (\phi^a_{jk})$ satisfies*

$$\lim_{(t_1,t_2)\to(+\infty,0)} \phi^a(t_1, t_2) = a(-\infty)I.$$

That is, for $\epsilon > 0$, there exists $\delta > 0$ and $N > 0$ such that $|\phi^a_{jk}(t_1, t_2) - \delta_{jk} a(-\infty)| < \epsilon$ whenever $0 < t_2 < \delta$ and $t_1 > N$. Analogously,

$$\lim_{(t_1,t_2)\to(-\infty,0)} \phi^a(t_1, t_2) = a(+\infty)I.$$

Proof. Suppose that $a(-\infty) = 0$. Let $\epsilon > 0$. Since $h_{j-1}(s_1)h_{k-1}(s_1) \in L^2(\mathbb{R})$, there exists $s_0 > 0$ such that

$$\|a\|_\infty \int_{s_0}^{\infty} |h_{j-1}(s_1)h_{k-1}(s_1)| ds_1 < \frac{\epsilon}{2}.$$

Let $C_{jk} = \int_{-\infty}^{\infty} |h_{j-1}(s_1)h_{k-1}(s_1)| dy_1 > 0$, then we have

$$\begin{aligned} |\phi^a_{jk}(t_1, t_2)| &= \left| \int_{-\infty}^{\infty} a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2+1)}}\right) h_{j-1}(s_1)h_{k-1}(s_1) ds_1 \right| \\ &\leq \int_{-\infty}^{s_0} \left| a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2+1)}}\right) h_{j-1}(s_1)h_{k-1}(s_1) \right| ds_1 \\ &\quad + \int_{s_0}^{\infty} \left| a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2+1)}}\right) h_{j-1}(s_1)h_{k-1}(s_1) \right| ds_1 \\ &\leq C_{jk} \max_{-\infty < s_1 < s_0} \left| a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2+1)}}\right) \right| + \frac{\epsilon}{2}. \end{aligned}$$

Since a converges to zero at $-\infty$, there exists $N_1 > 0$ such that $C_{jk}|a(s)| < \epsilon/2$ for $-s > N_1$. Take $\delta = 1/(16N_1^2)$, then we have $\frac{1}{2\sqrt{t_2}} > 2N_1$ for $0 < t_2 < \delta$. On the other hand, assume $t_1 > s_0$ and $-\infty < s_1 < s_0$, then

$$\frac{t_1 - s_1}{\sqrt{t_1^2 + 1}} > \frac{t_1 - s_0}{\sqrt{t_1^2 + 1}}.$$

The righthand side of this inequality converges to 1 when t_1 tends to $+\infty$, thus there exists $N_2 > s_0$ such that $(t_1 - s_0)/\sqrt{t_1^2 + 1} > 1/2$ for $t_1 > N_2$. Consequently,

$$N_1 = 2N_1\frac{1}{2} < \frac{1}{2\sqrt{t_2}}\frac{t_1 - s_0}{\sqrt{(t_1^2 + 1)}} < \frac{t_1 - s_1}{2\sqrt{t_2(t_1^2 + 1)}}.$$

For $0 < t_2 < \delta$ and $t_1 > N := \max\{s_0, N_2\}$ we have

$$C_{jk}\left|a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2 + 1)}}\right)\right| < \frac{\epsilon}{2}.$$

We define $\hat{a}(s) = a(s) - a_2$ in the case $a(-\infty) \neq 0$, where $a_2 := a(-\infty)$. Note that $\hat{a}$ converges to zero at $-\infty$ and $\phi^{a_1+a_2} = \phi^{a_1} + \phi^{a_2}$ for any nilpotent symbols a_1 and a_2, then

$$\begin{aligned}\lim_{(t_1,t_2)\to(+\infty,0)} \phi^a_{jk}(t_1, t_2) &= \lim_{(t_1,t_2)\to(+\infty,0)} \phi^{\hat{a}+a_2}_{jk}(t_1, t_2)\\ &= \lim_{(t_1,t_2)\to(+\infty,0)} \phi^{\hat{a}}_{jk}(t_1, t_2)\\ &\quad + a_2\int_{-\infty}^{\infty} h_{j-1}(s_1)h_{k-1}(s_1)ds_1\\ &= a(-\infty)\delta_{jk}.\end{aligned}$$

Finally, the limit of ϕ^a at $(-\infty, 0)$ can be proved analogously. □

Lemma 4.5. *Let $t_0 \in \mathbb{R}_+$. If $a \in L^\infty(\mathbb{R})$ is continuous at $-1/(2\sqrt{t_0})$, then the spectral matrix-valued function ϕ^a satisfies*

$$\lim_{(t_1,t_2)\to(+\infty,t_0)} \phi^a(t_1, t_2) = a\left(-\frac{1}{2\sqrt{t_0}}\right)I.$$

Analogously, if a is continuous at $1/(2\sqrt{t_0})$, then

$$\lim_{(t_1,t_2)\to(-\infty,t_0)} \phi^a(t_1, t_2) = a\left(\frac{1}{2\sqrt{t_0}}\right)I.$$

Proof. Suppose that a converges to zero at $-1/(2\sqrt{t_0})$. Let $\epsilon > 0$. Since $h_{j-1}(s_1), h_{k-1}(s_1) \in L^2(\mathbb{R}, ds_1)$, there exists $s_0 > 0$ such that

$$||a||_\infty \int_{-\infty}^{-s_0} |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 < \frac{\epsilon}{3}, \tag{4.6}$$

$$\|a\|_\infty \int_{s_0}^{\infty} |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 < \frac{\epsilon}{3}.$$

Take into account $C_{jk} = \int_{-\infty}^{\infty} |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 > 0$ in the following computation

$$\begin{aligned}
|\phi^a_{jk}(t_1,t_2)| &\leq \int_{-\infty}^{-s_0} \left| a\left(\frac{-t_1+s_1}{2\sqrt{t_2(t_1^2+1)}}\right)\right| |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 \\
&\quad + \int_{-s_0}^{s_0} \left| a\left(\frac{-t_1+s_1}{2\sqrt{t_2(t_1^2+1)}}\right)\right| |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 \\
&\quad + \int_{s_0}^{\infty} \left| a\left(\frac{-t_1+s_1}{2\sqrt{t_2(t_1^2+1)}}\right)\right| |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 \\
&< \frac{2\epsilon}{3} + C_{jk} \max_{-s_0<s_1<s_0} \left| a\left(\frac{-t_1+s_1}{2\sqrt{t_2(t_1^2+1)}}\right)\right|.
\end{aligned}$$

Because of the continuity of a at $-1/(2\sqrt{t_0})$, there exists $\delta_1 > 0$ such that $C_{jk}|a(s)| < \epsilon/3$ for $|s - \frac{-1}{2\sqrt{t_0}}| < \delta_1$. Let us estimate the value of the argument of a:

$$\begin{aligned}
I &:= \left| \frac{1}{2\sqrt{t_2(t_1^2+1)}}(-t_1+s_1) - \frac{-1}{2\sqrt{t_0}} \right| \\
&\leq \left| -\frac{1}{2\sqrt{t_2}} + \frac{1}{2\sqrt{t_0}} \right| \left| \frac{t_1}{\sqrt{t_1^2+1}} \right| + \frac{1}{2\sqrt{t_0}} \left| 1 - \frac{t_1}{\sqrt{t_1^2+1}} \right| \\
&\quad + \left| \frac{s_1}{2\sqrt{t_2(t_1^2+1)}} \right|.
\end{aligned}$$

Choose $\delta > 0$ such that $\left| -\frac{1}{2\sqrt{t_2}} + \frac{1}{2\sqrt{t_0}} \right| < \delta_1/3$ for $|t_2 - t_0| < \delta$. Pick $N_1 > 0$ such that $\left| 1 - \frac{t_1}{\sqrt{t_1^2+1}} \right| < (2\sqrt{t_0}\delta_1)/3$ whenever $t_1 > N_1$. Now, assume that $|t_2 - t_0| < \delta$ and $|s_1| < s_0$, then $|\frac{1}{2\sqrt{t_2}}| < \frac{1}{2\sqrt{t_0}} + \frac{\delta_1}{3}$. Thus, $\frac{|s_1|}{2\sqrt{t_2(t_1^2+1)}}$ converges to 0 when t_1 tends to $+\infty$. Therefore, there exists $N > N_1$ such that $\frac{|s_1|}{2\sqrt{t_2(t_1^2+1)}} < \delta_1/3$ for $t_1 > N$. The additional condition $t_1 > N$ implies

$$C_{jk} \left| a\left(\frac{-t_1+s_1}{2\sqrt{t_2(t_1^2+1)}}\right)\right| < \epsilon/3.$$

Hence, $|\phi^a_{jk}(t_1,t_2)| < \epsilon$ if $|t_2 - t_0| < \delta$ and $t_1 > N$.

If a does not converge to zero at $-\frac{1}{2\sqrt{t_0}}$, then take the function $\hat{a}(s) = a(s) - a_2$ and proceed as in the proof of Lemma 4.4, where $a_2 = a\left(-\frac{1}{2\sqrt{t_0}}\right)$.

Finally, the justification of the limit of ϕ^a at $(-\infty, t_0)$ can be done analogously. □

Lemma 4.6. *Let $a \in L^\infty(\mathbb{R})$ be continuous at $0 \in \mathbb{R}$. For $t_0 \in \overline{\mathbb{R}}$, the spectral matrix-valued function ϕ^a satisfies*

$$\lim_{(t_1,t_2)\to(t_0,+\infty)} \phi^a(t_1, t_2) = a(0)I.$$

Actually, we have uniform convergence of $\phi^a(t_1, t_2)$ at $(t_0, +\infty)$; that is, for $\epsilon > 0$, there exists $N > 0$ such that $|\phi^a_{jk}(t_1, t_2) - a(0)| < \epsilon$ for all $t_2 > N$ and for all $t_1 \in \overline{\mathbb{R}}$.

Proof. Suppose that $a(0) = 0$. Let $\epsilon > 0$, and choose $s_0 > 0$ such that Eq (4.6) holds. Let $C_{jk} = \int_{-\infty}^{\infty} |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 > 0$, then

$$\begin{aligned}
|\phi^a_{jk}(t_1, t_2)| \leq\ & \int_{-\infty}^{-s_0} \left| a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2 + 1)}}\right)\right| |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 \\
& + \int_{-s_0}^{s_0} \left| a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2 + 1)}}\right)\right| |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 \\
& + \int_{s_0}^{\infty} \left| a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2 + 1)}}\right)\right| |h_{j-1}(s_1)h_{k-1}(s_1)|ds_1 \\
<\ & \frac{2\epsilon}{3} + C_{jk} \max_{-s_0<s_1<s_0} \left| a\left(\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2 + 1)}}\right)\right|.
\end{aligned}$$

By the continuity of a at 0, there exists $\delta_1 > 0$ such that $|a(s)| < \epsilon/(3C_{jk})$ for $|s| < \delta_1$. For $-s_0 < s_1 < s_0$, we have

$$\left|\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2 + 1)}}\right| \leq \frac{1}{2\sqrt{t_2}}\left(\left|\frac{t_1}{\sqrt{t_1^2 + 1}}\right| + \frac{|s_1|}{\sqrt{t_1^2 + 1}}\right) < \frac{1}{2\sqrt{t_2}}(1 + s_0).$$

Take $N = (1 + s_0)^2/(4\delta_1^2)$. The inequality $t_2 > N$ implies $\frac{1}{2\sqrt{t_2}} < \frac{\delta_1}{(1+s_0)}$. Thus, if $t_2 > N$, $t_1 \in \overline{\mathbb{R}}$ and $-s_0 < s_1 < s_0$, then

$$\left|\frac{-t_1 + s_1}{2\sqrt{t_2(t_1^2 + 1)}}\right| < \delta_1.$$

Consequently, $|\phi^a_{jk}(t_1, t_2)| < \epsilon$ for all $t_2 > N$ and $t_1 \in \overline{\mathbb{R}}$.

Finally, in the case $a(0) \neq 0$, the proof can be carried out by considering the symbol $\hat{a}(s) = a(s) - a(0)$. □

For each nilpotent symbol $a \in C(\overline{\mathbb{R}})$, the spectral function ϕ^a is continuous on $\overline{\Pi}$ and is constant along $\overline{\mathbb{R}} \times \{+\infty\}$. In order to obtain a larger algebra, we now consider symbols $a \in PC(\overline{\mathbb{R}}, \{0\})$, where $PC(\overline{\mathbb{R}}, \{0\})$ is the set of continuous functions on $\overline{\mathbb{R}}$ with one-sided limits at 0.

Consider the indicator function $\chi_+ = \chi_{[0,+\infty]}$, for which

$$\phi^{\chi_+}(t_1, t_2) = \int_{t_1}^{\infty} H(s_1)[H(s_1)]^T ds_1. \tag{4.7}$$

Theorem 4.7. *Let $a \in PC(\overline{\mathbb{R}}, \{0\})$, then the spectral matrix-valued function ϕ^a can be extended continuously to $\overline{\Pi}$.*

Proof. For $a \in PC(\overline{\mathbb{R}}, \{0\})$, we have

$$a(s) = \hat{a}(s) + [a(0_+) - a(0_-)]\chi_+(s),$$

where $a(0_-)$ and $a(0_+)$ are the one-side limits of a at 0, and $\hat{a}(s) = a(s) + [a(0_-) - a(0_+)]\chi_+(s)$. Since $\hat{a} \in C(\overline{\mathbb{R}})$, the spectral function $\phi^{\hat{a}}$ is continuous on $\overline{\Pi}$. According to (4.7), ϕ^{χ_+} is obviously continuous on $\overline{\Pi}$. □

The spectral matrix-valued function ϕ^{χ_+} depends only the real variable t_1; thus, it can be identified with the one-variable function

$$\phi^+(t) := \int_t^{\infty} H(s)[H(s)]^T ds. \tag{4.8}$$

Lemma 4.8. *The matrix-valued function $\phi^+ = (\phi^+_{jk})$ satisfies:*

(1) $\phi^+(-\infty) = I$ and $\phi^+(+\infty) = 0$.
(2) For each $t \in \mathbb{R}$, $\phi^+(t)$ is symmetric positive definite and $\|\phi^+(t)\| \le 1$, where $\|\cdot\|$ is the uniform norm.
(3) There exists $E \in M_n(\mathbb{C})$ such that for all $t \in \overline{\mathbb{R}}$, one has that $\phi^+(t) = EM_tE^T$, where $E \in M_n(\mathbb{C})$ and

$$M_t = \int_t^{\infty} e^{-s^2} S S^T ds, \quad S = (1, s, ..., s^{n-1})^T.$$

(4) For each $t \in \mathbb{R}$ and $\lambda \in \mathbb{C}$, $\det(\lambda I - \phi^+(t)) = 0$ if, and only if, $\det(\lambda M_{-\infty} - M_t) = 0$.

Proof. Part (1) follows since $\{h_j\}_{j=0}^{\infty}$ is an orthonormal basis for $L_2(\mathbb{R})$. The matrix $\phi^+(t)$ is symmetric for all t because $H(s)H(s)^T$ is symmetric for all s. Let $v \in \mathbb{C}^n$ be a unit vector, then

$$\langle \phi^+(t)v, v\rangle = \int_t^{\infty} |\langle H(s), v\rangle|^2 ds, \tag{4.9}$$

where $e^{s^2}|\langle H(s), v\rangle|^2$ is a nonzero polynomial, thus $\langle \phi^+(t)v, v\rangle > 0$. Now, we note that

$$\langle \phi^+(t)v, v\rangle < \int_{-\infty}^{\infty} |\langle H(s), v\rangle|^2 ds = \langle \phi^+(-\infty)v, v\rangle = \langle Iv, v\rangle = 1,$$

hence, $\|\phi^+(t)\| \le 1$. This proves (2).

The Hermite function is given by

$$\begin{aligned} h_k(s) &= e^{-s^2/2}\sum_{m=0}^{[k/2]} d_{km}\, s^{k-2m}, \quad d_{km} = \frac{1}{\sqrt{2^k k!\,\sqrt{\pi}}}\frac{(-1)^m k! 2^{k-2m}}{m!(k-2m)!} \\ &= e^{-s^2/2}\sum_{m=0}^{k} c_{km}\, s^m. \end{aligned}$$

Define

$$E = \begin{pmatrix} c_{00} & 0 & \cdots & 0 \\ c_{10} & c_{11} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{n-1,0} & c_{n-1,1} & \cdots & c_{n-1,n-1} \end{pmatrix},$$

then $H(s) = (h_0(s), ..., h_{n-1}(s))^T = e^{-s^2/2} ES$ and

$$H(s)H(s)^T = e^{-s^2} ES(ES)^T = e^{-s^2} ESS^T E^T.$$

Therefore, $\phi^+(t) = EM_t E^T$. Also, $\det E \neq 0$ since E is a lower triangular matrix and the scalars c_{jj} are nonzero. This proves (3).

Finally, let $\lambda \in \mathbb{C}$. We have $I = \phi^+(-\infty) = EM_{-\infty}E^T$, then

$$\begin{aligned} \lambda I - \phi^+(t) &= \lambda EM_{-\infty}E^T - EM_t E^T \\ &= E(\lambda M_{-\infty} - M_t)E^T. \end{aligned}$$

Thus, $\det(\lambda I - EM_t E^T) = 0$ if, and only if, $\det(\lambda M_{-\infty} - M_t) = 0$. □

4.2. The algebra generated by the Toeplitz operator T_{χ^+}

The C^*-algebra generated by T_{χ^+} is isomorphic to the C^*-algebra generated by ϕ^+. Let $\mathcal{D}_n$ be C^*-algebra generated by I and ϕ^+, which is a subalgebra of $M_n(\mathbb{C}) \otimes C(\overline{\mathbb{R}})$, where the metric is given by $\|M\| = \max_{t\in\overline{\mathbb{R}}} \|M(t)\|$.

According to Lemma 4.8, the matrix $\phi^+(t)$ is diagonalizable for each $t \in \mathbb{R}$ and its spectrum $\sigma(\phi^+(t))$ lies in $[0, 1]$. The eigenvalues are given by the equation $\det(\lambda M_{-\infty} - M_t) = 0$. There exists an orthogonal matrix $B(t)$ such that

$$D(t) := B(t)^T \phi^+(t)B(t) = \operatorname{diag}\{\lambda_1(t), ..., \lambda_n(t)\},$$

that is, if $B(t) = [v_1(t)\cdots v_n(t)]$, then $\phi^+(t)v_j(t) = \lambda_j(t)v_j(t)$ for $j = 1, ..., n$. We may assume that B and λ_j are continuous on $\overline{\mathbb{R}}$, and $\lambda_1(t) \leq \lambda_2(t) \leq \cdots \leq \lambda_n(t)$. We have $\lambda_j(-\infty) = 1$ and $\lambda_j(+\infty) = 0$.

Up to isomorphism, the C^*-algebra $\mathcal{D}_n$ is equal to the C^*-algebra generated by D, that is, each element $\varphi \in \mathcal{D}_n$ is a uniform limit of polynomials on D:

$$\varphi(t) = \lim_{m\to\infty} \operatorname{diag}\{p_m(\lambda_1(t)), ..., p_m(\lambda_n(t))\}.$$

Therefore, we conclude

Theorem 4.9. *Let $C_n(\overline{\mathbb{R}})$ be the C^*-subalgebra of $(C(\overline{\mathbb{R}}))^n$ that consists of all n-tuples $f = (f_1, ..., f_n)$ such that*

$$f_j(t) = f_k(x)$$

when $\lambda_j(t) = \lambda_k(x)$. We can identify f with diag $\{f_1, ..., f_n\}$, then the C^*-algebra $\mathcal{D}_n$ generated by ϕ^+ is isomorphic to $C_n(\overline{\mathbb{R}})$, where the isomorphism is given by

$$\varphi \longmapsto B^T \varphi B.$$

4.3. The algebra generated by the Toeplitz operators with symbols $a \in PC(\overline{\mathbb{R}}, \{0\})$

In this section, we describe the C^*-algebra generated by all the Toeplitz operators T_a or, equivalently, the C^*-algebra generated by the matrix-valued functions $\phi^a : \overline{\Pi} \to \mathbb{C}$ with $a \in PC(\overline{\mathbb{R}}, \{0\})$. Let $\mathfrak{B}$ be the C^*-algebra generated by all the matrix-valued functions ϕ^a with $a \in PC(\overline{\mathbb{R}}, \{0\})$, and let $\mathcal{T}$ be the C^*-subalgebra of $M_n(C(\overline{\Pi})) = M_n(\mathbb{C}) \otimes C(\overline{\Pi})$ consisting of all M such that $M(\pm\infty, t_2) \in \mathbb{C}I$ for each $t_2 \in \overline{\mathbb{R}}_+$ and

$$B^T M(\cdot, 0)B,\ B^T M(\cdot, +\infty)B \in C_n(\overline{\mathbb{R}}).$$

We will prove that $\mathfrak{B} = \mathcal{T}$ by using a Stone-Weierstrass theorem for C^*-algebras. Recall that a C^*-algebra $\mathcal{A}$ is said to be a CCR algebra if for every non-cero irreducible representation (H, π) of $\mathcal{A}$ we have $\pi(\mathcal{A}) \subset K(H)$, where $K(H)$ is the ideal of all compact operators acting on the Hilbert space H.

Theorem 4.10. *[26] Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras such that $\mathcal{B} \subset \mathcal{A}$. If $\mathcal{A}$ is a CCR algebra and $\mathcal{B}$ separates the pure state space of $\mathcal{A}$, then $\mathcal{B} = \mathcal{A}$.*

Our main result of this section is the following:

Theorem 4.11. *The C^*-algebra generated by all matrix-valued functions ϕ^a, with $a \in PC(\overline{\mathbb{R}}, \{0\})$, equals $\mathcal{T}$. That is, the C^*-algebra generated by all Toeplitz operators T_a is isomorphic and isometric to the algebra $\mathcal{T}$, where the isomorphism is defined on the generators by the rule*

$$T_a \mapsto \phi^a.$$

Proof. $\mathfrak{B} = \mathcal{T}$ follows from Theorem 4.10. That is, $\mathfrak{B}$ separates the pure state space of $\mathcal{T}$ according to Lemmas 4.12–4.15, 4.17, and 4.19. □

It is easy to see that $\mathfrak{B}$ is contained in $\mathcal{T}$. Let $\langle \cdot, \cdot \rangle$ denote the usual inner product on $\mathbb{C}^n$. Now, the pure state space of the C^*-algebra $\mathcal{T}$ consists of all functionals having the form:

1) $f_{(x_1,x_2),v}(M) = \langle M(x_1, x_2)v, v \rangle$ for $(x_1, x_2) \in \Pi$, $v \in \mathbb{C}^n$ a unit vector,
2) $f_{(\pm\infty,t_2)}(M) = \lambda_{\pm t_2}$ for $0 \le t_2 \le +\infty$, where $\lambda_{\pm t_2} I = M(\pm\infty, t_2)$,
3) $f_{(t_1,\pm\infty),j}(M) = \langle M(t_1, \pm\infty)v_j(t_1), v_j(t_1) \rangle$ for $t_1 \in \overline{\mathbb{R}}$ and $j =, ..., n$,

where $M \in \mathcal{T}$ is arbitrary. The remainder of this section is devoted to separate all the pure states of $\mathcal{T}$.

Lemma 4.12. *Let $t_2, \tau_2 \in \overline{\mathbb{R}}_+$. We have $f_{(-\infty,t_2)}(\phi^{\chi_+}) \neq f_{(+\infty,\tau_2)}(\phi^{\chi_+})$. If $t_2 \neq \tau_2$, then there exists $a \in C(\overline{\mathbb{R}})$ such that $f_{(\pm\infty,t_2)}(\phi^a) \neq f_{(\pm\infty,\tau_2)}(\phi^a)$.*

Proof. The pure states $f_{(-\infty,t_2)}$ and $f_{(+\infty,\tau_2)}$ are separated by ϕ^{χ_+} since $\phi^{\chi_+}(-\infty,+\infty) = I$ and $\phi^{\chi_+}(+\infty,+\infty) = 0$. If $a \in C(\overline{\mathbb{R}})$, then $\phi^a(\pm\infty,t_2) = a(\mp 1/(2\sqrt{t_2}))I$ for $t_2 \in \mathbb{R}+$, $\phi^a(\pm\infty,0) = a(\mp\infty)I$, and $\phi^a(\pm\infty,+\infty) = a(0)I$. Thus, taking $a(s) = s/\sqrt{s^2+1}$, we have

$$f_{(\pm\infty,t_2)}(\phi^a) = \mp\frac{1}{\sqrt{1+4t_2}}.$$

Therefore, the pure states $f_{(\pm\infty,t_2)}$ and $f_{(\pm\infty,\tau_2)}$ are separated by ϕ^a when $t_2 \neq \tau_2$. □

We shall continue separating the rest of pure states using continuous functions on $\overline{\mathbb{R}}$ and the indicator function χ_+.

Let $v \in \mathbb{C}^n$ be a unit vector. Consider the function $h_v(s) = |\langle H(s), v\rangle|^2$. This can be written as $h_v(s) = q_v(s)e^{-s^2}$, where

$$q_v(s) = |v_0 d_0 H_0(s) + \cdots + v_{n-1}d_{n-1}H_{n-1}(s)|^2$$

is a polynomial of degree at most $2n-2$ taking nonnegative values.

Lemma 4.13. *Let $v \in \mathbb{C}^n$ be a unit vector and $(\pm\infty,t_2)$, (x_1,x_2) be points with $x_1 \in \mathbb{R}$ and $x_2, t_2 \in \overline{\mathbb{R}}_+$, then there exists a symbol $a \in PC(\overline{\mathbb{R}},\{0\})$ such that*

$$f_{(\pm\infty,t_2)}(\phi^a) \neq f_{(x_1,x_2),v}(\phi^a).$$

Proof. For $x_1 \in \mathbb{R}$ and $x_2 \in \overline{\mathbb{R}}_+$, we have

$$f_{(x_1,x_2),v}(\phi^{\chi_+}) = \int_{x_1}^{\infty} |\langle H(s), v\rangle|^2 ds = \int_{x_1}^{\infty} q_v(s)e^{-s^2}ds.$$

Since q_v is not zero and nonnegative, $f_{(x_1,x_2),v}(\phi^{\chi_+}) > 0$. On the other hand, $f_{(+\infty,t_2)}(\phi^{\chi_+}) = \chi_+(-1/(2\sqrt{t_2})) = 0$ for $t_2 \in [0,+\infty]$. Hence,

$$f_{(+\infty,t_2)}(\phi^{\chi_+}) \neq f_{(x_1,x_2),v}(\phi^{\chi_+}).$$

We now take $\chi_- = 1 - \chi_+$, then

$$f_{(x_1,x_2),v}(\phi^{\chi_-}) = \int_{-\infty}^{x_1} |\langle H(s), v\rangle|^2 ds = \int_{-\infty}^{x_1} q_v(s)e^{-s^2} > 0.$$

Also, $f_{(-\infty,t_2)}(\phi^{\chi_-}) = \chi_-(1/(2\sqrt{t_2})) = 0$, then

$$f_{(-\infty,t_2)}(\phi^{\chi_-}) \neq f_{(x_1,x_2),v}(\phi^{\chi_-}).$$

□

Lemma 4.14. *Let $v, w \in \mathbb{C}^n$ be unit vectors. Take $(t_1,0),(x_1,x_2) \in \overline{\Pi}$, with $t_1 \in \mathbb{R}$, $x_1 \in \mathbb{R}$ and $0 < x_2 \leq +\infty$, then there exists a symbol $a \in C(\overline{\mathbb{R}})$ such that*

$$f_{(t_1,0),w}(\phi^a) \neq f_{(x_1,x_2),v}(\phi^a).$$

Proof. Consider $a(s) = 1/(s^2 + 1)$ for which $a(-\infty) = 0 = a(+\infty)$, then

$$f_{(t_1,0),w}(\phi^a) = a(-\infty)\int_{-\infty}^{t_1} |\langle H(s), w\rangle|^2 ds + a(+\infty)\int_{t_1}^{\infty} |\langle H(s), w\rangle|^2 ds = 0.$$

Since $a(s) > 0$ for all $s \in \mathbb{R}$, and $q_v(s) = 0$ at most at finite number of values of s, we have that $a(s)q_v(s)e^{-s^2} > 0$ almost everywhere, then

$$f_{(x_1,x_2),v}(\phi^a) = \int_{-\infty}^{\infty} a\left(\frac{-x_1 + s}{2\sqrt{x_2(x_1^2 + 1)}}\right) q_v(s)e^{-s^2} ds > 0 \quad \text{if } x_2 \in \mathbb{R}.$$

Moreover, $\phi^a(x_1, +\infty) = a(0)I = 1 \cdot I$. Thus, $f_{(x_1,+\infty),v}(\phi^a) = 1$. We have proved that $f_{(t_1,0),w}(\phi^a) \neq f_{(x_1,x_2),v}(\phi^a)$. □

Lemma 4.15. *Let $v, w \in \mathbb{C}^n$ be unit vectors, $(x_1, x_2) \in \Pi$, and $(t_1, +\infty) \in \overline{\Pi}$ with $t_1 \in \mathbb{R}$, then there exists $a \in C(\overline{\mathbb{R}})$ such that*

$$f_{(x_1,x_2),v}(\phi^a) \neq f_{(t_1,+\infty),w}(\phi^a).$$

Proof. Let $a(s) = |s|/(s^2 + 1)$ so that $a(0) = 0$ and $f_{(t_1,+\infty),w}(\phi^a) = a(0) = 0$. Since $q_v(s) = 0$ at most at finite number of points,

$$f_{(x_1,x_2),v}(\phi^a) = \int_{-\infty}^{\infty} a\left(\frac{-x_1 + s}{2\sqrt{x_2(x_1^2 + 1)}}\right) q_v(s)e^{-s^2} ds > 0.$$

□

Next, we will separate the pure states associated to the points $(t_1, t_2) \in \Pi$ using continuous symbols indexed by $\alpha > 0$ and $r \in \mathbb{R}$. We introduce

$$a_\alpha^r(y) = \frac{1}{\alpha} a([y - r]/\alpha)$$

where

$$a(y) = \begin{cases} 0 & \text{if } \; y \notin [-1, 1] \\ 1 + y & \text{if } \; y \in [-1, 0] \\ 1 - y & \text{if } \; y \in [0, 1]. \end{cases}$$

Note that the family of functions $a_\alpha := a_\alpha^0$ is an approximate identity in $L^1(\mathbb{R}, d\mu)$. Since $h_j h_k \in L^1(\mathbb{R})$, we have pointwise convergence in

$$\lim_{\alpha\to 0}(a_\alpha * h_j h_k)(y) = (h_j h_k)(y)$$

because $h_j h_k$ is continuous.

Since $\Phi : \Pi \to \Pi$ is a homeomorphism and $\phi^a = \gamma^a \circ \Phi^{-1}$, we consider the matrix-valued function γ^a in order to carry out the separation of pure states associated to the points in Π.

Lemma 4.16. *Let $(x_1, x_2) \in \Pi$ and $a_{r,\alpha} = a^r_{\frac{\alpha}{2\sqrt{x_2}}}$, with $\alpha > 0$ and $r \in \mathbb{R}$, then the matrix-valued function $\gamma^{a_{r,\alpha}}$ satisfies*

$$\lim_{\alpha\to 0} \gamma^{a_{r,\alpha}}(x_1, x_2) = 2\sqrt{x_2} H(x_1 + 2\sqrt{x_2} r)[H(x_1 + 2\sqrt{x_2} r)]^T. \tag{4.10}$$

Further,

$$\lim_{\alpha\to 0} f_{(x_1,x_2),v}(\gamma^{a_{r_1,\alpha}} \gamma^{a_{r_2,\alpha}}) = 4x_2\ [H(\beta_1)]^T H(\beta_2)\ \langle H(\beta_1), v\rangle\langle v, H(\beta_2)\rangle, \tag{4.11}$$

where $\beta_i = x_1 + 2\sqrt{x_2} r_i$ for $i = 1, 2$.

Proof. Take into account that $\{a_\alpha\}$ is an approximate identity and $a_\alpha(y - x) = a_\alpha(x - y)$ for any $x, y \in \mathbb{R}$. Calculate the entries of $\gamma^{a_{r,\alpha}}$:

$$\begin{aligned}
\gamma^{a_{r,\alpha}}_{jk}(x_1, x_2) &= \int_{-\infty}^{\infty} a_{r,\alpha}\left(\frac{-x_1 + y}{2\sqrt{x_2}}\right)(h_{j-1}h_{k-1})(y)dy \\
&= 2\sqrt{x_2} \int_{-\infty}^{\infty} \frac{1}{\alpha} a\left(\frac{y - (x_1 + 2\sqrt{x_2} r)}{\alpha}\right)(h_{j-1}h_{k-1})(y)dy \\
&= 2\sqrt{x_2} \int_{-\infty}^{\infty} a_\alpha(y - (x_1 + 2\sqrt{x_2} r))(h_{j-1}h_{k-1})(y)dy \\
&= 2\sqrt{x_2} \int_{-\infty}^{\infty} a_\alpha((x_1 + 2\sqrt{x_2} r) - y)(h_{j-1}h_{k-1})(y)dy \\
&= 2\sqrt{x_2}((h_{j-1}h_{k-1}) * a_\alpha)(x_1 + 2\sqrt{x_2} r).
\end{aligned}$$

Since a_α is an approximate identity, we have that

$$\begin{aligned}
\lim_{\alpha\to 0} \gamma^{a_{r,\alpha}}_{jk}(x_1, x_2) &= \lim_{\alpha\to 0} 2\sqrt{x_2}((h_{j-1}h_{k-1}) * a_\alpha)(x_1 + 2\sqrt{x_2} r) \\
&= 2\sqrt{x_2} \lim_{\alpha\to 0}((h_{j-1}h_{k-1}) * a_\alpha)(x_1 + 2\sqrt{x_2} r) \\
&= 2\sqrt{x_2}(h_{j-1}h_{k-1})(x_1 + 2\sqrt{x_2} r).
\end{aligned}$$

This completes the proof of (4.10). Finally,

$$\begin{aligned}
I &:= \lim_{\alpha\to 0} f_{(x_1,x_2),v}(\gamma^{a_{r_1,\alpha}} \gamma^{a_{r_2,\alpha}}) \\
&= \lim_{\alpha\to 0}\langle \gamma^{a_{r_1,\alpha}}(x_1, x_2)\gamma^{a_{r_2,\alpha}}(x_1, x_2)v, v\rangle \\
&= 4x_2\langle H(\beta_1)[H(\beta_1)]^T H(\beta_2)[H(\beta_2)]^T v, v\rangle \\
&= 4x_2\ [H(\beta_1)]^T H(\beta_2)\ \langle H(\beta_1), v\rangle\langle v, H(\beta_2)\rangle.
\end{aligned}$$

□

For v and w unit vectors in $\mathbb{C}^n$ and $(t_1, t_2) \neq (x_1, x_2) \in \Pi$, the following result says that the pure states $f_{(t_1,t_2),w}$ and $f_{(x_1,x_2),v}$ can be separated by $\mathfrak{B}$.

Lemma 4.17. *Let v, $w \in \mathbb{C}^n$ be unit vectors, (t_1, t_2), $(x_1, x_2) \in \Pi$, and $a_{r,\alpha,t_2} = a^r_{\frac{\alpha}{4\sqrt{x_2 t_2}}}$, with $\alpha > 0$ and $r \in \mathbb{R}$. If*

$$f_{(x_1,x_2),v}(\gamma^{a_{r,\alpha,t_2}}) = f_{(t_1,t_2),w}(\gamma^{a_{r,\alpha,t_2}}) \quad \forall\alpha > 0, r \in \mathbb{R}, \tag{4.12}$$

then $(t_1, t_2) = (x_1, x_2)$.

Proof. It is easy to see that

$$\gamma_{jk}^{a_{r,\alpha,t_2}}(x_1,x_2) = 2\sqrt{x_2}((h_{j-1}h_{k-1}) * a_{\frac{\alpha}{2\sqrt{t_2}}})(x_1 + 2\sqrt{x_2}r)$$

and

$$\gamma_{jk}^{a_{r,\alpha,t_2}}(t_1,t_2) = 2\sqrt{t_2}((h_{j-1}h_{k-1}) * a_{\frac{\alpha}{2\sqrt{x_2}}})(t_1 + 2\sqrt{t_2}r).$$

Since Eq (4.12) holds for all $\alpha > 0$, we can take the limit in both sides of it when $\alpha \to 0$; then for all $r \in \mathbb{R}$, we have

$$\begin{aligned} \lim_{\alpha\to 0} f_{(x_1,x_2),v}(\gamma^{a_{r,\alpha,t_2}}) &= \lim_{\alpha\to 0} f_{(t_1,t_2),w}(\gamma^{a_{r,\alpha,t_2}}), \\ 2\sqrt{x_2}|\langle H(x_1 + 2\sqrt{x_2}r), v\rangle|^2 &= 2\sqrt{t_2}|\langle H(t_1 + 2\sqrt{t_2}r), w\rangle|^2, \\ \sqrt{x_2}e^{-(x_1+2\sqrt{x_2}r)^2}q_v(x_1 + 2\sqrt{x_2}r) &= \sqrt{t_2}e^{-(t_1+2\sqrt{t_2}r)^2}q_w(t_1 + 2\sqrt{t_2}r), \end{aligned}$$

where q_v and q_w are polynomials of degree at most $2n - 2$. Thus, there is a constant $C \in \mathbb{R}$ such that

$$e^{4(x_2-t_2)r^2+4(x_1\sqrt{x_2}-t_1\sqrt{t_2})r+x_1^2-t_1^2} = C \quad \forall r \in \mathbb{R}. \tag{4.13}$$

Therefore, (4.13) holds if, and only if, $x_1 = t_1$ and $x_2 = t_2$. □

For the separation of pure states attached to the same fiber, we will use the following lemma.

Lemma 4.18. *[19] Let $y_1, ..., y_n$ be real numbers different from each other, then $\{H(y_1), ..., H(y_n)\}$ is a basis for $\mathbb{C}^n$.*

Lemma 4.19. *Let $w, v \in \mathbb{C}^n$ be unit vectors and $(x_1, x_2) \in \Pi$. Take the matrix-valued functions $\gamma^{a_{r_1,\alpha}}$ and $\gamma^{a_{r_2,\alpha}}$ with symbols as defined in Lemma 4.16, where $r_1, r_2 \in \mathbb{R}$ and $\alpha > 0$. Suppose that*

$$f_{(x_1,x_2),w}(\gamma^{a_{r_1,\alpha}}\gamma^{a_{r_2,\alpha}}) = f_{(x_1,x_2),v}(\gamma^{a_{r_1,\alpha}}\gamma^{a_{r_2,\alpha}}) \quad \forall \alpha > 0, r_1, r_2 \in \mathbb{R}. \tag{4.14}$$

Then $w = \lambda v$, where λ is a uni-modular complex number; that is, $f_{(x_1,x_2),w} = f_{(x_1,x_2),v}$.

Proof. Define $\beta_i = 2\sqrt{x_2}r_i + x_1$ for $i = 1, 2$. The real number $[H(\beta_1)]^T H(\beta_2)$ could be zero only for a finite number of values of β_1 and β_2. We also have $x_2 > 0$. By continuity and (4.11), the following equality

$$\lim_{\alpha\to 0} f_{(x_1,x_2),w}(\gamma^{a_{r_1,\alpha}}\gamma^{a_{r_2,\alpha}}) = \lim_{\alpha\to 0} f_{(x_1,x_2),v}(\gamma^{a_{r_1,\alpha}}\gamma^{a_{r_2,\alpha}})$$

is reduced to

$$\langle H(\beta_1), w\rangle\langle w, H(\beta_2)\rangle = \langle H(\beta_1), v\rangle\langle v, H(\beta_2)\rangle.$$

Without loss of generality, we can assume that $x_1 = 0$ and $x_2 = 1/4$, then

$$\overline{\langle w, H(r_1)\rangle}\langle w, H(r_2)\rangle = \overline{\langle v, H(r_1)\rangle}\langle v, H(r_2)\rangle.$$

This equality holds for all r_1 and r_2. In particular, take $r = r_2 = r_1$; thus, $|\langle w, H(r)\rangle| = |\langle v, H(r)\rangle|$ for all r. We can write $\langle w, H(r)\rangle = \langle v, H(r)\rangle e^{i\theta(r)}$ with $\theta(r) \in \mathbb{R}$, then

$$\overline{\langle v, H(r_1)\rangle}\langle v, H(r_2)\rangle e^{i\theta(r_2)-i\theta(r_1)} = \overline{\langle v, H(r_1)\rangle}\langle v, H(r_2)\rangle.$$

Thus, $e^{i\theta(r_2)-i\theta(r_1)} = 1$ for all r_1, r_2, which means that $\langle w, H(y)\rangle = e^{i\theta_0}\langle v, H(y)\rangle$ for all $y \in \mathbb{R}$ and some constant θ_0. Take $u = w - e^{i\theta_0}v$, then $\langle u, H(y)\rangle = 0$. According to Lemma 4.18, the set $\{H(y_k)\}_{k=1}^n$ is a basis for $\mathbb{C}^n$ and

$$\langle u, H(y_k)\rangle = 0, \quad k = 1, ..., n.$$

Therefore, u must be the zero vector. □

5. Conclusions

Recall that a nilpotent symbol for the Siegel domain D_2 has the form $c(\operatorname{Im}\zeta_1, \operatorname{Im}\zeta_2 - |\zeta_1|^2)$. Certainly each Toeplitz operator $T_c : \mathcal{A}^2_L(D_2) \to \mathcal{A}^2_L(D_2)$ can be unitarily identified with a multiplication operator $\gamma^c I$, but the C^*-algebra generated by all of them is large enough to fully describe its space of irreducible representations. The problem arises because γ^c admits a continuous extension to the spectrum of the algebra and such spectrum is uknown in general. For this reason, we confine ourselves to two subclasses of nilpotent symbols in two particular cases of poly-Bergman-type spaces.

In the case of the poly-Bergman-type space $\mathcal{A}^2_{(1,n)}(D_2)$, in Theorem 3.3 we described the C^*-algebra generated by all Toeplitz operators with symbols of the form $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$, whereas in Theorem 3.4 we used symbols of the form $a(\operatorname{Im}\zeta_1)$. Concerning the poly-Bergman-type space $\mathcal{A}^2_{(n,1)}(D_2)$, Theorem 4.11 is our main result, where we described the C^*-algebra generated by all Toeplitz operators with symbols of the form $a(\operatorname{Im}\zeta_1)$ using the Stone-Weierstrass theorem for noncommutative C^*-algebras [26]. The C^*-algebra generated by all Toeplitz operators with symbols of the form $\tilde{b}(\zeta) = b(\operatorname{Im}\zeta_2 - |\zeta_1|^2)$ was studied in [16]. The description of the C^*-algebra generated by all Toeplitz operators with nilpotent symbols without restrictions is still an open problem.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare no conflicts of interest in this paper.

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