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Research article

The sum of a hybrid arithmetic function over a sparse sequence

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Abstract: Let $\lambda_f(n)$ be the n-th normalized Fourier coefficient of f, which is a primitive holomorphic cusp form of even integral weight $k \geq 2$ for the full modular group $SL_2(\mathbb{Z})$. Let also $\sigma(n)$ and $\phi(n)$ be the sum-of-divisors function and the Euler totient function, respectively. In this paper, we are able to establish the asymptotic formula of the sum of the hybrid arithmetic function $\lambda_f^l(n)\sigma^c(n)\phi^d(n)$ over the sparse sequence $\{n : n = a^2 + b^2\}$, namely, $\sum_{n \leq x} \lambda_f^l(n)\sigma^c(n)\phi^d(n)r_2(n)$ for $1 \leq l \leq 8$, where x is a sufficiently large real number, the function $r_2(n)$ denotes the number of representations of n as $n = a^2 + b^2$, $a, b, l \in \mathbb{Z}$ and $c, d \in \mathbb{R}$.

Keywords: Fourier coefficients; cusp forms; automorphic L-function
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1. Introduction

Let H_k^* be the set of all normalized primitive holomorphic cusp form of even integral weight $k \geq 2$ for the full modular group $SL_2(\mathbb{Z})$. The primitive holomorphic cusp form $f \in H_k^*$ at the cusp $z = \infty$ has the Fourier expansion

$$f(z) = \sum_{n=1}^{\infty} n^{\frac{k-1}{2}} \lambda_f(n) e^{2\pi i n z}, \quad \mathrm{Im}(z) > 0,$$

where $\lambda_f(n)$ is the n-th normalized Fourier coefficient. $\lambda_f(n)$ is real-valued and has the following multiplicative property

$$\lambda_f(m)\lambda_f(n) = \sum_{d|(m,n)} \lambda_f\left(\frac{mn}{d^2}\right)$$

with $m, n \in \mathbb{N}^+$. In number theory, the study of the Fourier coefficient $\lambda_f(n)$ is of great significance and has attracted attention of many mathematicians. Let $d(n)$ be the Dirichlet divisor function. In 1974, Deligne [1] proved the Ramanujan-Petersson conjecture

$$|\lambda_f(n)| \leq d(n).$$

In 1927, Hecke [2] established that

$$\sum_{n\le x}\lambda_f(n)\ll x^{\frac{1}{2}}.$$

Subsequently, Hecke's result was refined by many scholars and the best result now is

$$\sum_{n\le x}\lambda_f(n)\ll x^{\frac{1}{3}}\log^{\rho}x,$$

where $\rho=-0.118\cdots$, proved by Wu [3]. In 1930, by their powerful method Rankin [4] and Selberg [5] proved

$$\sum_{n\le x}\lambda_f^2(n)=cx+O(x^{\frac{3}{5}}),$$

where c is a positive constant depending on f. Recently, the exponent $\frac{3}{5}$ was improved to $\frac{3}{5}-\Delta$ with $\Delta\le\frac{1}{560}$ by Huang [6]. There is also a long history on higher powers sums $\sum_{n\le x}\lambda_f^l(n)$ with $l\ge 3$, and here we refer to the references [7, 8] and the references therein for detailed historical descriptions.

Let $a,b,l\in\mathbb{Z}$. Let also $r_2(n)$ denote the number of representations of n as $n=a^2+b^2$, i.e.,

$$r_2(n)=\sharp\left\{n=a^2+b^2,(a,b)\in\mathbb{Z}^2\right\}. \tag{1.1}$$

In 2013, Zhai [9] studied a related power sum over a sum of two squares and established the following asymptotic formula

$$\sum_{n\le x}\lambda_f^l(n)r_2(n)=xP_l(\log x)+O_{f,\varepsilon}(x^{\theta_l+\varepsilon}),$$

where $P_2(t),P_4(t),P_6(t),P_8(t)$ are polynomials of degree $0,1,4,13$, respectively,

$$P_l(t)\equiv 0$$

for $l=3,5,7$, and

$$\theta_2=\frac{8}{11},\ \theta_3=\frac{17}{20},\ \theta_4=\frac{43}{46},\ \theta_5=\frac{83}{86},\ \theta_6=\frac{184}{187},\ \theta_7=\frac{355}{358},\ \theta_8=\frac{752}{755}.$$

Later, Xu [8] refined and generalized the results of Zhai [9]. Recently, Liu [10] further improved Zhai and Xu's results.

Let $\sigma(n)$ and $\phi(n)$ be the sum-of-divisors function and the Euler totient function, respectively. In 2015, Manski et al. [11] proved that

$$\sum_{n\le x}d^{a'}(n)\sigma^{b'}(n)\phi^{c'}(n)=x^{b'+c'+1}P_{2^{a'}-1}(\log x)+O\left(x^{b'}+c'+r_{a'}+\varepsilon\right),$$

where $a',b',c'\in\mathbb{R}$, $2^{a'}\in\mathbb{N}$, $b'+c'>-r_{a'}$, $P_m(t)$ is a polynomial in t of degree m and $r_{a'}$ takes specific values as in [11, (2)].

Let $c, d \in \mathbb{R}$. Many scholars also studied the mean values of the arithmetic function $\lambda_f^l(n)\sigma^c(n)\phi^d(n)$ and we refer to [12] for historical results. In detail, Wei and Lao [12] proved that

$$\sum_{n\leq x} \lambda_f^l(n)\sigma^c(n)\phi^d(n) = x^{c+d+1}P_l(\log x) + O\left(x^{c+d+\theta_l+\varepsilon}\right),$$

where $P_2(t)$, $P_4(t)$, $P_6(t)$, $P_8(t)$ are polynomials in t of degree 0, 1, 4, 13, respectively, $P_7(t) \equiv 0$, and

$$\theta_2 = \frac{23}{37}, \quad \theta_4 = \frac{257}{299}, \quad \theta_6 = \frac{201}{208}, \quad \theta_7 = \frac{67}{68}, \quad \theta_8 = \frac{117}{118}.$$

In this paper, motivated by the above results we study the asymptotic behavior of the hybrid arithmetic function $\lambda_f^l(n)\sigma^c(n)\phi^d(n)$ over the sparse sequence $\{n : n = a^2 + b^2\}$. Define

$$S_l(f; x) = \sum_{n\leq x} \lambda_f^l(n)\sigma^c(n)\phi^d(n)r_2(n), \tag{1.2}$$

where $1 \leq l \leq 8$, x is a sufficiently large real number, $a, b, l \in \mathbb{Z}$ and $c, d \in \mathbb{R}$. By combining some analytic methods with properties of some primitive automorphic L-functions we establish the following theorem.

Theorem 1.1. *Let $f \in H_k^*$ and $\lambda_f(n)$ be the n-th normalized Fourier coefficient of f. Under the notations above, for any $\varepsilon > 0$, we have*

$$S_l(f; x) = x^{c+d+1}A_l(\log x) + O_{f,\varepsilon}\left(x^{c+d+\theta_l+\varepsilon}\right), \tag{1.3}$$

where

$$A_l(t) \equiv 0$$

for $l = 1, 3, 5, 7$, $A_2(t)$, $A_4(t)$, $A_6(t)$, $A_8(t)$ are polynomials in t of degree 0, 1, 4, 13, respectively, and

$$\begin{aligned} &\theta_1 = \frac{1}{2} = 0.5, && \theta_2 = \frac{12}{17} = 0.7058\cdots, && \theta_3 = \frac{17}{20} = 0.85, \\ &\theta_4 = \frac{5209}{5629} = 0.9254\cdots, && \theta_5 = \frac{83}{86} = 0.9651\cdots, && \theta_6 = \frac{6487}{6607} = 0.9818\cdots, \\ &\theta_7 = \frac{353}{356} = 0.9915\cdots, && \theta_8 = \frac{48857}{49067} = 0.9957\cdots. \end{aligned}$$

For a random variable X defined on a countable sample space $\mathbb{V}$, let $E(X)$ denote the mathematical expectation of X. With the help of Theorem 1.1, we can obtain the asymptotic mathematical expectation, denoted by $E\left(\lambda_f^l(n)\sigma^c(n)\phi^d(n)r_2(n)\right)_{1\leq n\leq x}$, of $\lambda_f^l(n)\sigma^c(n)\phi^d(n)$ over the sample space

$$1 \leq n \leq x, \quad n = a^2 + b^2.$$

Theorem 1.2. *Under the same notations as in Theorem 1.1, we have*

$$E\left(\lambda_f^l(n)\sigma^c(n)\phi^d(n)r_2(n)\right)_{1\leq n\leq x} = \pi^{-1}x^{c+d}A_l(\log x) + O_{f,\varepsilon}\left(x^{c+d+\theta_l-1+\varepsilon}\right).$$

In the following Section 2, we give some preliminary lemmas. In Sections 3 and 4, we complete the proofs of Theorems 1.1 and 1.2, respectively.

Notation. *Throughout this paper, we apply the letter ε to represent a sufficiently small positive constant, whose value may change from statement to statement. The constants, both explicit and implicit, in Vinogradov symbols may depend on ε and f.*

2. Preliminary lemmas

We first introduce some L-functions and then give some necessary lemmas. As usual, we define Riemann zeta function $\zeta(s)$ and Dirichlet L-function $L(s,\chi)$ as

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\quad\text{and}\quad L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}\tag{2.1}$$

for $\mathrm{Re}(s)>1$, respectively. For the n-th normalized Fourier coefficient $\lambda_f(n)$, Deligne [1] showed that for any prime p there are two complex numbers $\alpha_f(p)$ and $\beta_f(p)$ satisfying

$$\lambda_f(p)=\alpha_f(p)+\beta_f(p),\ \ |\alpha_f(p)|=|\beta_f(p)|=\alpha_f(p)\beta_f(p)=1.\tag{2.2}$$

Thus, the Hecke L-function associated to $f\in H_k^*$ can be represented as

$$L(s,f)=\sum_{n=1}^{\infty}\frac{\lambda_f(n)}{n^s}=\prod_p(1-\alpha_f(p)p^{-s})^{-1}(1-\beta_f(p)p^{-s})^{-1},\quad \mathrm{Re}(s)>1.\tag{2.3}$$

Then, the j-th symmetric power L-function with $f\in H_k^*$ can be defined as, for $\mathrm{Re}(s)>1$,

$$\begin{aligned}L(s,\mathrm{sym}^j f)&:=\prod_p\prod_{m=0}^{j}\left(1-\alpha_f^{j-m}(p)\beta_f^m(p)p^{-s}\right)^{-1}\\&=\sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f}(n)}{n^s}.\end{aligned}\tag{2.4}$$

Note that

$$L(s,\mathrm{sym}^0 f)=\zeta(s)$$

and

$$L(s,\mathrm{sym}^1 f)=L(s,f).$$

For $\mathrm{Re}(s)>1$, the j-th symmetric power L-function twisted by the Dirichlet character χ is defined as

$$\begin{aligned}L(s,\mathrm{sym}^j f\times\chi)&:=\prod_p\prod_{m=0}^{j}\left(1-\alpha_f^{j-m}(p)\beta_f^m(p)\chi(p)p^{-s}\right)^{-1}\\&=\sum_{n=1}^{\infty}\frac{\lambda_{\mathrm{sym}^j f}(n)\chi(n)}{n^s}.\end{aligned}\tag{2.5}$$

Recall (1.1). It was showed by Iwaniec [13] that

$$r_2(n)=4\sum_{d|n}\chi(d),$$

where $\chi(d)$ is the non-trivial Dirichlet character modulo 4. Let $r(n)$ denote $r_2(n)/4$. Since $\chi(n)$ is completely multiplicative, one has

$$r(p)=\sum_{d|p}\chi(d)=1+\chi(p).$$

Therefore, we can write

$$S_l(f;x) = \sum_{n\le x} \lambda_f^l(n)\sigma^c(n)\phi^d(n)r_2(n) = 4\sum_{n\le x} \lambda_f^l(n)\sigma^c(n)\phi^d(n)r(n).$$

Now, we turn to give some necessary lemmas. From the recent deep results of Newton and Thorne [14, 15], we know that all $\mathrm{sym}^j f$ with $j \in \mathbb{N}^+$ are automorphic cuspidal representations of GL_{j+1}. That is, the j-th symmetric power L-function with $L(s, \mathrm{sym}^j f)$ with $j \in \mathbb{N}^+$ has analytic continuation as an entire function in the whole plane and certain functional equations. Thus, $L(s, \mathrm{sym}^j f)$ with $j \in \mathbb{N}^+$ are general L-functions in sense of Perelli [16].

Lemma 2.1. *For any*

$$\varepsilon > 0, \ \ \frac{1}{2} \le \sigma \le 1 \ \ \textit{and} \ \ |t| \ge 1,$$

we have

$$\begin{aligned}
\zeta(\sigma+it) &\ll (1+|t|)^{\frac{13}{42}(1-\sigma)+\varepsilon},\\
L(\sigma+it, f) &\ll (1+|t|)^{\frac{2}{3}(1-\sigma)+\varepsilon},\\
L(\sigma+it, \mathrm{sym}^2 f) &\ll (1+|t|)^{\frac{6}{5}(1-\sigma)+\varepsilon},\\
L(\sigma+it, \mathrm{sym}^j f) &\ll (1+|t|)^{\frac{j+1}{2}(1-\sigma)+\varepsilon}, \quad j = 3,4,5,\cdots.
\end{aligned}$$

Proof. The former three results can be found in the works [17, Theorem 5], [9, Lemma 2.3] and [18, Corollary 1.2], respectively. The last result follows from [16, Theorem 4] and [19, Proposition 1] plainly. □

Lemma 2.2. *Let χ be the non-trivial Dirichlet character modulo 4. For any*

$$\varepsilon > 0, \ \ \frac{1}{2} \le \sigma \le 1 \ \ \textit{and} \ \ |t| \ge 1,$$

one has

$$\begin{aligned}
L(\sigma+it, \chi) &\ll (1+|t|)^{\frac{13}{42}(1-\sigma)+\varepsilon},\\
L(\sigma+it, f\times\chi) &\ll (1+|t|)^{\frac{2}{3}(1-\sigma)+\varepsilon},\\
L(\sigma+it, \mathrm{sym}^2 f\times\chi) &\ll (1+|t|)^{\frac{6}{5}(1-\sigma)+\varepsilon},\\
L(\sigma+it, \mathrm{sym}^j f\times\chi) &\ll (1+|t|)^{\frac{j+1}{2}(1-\sigma)+\varepsilon}, \quad j = 3,4,5,\cdots.
\end{aligned}$$

Proof. Since χ is the non-trivial Dirichlet character modulo 4, twisting by the character χ does not affect subconvexity bounds and convexity bounds of L-functions in the t's aspect. □

Lemma 2.3. *Let $f \in H_k^*$ and χ be the non-trivial Dirichlet character modulo 4. Then, for any $\varepsilon > 0$, $j \in \mathbb{N}^+$ and $|t| \ge 1$, we have*

$$\int_T^{2T} \left|L\left(\sigma+it, \mathrm{sym}^j f\right)\right|^2 dt \ll |T|^{(j+1)(1-\sigma)+\varepsilon}$$

and

$$\int_T^{2T} \left|L\left(\sigma+it, \mathrm{sym}^j f\times\chi\right)\right|^2 dt \ll |T|^{(j+1)(1-\sigma)+\varepsilon}.$$

Proof. The first result is in [16, Lemma 13]. The second result follows from the first result by the same reason as in Lemma 2.2. □

Lemma 2.4. *For any $U \geq U_0$, where U_0 is a sufficiently large constant, there exists $T^* \in (U, 2U)$, such that*

$$\max_{\sigma \geq \frac{1}{2}} |\zeta(\sigma \pm iT^*)| \leq \exp\left(C(\log\log U)^2\right),$$

where $C > 0$ is an absolute constant.

Proof. This result is proved by Ramachandra and Sankaranarayanan [20, Lemma 2]. □

Lemma 2.5. *For any $\varepsilon > 0$, we have*

$$\int_0^T \left|\zeta\left(\frac{5}{7} + it\right)\right|^{12} dt \ll T^{1+\varepsilon},$$

uniformly for $T \geq 1$.

Proof. This result was established by Ivić [21, Theorem 8.4 and (8.87)]. □

Lemma 2.6. *Let*

$$\mathfrak{F}(s) := \sum_{n \geq 1} \frac{a_n}{n^s}$$

be a Dirichlet series with a finite abscissa of absolute convergence σ_a. Suppose there exists a real number $\alpha \geq 0$ such that
(i)

$$\sum_{n \geq 1} |a_n| n^{-\sigma} \ll (\sigma - \sigma_a)^{-\alpha},$$

where $\sigma_a < \sigma \leq \sigma_a + 1$, and that B is a non-decreasing function satisfying
(ii)

$$|a_n| \leq B(n),$$

where $n \geq 1$.

Then, for

$$x \geq 2, \quad T \geq 2, \quad \text{and} \quad \sigma \leq \sigma_a,$$

$$\kappa := \sigma_a - \sigma + \frac{1}{\log x},$$

we have

$$\sum_{n \leq x} \frac{a_n}{n^s} = \frac{1}{2\pi i} \int_{\kappa - iT}^{\kappa + iT} \mathfrak{F}(s + w) \frac{x^w}{w} dw + O\left(x^{\sigma_a - \sigma} \frac{(\log x)^\alpha}{T} + \frac{B(2x)}{x^\sigma}\left(1 + x\frac{\log T}{T}\right)\right).$$

Proof. This is the well-known Perron's formula, which can be found in [22, Corollary 2.4]. □

Lemma 2.7. *Let*

$$F_l(s) = \sum_{n=1}^{\infty} \frac{\lambda_f^l(n)\sigma^c(n)\phi^d(n)r(n)}{n^s}.$$

Then, for $l = 1, \cdots, 8$*, we have*

$$F_l(s) = G_l(s-c-d)H_l(s),$$

where

$$\begin{aligned}
G_1(s) =& L(s,f)L(s,f\times\chi),\\
G_2(s) =& \zeta(s)L(s,\chi)L(s,\mathrm{sym}^2 f)L(s,\mathrm{sym}^2 f\times\chi),\\
G_3(s) =& L^2(s,f)L^2(s,f\times\chi)L(s,\mathrm{sym}^3 f)L(s,\mathrm{sym}^3 f\times\chi),\\
G_4(s) =& \zeta^2(s)L^2(s,\chi)L^3(s,\mathrm{sym}^2 f)L^3(s,\mathrm{sym}^2 f\times\chi)L(s,\mathrm{sym}^4 f)L(s,\mathrm{sym}^4 f\times\chi),\\
G_5(s) =& L^5(s,f)L^5(s,f\times\chi)L^4(s,\mathrm{sym}^3 f)L^4(s,\mathrm{sym}^3 f\times\chi)L(s,\mathrm{sym}^5 f)\times L(s,\mathrm{sym}^5 f\times\chi),\\
G_6(s) =& \zeta^5(s)L^5(s,\chi)L^9(s,\mathrm{sym}^2 f)L^9(s,\mathrm{sym}^2 f\times\chi)L^5(s,\mathrm{sym}^4 f)L^5(s,\mathrm{sym}^4 f\times\chi)\\
&\times L(s,\mathrm{sym}^6 f)L(s,\mathrm{sym}^6 f\times\chi),\\
G_7(s) =& L^{14}(s,f)L^{14}(s,f\times\chi)L^{14}(s,\mathrm{sym}^3 f)L^{14}(s,\mathrm{sym}^3 f\times\chi)L^6(s,\mathrm{sym}^5 f)\\
&\times L^6(s,\mathrm{sym}^5 f\times\chi)L(s,\mathrm{sym}^7 f)L(s,\mathrm{sym}^7 f\times\chi),\\
G_8(s) =& \zeta^{14}(s)L^{14}(s,\chi)L^{28}(s,\mathrm{sym}^2 f)L^{28}(s,\mathrm{sym}^2 f\times\chi)L^{20}(s,\mathrm{sym}^4 f)\\
&\times L^{20}(s,\mathrm{sym}^4 f\times\chi)L^7(s,\mathrm{sym}^6 f)L^7(s,\mathrm{sym}^6 f\times\chi)L(s,\mathrm{sym}^8 f)\times L(s,\mathrm{sym}^8 f\times\chi),
\end{aligned}$$

where χ *is the non-trivial Dirichlet character modulo 4 and* $H_l(s)$ *is absolutely convergent for*

$$\mathrm{Re}(s) \geq c+d+\frac{1}{2}.$$

Proof. Here, we give the detailed proof for $l = 7$ as an example, since the remaining cases can be proven by following a similar argument.

For $l = 7$, due to the multiplicative property of $\lambda_f(n)$, $\sigma(n)$, $\phi(n)$ and $r(n)$, we have

$$\begin{aligned}
F_7(s) =& \prod_p \sum_{k=0}^{\infty} \frac{\lambda_f^7(p^k)\sigma^c(p^k)\phi^d(p^k)r(p^k)}{p^{ks}}\\
=& \prod_p \left(1 + \frac{\lambda_f^7(p)\sigma^c(p)\phi^d(p)r(p)}{p^s} + \frac{\lambda_f^7(p^2)\sigma^c(p^2)\phi^d(p^2)r(p^2)}{p^{2s}} + \cdots\right)\\
=& \prod_p \left(1 + \frac{(\alpha_f(p)+\beta_f(p))^7(p+1)^c(p-1)^d r(p)}{p^s}\right.\\
&\left. + \frac{\left(\frac{\alpha_f^3(p)-\beta_f^3(p)}{\alpha_f(p)-\beta_f(p)}\right)^7 (p^2+p+1)^c(p^2-p)^d r^2(p)}{p^{2s}} + \cdots\right)\\
=& \prod_p \left(1 + \frac{\left(\alpha_f(p)+\beta_f(p)\right)^7(1+\chi(p))}{p^{s-c-d}} + O\left(p^{2(c+d-\sigma)} + p^{c+d-\sigma-1}\right)\right).
\end{aligned}$$

Further, by the binomial theorem, (2.2) and (2.4) we have

$$
\begin{aligned}
F_7(s) = &\prod_p \left(1 + \left(\alpha_f(p) + \beta_f(p)\right)^7 (1+\chi(p))\, p^{-(s-c-d)} + O\left(p^{2(c+d-\sigma)} + p^{c+d-\sigma-1}\right)\right) \\
= &\prod_p \Big(1 + \left(\alpha_f^7(p) + 7\alpha_f^5(p) + 21\alpha_f^3(p) + 35\alpha_f(p) + 35\beta_f(p) + 21\beta_f^3(p) + 7\beta_f^5(p)\right. \\
&\left. + \beta_f^7(p)\right)p^{-(s-c-d)} + \left(\alpha_f^7(p) + 7\alpha_f^5(p) + 21\alpha_f^3(p) + 35\alpha_f(p) + 35\beta_f(p) + 21\beta_f^3(p)\right. \\
&\left. + 7\beta_f^5(p) + \beta_f^7(p)\right)p^{-(s-c-d)}\chi(p) + O\left(p^{2(c+d-\sigma)} + p^{c+d-\sigma-1}\right)\Big) \\
=&L(s-c-d, \mathrm{sym}^7 f)L(s-c-d, \mathrm{sym}^7 f \times \chi) \\
&\times \prod_p \Big(1 + \left(6\alpha_f^5(p) + 20\alpha_f^3(p) + 34\alpha_f(p) + 34\beta_f(p) + 20\beta_f^3(p) + 6\beta_f^5(p)\right) \\
&\times p^{-(s-c-d)} + \left(6\alpha_f^5(p) + 20\alpha_f^3(p) + 34\alpha_f(p) + 34\beta_f(p) + 20\beta_f^3(p) + 6\beta_f^5(p)\right) \\
&\times p^{-(s-c-d)}\chi(p) + O\left(p^{2(c+d-\sigma)} + p^{c+d+1+\sigma}\right)\Big) \\
=&L(s-c-d, \mathrm{sym}^7 f)L(s-c-d, \mathrm{sym}^7 f \times \chi)L^6(s-c-d, \mathrm{sym}^5 f) \\
&\times L^6(s-c-d, \mathrm{sym}^5 f \times \chi) \prod_p \Big(1 + \left(14\alpha_f^3(p) + 28\alpha_f(p) + 28\beta_f(p)\right. \\
&\left. + 14\beta_f^3(p)\right)p^{-(s-c-d)} + \left(14\alpha_f^3(p) + 28\alpha_f(p) + 28\beta_f(p) + 14\beta_f^3(p)\right)p^{-(s-c-d)} \\
&\times \chi(p) + O\left(p^{2(c+d-\sigma)} + p^{c+d-\sigma-1}\right)\Big) \\
=&L(s-c-d, \mathrm{sym}^7 f)L(s-c-d, \mathrm{sym}^7 f \times \chi)L^6(s-c-d, \mathrm{sym}^5 f) \\
&\times L^6(s-c-d, \mathrm{sym}^5 f \times \chi)L^{14}(s-c-d, \mathrm{sym}^3 f)L^{14}(s-c-d, \mathrm{sym}^3 f \times \chi) \\
&\times \prod_p \Big(1 + \left(14\alpha_f(p) + 14\beta_f(p)\right)p^{-(s-c-d)} + \left(14\alpha_f(p) + 14\beta_f(p)\right)p^{-(s-c-d)} \\
&\times \chi(p) + O\left(p^{2(c+d-\sigma)} + p^{c+d-\sigma-1}\right)\Big) \\
=&L(s-c-d, \mathrm{sym}^7 f)L(s-c-d, \mathrm{sym}^7 f \times \chi)L^6(s-c-d, \mathrm{sym}^5 f) \\
&\times L^6(s-c-d, \mathrm{sym}^5 f \times \chi)L^{14}(s-c-d, \mathrm{sym}^3 f)L^{14}(s-c-d, \mathrm{sym}^3 f \times \chi) \\
&\times L^{14}(s, f)L^{14}(s, f \times \chi) \prod_p \left(1 + O\left(p^{2(c+d-\sigma)} + p^{c+d-\sigma-1}\right)\right) \\
=&G_7(s-c-d)H_7(s),
\end{aligned}
$$

where $H_7(s)$ converges absolutely and uniformly for $\mathrm{Re}(s) > c + d + \frac{1}{2}$. □

3. Proof of Theorem 1.1

In this section, we shall give the proof of Theorem 1.1. Here, we shall give the detailed proofs for the cases $l = 7, 8$. For the cases $l = 1, 3, 5$, the proofs are similar to the proof of $l = 7$. For the cases $l = 2, 4, 6$, the proofs are similar to the proof of $l = 8$.

We first handle the case $l = 7$. Using Lemma 2.6 to $\sum_{n\le x} \lambda_f^7(n)\sigma^c(n)\phi^d(n)r(n)$, we get

$$\sum_{n\le x} \lambda_f^7(n)\sigma^c(n)\phi^d(n)r(n) = \frac{1}{2\pi i}\int_{c+d+1+\varepsilon-iT}^{c+d+1+\varepsilon+iT} G_7(s-c-d)H_7(s)\frac{x^s}{s}ds + O\left(x^{c+d+1+\varepsilon}T^{-1}\right).$$

Since, from Lemma 2.7, $G_7(s-c-d)H_7(s)\frac{x^s}{s}$ has no poles in the range

$$c+d+\frac{1}{2}+\varepsilon \le \sigma \le c+d+1+\varepsilon$$

and $|t| \le T$, by Cauchy's Residue Theorem we obtain

$$\begin{aligned}
&\sum_{n\le x} \lambda_f^7(n)\sigma^c(n)\phi^d(n)r(n)\\
&= \frac{1}{2\pi i}\left(\int_{c+d+\frac{1}{2}+\varepsilon+iT}^{c+d+1+\varepsilon+iT} + \int_{c+d+\frac{1}{2}+\varepsilon-iT}^{c+d+\frac{1}{2}+\varepsilon+iT} + \int_{c+d+1+\varepsilon-iT}^{c+d+\frac{1}{2}+\varepsilon-iT}\right)G_7(s-c-d)H_7(s)\frac{x^s}{s}ds + O\left(x^{c+d+1+\varepsilon}T^{-1}\right)\\
&:= \frac{1}{2\pi i}(I_{71}+I_{72}+I_{73}) + O\left(x^{c+d+1+\varepsilon}T^{-1}\right).
\end{aligned}$$

For the horizontal segments, since $H_7(s)$ is absolutely convergent in $\mathrm{Re}(s) > c+d+\frac{1}{2}$, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
|I_{71}+I_{73}| &\ll \int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} |G_7(s)x^{c+d+\sigma}T^{-1}|d\sigma\\
&\ll x^{c+d}\int_{\frac{1}{2}+\varepsilon}^{1+\varepsilon} |G_7(s)x^{\sigma}T^{-1}|d\sigma\\
&\ll x^{c+d+\varepsilon}\max_{\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon} x^{\sigma}T^{(\frac{2}{3}\times14+\frac{4}{2}\times14+\frac{6}{2}\times6+\frac{8}{2})\times(1-\sigma)\times2-1}\\
&\ll x^{c+d+\varepsilon}\max_{\frac{1}{2}+\varepsilon\le\sigma\le1+\varepsilon} T^{\frac{353}{3}}\left(\frac{x}{T^{\frac{356}{3}}}\right)^{\sigma}\\
&\ll x^{c+d+1+\varepsilon}T^{-1} + x^{c+d+\frac{1}{2}+\varepsilon}T^{\frac{175}{3}}.
\end{aligned} \tag{3.1}$$

Then, for the vertical segment, by Lemmas 2.1–2.3 and Cauchy's inequality, we have

$$\begin{aligned}
|I_{72}| \ll& \int_1^T \left|G_7\left(\frac{1}{2}+\varepsilon+it\right)\frac{x^{c+d+\frac{1}{2}+\varepsilon}}{c+d+\frac{1}{2}+\varepsilon+it}\right|dt\\
\ll& x^{c+d+\frac{1}{2}+\varepsilon} + x^{c+d+\frac{1}{2}+\varepsilon}\int_1^T \left|G_7\left(\frac{1}{2}+\varepsilon+it\right)\frac{1}{t}\right|dt\\
\ll& x^{c+d+\frac{1}{2}+\varepsilon} + x^{c+d+\frac{1}{2}+\varepsilon}\log T\max_{1\le T_1\le T}\frac{1}{T_1}\left(\max_{\frac{T_1}{2}\le t\le T_1}\left|L^{14}\left(\frac{1}{2}+\varepsilon+it,f\right)\right.\right.\\
&\times L^{14}\left(\frac{1}{2}+\varepsilon+it,f\times\chi\right)L^{14}\left(\frac{1}{2}+\varepsilon+it,\mathrm{sym}^3 f\right)
\end{aligned}$$

$$
\begin{aligned}
&\times L^{14}\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^3 f\times\chi\right) L^{6}\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^5 f\right)\\
&\times L^{6}\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^5 f\times\chi\right)\Bigg|\Bigg)\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^7 f\right)\right|^2 dt\right)^{\frac{1}{2}}\\
&\times\left(\int_{\frac{T_1}{2}}^{T_1}\left|L\left(\frac{1}{2}+\varepsilon+it, \mathrm{sym}^7 f\times\chi\right)\right|^2 dt\right)^{\frac{1}{2}}\\
\ll& x^{c+d+\frac{1}{2}+\varepsilon} + x^{c+d+\frac{1}{2}+\varepsilon}\max_{1\le T_1\le T} T_1^{-1+(\frac{2}{3}\times 14+\frac{4}{2}\times 14+\frac{6}{2}\times 6)\times\frac{1}{2}\times 2+8\times\frac{1}{2}\times\frac{1}{2}\times 2}\\
\ll& x^{c+d+\frac{1}{2}+\varepsilon} + x^{c+d+\frac{1}{2}+\varepsilon}T^{\frac{175}{3}}\\
\ll& x^{c+d+\frac{1}{2}+\varepsilon}T^{\frac{175}{3}}.
\end{aligned}\tag{3.2}
$$

Thus, according to (3.1) and (3.2), we get

$$\sum_{n\le x}\lambda_f^7(n)\sigma^c(n)\phi^d(n)r(n) = O\left(x^{c+d+1+\varepsilon}T^{-1} + x^{c+d+\frac{1}{2}+\varepsilon}T^{\frac{175}{3}}\right).$$

Taking

$$T = x^{\frac{3}{356}},$$

we have

$$S_7(f;x) = O\left(x^{c+d+\frac{353}{356}+\varepsilon}\right).$$

Then, we turn to the case $l = 8$. Using Lemma 2.6 to $\sum_{n\le x}\lambda_f^8(n)\sigma^c(n)\phi^d(n)r(n)$, we get

$$\sum_{n\le x}\lambda_f^8(n)\sigma^c(n)\phi^d(n)r(n) = \frac{1}{2\pi i}\int_{c+d+1+\varepsilon-iT}^{c+d+1+\varepsilon+iT} G_8(s-c-d)H_8(s)\frac{x^s}{s}ds + O\left(x^{c+d+1+\varepsilon}T^{-1}\right).$$

Since, from Lemma 2.7, $G_8(s-c-d)H_8(s)\frac{x^s}{s}$ only has one pole at $s = c+d+1$ of order 14 in the range

$$c+d+\frac{1}{2}+\varepsilon\le\sigma\le c+d+1+\varepsilon$$

and $|t|\le T$, by Cauchy's Residue Theorem again we obtain

$$
\begin{aligned}
&\sum_{n\le x}\lambda_f^8(n)\sigma^c(n)\phi^d(n)r(n)\\
=&\operatorname*{Res}_{s=c+d+1}\left\{F_8(s)\frac{x^s}{s}\right\} + \frac{1}{2\pi i}\left(\int_{c+d+\frac{5}{7}+\varepsilon+iT}^{c+d+1+\varepsilon+iT} + \int_{c+d+\frac{5}{7}+\varepsilon-iT}^{c+d+\frac{5}{7}+\varepsilon+iT} + \int_{c+d+1+\varepsilon-iT}^{c+d+\frac{5}{7}+\varepsilon-iT}\right)\\
&G_8(s-c-d)H_8(s)\frac{x^s}{s}ds + O\left(x^{c+d+1+\varepsilon}T^{-1}\right)\\
:=& x^{c+d+1}A'_{13}(\log x) + \frac{1}{2\pi i}(I_{81}+I_{82}+I_{83}) + O\left(x^{c+d+1+\varepsilon}T^{-1}\right),
\end{aligned}
$$

where $A'_{13}(t)$ is a polynomial in t of degree 13.

For the horizontal segments, since $H_8(s)$ is absolutely convergent in

$$\mathrm{Re}(s) > c + d + \frac{5}{7},$$

by Lemmas 2.1, 2.2 and 2.4, we have

$$\begin{aligned}
|I_{81} + I_{83}| &\ll \int_{\frac{5}{7}+\varepsilon}^{1+\varepsilon} |G_8(s) x^{c+d+\sigma} T^{-1}| d\sigma \\
&\ll x^{c+d} \int_{\frac{5}{7}+\varepsilon}^{1+\varepsilon} |G_8(s) x^{\sigma} T^{-1}| d\sigma \\
&\ll x^{c+d+\varepsilon} \max_{\frac{5}{7}+\varepsilon \le \sigma \le 1+\varepsilon} x^{\sigma} T^{\frac{13}{42}\times(1-\sigma)\times 14 + (\frac{6}{5}\times 28 + \frac{5}{2}\times 20 + \frac{7}{2}\times 7 + \frac{9}{2})\times(1-\sigma)\times 2 - 1} \\
&\ll x^{c+d+\varepsilon} \max_{\frac{5}{7}+\varepsilon \le \sigma \le 1+\varepsilon} T^{\frac{3428}{15}} \left(\frac{x}{T^{\frac{3443}{15}}} \right)^{\sigma} \\
&\ll x^{c+d+1+\varepsilon} T^{-1} + x^{c+d+\frac{5}{7}+\varepsilon} T^{\frac{6781}{105}}.
\end{aligned} \tag{3.3}$$

Then, for the vertical segment, by Lemmas 2.1–2.3 and 2.5, we have

$$\begin{aligned}
|I_{82}| &\ll \int_1^T \left| G_8\left(\frac{5}{7} + \varepsilon + it\right) \frac{x^{c+d+\frac{5}{7}+\varepsilon}}{c + d + \frac{5}{7} + \varepsilon + it} \right| dt \\
&\ll x^{c+d+\frac{5}{7}+\varepsilon} + x^{c+d+\frac{5}{7}+\varepsilon} \int_1^T \left| G_8\left(\frac{5}{7} + \varepsilon + it\right) \frac{1}{t} \right| dt \\
&\ll x^{c+d+\frac{5}{7}+\varepsilon} + x^{c+d+\frac{5}{7}+\varepsilon} \log T \max_{1 \le T_1 \le T} \frac{1}{T_1} \Bigg(\max_{\frac{T_1}{2} \le t \le T_1} \Bigg| \zeta^2\left(\frac{5}{7} + \varepsilon + it\right) \\
&\quad \times L^{14}\left(\frac{5}{7} + \varepsilon + it, \chi\right) L^{28}\left(\frac{5}{7} + \varepsilon + it, \mathrm{sym}^2 f\right) \\
&\quad \times L^{28}\left(\frac{5}{7} + \varepsilon + it, \mathrm{sym}^2 f \times \chi\right) L^{20}\left(\frac{5}{7} + \varepsilon + it, \mathrm{sym}^4 f\right) \\
&\quad \times L^{20}\left(\frac{5}{7} + \varepsilon + it, \mathrm{sym}^4 f \times \chi\right) L^{7}\left(\frac{5}{7} + \varepsilon + it, \mathrm{sym}^6 f\right) \\
&\quad \times L^{7}\left(\frac{5}{7} + \varepsilon + it, \mathrm{sym}^6 f \times \chi\right) L\left(\frac{5}{7} + \varepsilon + it, \mathrm{sym}^8 f\right) \\
&\quad \times L\left(\frac{5}{7} + \varepsilon + it, \mathrm{sym}^8 f \times \chi\right) \Bigg| \Bigg) \int_{\frac{T_1}{2}}^{T_1} \left| \zeta\left(\frac{5}{7} + \varepsilon + it\right) \right|^{12} dt \\
&\ll x^{c+d+\frac{5}{7}+\varepsilon} + x^{c+d+\frac{5}{7}+\varepsilon} \max_{1 \le T_1 \le T} T_1^{-1 + \frac{13}{42}\times\frac{2}{7}\times 16 + (\frac{6}{5}\times 28 + \frac{5}{2}\times 20 + \frac{7}{2}\times 7 + \frac{9}{2})\times\frac{2}{7}\times 2 + 1} \\
&\ll x^{c+d+\frac{5}{7}+\varepsilon} + x^{c+d+\frac{5}{7}+\varepsilon} T^{\frac{48332}{735}} \\
&\ll x^{c+d+\frac{5}{7}+\varepsilon} T^{\frac{48332}{735}}.
\end{aligned} \tag{3.4}$$

Thus, according to (3.3) and (3.4), we get

$$\sum_{n \le x} \lambda_f^8(n) \sigma^c(n) \phi^d(n) r(n) = x^{c+d+1} A'_{13}(\log x) + O\left(x^{c+d+1+\varepsilon} T^{-1} + x^{c+d+\frac{5}{7}+\varepsilon} T^{\frac{48332}{735}} \right).$$

Taking

$$T = x^{\frac{210}{49067}},$$

we have

$$S_8(f;x) = x^{c+d+1}A_{13}(\log x) + O\left(x^{c+d+\frac{48857}{49067}+\varepsilon}\right).$$

4. Proof of Theorem 1.2

For the Gauss circle problem, we have the following famous result

$$\sum_{\substack{n\le x\\ n=a^2+b^2}} 1 = \pi x + O\left(x^{\frac{1}{3}}\right),$$

where $a, b \in \mathbb{Z}$. This result can be found in [13, Corollary 4.9]. Then, by the definition of mathematical expectation and Theorem 1.1, for $l = 1, 2, \cdots, 8$, one has

$$\begin{aligned} E\left(\lambda_f^l(n)\sigma^c(n)\phi^d(n)\right)_{\substack{1\le n\le x\\ n=a^2+b^2}} &= \frac{\sum_{\substack{n\le x\\ n=a^2+b^2}} \lambda_f^l(n)\sigma^c(n)\phi^d(n)}{\sum_{\substack{n\le x\\ n=a^2+b^2}} 1} \\ &= \frac{x^{c+d+1}A_l(\log x) + O\left(x^{c+d+\theta_l+\varepsilon}\right)}{\pi x + O(x^{\frac{1}{3}})} \\ &= \pi^{-1}x^{c+d}A_l(\log x) + O\left(x^{c+d+\theta_l-1+\varepsilon}\right), \end{aligned}$$

where the notations of $A_l(\log x)$ and θ_l are the same as ones in Theorem 1.1. Thus, we complete the proof of Theorem 1.2.

5. Conclusions

In this paper, we establish the asymptotic formula of the sum of the hybrid arithmetic function $\lambda_f^l(n)\sigma^c(n)\phi^d(n)$ over the sparse sequence $\{n : n = a^2+b^2\}$, i.e., $\sum_{n\le x} \lambda_f^l(n)\sigma^c(n)\phi^d(n)r_2(n)$ for $1 \le l \le 8$, where x is a sufficiently large real number, $\lambda_f(n)$ is the n-th normalized Fourier coefficient of the primitive holomorphic cusp form f of even integral weight $k \ge 2$ for $SL_2(\mathbb{Z})$, $\sigma(n)$ and $\phi(n)$ are the sum-of-divisors function and the Euler totient function, $r_2(n)$ denotes the number of representations of n as $n = a^2+b^2$, $a, b, l \in \mathbb{Z}$ and $c, d \in \mathbb{R}$. In addition, we also study the mathematical expectation of this hybrid arithmetic function. With the help of Theorems 1.1 and 1.2, we can understand the asymptotic behaviors of the hybrid arithmetic function $\lambda_f^l(n)\sigma^c(n)\phi^d(n)$ over the sparse sequence $\{n : n = a^2 + b^2\}$ more precisely.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflicts of interest.

References

1. P. Deligne, La conjecture de Weil. I, *Publ. Math. Inst. Hautes Études Sci.*, **43** (1974), 273–307. https://doi.org/10.1007/BF02684373
2. E. Hecke, Theorie der eisensteinsche reihen höherer stufe und ihre anwendung auf funktionentheorie und arithmetik, *Abhandlungen Math. Semin. Univ. Hamb.*, **5** (1927), 199–224. https://doi.org/10.1007/BF02952521
3. J. Wu, Power sums of Hecke eigenvalues and application, *Acta Arith.*, **137** (2009), 333–344. https://doi.org/10.4064/aa137-4-3
4. R. A. Rankin, Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions, *Math. Proc. Cambridge Philos. Soc.*, **35** (1939), 351–372. https://doi.org/10.1017/S0305004100021101
5. A. Selberg, V. Bjerknes, J. Molland, *Bemerkungen über eine Dirichletsche Reihe, die mit der theorie der modulformen nahe verbunden ist*, Cammermeyer i Komm, 1940.
6. B. R. Huang, On Rankin-Selberg problem, *Math. Ann.*, **381** (2021), 1217–1251. https://doi.org/10.1007/s00208-021-02186-7
7. Y. K. Lau, G. S. Lü, Sums of Fourier coefficients of cusp forms, *Q. J. Math.*, **62** (2011), 687–716. https://doi.org/10.1093/qmath/haq012
8. C. R. Xu, General asymptotic formula of Fourier coefficients of cusp forms over sum of two squares, *J. Number Theory*, **236** (2022), 214–229. https://doi.org/10.1016/j.jnt.2021.07.017
9. S. Zhai, Average behavior of Fourier coefficients of cusp forms over sum of two squares, *J. Number Theory*, **133** (2013), 3862–3876. https://doi.org/10.1016/j.jnt.2013.05.013
10. H. F. Liu, On the asymptotic distribution of Fourier coefficients of cusp forms, *Bull. Braz. Math. Soc.*, **54** (2023), 21. https://doi.org/10.1007/s00574-023-00335-x
11. S. Manski, J. Mayle, N. Zbacnik, The asymptotic distribution of a hybrid arithmetic function, *Integers*, **15** (2015), A28. https://doi.org/10.5281/zenodo.10456207
12. L. L. Wei, H. X. Lao, The mean value of a hybrid arithmetic function associated to Fourier coefficients of cusp forms, *Integers*, **19** (2019), A44.
13. H. Iwaniec, *Topics in classical automorphic forms*, Providence, Rhode lsland: American Mathematical Society, 1997. https://doi.org/10.1090/gsm/017
14. J. Newton, J. A. Thorne, Symmetric power functoriality for holomorphic modular forms, *Publ. Math. l'IHÉS*, **134** (2021), 1–116. https://doi.org/10.1007/s10240-021-00127-3
15. J. Newton, J. A. Thorne, Symmetric power functoriality for holomorphic modular forms, *Publ. Math. l'IHÉS*, **134** (2021), 117–152. https://doi.org/10.1007/s10240-021-00126-4
16. A. Perelli, General *L*-functions, *Ann. Mat. Pura Appl.*, **4** (1982), 287–306. https://doi.org/10.1007/BF01761499

17. J. Bourgain, Decoupling, exponential sums and the Riemann zeta function, *J. Amer. Math. Soc.*, **30** (2017), 205–224. https://doi.org/10.1090/jams/860

18. Y. Lin, R. Nunes, Z. Qi, Strong subconvexity for self-dual $GL(3)$ L-functions, *Int. Math. Res. Not.*, **2023** (2023), 11453–11470. https://doi.org/10.1093/imrn/rnac153

19. K. Matsumoto, The mean values and the university of Rankin-Selberg L-functions, *Number Theory*, 2001, 201–221.

20. K. Ramachandra, A. Sankaranarayanan, Notes on the Riemann zeta-function, *J. Indian Math. Soc.*, **57** (1991), 67–77.

21. A. Ivić, Exponent pairs and the zeta function of Riemann, *Studia Sci. Math. Hunger.*, **15** (1980), 157–181.

22. G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge University Press, 1995. https://doi.org/10.1090/gsm/163