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Research article

Distribution of values of Hardy sums over Chebyshev polynomials

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Abstract: This paper mainly studied the distribution of values of Hardy sums involving Chebyshev polynomials. By using the method of analysis and the arithmetic properties of Hardy sums and Chebyshev polynomials of the first kind, we obtained a sharp asymptotic formula for the hybrid mean value of Hardy sums $S_5(h,q)$ involving Chebyshev polynomials of the first kind. In addition, we also gave the value of Hardy sums $S(h,q)$ and $S_3(h,q)$ involving Chebyshev polynomials. Finally, we found the reciprocal formulas of $S_3(h,q)$ and $S_4(h,q)$ involving Chebyshev polynomials of the first kind.

Keywords: Hardy sums; Chebyshev polynomials; values
Mathematics Subject Classification: 11F20, 11B83

1. Introduction

Let q be an integer with $q > 0$. For any integer h, we have Dedekind sums $s(h,q)$

$$s(h,q) = \sum_{a=1}^{q-1} \left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right)$$

where

$$((x)) = \begin{cases} x - [x] - \frac{1}{2}, & \text{if } x \text{ is not an integer;} \\ 0, & \text{if } x \text{ is an integer,} \end{cases}$$

$[x]$ is the largest integer not exceeding x. The Dedekind sums, named after Richard Dedekind himself, are finite sums of products of the sawtooth functions. They arise in the functional equation resulting from the action of the Dedekind eta function under modular groups (see [1, 3]). Dedekind sums have been found in applications in the analytic number theory, topology, and other branches of mathematics. Many scholars gave the various interesting properties of Dedekind sums (see [5, 16]), the most famous of which is the reciprocal formula of Dedekind sums

$$s(h,q) + s(q,h) = \frac{h^2+q^2+1}{12hq} - \frac{1}{4}$$

where h, q are two positive integers such that $(h, q) = 1$.

Hardy sums analogue to the Dedekind sums are defined by

$$S(h,q) = \sum_{i=1}^{q-1} (-1)^{i+1+\left[\frac{hi}{q}\right]}, \qquad S_1(h,q) = \sum_{i=1}^{q} (-1)^{\left[\frac{hi}{q}\right]} \left(\left(\frac{i}{q}\right)\right),$$

$$S_2(h,q) = \sum_{i=1}^{q} (-1)^{i} \left(\left(\frac{i}{q}\right)\right)\left(\left(\frac{hi}{q}\right)\right), \qquad S_3(h,q) = \sum_{i=1}^{q} (-1)^{i} \left(\left(\frac{hi}{q}\right)\right),$$

$$S_4(h,q) = \sum_{i=1}^{q-1} (-1)^{\left[\frac{hi}{q}\right]}, \qquad S_5(h,q) = \sum_{i=1}^{q} (-1)^{i+\left[\frac{hi}{q}\right]} \left(\left(\frac{i}{q}\right)\right).$$

Hardy sums play important roles in the transformation formula of the logarithm of the classical theta function [4] and they are closely connected with Dedekind sums [17]. Many scholars studied the mean square value of Hardy sums (see [13, 19]). Many scholars also considered the connection between Hardy sums and some famous sums. Guo et al. [12] and Peng and Zhang [15] gave some identities involving certain Hardy sums and Kloosterman sums. Dağlı [6] studied a computational problem of the mean values involving certain Hardy sums and two-term exponential sums. Further, Dağlı and Sever [7] and Tian and Wang [18] considered the mean value of generalized Hardy sums weighted by Kloosterman sums.

For any integer $n \geq 0$, Chebyshev polynomials of the first kind $T_n(x)$ and Chebyshev polynomials of the second kind $U_n(x)$ are defined by $T_0(x) = 1$, $T_1(x) = x$ and $T_{n+2}(x) = 2xT_{n+1}(x) - T_n(x)$, $U_0(x) = 1$, $U_1(x) = 2x$, and $U_{n+2}(x) = 2xU_{n+1}(x) - U_n(x)$. Let $\alpha = x + \sqrt{x^2 - 1}$ and $\beta = x - \sqrt{x^2 - 1}$, and we know that

$$T_n(x) = \frac{1}{2}(\alpha^n + \beta^n) = \frac{n}{2} \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{(n-k-1)!}{k!(n-2k)!} (2x)^{(n-2k)}$$

and

$$U_n(x) = \frac{1}{\alpha - \beta}(\alpha^{n+1} - \beta^{n+1}) = \sum_{k=0}^{\left[\frac{n}{2}\right]} (-1)^k \frac{(n-k)!}{k!(n-2k)!} (2x)^{(n-2k)}.$$

Many authors have studied the elementary properties of Chebyshev polynomials and obtained a series of interesting conclusions (see [2, 9, 14, 20]). Guan and Li [11] studied the problem of asymptotic analysis of the mean value involving Dedekind sums and Chebyshev polynomials of the first kind, and they obtained

$$\sum_{m \leq N} \frac{s(T_m(x), T_{m+1}(x))}{m} = \frac{2x-3}{12} \cdot N + \frac{1}{12}\left(2x - 3 - \sqrt{x^2-1} + C(x) + \frac{1}{x}\right) \cdot \ln N + O(1)$$

where

$$C(x) = \sum_{n=1}^{\infty} \frac{1}{T_n(x) T_{n+1}(x)}.$$

The distribution of values in Dedekind-type sums, such as Dedekind sums and Hardy sums involving the second-order linear recurrence polynomials, is an intriguing area of study, offering insights into the behavior and properties of these mathematical objects (see [8, 21]). Understanding the distribution of these values is crucial for various applications in number theory and analysis. In this paper, we delve into the investigation of the distribution of values in Hardy sums, focusing on Chebyshev polynomials of the first kind. By leveraging the method of analysis and exploiting the arithmetic properties of Hardy sums and Chebyshev polynomials, we aim to provide a comprehensive understanding of the distribution patterns and establish precise formulas for hybrid mean values and other relevant quantities. This research contributes to the broader understanding of the interplay between Hardy sums and Chebyshev polynomials, shedding light on their intricate properties and applications in mathematical analysis.

First, we give a sharp asymptotic formula for the mean value of Hardy sums $S_5(h, q)$ involving Chebyshev polynomials of the first kind.

Theorem 1.1. *Let $N > 2$ be an integer. For any odd integer $x \geq 3$, we have*

$$\sum_{m \leq N} \frac{S_5(T_m(x), T_{m+1}(x))}{m} = \frac{N}{2} + \left(\frac{x-1}{2x} - \frac{C(x)}{2}\right) \cdot \ln N + O(1)$$

where $C(x) = \sum_{n=1}^{\infty} \frac{1}{T_n(x)T_{n+1}(x)}$.

Remark. By using the same methods as used in this paper, for any real number $1 < s < 2$, we also have

$$\sum_{m \leq N} \frac{S_5(T_m(x), T_{m+1}(x))}{m^s} = \frac{N^{2-s}}{2(2-s)} + \zeta(s-1) + O(1)$$

where $\zeta(s)$ denotes the Riemann zeta function. If $s = 2$, then we have

$$\sum_{m \leq N} \frac{S_5(T_m(x), T_{m+1}(x))}{m^2} = \frac{\ln N}{2} + O(1).$$

From Lemma 2.4, we know that $S_5(T_m(x), T_{m+1}(x)) \sim m$ when m is large, so when $s > 1$, the asymptotic formula does not reflect the effect of $\sum_{n=1}^{m} \frac{1}{T_n(x)T_{n+1}(x)}$.

We also give the values of $S(h, q)$ with Chebyshev polynomials.

Theorem 1.2. *Let m be a nonnegative integer. For any positive integer x, we have*

$$S(U_m(x), U_{m+1}(x)) = m + 1.$$

Note. Taking $x = 1$ in Theorem 1.2, we have $U_m(1) = m + 1$ and $S(m + 1, m + 2) = m + 1$.

Theorem 1.3. *Let m be a nonnegative integer. For any even integer $x \geq 2$, we have*

$$S(T_m(x), T_{m+1}(x)) = m + 1.$$

Note. Taking $x = 2$ in Theorem 1.3, we have $T_3(2) = 26$, $T_4(2) = 97$, and $T_5(2) = 362$. We also know that $S(26, 97) = 4$ and $S(97, 362) = 5$.

Moreover, we have the value of $S_3(h, q)$ involving Chebyshev polynomials of the first kind as follows:

Theorem 1.4. *Let m be a nonnegative integer. For any odd integer $x \geq 3$, we have*

$$S_3\left(\overline{2}T_m(x), T_{m+1}(x)\right) = \frac{1-x}{2}m + \frac{2-x}{4} - \frac{T_m(x)}{4T_{m+1}(x)}$$

where $\overline{2}$ satisfies $\overline{2}2 \equiv 1 \pmod{T_{m+1}(x)}$.

Finally, we also have the reciprocal formulas of $S_3(h,q)$ and $S_4(h,q)$ involving Chebyshev polynomials of the first kind by using Theorem 1.4.

Theorem 1.5. *Let m be a nonnegative integer. For any odd integer $x \geq 3$, we have*

$$S_3\left(T_m(x), T_{m+1}(x)\right) - S_3\left(T_{m+1}(x), T_m(x)\right) = \frac{T_{m+1}(x)}{2T_m(x)} - \frac{T_m(x)}{2T_{m+1}(x)} - \frac{(2m+1)(x-1)}{2}.$$

Theorem 1.6. *Let m be a nonnegative integer. For any odd integer $x \geq 3$, we have*

$$S_4\left(T_m(x), T_{m+1}(x)\right) - S_4\left(T_{m+1}(x), T_m(x)\right) = (2m+1)(x-1).$$

Note. Taking $x = 3$, we have $T_1(3) = 3$, $T_2(3) = 17$, $T_3(3) = 99$, $T_4(3) = 577$, and $T_5(3) = 3363$. We also know that $S_3(27,17) = -\frac{22}{17}$, $S_3(850,99) = -\frac{227}{99}$, $S_3(28611,577) = -\frac{1900}{577}$, $S_3(3,17) = -\frac{10}{17}$, $S_3(17,99) = -\frac{157}{99}$, $S_3(99,577) = -\frac{1492}{577}$, $S_4(3,17) = 4$, $S_4(17,99) = 6$, $S_4(99,577) = 8$, and $S_4(577,3363) = 10$. From Theorem 1.6 and the fact that $S_4\left(T_{m+2}(x), T_{m+1}(x)\right) = -S_4\left(T_m(x), T_{m+1}(x)\right)$, we have Corollary 1.7 as follows:

Corollary 1.7. *For any nonnegative integer m, we have*

$$S_4\left(T_m(3), T_{m+1}(3)\right) = 2m + 2.$$

From Corollary 1.7 and Lemma 2.3, we also give the following corollary:

Corollary 1.8. *For any nonnegative integer m, we have*

$$S_3\left(T_m(3), T_{m+1}(3)\right) = \frac{1}{2} - m - \frac{T_m(3)}{2T_{m+1}(3)}.$$

By using similar methods as used in this paper, we can also consider Dedekind-type sums involving the second-order linear recurrence polynomials, for example, Fibonacci polynomials, Lucas polynomials, Pell polynomials, etc. Fibonacci polynomials $F_n(x)$ are defined by $F_0(x) = 0$, $F_1(x) = 1$, and $F_{n+1}(x) = xF_n(x) + F_{n-1}(x)$ for any positive integer n. We have the following result:

Theorem 1.9. *For any positive integer n and any even integer $x \geq 2$, we have*

$$S\left(F_{2n-1}(x), F_{2n}(x)\right) = 1, \quad S\left(F_{2n}(x), F_{2n+1}(x)\right) = 0.$$

For Bessel polynomials, which are defined by $y_0(x) = 1$, $y_1(x) = x+1$, and $y_{n+1}(x) = (2n+1)xy_n(x) + y_{n-1}(x)$, we can get the following result similar to Theorem 1.4:

Theorem 1.10. *For any nonnegative integer n and any even integer $x \geq 2$, we have*

$$S_3(\overline{2}y_n(x), y_{n+1}(x)) = \frac{1}{4} - \frac{y_n(x)}{4y_{n+1}(x)} - \frac{(n+1)x}{4}$$

where $\overline{2}$ satisfies $\overline{2}2 \equiv 1 \pmod{y_{n+1}(x)}$.

Remark. Restricted by Lemma 2.1, we can draw results similar to Theorems 1.4–1.6 for Hardy sums involving the second-order linear recurrence polynomials when the values of these polynomials are odd integers, and it is easy to obtain results similar to Theorems 1.5 and 1.6 for Hardy sums involving Bessel polynomials from Theorem 1.10.

2. Some lemmas

To prove theorems, we need the following several lemmas.

Lemma 2.1. *Let $h, q > 0$ be odd integers with $(h, q) = 1$, then we have*

$$s\left(\frac{q+1}{2}h, q\right) + s\left(\frac{h+1}{2}q, h\right) = \frac{h^2 + q^2 + 4}{24hq} - \frac{1}{4}$$

and

$$s(h, 2q) + s(2h, q) + s\left(2\overline{h}, q\right) = 3s(h, q)$$

where $\overline{h}$ satisfies $h\overline{h} \equiv 1 \pmod{q}$.

Proof. See Theorem 1 and Lemma 2 of [10]. □

Lemma 2.2. *Let $h, q > 0$ be integers with $(h, q) = 1$, then we have*

$$\begin{aligned} S_3(h,q) &= 2s(h,q) - 4s(2h,q), && if\ q\ is\ odd; \\ S_4(h,q) &= -4s(h,q) + 8s(h,2q), && if\ h\ is\ odd. \end{aligned}$$

Proof. See [17]. □

Lemma 2.3. *Let h, q be positive integers with $(h, q) = 1$. If $h + q$ is odd, then we have*

$$S(h, q) + S(q, h) = 1;$$

if $h + q$ is even, then we have

$$S_5(h, q) + S_5(q, h) = \frac{1}{2} - \frac{1}{2hq};$$

and if q is odd, then we have

$$2S_3(h, q) - S_4(q, h) = 1 - \frac{h}{q}.$$

Proof. See [17]. □

Lemma 2.4. *Let m be a positive integer. For any odd integer $x \geq 3$, we have*

$$S_5(T_m(x), T_{m+1}(x)) = \frac{m}{2} - \frac{1}{2}\sum_{n=1}^{m} \frac{1}{T_n(x)T_{n+1}(x)} + \frac{x-1}{2x}.$$

Proof. From the recursive formula of Chebyshev polynomials of the first kind, if $x \geq 3$ is any odd integer, then we have that $T_m(x)$ is an odd integer. According to the recursive formula of $T_m(x)$ and the properties of the greatest common divisor of $(T_{m+1}(x), T_{m+2}(x))$, we also have

$$\begin{aligned} (T_{m+1}(x), T_{m+2}(x)) &= (T_{m+1}(x), 2xT_{m+1}(x) - T_m(x)) \\ &= (T_{m+1}(x), T_m(x)) \\ &= \cdots = (T_1(x), T_0(x)) \\ &= 1. \end{aligned}$$

From Lemma 2.3, we obtain

$$S_5(T_n(x), T_{n+1}(x)) + S_5(T_{n+1}(x), T_n(x)) = \frac{1}{2} - \frac{1}{2}\frac{1}{T_n(x)T_{n+1}(x)}$$

and it is obvious that

$$S_5(T_{n+1}(x), T_n(x)) = S_5(2xT_n(x) - T_{n-1}(x), T_n(x)) = -S_5(T_{n-1}(x), T_n(x)),$$

$$S_5(T_0(x), T_1(x)) = S_5(1, x) = \frac{x-1}{2x}.$$

Thus,

$$\sum_{n=1}^{m}[S_5(T_n(x), T_{n+1}(x)) - S_5(T_{n-1}(x), T_n(x))] = \sum_{n=1}^{m}\frac{1}{2} - \frac{1}{2}\sum_{n=1}^{m}\frac{1}{T_n(x)T_{n+1}(x)}$$

and

$$S_5(T_m(x), T_{m+1}(x)) - S_5(T_0(x), T_1(x)) = \frac{m}{2} - \frac{1}{2}\sum_{n=1}^{m}\frac{1}{T_n(x)T_{n+1}(x)}.$$

In summary, we have

$$S_5(T_m(x), T_{m+1}(x)) = \frac{m}{2} - \frac{1}{2}\sum_{n=1}^{m}\frac{1}{T_n(x)T_{n+1}(x)} + \frac{x-1}{2x}.$$

□

Lemma 2.5. *Let $N \geq 3$ be an integer, then for any integer $x \geq 2$, we have*

$$\sum_{m\leq N}\frac{1}{m}\sum_{n=1}^{m}\frac{1}{T_n(x)T_{n+1}(x)} = C(x)\ln N + O(1)$$

where $C(x) = \sum_{n=1}^{\infty}\frac{1}{T_n(x)T_{n+1}(x)}$.

Proof. See [11]. □

3. Proofs of theorems

Now, we prove our theorems by using the above lemmas.

Proof of Theorem 1.1. Combining Lemmas 2.4 and 2.5, we see that

$$\begin{aligned}\sum_{m\leq N}\frac{S_5(T_m(x), T_{m+1}(x))}{m} &= \sum_{m\leq N}\frac{1}{2} - \sum_{m\leq N}\frac{1}{2m}\sum_{n=1}^{m}\frac{1}{T_n(x)T_{n+1}(x)} + \frac{x-1}{2x}\sum_{m\leq N}\frac{1}{m}\\ &= \frac{N}{2} - \frac{C(x)}{2}\cdot\ln N + \frac{x-1}{2x}\cdot\ln N + O(1)\\ &= \frac{N}{2} + \left(\frac{x-1}{2x} - \frac{C(x)}{2}\right)\cdot\ln N + O(1).\end{aligned}$$

□

Proof of Theorems 1.2 and 1.3. Similar to the proof of Lemma 2.4, for any nonnegative integer m from the recursive formula of Chebyshev polynomials of the second kind, we have that $U_{2m}(x)$ is an odd integer and $U_{2m+1}(x)$ is an even integer for any positive integer x. We also have

$$\begin{aligned}(U_{m+1}(x), U_{m+2}(x)) &= (U_{m+1}(x), 2xU_{m+1}(x) - U_m(x)) \\ &= (U_{m+1}(x), U_m(x)) = \cdots = (U_1(x), U_0(x)) \\ &= 1.\end{aligned}$$

From Lemma 2.3, we obtain

$$\begin{aligned}&S(U_n(x), U_{n+1}(x)) + S(U_{n+1}(x), U_n(x)) \\ &\quad = S(U_n(x), U_{n+1}(x)) - S(U_{n-1}(x), U_n(x)) = 1\end{aligned}$$

and we know that

$$S(U_0(x), U_1(x)) = S(1, 2x) = 1.$$

Thus,

$$\begin{aligned}&S(U_m(x), U_{m+1}(x)) \\ &= \sum_{n=1}^{m} [S(U_n(x), U_{n+1}(x)) - S(U_{n-1}(x), U_n(x))] + S\ (U_0(x), U_1(x)) \\ &= m + 1.\end{aligned}$$

This proves Theorem 1.2.

Similarly, we also have Theorem 1.3. □

Proof of Theorem 1.4. For any odd integer $x \geq 3$, from the proof of Lemma 2.4 we have that $T_m(x)$ is an odd integer and $(T_m(x), T_{m+1}(x)) = 1$. By Lemma 2.1, we can get

$$\begin{aligned}&s\left(\frac{T_{n+1}(x)+1}{2}T_n(x), T_{n+1}(x)\right) - s\left(\frac{T_n(x)+1}{2}T_{n-1}(x), T_n(x)\right) \\ &\qquad = \frac{T_n(x)^2 + T_{n+1}(x)^2 + 4}{24T_n(x)T_{n+1}(x)} - \frac{1}{4},\end{aligned}$$

then we have

$$\begin{aligned}&s\left(\frac{T_{m+1}(x)+1}{2}T_m(x), T_{m+1}(x)\right) \\ &= \sum_{n=1}^{m} \frac{T_n(x)^2 + T_{n+1}(x)^2 + 4}{24T_n(x)T_{n+1}(x)} + s\left(\frac{T_1(x)+1}{2}T_0(x), T_1(x)\right) - \frac{m}{4} \\ &= \sum_{n=1}^{m} \left\{\frac{T_n(x)}{24T_{n+1}(x)} - \frac{T_{n-1}(x)}{24T_n(x)}\right\} + \frac{1}{6}\sum_{n=1}^{m} \frac{1}{T_n(x)T_{n+1}(x)} + s(2, x) + \frac{x-3}{12}m \\ &= \frac{T_m(x)}{24T_{m+1}(x)} + \frac{1}{6}\sum_{n=1}^{m} \frac{1}{T_n(x)T_{n+1}(x)} + \frac{x^2+4}{24x} - \frac{1}{4} + \frac{x-3}{12}m. \qquad (3.1)\end{aligned}$$

From Lemma 2.1 of [11]:

$$\begin{aligned}s\left(T_m(x), T_{m+1}(x)\right) =& \frac{T_m(x)}{12T_{m+1}(x)} + \frac{1}{12}\sum_{n=1}^{m}\frac{1}{T_n(x)T_{n+1}(x)}\\ &+ \frac{1}{12x} + \frac{x-3}{12} + \frac{2x-3}{12}m. \end{aligned}\tag{3.2}$$

Using the fact that $s(h,q) = s(\overline{h}, q)$, where $\overline{h}$ satisfies $\overline{h}h \equiv 1 \pmod q$, we have

$$s\left(\frac{T_{m+1}(x)+1}{2}T_m(x), T_{m+1}(x)\right) = s\left(2\overline{T_m(x)}, T_{m+1}(x)\right).$$

Let $2\times(3.2)$ subtract (3.1), then we have

$$2s\left(T_m(x), T_{m+1}(x)\right) - s\left(2\overline{T_m(x)}, T_{m+1}(x)\right) = \frac{T_m(x)}{8T_{m+1}(x)} + \frac{x-1}{4}m + \frac{x-2}{8}.\tag{3.3}$$

By Lemma 2.2, we can write

$$S_3\left(\overline{2}T_m(x), T_{m+1}(x)\right) = -2\left(2s\left(T_m(x), T_{m+1}(x)\right) - s\left(2\overline{T_m(x)}, T_{m+1}(x)\right)\right).$$

According to (3.3), we have Theorem 1.4. □

Proof of Theorems 1.5 and 1.6. Combining Lemmas 2.1 and 2.2, we see that

$$\begin{aligned}S_4(h,q) - 2S_3(h,q) &= -4s(h,q) + 8s(h,2q) - 2(2s(h,q) - 4s(2h,q))\\ &= -8s(h,q) + 8s(h,2q) + 8s(2h,q)\\ &= -8s(h,q) + 8(3s(h,q) - s(2\overline{h},q))\\ &= 16s(h,q) - 8s(2\overline{h},q)\\ &= -4S_3(\overline{2}h,q).\end{aligned}$$

According to Theorem 1.4 and Lemma 2.3, we can get

$$S_4\left(T_m(x), T_{m+1}(x)\right) - 2S_3\left(T_m(x), T_{m+1}(x)\right) = \frac{T_m(x)}{T_{m+1}(x)} + 2(x-1)m + x - 2,$$

$$2S_3\left(T_m(x), T_{m+1}(x)\right) - S_4\left(T_{m+1}(x), T_m(x)\right) = 1 - \frac{T_m(x)}{T_{m+1}(x)}$$

and

$$2S_3\left(T_{m+1}(x), T_m(x)\right) - S_4\left(T_m(x), T_{m+1}(x)\right) = 1 - \frac{T_{m+1}(x)}{T_m(x)},$$

then we immediately have

$$S_4\left(T_m(x), T_{m+1}(x)\right) - S_4\left(T_{m+1}(x), T_m(x)\right) = (2m+1)(x-1)$$

and

$$S_3\left(T_m(x), T_{m+1}(x)\right) - S_3\left(T_{m+1}(x), T_m(x)\right) = \frac{T_{m+1}(x)}{2T_m(x)} - \frac{T_m(x)}{2T_{m+1}(x)} - \frac{(2m+1)(x-1)}{2}.$$

□

4. Conclusions

Hardy sums play important roles in analytic number theory and it is of interest to study the properties of Hardy sums. Generally speaking, it is difficult to give explicit formulas for the values of Hardy sums, especially when (h, q) is large. Therefore, we want to give as many of the values of Hardy sums as possible in special cases, such as Hardy sums involving polynomials. In this paper, we obtained an explicit asymptotic formula for the mean value of $S_5(T_m(x), T_{m+1}(x))$. Moreover, we also found the values of $S(T_m(x), T_{m+1}(x))$; $S(U_m(x), U_{m+1}(x))$; and $S_3(T_m(x), T_{m+1}(x))$. These results are useful for understanding the behavior and properties of Hardy sums. For example, for any integers $x > 0$ and $m \geq 0$, we have

$$S(U_m(x), U_{m+1}(x)) = m + 1.$$

If we know the values of Chebyshev polynomials of the second kind $U_m(x)$ and $U_{m+1}(x)$, then we can immediately find the values of Hardy sums $S(U_m(x), U_{m+1}(x))$. Many scholars also considered the properties of generalized Hardy sums (see [7, 18]). In the future, we hope to study the values of generalized Hardy sums involving Chebyshev polynomials by using similar methods as used in this paper.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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