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Research article

A prime number theorem in short intervals for dihedral Maass newforms

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Abstract: In this paper, we prove a prime number theorem in short intervals for the Rankin-Selberg L-function $L(s, \phi \times \phi)$, where ϕ is a fixed dihedral Maass newform. As an application, we give a lower bound for the proportion of primes in a short interval at which the Hecke eigenvalues of the dihedral form are greater than a given constant.

Keywords: prime number theorem; Hecke eigenvalue; Rankin-Selberg L-function; zero-free region; zero density estimate
Mathematics Subject Classification: 11F30, 11N05

1. Introduction

The classical prime number theorem asserts that

$$\sum_{n \le x} \Lambda(n) \sim x,$$

where $\Lambda(n)$ is the von Mangoldt function. One can deduce from this (cf. [1, §VII.2] for example) a prime number theorem for short intervals:

$$\sum_{x<p\le x+h} \log p = h + o(h), \tag{1.1}$$

where the sum ranges over prime numbers, provided that h is not too small. The proof of this result relies on the zero-free region

$$s = \sigma + iT, \quad \sigma \ge 1 - \frac{c}{(\log T)^{2/3}(\log\log T)^{1/3}} \tag{1.2}$$

of the Riemann zeta-function $\zeta(s)$, and on the zero density estimate of the form

$$N_1(\sigma, T) := \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\ \beta \ge \sigma,\ |\gamma| \le T\} \ll T^{A(1-\sigma)}(\log T)^B, \tag{1.3}$$

where c, A, B are positive constants.

Let π be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_\mathbb{Q})$ with unitary central character, and $\tilde{\pi}$ be its contragredient. To generalize (1.1) to higher ranks, consider the Rankin-Selberg L-function

$$L(s, \pi \times \tilde{\pi}) = \sum_n \frac{\lambda_{\pi\times\tilde{\pi}}(n)}{n^s} = \prod_p \prod_{j=1}^{m} \prod_{j'=1}^{m} \left(1 - \alpha_{j,j'}^{\pi\times\tilde{\pi}}(p) p^{-s}\right)^{-1},$$

which has an analytic continuation and a functional equation (see Section 2.1 for more details). Here, $\alpha_{j,j'}^{\pi\times\tilde{\pi}}(p)$ depends on the Satake parameter $\alpha_j^\pi(p)$ and is equal to $\alpha_j^\pi(p)\overline{\alpha_{j'}^\pi(p)}$ at unramified primes. We define the generalized von Mangoldt function $\Lambda_{\pi\times\tilde{\pi}}(n)$ by the Dirichlet series identity

$$-\frac{L'}{L}(s, \pi \times \tilde{\pi}) = \sum_n \frac{\Lambda_{\pi\times\tilde{\pi}}(n)}{n^s},$$

where $\Lambda_{\pi\times\tilde{\pi}}(p) = |\lambda_\pi(p)|^2 \log p$ when p does not divide the conductor $\mathfrak{q}_\pi$ of π. Here, $\lambda_\pi(p)$ is the eigenvalue of the Hecke operator at p.

It is well known that we have a prime number theorem for $L(s, \pi \times \tilde{\pi})$ in the form

$$\sum_{n \le x} \Lambda_{\pi\times\tilde{\pi}}(n) \sim x,$$

following from standard Rankin-Selberg theory and the Wiener-Ikehara Tauberian theorem (cf. [2, Lemma 5.2]). It is reasonable to expect that

$$\sum_{x<n\le x+h} \Lambda_{\pi\times\tilde{\pi}}(n) \sim h, \tag{1.4}$$

when h is not too small. For example, Motohashi [3] shows that, when π is the cuspidal automorphic representation corresponding to a Hecke-Maass cusp form for $\mathrm{SL}(2, \mathbb{Z})$, there exist constants $c_0, \theta_0 > 0$ such that uniformly for $(\log x)^{-1/2} \le \theta \le \theta_0$,

$$\sum_{x-h\le p\le x} \lambda_\pi(p)^2 = \frac{h}{\log x}(1 + O(\exp(-c_0/\theta))), \quad h = x^{1-\theta}$$

for sufficiently large x. Unfortunately, a zero-free region for $L(s, \pi \times \tilde{\pi})$ of the shape (1.2) does not yet exist for all π, so it seems impossible to prove a generalization of (1.1) except in special situations.

In this paper, we study the case when π is the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ corresponding to a dihedral Maass newform ϕ_k (see Section 2.2 for the definition). In Section 3.2 we will show the following asymptotic formula.

Theorem 1.1. *Let π be the automorphic representation of* $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ *corresponding to the dihedral form ϕ_k. Then, for x sufficiently large, we have*

$$\sum_{x<n\le x+h} \Lambda_{\pi\times\tilde{\pi}}(n) = h + O(h \exp(-(\log x)^{1/10}))$$

for $x^{13/16} \exp\left((\log x)^{4/5}\right) \le h \le x$.

The key ingredient of the proof is the factorization of the Rankin-Selberg L-function

$$L(s, \phi_k \times \phi_k) = L(s, \chi_q)L(s, \mathrm{Sym}^2 \phi_k) = \zeta(s)L(s, \chi_q)L(s, \phi_{2k}),$$

where the q is the level of ϕ_k, and χ_q is the Dirichlet character defined by the Kronecker symbol $(\frac{q}{\cdot})$. This factorization is why the zero-free region and the zero density estimate are both good enough to generalize (1.1): The L-function of ϕ_k coincides with that of a specific Hecke Grössencharacter, which has a zero-free region of Korobov-Vinogradov type according to [4]; and the zero density estimate of $L(s, \phi_k)$ is given by [5] and [6].

One can also write Theorem 1.1 as an estimate for sums of Hecke-Maass eigenvalues squared over primes in short intervals.

Corollary 1.2. *Let $\lambda_k(p)$ be the Hecke eigenvalue of the dihedral form ϕ_k. Then, for x sufficiently large (in particular $x > q$), and for $x^{13/16} \exp\left((\log x)^{4/5}\right) \leq h \leq x$,*

$$\sum_{x<p\leq x+h} \lambda_k(p)^2 = \frac{h}{\log x}\left(1 + O(\frac{1}{\log x})\right).$$

As an application, we give a lower bound for the proportion of primes in a short interval at which the Hecke eigenvalues of the dihedral form are greater than a given constant.

Corollary 1.3. *Under the assumption of Corollary 1.2, for x sufficiently large, there exists a prime $p \in (x, x + h]$ such that $|\lambda_k(p)| \geq 1$. In general, for any $0 \leq \delta \leq 1$,*

$$\frac{\#\{x < p \leq x + h : |\lambda_k(p)| \geq \delta\}}{\#\{x < p \leq x + h\}} > 1 - \frac{3}{4 - \delta^2} + o(1).$$

This paper is set up as follows. We begin by introducing our notation and defining the standard L-functions, Rankin-Selberg L-functions and dihedral Maass forms in Section 2. In Section 3.1 we recall some results on the zero density estimates, and in the rest of Section 3 we prove all the above theorem and corollaries.

2. Priliminaries

2.1. L-functions and the generalized von Mangoldt functions

We consider the following general setup. Let F be a number field, O_F be its ring of integers and $\mathbb{A}_F$ the ring of adeles, and let $\pi = \otimes'_{\mathfrak{p}}\pi_{\mathfrak{p}}$ be a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_F)$. Assume that π has a unitary central character. The local standard L-factor $L(s, \pi_{\mathfrak{p}})$ at a prime ideal $\mathfrak{p} \subset O_F$ is given by (cf. [7] or [8, §3.1] for example)

$$L(s, \pi_{\mathfrak{p}}) = \prod_{j=1}^{m} \left(1 - \alpha_j^{\pi}(\mathfrak{p})\mathrm{N}\mathfrak{p}^{-s}\right)^{-1}, \tag{2.1}$$

where $\alpha_1^{\pi}(\mathfrak{p}), \ldots, \alpha_m^{\pi}(\mathfrak{p})$ are the Satake parameters, and $\mathrm{N}\mathfrak{p} = \mathrm{N}_{F/\mathbb{Q}}\mathfrak{p} := \#O_F/\mathfrak{p}$ is the absolute norm of $\mathfrak{p}$. Denote by $\tilde{\pi}$ the contragredient of π. Then they have the same conductor $\mathfrak{q}_\pi = \mathfrak{q}_{\tilde{\pi}}$, and for each $\mathfrak{p} \nmid \mathfrak{q}_\pi$, we have $\alpha_j^{\tilde{\pi}}(\mathfrak{p}) = \overline{\alpha_j^{\pi}(\mathfrak{p})}$ (up to rearrangement of the parameters).

The global standard L-function $L(s,\pi)$ attached to π is defined by

$$L(s,\pi)=\sum_{\mathfrak{n}}\lambda_\pi(\mathfrak{n})\mathrm{N}\mathfrak{n}^{-s}=\prod_{\mathfrak{p}}L(s,\pi_{\mathfrak{p}}),\quad \mathrm{Re}(s)>1,$$

where the sum runs over the non-zero integral ideals of F, the product runs over the prime ideals of F, and $\mathrm{N}\mathfrak{n}:=\#\mathcal{O}_F/\mathfrak{n}$ is the absolute norm of $\mathfrak{n}$. It has an analytic continuation apart from a simple pole at $s=1$ when π is trivial, and it satisfies the functional equation $\Lambda(s,\pi)=\varepsilon(\pi)\Lambda(1-s,\tilde{\pi})$ for all $s\in\mathbb{C}$ and for some complex number $\varepsilon(\pi)$ with $|\varepsilon(\pi)|=1$. Here, the complete L-function $\Lambda(s,\pi)$ is defined by $L(s,\pi)$ times the infinite L-factors given by some Gamma functions, times a term depending on the conductor $\mathfrak{q}_\pi$. See [8, §3.1] for more details.

In particular, when $n=2$ and $F=\mathbb{Q}$, each Hecke-Maass form ϕ corresponds to a cuspidal automorphic representation π_ϕ of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$. See, for instance, [9, §1.4] for the detailed description of the local representations $(\pi_\phi)_v$ and Satake parameters for each place v. By comparing the local L-factors, we notice that they have the same L-function $L(s,\phi)=L(s,\pi_\phi)$, where the Hecke L-function of ϕ (with Hecke eigenvalues $\lambda_\phi(n)$ and nebentypus χ) is defined by (cf. [10, §5.11])

$$L(s,\phi)=\sum_{n\geq 1}\lambda_\phi(n)n^{-s}=\prod_p(1-\lambda_\phi(p)p^{-s}+\chi(p)p^{-2s})^{-1}.$$

From now on, we consider the case when $F=\mathbb{Q}$. The Rankin-Selberg L-function $L(s,\pi\times\tilde{\pi})$ can be defined by (cf. [11, §3.9] or [8, §3.2] for example)

$$L(s,\pi\times\tilde{\pi})=\sum_n\lambda_{\pi\times\tilde{\pi}}(n)n^{-s}=\prod_p L(s,\pi_p\times\tilde{\pi}_p),$$

where the local L-factors are given by

$$L(s,\pi_p\times\tilde{\pi}_p)=\prod_{j=1}^m\prod_{j'=1}^m\left(1-\alpha_{j,j'}^{\pi\times\tilde{\pi}}(p)p^{-s}\right)^{-1}.$$

For example $\alpha_{j,j'}^{\pi\times\tilde{\pi}}(p)=\alpha_j^\pi(p)\overline{\alpha_{j'}^\pi(p)}$ when $p\nmid\mathfrak{q}_\pi$. (See [12, §5.2] or [13, §A.1] for the definition of Satake parameters $\alpha_{j,j'}^{\pi\times\tilde{\pi}}(p)$ at ramified primes.) The above Dirichlet series and Euler product of $L(s,\pi\times\tilde{\pi})$ both converge absolutely when $\mathrm{Re}(s)>1$, and the Rankin-Selberg L-function $L(s,\pi\times\tilde{\pi})$ also has an analytic continuation and functional equation, and has simple poles at $s=0$ and $s=1$.

The generalized von Mangoldt function $\Lambda_{\pi\times\tilde{\pi}}(n)$ is defined by the logarithmic derivative

$$-\frac{L'}{L}(s,\pi\times\tilde{\pi})=\sum_n\Lambda_{\pi\times\tilde{\pi}}(n)n^{-s}=\sum_p\sum_{r=1}^\infty\Lambda_{\pi\times\tilde{\pi}}(p^r)p^{-rs},\quad \mathrm{Re}(s)>1.$$

By [14, Lemma a] we know that $\Lambda_{\pi\times\tilde{\pi}}(n)\geq 0$, and one can check that

$$\Lambda_{\pi\times\tilde{\pi}}(n)=\begin{cases}\sum_{j=1}^m\sum_{j'=1}^m\alpha_{j,j'}^{\pi\times\tilde{\pi}}(p)^r\log p & \text{if } n=p^r,\\ 0 & \text{otherwise.}\end{cases}\tag{2.2}$$

In particular, we have

$$\Lambda_{\pi\times\tilde{\pi}}(p)=\lambda_{\pi\times\tilde{\pi}}(p)\log p=|\lambda_\pi(p)|^2\log p,$$

where the last identity holds when $p\nmid\mathfrak{q}_\pi$.

2.2. *Dihedral Maass forms*

Let $K = \mathbb{Q}(\sqrt{q})$ with fundamental discriminant q, where $q > 0$ is squarefree and $q \equiv 1 \pmod 4$. Let χ_q be the quadratic character modulo q associated to the extension $K/\mathbb{Q}$ via class field theory, i.e., the Dirichlet character defined by the Kronecker symbol $(\frac{q}{\cdot})$.

For each integer $k \neq 0$, a dihedral Maass newform ϕ_k is the automorphic induction of a Hecke Grössencharacter Ξ_k of K for which Ξ_k does not factor through the norm map $\mathrm{N}_{K/\mathbb{Q}}$. We consider the characters Ξ_k with conductor O_K, which satisfy

$$\Xi_k((\alpha)) := |\alpha/\alpha'|^{\pi i k/\log \epsilon_q}$$

for any principal ideal $(\alpha) \subset O_K$, where $\alpha \mapsto \alpha'$ is the nontrivial automorphism in $\mathrm{Gal}(K/\mathbb{Q})$. Let $K_\nu(z)$ be the modified Bessel function. Then the dihedral Maass newform ϕ_k is given by

$$\phi_k(x+iy) := \rho_k(1) y^{1/2} \sum_{\substack{\mathfrak{n} \subset O_K \\ \mathfrak{n} \neq \{0\}}} \Xi_k(\mathfrak{n}) K_{i\frac{k\pi}{\log \epsilon_q}}(2\pi \mathrm{N}\mathfrak{n} y)(e(\mathrm{N}\mathfrak{n}x) + e(-\mathrm{N}\mathfrak{n}x)),$$

where $e(\xi) := e^{2\pi i \xi}$, $\epsilon_q > 0$ is the fundamental unit of K, $\mathrm{N}\mathfrak{n}$ is the absolute norm of $\mathfrak{n}$, and $\rho_k(1)$ is the positive real number such that ϕ_k is L^2-normalized. It is well known (cf. [15], [16, §2], and [17, §1.1] for example) that ϕ_k is a Hecke-Maass cusp form on $\Gamma_0(q)$. It has level $\mathfrak{q}_{\phi_k} = q$, weight 0, nebentypus χ_q, and eigenvalue $\frac{1}{4} + (\frac{k\pi}{\log \epsilon_q})^2$.

The Hecke eigenvalue of ϕ_k at any n can be calculated by $\lambda_k(n) = \sum_{\mathrm{N}\mathfrak{n}=n} \Xi_k(\mathfrak{n})$. The Satake parameters $\alpha_1^{\phi_k}, \alpha_2^{\phi_k}$ of ϕ_k at a prime p are related to the Hecke eigenvalue $\lambda_k(p)$ and nebentypus $\chi_q(p)$ via

$$\alpha_1^{\phi_k}(p) + \alpha_2^{\phi_k}(p) = \lambda_k(p), \quad \alpha_1^{\phi_k}(p)\alpha_2^{\phi_k}(p) = \chi_q(p). \tag{2.3}$$

Moreover, they are described in detail by the following fact.

Fact 2.1 ([18, §A.1]). *The relationship between the Satake parameters $\alpha_1^{\phi_k}, \alpha_2^{\phi_k}$ of ϕ_k at a prime p and the values of the Hecke Grössencharacter Ξ_k on prime ideals $\mathfrak{p} \mid pO_K$ is as follows:*

(i) If $\chi_q(p) = 1$, i.e., p splits in K with $pO_K = \mathfrak{p}\mathfrak{p}'$, then the Satake parameters are

$$\alpha_1^{\phi_k}(p) = \Xi_k(\mathfrak{p}), \quad \alpha_2^{\phi_k}(p) = \Xi_k(\mathfrak{p}') = \overline{\Xi_k(\mathfrak{p})}.$$

(ii) If $\chi_q(p) = -1$, i.e., p is inert in K with $pO_K = \mathfrak{p}$, then

$$\alpha_1^{\phi_k}(p) = 1, \quad \alpha_2^{\phi_k}(p) = -1.$$

(iii) If $\chi_q(p) = 0$, i.e., $p \mid q$, p ramifies in K with $pO_K = \mathfrak{p}^2$, then

$$\alpha_1^{\phi_k}(p) = \Xi_k(\mathfrak{p}), \quad \alpha_2^{\phi_k}(p) = 0.$$

A direct corollary of the above fact is that the Hecke eigenvalue $\lambda_k(p)$ is real. One can also see this from that $\overline{\phi_k} = \phi_k$ is real by definition.

By comparing the local L-factors we have that: the Hecke L-function $L(s, \Xi_k)$ coincides with the classical L-function $L(s, \phi_k)$ for Maass forms; moreover, the Rankin-Selberg L-function satisfies the following factorization (cf. [10, §5.12] for example)

$$L(s, \phi_k \times \phi_k) = L(s, \chi_q) L(s, \mathrm{Sym}^2 \phi_k) = \zeta(s) L(s, \chi_q) L(s, \phi_{2k}) \tag{2.4}$$

(in particular, we mention that $L_p(s, \phi_k \times \phi_k) = (1 - p^{-s})^{-2}$ when $p \mid q$ by the the explicit descriptions in [12, §5.2]).

3. Proof of the main theorem

3.1. Zero density estimates

Let χ be any Dirichlet character modulo q and ϕ be any Hecke-Maass newform. For $0 \le \sigma \le 1$ and $T \ge 2$, we define

$$\begin{aligned} N_{\mathbf{1}}(\sigma, T) &:= \#\{\rho = \beta + i\gamma \mid \zeta(\rho) = 0,\ \sigma \le \beta < 1,\ \text{and } |\gamma| \le T\}, \\ N_{\chi}(\sigma, T) &:= \#\{\rho = \beta + i\gamma \mid L(\rho,\chi) = 0,\ \sigma \le \beta < 1,\ \text{and } |\gamma| \le T\}, \\ N_{\phi}(\sigma, T) &:= \#\{\rho = \beta + i\gamma \mid L(\rho,\phi) = 0,\ \sigma \le \beta < 1,\ \text{and } |\gamma| \le T\}. \end{aligned}$$

In other words, $N_*(\sigma, T)$ is the number of zeros of the corresponding L-function (or zeta-function) in the rectangle $|\mathrm{Im}\,\rho| \le T$, $\sigma \le \mathrm{Re}\,\rho < 1$. The GRH predicts that the non-trivial zeros of an L-function all lie on the critical line $\mathrm{Re}(s) = 1/2$, therefore $N_*(\sigma, T) = 0$ for $\sigma > 1/2$; and the zero-density conjecture states $N_*(\sigma, T) \ll T^{2(1-\sigma)}(\log T)^{B_*}$ for some $B_* > 0$.

In the proof of Theorem 1.1, we use the following uniform bounds for $N_*(\sigma, T)$ in the range $1/2 \le \sigma \le 1$.

Lemma 3.1. *Let χ be any (fixed) Dirichlet character modulo q and ϕ be any (fixed) Hecke-Maass newform. For $1/2 \le \sigma \le 1$, we have*

$$\begin{aligned} N_{\mathbf{1}}(\sigma, T) &\ll T^{\frac{12}{5}(1-\sigma)}(\log T)^{44}, \\ N_{\chi}(\sigma, T) &\ll_q T^{\frac{5}{2}(1-\sigma)}(\log T)^{13}, \\ N_{\phi}(\sigma, T) &\ll_{\mathfrak{q}_\phi} T^{\frac{8}{3}(1-\sigma)}(\log T)^{57}. \end{aligned}$$

Proof. These uniform bounds come from the following results of zero-density estimates: for $* = \mathbf{1}, \chi$ or ϕ we have $N_*(\sigma, T) \ll T^{A_*(\sigma)(1-\sigma)}(\log T)^{B_*}$ with

$$\begin{aligned} A_{\mathbf{1}}(\sigma) &= \frac{3}{2-\sigma}, \quad B_{\mathbf{1}} = 5 \quad \text{for } \tfrac{1}{2} \le \sigma \le \tfrac{3}{4} \quad ([19]), \\ A_{\mathbf{1}}(\sigma) &= \frac{3}{3\sigma-1}, \quad B_{\mathbf{1}} = 44 \quad \text{for } \tfrac{3}{4} \le \sigma \le 1 \quad ([20]); \\ A_{\chi}(\sigma) &= \frac{3}{2-\sigma}, \quad B_{\chi} = 13 \quad \text{for } \tfrac{1}{2} \le \sigma \le \tfrac{4}{5} \quad ([21]), \\ A_{\chi}(\sigma) &= \frac{2}{\sigma}, \quad B_{\chi} = 13 \quad \text{for } \tfrac{4}{5} \le \sigma \le 1 \quad ([21]). \end{aligned}$$

For any (fixed) Hecke-Maass newform ϕ, one can follow the proofs in [5] and [6] to show that

$$N_{\phi}(\sigma, T) \ll_{\mathfrak{q}_\phi} \begin{cases} T^{\frac{4}{3-2\sigma}(1-\sigma)}(\log T)^{26} & \text{for } \frac{1}{2} + \frac{1}{\log T} \le \sigma \le \frac{3}{4}, \\ T^{\frac{8\sigma-5}{-2\sigma^2+6\sigma-3}(1-\sigma)}(\log T)^{57} & \text{for } \frac{3}{4} \le \sigma \le 1. \end{cases}$$

□

3.2. Proof of Theorem 1.1

For any cuspidal automorphic representation π of $\mathrm{GL}_m(\mathbb{A}_{\mathbb{Q}})$ whose central character is unitary, [8, (6.2)] shows the following explicit formula

$$\sum_{n\le x}\Lambda_{\pi\times\tilde{\pi}}(n)=x-\sum_{\substack{0<\beta<1\\|\gamma|\le T}}\frac{x^{\rho}}{\rho}+O\left(\frac{x(\log x)^2}{\sqrt{T}}\right),$$

where $\rho=\beta+i\gamma$ denotes the nontrivial zeros of $L(s,\pi\times\tilde{\pi})$. It follows that, for $2\le h\le x$, we have

$$\sum_{x<n\le x+h}\Lambda_{\pi\times\tilde{\pi}}(n)=h-\sum_{\substack{0<\beta<1\\|\gamma|\le T}}\frac{(x+h)^{\rho}-x^{\rho}}{\rho}+O\left(\frac{x(\log x)^2}{\sqrt{T}}\right).$$

To estimate the sum over ρ, we observe that

$$\left|\frac{(x+h)^{\rho}-x^{\rho}}{\rho}\right|=\left|\int_x^{x+h}\tau^{\rho-1}\,d\tau\right|\le\int_x^{x+h}\tau^{\beta-1}\,d\tau\le hx^{\beta-1}.$$

Hence,

$$\frac{1}{h}\sum_{x<n\le x+h}\Lambda_{\pi\times\tilde{\pi}}(n)=1+O\left(\frac{1}{x}\sum_{\substack{0<\beta<1\\|\gamma|\le T}}x^{\beta}\right)+O\left(\frac{x(\log x)^2}{h\sqrt{T}}\right).\tag{3.1}$$

Furthermore,

$$\sum_{\substack{0<\beta<1\\|\gamma|\le T}}x^{\beta}=\sum_{\substack{0<\beta<1\\|\gamma|\le T}}\left(\log x\int_0^{\beta}x^u\,du+1\right)=N_{\pi\times\tilde{\pi}}(0,T)+\log x\sum_{\substack{0<\beta<1\\|\gamma|\le T}}\int_0^1 x^uF(u,\beta)\,du,$$

where $N_{\pi\times\tilde{\pi}}(u,T)$ is the number of nontrivial zeros ρ of $L(s,\pi\times\tilde{\pi})$ in the rectangle $|\mathrm{Im}\,\rho|\le T$, $u\le\mathrm{Re}\,\rho<1$, and

$$F(u,\beta):=\begin{cases}1 & \text{if } 0\le u\le\beta,\\ 0 & \text{if } \beta<u\le 1.\end{cases}$$

By definition, we have that

$$\sum_{\substack{0<\beta<1\\|\gamma|\le T}}F(u,\beta)=N_{\pi\times\tilde{\pi}}(u,T),$$

and hence,

$$\sum_{\substack{0<\beta<1\\|\gamma|\le T}}x^{\beta}=N_{\pi\times\tilde{\pi}}(0,T)+\log x\int_0^1 x^uN_{\pi\times\tilde{\pi}}(u,T)\,du.\tag{3.2}$$

If $0\le u<1/2$, then we use the trivial bound (cf. [10, Theorem 5.8])

$$N_{\pi\times\tilde{\pi}}(u,T)\ll N_{\pi\times\tilde{\pi}}(0,T)\ll_{\mathfrak{q}_\pi}T\log T.$$

Assume that $N_{\pi\times\tilde{\pi}}(u,T) \ll T^{A(1-u)}(\log T)^B$ if $1/2 \le u < 1$, for some constants $A > 2$ and $B > 0$. In fact, when π is the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to the dihedral form ϕ_k, one can take $A = 8/3$ and $B = 57$, by Lemma 3.1 and the factorization (2.4).

Recall that there exists a constant $c = c(q) > 0$ such that

$$N_{\mathbf{1}}(\sigma,T) = N_{\chi_q}(\sigma,T) = N_{\phi_k}(\sigma,T) = 0 \quad \text{if } \sigma \ge 1 - \frac{c}{(\log T)^{2/3}(\log\log T)^{1/3}}$$

for sufficiently large T (cf. [22, §9.5] and [4, Theorem 2] for example, recall that $L(s,\phi_k) = L(s,\Xi_k)$ for the Hecke Grössencharacter Ξ_k). Assume $x \ge e \cdot T^A$ with e the base of natural logarithms, and let $\theta(T) = c(\log T)^{-2/3}(\log\log T)^{-1/3}$. Again by the factorization (2.4) we have $N_{\pi\times\tilde{\pi}}(\sigma,T) \le N_{\mathbf{1}}(\sigma,T) + N_{\chi_q}(\sigma,T) + N_{\phi_{2k}}(\sigma,T) = 0$ for $\sigma \ge 1-\theta(T)$. Then, we obtain the estimate for (3.2):

$$\begin{aligned}\sum_{\substack{0<\beta<1\\ |\gamma|\le T}} x^\beta &\ll T\log T + \log x\int_0^{1/2} x^u T\log T\,du + \log x\int_{1/2}^{1-\theta(T)} x^u T^{A(1-u)}(\log T)^B\,du\\ &\ll x^{1/2}T\log T + (xT^{-A})^{1-\theta(T)}T^A(\log T)^B\log x.\end{aligned}$$

From this and (3.1) we have that when π is the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ corresponding to ϕ_k,

$$\frac{1}{h}\sum_{x<n\le x+h}\Lambda_{\pi\times\tilde{\pi}}(n) = 1 + O\left(\frac{T\log T}{x^{1/2}}\right) + O\left(\left(\frac{T^A}{x}\right)^{\theta(T)}(\log T)^B\log x\right) + O\left(\frac{x(\log x)^2}{T^{1/2}h}\right).$$

If we set

$$T^A = x\exp(-(\log x)^\alpha) \quad \text{for some } 2/3 < \alpha < 1,$$

then we have that, for any

$$h \ge x^{1-\frac{1}{2A}}\exp((\log x)^\alpha),$$

the remainder term is $O(\exp(-(\log x)^{\alpha-\frac{2}{3}-\varepsilon}))$: more precisely,

$$\begin{aligned}\frac{T\log T}{x^{1/2}} &= x^{1/A}\exp(-\tfrac{1}{A}(\log x)^\alpha)\cdot\tfrac{1}{A}(\log x - (\log x)^\alpha)\cdot x^{-1/2}\\ &\ll x^{\frac{1}{A}-\frac{1}{2}} \ll \exp(-(\log x)^{1-\varepsilon});\end{aligned}$$

$$\begin{aligned}\left(\frac{T^A}{x}\right)^{\theta(T)}(\log T)^B\log x &= \exp(-(\log x)^\alpha\theta(T))\cdot(\tfrac{1}{A}(\log x - (\log x)^\alpha))^B\log x\\ &\ll \exp\big(-(\log x)^\alpha\cdot c(\log x)^{-2/3}(\log\log x)^{-1/3}\big)\cdot(\log x)^{B+1}\\ &\ll \exp(-(\log x)^{\alpha-\varepsilon-2/3});\end{aligned}$$

$$\begin{aligned}\frac{x(\log x)^2}{T^{1/2}h} &\ll x(\log x)^2\cdot x^{-\frac{1}{2A}}\exp(\tfrac{1}{2A}(\log x)^\alpha)\cdot x^{\frac{1}{2A}-1}\exp(-(\log x)^\alpha)\\ &\ll \exp\big(-(1-\tfrac{1}{2A})(\log x)^{\alpha-\varepsilon}\big) \ll \exp(-(\log x)^{\alpha-\varepsilon}).\end{aligned}$$

At last, we take $A = 8/3$ and $\alpha = 4/5$ to complete the proof of Theorem 1.1.

3.3. Proof of Corollary 1.2

Let π be the automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ corresponding to the dihedral form ϕ_k. Recall (cf. (2.3) and Fact 2.1) that $|\alpha_j^\pi(p)| \le 1$ and $|\lambda_k(p)| \le 2$ for any prime p, which depend on the value $\Xi_k(\mathfrak{p})$ of the Hecke Grössencharacter Ξ_k at the prime ideals $\mathfrak{p} \mid p\mathcal{O}_K$. Moreover, we have $|\alpha_{j,j'}^{\pi\times\tilde{\pi}}(p)| \le 1$ because of the factorization (2.4). Therefore by the definition (2.2),

$$\sum_{p^r \le x,\ r\ge 2} \Lambda_{\pi\times\tilde{\pi}}(p^r) \le \sum_{p\le\sqrt{x}} \sum_{1\le r\le \frac{\log x}{\log p}} 4\log p \le \sum_{p\le\sqrt{x}} 4\log x \le 4x^{1/2}\log x.$$

By Theorem 1.1 we have, for $x > q$ and $x^{13/16}\exp\left((\log x)^{4/5}\right) \le h \le x$,

$$\sum_{x<p\le x+h} \lambda_k(p)^2 \log p = \sum_{x<n\le x+h} \Lambda_{\pi\times\tilde{\pi}}(n) + O(x^{1/2}\log x) = h\Big(1 + O(\exp(-(\log x)^{1/10}))\Big). \tag{3.3}$$

Here $\lambda_k(p)$ is real via Fact 2.1. Note that in this short interval $p \in (x, x+h]$ we have that $\log x < \log p \le \log 2x$, therefore $\log p = \log x + O(1)$. Hence,

$$\sum_{x<p\le x+h} \lambda_k(p)^2 = \frac{h\Big(1 + O(\exp(-(\log x)^{1/10}))\Big)}{\log x\Big(1 + O(\frac{1}{\log x})\Big)} = \frac{h}{\log x}\Big(1 + O(\frac{1}{\log x})\Big).$$

3.4. Proof of Corollary 1.3

We have

$$\begin{aligned}\sum_{x<p\le x+h} \lambda_k(p)^2 \log p &\le \left(\sum_{\substack{x<p\le x+h \\ |\lambda_k(p)|<\delta}} \lambda_k(p)^2 + \sum_{\substack{x<p\le x+h \\ \delta\le|\lambda_k(p)|\le 2}} \lambda_k(p)^2\right)\log(x+h) \\ &< (\delta^2\mathcal{N}_\delta + 4(\mathcal{N} - \mathcal{N}_\delta))\log(x+h),\end{aligned} \tag{3.4}$$

with

$$\begin{aligned}\mathcal{N}_\delta &:= \#\{x < p \le x+h : |\lambda_k(p)| < \delta\}, \\ \mathcal{N} &:= \#\{x < p \le x+h\} = \frac{h}{\log x}(1 + o(1)).\end{aligned}$$

Combining (3.3) and (3.4), we get

$$(4-\delta^2)\frac{\mathcal{N}_\delta}{\mathcal{N}} < 4 - \frac{h(1+o(1))}{\mathcal{N}\log(x+h)} = 3 + o(1),$$

i.e.,

$$\frac{\mathcal{N}_\delta}{\mathcal{N}} < \frac{3}{4-\delta^2} + o(1).$$

This implies Corollary 1.3.

4. Conclusions

In this paper, we study the prime number theorem in short intervals for L-functions of higher ranks. It is not easy to prove a prime number theorem in short intervals of the form (1.4) for the Rankin-Selberg L-function $L(s, \pi \times \tilde{\pi})$ when π is any cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. But if π corresponds to a dihedral Maass newform ϕ_k, the L-function $L(s, \phi_k \times \phi_k)$ has a factorization (2.4) which leads to a nice zero-free region and a zero density estimate. Theorem 1.1 and Corollary 1.2 give two equivalent asymptotic formulas for the prime number theorem in short intervals for dihedral Maass newforms. As an application, in Corollary 1.3 we show a lower bound for the proportion of primes in a short interval at which the Hecke eigenvalues of the dihedral form are greater than a given constant.

Use of AI tools declaration

The author declares that he/she has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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