Mathematics

http://www.aimspress.com/journal/Math

AIMS Mathematics, 9(2): 3740–3762.
DOI: 10.3934/math.2024184
Received: 07 October 2023
Revised: 13 September 2023
Accepted: 27 September 2023
Published: 10 January 2024

Research article

Analysis of a new jointly hybrid censored Rayleigh populations

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Abstract: When a researcher wants to perform a life-test comparison study of items made by two separate lines inside the same institution, joint censoring strategies are particularly important. In this paper, we present a new joint Type-I hybrid censoring that enables an experimenter to stop the investigation as soon as a pre-specified number of failures or time is first achieved. In the context of newly censored data, the estimates of the unknown mean lifetimes of two different Rayleigh populations are acquired using maximum likelihood and Bayesian inferential techniques. The normality characteristic of classical estimators is used to offer asymptotic confidence interval bounds for each unknown parameter. Against gamma conjugate priors, the Bayes estimators and related credible intervals are gathered about symmetric and asymmetric loss functions. Since classical and Bayes estimators are acquired in closed form, simulation tests can be easily made to evaluate the effectiveness of the proposed methodologies. The efficiency of the suggested approaches is examined in terms of four metrics, namely: Root mean squared error, average relative absolute bias, average confidence length, and coverage probability. To demonstrate the applicability of the offered approaches to real events, two real applications employing data sets from the engineering area are analyzed. As a result, when the experimenter's primary goal is to complete the test as soon as the total number of failures or the threshold period is recorded, the numerical results reveal that the recommended strategy is adaptable and very helpful in completing the study.

Keywords: jointly Type-I hybrid censoring; Rayleigh populations; maximum likelihood; Bayes estimation; confidence intervals
Mathematics Subject Classification: 62F10, 62F15, 62N01, 62N02, 62N05

1. Introduction

If the lifespan of particular items is relatively lengthy and/or the sample size n is large, it is challenging to continue the examination until all n observations are recorded (full data). The censored mechanism is a less costly choice for ending the test under predetermined conditions. The hybrid censoring (HC), by Balakrishnan and Kundu [1], is a combination of Type-I (time) and Type-II (failure) censoring methods in which the life-testing experiment is stopped when a pre-defined number m (of n) items fail or a certain time T on the test is reached. The key advantage of employing a HC plan over standard Type-I or Type-II censoring is that it saves time and money by reducing the projected testing time and failures seen.

In reliability studies, to evaluate the performance of the lifetime of two competing units coming from different production lines in the same facility, joint censoring schemes become quite useful for performing comparative life tests. In this scenario, Balakrishnan and Rasouli [2] proposed joint Type-II censoring; Su [3] developed jointly Type-I hybrid censoring (JHC-T1) arising from multiple independent exponential populations; Su and Zhu [4] proposed jointly generalized Type-I HC; Shafay [5] proposed jointly Type-II HC; and recently, Elshahhat and Abo-Kasem [6] proposed a jointly generalized Type-II HC plan.

Briefly, the JHC-T1 can be stated as follows: Assume that two items (say, A and B) are created in the identical factory by two distinct lines of operation (say, L_1 and L_2). Assume that $\mathbf{X} = \{X_1, X_2, ..., X_m\}$ and $\mathbf{Y} = \{Y_1, Y_2, ..., Y_n\}$ are two separate samples of m and n pieces of product A and B, respectively, that get placed in a life-testing experiment at the same time. Here, we have $N = m + n$. Assume that $\mathbf{X}$ ($\mathbf{Y}$) are independent and identically distributed (iid) variables from a population with cumulative distribution function (CDF), $F_X(\cdot)$ ($F_Y(\cdot)$), and probability density function (PDF), $f_X(\cdot)$ ($f_Y(\cdot)$). Regarding the manufacturing information on the product of interest, the threshold point T is also assumed to be fixed beforehand. Let $G_{(1)} \leqslant G_{(2)} \leqslant \cdots \leqslant G_{(N)}$ be the order statistics of N random variables $\{\mathbf{X}; \mathbf{Y}\} = \{X_1, X_2, ..., X_m; Y_1, Y_2, ..., Y_n\}$.

To perform a life test that doesn't take more time to complete due to cost constraints, the investigator decides to perform the proposed censoring. As a result, the collected data ($\mathbf{g}$,$\mathbf{z}$) will consist of one of the following data forms:

$$(\mathbf{g}; \mathbf{z}) = \begin{cases} \{g_{(1)}, g_{(2)}, ..., g_{(r)}; z_1, z_2, ..., z_r\}, & \text{if } g_{(r)} \leqslant T \text{ (Case-1)}; \\ \{g_{(1)}, g_{(2)}, ..., g_{(d)}; z_1, z_2, ..., z_d\}, & \text{if } g_{(r)} > T \text{ (Case-2)}, \end{cases} \quad \text{and} \quad z_i = \begin{cases} 1, & \text{if } g_{(i)} \in \mathbf{X}; \\ 0, & \text{otherwise} \end{cases}$$

where d represents the size of observed failures before T. More specifically, let $m_D = \sum_{i=1}^{D} z_i$ and $n_D = \sum_{i=1}^{D}(1 - z_i)$ be the number of subjects that failed in $\mathbf{X}$ and $\mathbf{Y}$, respectively. Also, let $D = m^* + n^*$, where $D \in \{1, 2, ..., N\}$, be the effective sample size. The main advantage of the JHC-T1 is that it can effectively guarantee that the experiment stops the test as soon as a certain number of failures (r) or a threshold time (T) is first achieved.

If the JHC-T1 failure times of N specimens are coming from two samples of continuous population with PDFs $f_i(\cdot)$ and CDFs $F_i(\cdot)$ for $i = 1, 2$, then the joint PDF of θ given $\mathbf{g}$ can be expressed as

$$L(\boldsymbol{\theta}|\,\mathbf{g}) = C_D \prod_{i=1}^{D} \left[f_1(g_{(i)}; \theta_1)^{z_i} f_2(g_{(i)}; \theta_2)^{1-z_i} \right] [1 - F_1(T^*; \theta_1)]^{m-m^*} [1 - F_2(T^*; \theta_2)]^{n-n^*}, \tag{1.1}$$

where $\boldsymbol{\theta}$ is a parameter vector, $C_D = \frac{m!n!}{(m-m^*)!(n-n^*)!}$, and $T^* = g_{(r)}$(or T) for Case-1 (or Case-2). In Figure 1, a schematic representation of the proposed plan is depicted.

Remark 1. From (1.1), when $T \to \infty$ and $r = N$, the joint Type-I HC (by Su [3]) reduced to the joint Type-II censoring (by Balakrishnan and Rasouli [2]).

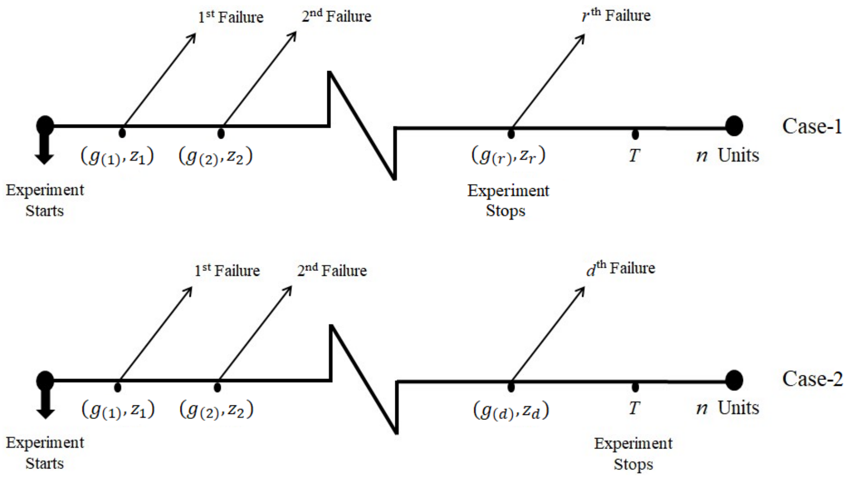

Figure 1. Schematic representation of the JHC-T1 plan.

In this paper, depending on unit capacity, experimental facilities, and cost restrictions, we explore a new scheme called JHC-T1, in which the experimenter terminates the test once the predetermined time (or the predetermined number of failures) is recorded first. Therefore, to the greatest extent of our understanding, no research has been presented on the estimation problem of population parameters when lifetime data are acquired via the proposed jointly hybrid censoring. Thus, the novelty of this work originates from the fact that, in the presence of new JHC-T1 failure times, it is the first time that two maximum likelihood and Bayesian approaches for the Rayleigh distribution's parameters of life have been compared since its inception.

The Rayleigh distribution is the simplest probability distribution of wind speed to represent the wind resource because it requires only knowledge of the average wind speed. It is also utilized in reliability analysis of radar and microwave networks, image recognition, wind energy modeling, electro-vacuum device design, wave height modeling in oceanography, acoustics, and magnetic resonance imaging, among other applications; see Chattamvelli and Shanmugam [7]. From a probability point of view, this is a subset of the Weibull lifespan model, which Rayleigh proposed while studying acoustic problems. For this purpose, we consider the Rayleigh lifetime model, which is commonly used in several areas of statistics. In the context of a hybrid censored sample, the proposed population has been discussed by Kwon et al. [8], Asgharzadeh and Azizpour [9], and Jeon and Kang [10], among others.

Our primary goal of this study is fourfold:

- Compute the maximum likelihood and asymptotic interval estimators for the unknown mean lifespan of the test subjects in two samples based on the Rayleigh population with varied scale parameters.
- Acquire both symmetric and asymmetric Bayes' estimators as well as two bounds of credible interval estimators for the same unknown quantities.

- Using various choices of n, m, r, and T, the performance of the proposed methods is compared through a Monte-Carlo simulation.
- Two genuine data set-based applications from the engineering sector highlight the ability of two different Rayleigh populations to accommodate various data types and adapt the given estimation approaches to actual practical situations.

The rest of the article is arranged as follows: In Section 2, the maximum likelihood estimators and the asymptotic intervals are obtained. The Bayesian inference along with the credible intervals are discussed in Section 3, where three loss functions are considered, the square error loss function, the linear exponential loss function, and the general entropy loss function. Numerical simulation with the Monte Carlo technique is implemented in Section 4 to assess the performance of the proposed estimates in terms of root mean square error and average relative absolute biases, and the use of the probability coverage to assess the interval estimation performance. In Section 5, illustrative examples from the engineering field are developed. Last, Section 6 concludes the results of the study.

2. Likelihood inference

This section presents the maximum likelihood estimators (MLEs) as well as asymptotic interval estimators (ACIs) of two Rayleigh population parameters in the presence of such proposed censored data.

2.1. Maximum likelihood estimators

Suppose the lifetimes $\mathbf{X}$ (of size m from facility-line L_1) are iid random variables from a Rayleigh population with a scale parameter ξ_1 having the following PDF $f_X(x;\xi_1)$ and CDF $F_X(x;\xi_1)$ as

$$f_X(x;\xi_1) = 2x\xi_1 e^{-\xi_1 x^2} \quad \text{and} \quad F_X(x;\xi_1) = 1 - e^{-\xi_1 x^2}, \quad \text{for } x,\xi_1 > 0, \tag{2.1}$$

respectively. Similarly, let the lifetimes $\mathbf{Y}$ (of size n from facility-line L_2) are iid random variables from another Rayleigh population with a scale parameter ξ_2 having the following PDF $f_Y(y;\xi_2)$ and CDF $F_Y(y;\xi_2)$ as

$$f_Y(y;\xi_2) = 2y\xi_2 e^{-\xi_2 y^2} \quad \text{and} \quad F_Y(y;\xi_2) = 1 - e^{-\xi_2 y^2}, \quad \text{for } y,\xi_2 > 0. \tag{2.2}$$

Substituting (2.1) and (2.2) into (1.1), the likelihood function of ξ_1 and ξ_2 can be written up to proportional as

$$L\left(\xi_1,\xi_2|\,\mathbf{g}\right) \propto \xi_1^{m^*}\xi_2^{n^*}\exp\left(-\left(\xi_1 q_1 + \xi_2 q_2\right)\right), \tag{2.3}$$

where $q_1 = \sum_{i=1}^{D} g_i^2 z_i + (m - m^*)\,T^{*2}$ and $q_2 = \sum_{i=1}^{D} g_i^2 (1 - z_i) + (n - n^*)\,T^{*2}$. The log-likelihood function, $\ell(\cdot) \propto \ln L(\cdot)$, of (2.3) becomes

$$\ell\,\left(\xi_1,\xi_2|\,\mathbf{g}\right) \propto m^* \ln(\xi_1) + n^* \ln(\xi_2) - \xi_1 q_1 - \xi_2 q_2. \tag{2.4}$$

Through the differentiation of (2.4) with regard to ξ_1 and ξ_2, then equating each outcome to zero, the MLEs $\hat{\xi}_1$ and $\hat{\xi}_2$ of ξ_1 and ξ_2 can be easily acquired respectively as

$$\hat{\xi}_1 = \frac{m^*}{q_1} \quad \text{and} \quad \hat{\xi}_2 = \frac{n^*}{q_2}. \tag{2.5}$$

Remark 2. From (2.5), it is obvious that when $\sum_{i=1}^{D} z_i$ equals zero (or D), the MLE $\hat{\xi}_1$ (or $\hat{\xi}_2$) does not exist. Thus, the existence of the MLEs $\hat{\xi}_i,\ i = 1, 2$ is conditioned on $1 \leqslant m^* \leqslant D-1$ and $1 \leqslant n^* \leqslant D-1$.
Remark 3. When $T \to \infty$ and $r = N$, we extend the results of Al-Matrafi and Abd-Elmougod [11] in the case of jointly Type-II censoring to jointly Type-I hybrid censoring.

2.2. Asymptotic intervals

To obtain two-sided $100(1-\omega)\%$ ACIs of ξ_1 and ξ_2, the asymptotic normality of their MLEs $\hat{\xi}_1$ and $\hat{\xi}_2$ is utilized. The 2×2 matrix of a Fisher information (say $\mathbf{I}(\cdot)$) is acquired by the negative-expected of the second-partial derivatives of (2.4) with regard to ξ_1 and ξ_2 as

$$\mathbf{I}_{ij}(\xi_1, \xi_2) = -E\left[\frac{\partial^2}{\partial\xi_i\partial\xi_j}\ell\left(\xi_1, \xi_2|\,\mathbf{g}\right)\right],\ i, j = 1, 2. \tag{2.6}$$

Following the invariance property of $\hat{\xi}_1$ and $\hat{\xi}_2$, the asymptotic variance-covariance matrix (say $\Sigma(\cdot) = \mathbf{I}^{-1}(\cdot)$) of the parameters ξ_1 and ξ_2 can be easily offered by locally at $\hat{\xi}_1$ and $\hat{\xi}_2$, respectively, as

$$\widehat{\Sigma}(\hat{\xi}_1, \hat{\xi}_2) = \begin{bmatrix} \widehat{var}(\hat{\xi}_1) & 0 \\ 0 & \widehat{var}(\hat{\xi}_2) \end{bmatrix}_{(\hat{\xi}_1, \hat{\xi}_2)}, \tag{2.7}$$

where $\widehat{var}(\hat{\xi}_1) = \frac{\hat{\xi}_1^2}{m^*}$ and $\widehat{var}(\hat{\xi}_2) = \frac{\hat{\xi}_2^2}{n^*}$. From (2.6), it is clear that $\mathbf{I}_{12}(\cdot) = \mathbf{I}_{21}(\cdot) = 0$.

Since the MLEs $\hat{\xi}_1$ and $\hat{\xi}_2$ are asymptotically normally distributed as $\hat{\xi}_1 \sim N(\xi_1, \widehat{var}(\hat{\xi}_1))$ and $\hat{\xi}_2 \sim N(\xi_2, \widehat{var}(\hat{\xi}_2))$, see Lawless [12], Thus, from (2.7), the $100(1-\omega)\%$ two-sided ACIs for ξ_1 and ξ_2 are given, respectively, by

$$\hat{\xi}_1 \mp z_{\omega/2}\sqrt{\widehat{var}(\hat{\xi}_1)} \quad \text{and} \quad \hat{\xi}_2 \mp z_{\omega/2}\sqrt{\widehat{var}(\hat{\xi}_2)},$$

where $z_{\omega/2}$ denotes the $(\omega/2)$th percentage point of the standard normal distribution.

3. Bayes paradigm

This section deals with obtaining the Bayes' point estimators as well as the corresponding credible intervals of the unknown Rayleigh parameters ξ_1 and ξ_2 against several loss functions.

3.1. Loss functions

The loss function is significant in statistical analysis since it focuses on estimating precision. However, the squared-error loss (SEL) is a particularly widely utilized symmetric loss in the literature for creating a Bayes estimator of an unknown parameter of interest. Njomen et al. [13] studied Bayesian estimation under different Loss Functions in competing Risks model, Hasan et al. [14] used loss functions to perform Bayesian estimation for the exponential distribution, and Ali et al. [15] studied the effect of loss functions on the performance of Bayesian inference under Lindley distribution. If ϑ is the parameter to be estimated using an estimator $\tilde{\vartheta}$, the SEL function (say $\Im_s(\cdot)$) is defined as

$$\Im_s(\vartheta, \tilde{\vartheta}) = (\tilde{\vartheta} - \vartheta)^2. \tag{3.1}$$

Using (3.1), the Bayes estimator $\tilde{\vartheta}_S$ of ϑ is acquired as

$$\tilde{\vartheta}_S = E_{\vartheta}\left[\vartheta|\,\mathbf{g}\right] = \int_{\Theta} \vartheta \cdot \pi\left(\vartheta|\,\mathbf{g}\right) d\vartheta,\ \vartheta \in \Theta,$$

where $\pi\left(\vartheta|\,\mathbf{g}\right)$ is the posterior PDF of ϑ.

Besides the traditional SEL function, we also use two well-known asymmetric loss functions called linear-exponential loss (LL) and general-entropy loss (GEL) functions because the SEL is often inappropriate in reliability estimation, especially when overestimation is more harmful than underestimation. It is well known that the Bayes estimator in the case of the SEL is the posterior mean where the overestimation and underestimation are treated equally; see Elshahhat et al. [16].

First, the LL (say $\Im_L(\cdot)$) is defined as

$$\Im_L(\vartheta, \tilde{\vartheta}) = e^{\upsilon(\tilde{\vartheta}-\vartheta)} - \upsilon(\tilde{\vartheta} - \vartheta) - 1,\ \upsilon \neq 0. \tag{3.2}$$

Using (3.2), the Bayes estimator $\tilde{\vartheta}_L$ of ϑ is given by

$$\tilde{\vartheta}_L = -\frac{1}{\upsilon} \ln\left(E_{\vartheta}\left[e^{-\upsilon\vartheta}\middle|\,\mathbf{g}\right]\right),\ \upsilon \neq 0,$$

where $\upsilon \to 0$, the LL (3.2) is quite approximately to the SEL (3.1). Another useful asymmetric loss function is the GEL function, $\Im_G(\cdot)$, is defined as

$$\Im_G(\vartheta, \tilde{\vartheta}) = \left(\frac{\tilde{\vartheta}}{\vartheta}\right)^{\tau} - \tau \ln\left(\frac{\tilde{\vartheta}}{\vartheta}\right) - 1,\ \tau \neq 0, \tag{3.3}$$

where τ is the shape parameter. Using (3.3), the Bayes estimator $\tilde{\vartheta}_G$ of ϑ is given by

$$\tilde{\vartheta}_G = \left(E_{\vartheta}\left[\vartheta^{-\tau}\middle|\,\mathbf{g}\right]\right)^{-1/\tau},\ \tau \neq 0,$$

where $\tau = -1$, both estimators $\tilde{\vartheta}_G$ and $\tilde{\vartheta}_S$ coincided. Further, the minimum error under the GEL function occurs at $\vartheta = \tilde{\vartheta}_G$.

3.2. Bayes estimators

The prior distribution, which reflects knowledge about an unknown parameter, is an important aspect of Bayesian inference. Thus, based on several reasons to consider gamma prior, such as: (i) It provides various density shapes; (ii) It is flexible; (iii) It is fairly straightforward, concise, and may not lead to a result with a complex estimation issue, we assumed that ξ_1 and ξ_2 are independently distributed with gamma random variables with PDFs $Gamma(a_1, b_1)$ and $Gamma(a_2, b_2)$, respectively. On the other hand, one can easily consider other prior information based on the Rayleigh parameter domain, e.g., Weibull, generalized-exponential, or others.

Hence, the joint prior PDF (say $\pi(\cdot)$) of ξ_1 and ξ_2 is

$$\pi(\xi_1, \xi_2) \ = \ \frac{b_1^{a_1} b_2^{a_2}}{\Gamma(a_1)\Gamma(a_2)}\, \xi_1^{a_1-1} \xi_2^{a_2-1} \exp\left(-\left[\xi_1 b_1 + \xi_2 b_2\right]\right),\ \xi_1, \xi_2 > 0, \tag{3.4}$$

where $\Gamma(\cdot)$ is the complete gamma function.

From (2.3) and (3.4), the joint posterior PDF (say $\Pi(\cdot)$) of ξ_1 and ξ_2, becomes

$$\Pi(\xi_1, \xi_2|\,\mathbf{g}) = C_D^* \xi_1^{a_1-m^*-1} \xi_2^{a_2-n^*-1} \exp\left(-\left[\xi_1(b_1+q_1) + \xi_2(b_2+q_2)\right]\right), \tag{3.5}$$

where $C_D^* = \frac{(b_1+q_1)^{a_1-m^*}(b_2+q_2)^{a_2-n^*}}{\Gamma(a_1-m^*)\Gamma(a_2-n^*)}$.

From (3.5), the joint posterior PDF of ξ_1 and ξ_2 is obtained as a product of two independent gamma probability densities as $\xi_1 \sim Gamma(a_1 - m^*, b_1 + q_1)$ and $\xi_2 \sim Gamma(a_2 - n^*, b_2 + q_2)$. As a result, if the improper prior knowledge of ξ_1 and ξ_2 is available, i.e., $a_i, b_i = 0,\ i = 1, 2$, Eq (3.5) reduced in proportion to Eq (2.3).

However, using the SEL function (3.1), the Bayes estimators $\tilde{\xi}_{1S}$ and $\tilde{\xi}_{2S}$ of ξ_1 and ξ_2 are given, respectively, by

$$\tilde{\xi}_{1S} = \frac{m^* + a_1}{q_1 + b_1} \quad \text{and} \quad \tilde{\xi}_{2S} = \frac{n^* + a_2}{q_2 + b_2}.$$

Similarly, using the LL function (3.2), the Bayes estimators $\tilde{\xi}_{1L}$ and $\tilde{\xi}_{2L}$ of ξ_1 and ξ_2 are given, respectively, by

$$\tilde{\xi}_{1L} = -\frac{m^* + a_1}{\upsilon} \ln\left(\frac{q_1 + b_1}{q_1 + b_1 + \upsilon}\right) \quad \text{and} \quad \tilde{\xi}_{2L} = -\frac{n^* + a_2}{\upsilon} \ln\left(\frac{q_2 + b_2}{q_2 + b_2 + \upsilon}\right), \text{ for } \upsilon \neq 0.$$

Again, using the GEL function (3.3), the Bayes estimators $\tilde{\xi}_{1G}$ and $\tilde{\xi}_{2G}$ of ξ_1 and ξ_2 are given, respectively, by

$$\tilde{\xi}_{1G} = \left[\frac{\Gamma(m^* + a_1 - \tau)}{(q_1 + b_1)^{-\tau}\Gamma(m^* + a_1)}\right]^{-1/\tau} \quad \text{and} \quad \tilde{\xi}_{2G} = \left[\frac{\Gamma(n^* + a_2 - \tau)}{(q_2 + b_2)^{-\tau}\Gamma(n^* + a_2)}\right]^{-1/\tau}, \text{ for } \tau \neq 0.$$

3.3. Credible intervals

To construct the two bounds $100(1-\omega)\%$ Bayes credible intervals (BCIs) of ξ_1 and ξ_2, the posterior density function (3.5) is used. First, let $\delta_1 = 2\xi_1(q_1 + b_1)$ and $\delta_2 = 2\xi_2(q_2 + b_2)$, where δ_1 and δ_2 are positive integers, be two pivots follow χ^2 distributions with $2(m^*+a_1)$ and $2(n^*+a_2)$ degrees of freedom, respectively, see Kundu and Joarder [17].

Hence, $100(1-\omega)\%$ BCIs of ξ_1 and ξ_2 are offered by

$$\left[\frac{\chi^2_{2(m^*+a_1),\frac{\omega}{2}}}{2(q_1+b_1)}, \frac{\chi^2_{2(m^*+a_1),1-\frac{\omega}{2}}}{2(q_1+b_1)}\right], \quad \text{and} \quad \left[\frac{\chi^2_{2(n^*+a_2),\frac{\omega}{2}}}{2(q_2+b_2)}, \frac{\chi^2_{2(n^*+a_2),1-\frac{\omega}{2}}}{2(q_2+b_2)}\right],$$

respectively, where $\chi^2_{\nu,\omega}$ is the ωth percentile the χ^2_ν distribution.

Remark 4. If the levels of freedom $2(m^* + a_1$ and $2(n^* + a_2$ cannot be obtained in integers, then the gamma PDF will be utilized instead of the χ^2 distribution to create the BCIs of ξ_1 and ξ_2 as

$$\left[F_G^{-1}\left(\frac{\omega}{2}, m^* + a_1, q_1 + b_1\right), F_G^{-1}\left(1 - \frac{\omega}{2}, m^* + a_1, q_1 + b_1\right)\right],$$

and

$$\left[F_G^{-1}\left(\frac{\omega}{2}, n^* + a_2, q_2 + b_2\right), F_G^{-1}\left(1 - \frac{\omega}{2}, n^* + a_2, q_2 + b_2\right)\right],$$

respectively, where $F_G^{-1}(\omega, \nu_1, \nu_2)$ is the ωth percentile of the gamma distribution with shape ν_1 and scale ν_2 parameters.

4. Monte-Carlo study

Comprehensive Monte-Carlo simulations are run to examine the behavior of the suggested estimation procedures developed in the preceding sections. For fixed $T = (1, 2)$, when ξ_1 and ξ_2 are taken as $(\xi_1, \xi_2) = (0.5, 0.5)$ and (0.75,1.5), we simulate 5,000 JHC-T1 samples based on various tests of n, m, and r such as $n = m = 10$, 30, and 50. We assign the level of r as a failure percentage (of each N) such as $(\frac{r}{N})100\% = 30$, 50 and 80%. We also assigned $(a_i, b_i) = (0.5, 1)$ for $i = 1, 2$, when $(\xi_1, \xi_2) = (0.5, 0.5)$ and $(a_1, a_2) = (1.5, 3)$ and $b_i = 2$ for $i = 1, 2$, when $(\xi_1, \xi_2) = (0.75, 1.5)$. Under SEL, LL($\upsilon(= -3, -0.03, +3)$), and GEL($\tau(= -2, -0.02, +2)$), all offered Bayes estimates are calculated.

For each scenario, the average estimates (AEs), root mean squared-errors (RMSEs), and average relative absolute biases (ARABs) of the unknown parameter ξ_1 (for instance) are calculated using the next formulae, respectively, as

$$\mathrm{AE}(\hat{\xi}_1) = \frac{1}{5000} \sum_{j=1}^{5000} \hat{\xi}_1^{(j)},$$

$$\mathrm{RMSE}(\hat{\xi}_1) = \sqrt{\frac{1}{5000} \sum_{j=1}^{5000} \left(\hat{\xi}_1^{(j)} - \xi_1\right)^2}$$

and

$$\mathrm{ARAB}\,(\xi_1) = \frac{1}{5000} \sum_{j=1}^{5000} \frac{1}{\xi_1} \left|\hat{\xi}_1^{(j)} - \xi_1\right|,$$

where $\hat{\xi}_1^{(j)}$ is the obtained estimate at the j^{th} sample of ξ_1.

Further, the corresponding average confidence lengths (ACLs) and coverage probabilities (CPs) related to the 95% ACI/BCI estimates of ξ_1 are acquired using the following formulae, respectively, as

$$\mathrm{ACL}(\xi_1) = \frac{1}{5000} \sum_{j=1}^{5000} \left(\mathsf{U}(\hat{\xi}_1^{(j)}) - \mathsf{L}(\hat{\xi}_1^{(j)})\right)$$

and

$$\mathrm{CP}(\xi_1) = \frac{1}{5000} \sum_{j=1}^{5000} \mathbf{1}_{\left(\mathsf{L}(\hat{\xi}_1^{(j)});\mathsf{U}(\hat{\xi}_1^{(j)})\right)}(\xi_1),$$

where $\mathbf{1}(\cdot)$ is the indicator operator and $(\mathsf{L}(\cdot), \mathsf{U}(\cdot))$ denotes the lower and upper sides, respectively, of the ACI (or BCI) of ξ_1. Clearly, in a similar fashion, the AEs, RMSEs, ARABs, ACLs, and CPs of ξ_2 can be easily offered.

A Heat-Map is a common tool for data visualization. It is known as a data visualization technique in which each value in a matrix is portrayed by specific colors. So, using this tool via the R programming language, Figures 2–5 display the simulated results (including RMSEs, ARABs, ACLs, and CPs) of ξ_i, $i = 1, 2$. All simulated findings of ξ_1 and ξ_2 are reported in the Supplementary File. Briefly, some numerical results of ξ_1 and ξ_2 are presented in Tables 1–3. In these tables, the AEs, RMSEs, and ARABs for each parameter are tabulated in the first, second, and third rows, respectively.

Table 1. Monte Carlo point results of ξ_1 and ξ_2 when $(\xi_1, \xi_2) = (0.5, 0.5)$.

Par.	T	(m, n)	r	MLE	SEL	LL1	LL2	LL3	GEL1	GEL2	GEL3
ξ_1	1	(10,10)	6	0.6720	0.6160	0.5940	0.4400	0.3640	0.6980	0.5320	0.3730
				0.4930	0.3450	0.4930	0.2620	0.2310	0.4020	0.3020	0.2710
				0.6540	0.5000	0.5930	0.4160	0.3940	0.5780	0.4540	0.4570
			10	0.6280	0.6040	0.5110	0.4240	0.3680	0.6670	0.5400	0.4170
				0.3930	0.3220	0.3320	0.2440	0.2300	0.3590	0.2950	0.2710
				0.5580	0.4730	0.4570	0.3940	0.3910	0.5160	0.4380	0.4430
			16	0.6230	0.5980	0.5050	0.4230	0.3700	0.6590	0.5370	0.4200
				0.3840	0.3170	0.3060	0.2430	0.2310	0.3510	0.2940	0.2750
				0.5500	0.4730	0.4440	0.3940	0.3930	0.5130	0.4410	0.4500
	2	(10,10)	6	0.6900	0.6290	0.5850	0.4360	0.3630	0.7110	0.5450	0.3770
				0.5060	0.3440	0.4900	0.2510	0.2260	0.4050	0.2970	0.2650
				0.6490	0.4940	0.5630	0.3990	0.3870	0.5790	0.4420	0.4480
			10	0.8390	0.6930	0.5330	0.4520	0.3990	0.7440	0.6420	0.5240
				0.4970	0.3470	0.3150	0.2290	0.2070	0.3890	0.3090	0.2460
				0.7190	0.4950	0.4380	0.3650	0.3500	0.5640	0.4370	0.3640
			16	0.9170	0.8590	0.6380	0.5590	0.5040	0.9080	0.8100	0.7050
				0.5830	0.4860	0.3930	0.2640	0.2080	0.5330	0.4400	0.3480
				0.8570	0.7390	0.4660	0.3610	0.3090	0.8290	0.6520	0.4890
ξ_2	1	(10,10)	6	0.6730	0.6210	0.5950	0.4390	0.3630	0.7040	0.5370	0.3750
				0.4980	0.3470	0.5200	0.2720	0.2380	0.4060	0.3030	0.2720
				0.6550	0.5030	0.6130	0.4320	0.4070	0.5830	0.4530	0.4560
			10	0.6330	0.6080	0.5190	0.4300	0.3740	0.6710	0.5440	0.4200
				0.3940	0.3200	0.3370	0.2490	0.2310	0.3580	0.2920	0.2690
				0.5640	0.4800	0.4650	0.3970	0.3920	0.5240	0.4440	0.4460
			16	0.6180	0.5970	0.5080	0.4260	0.3730	0.6580	0.5360	0.4150
				0.3750	0.3110	0.3040	0.2390	0.2270	0.3440	0.2870	0.2730
				0.5440	0.4600	0.4340	0.3840	0.3850	0.4970	0.4290	0.4480
	2	(10,10)	6	0.6920	0.6270	0.5860	0.4370	0.3630	0.7090	0.5430	0.3770
				0.5180	0.3490	0.4700	0.2550	0.2270	0.4090	0.3020	0.2680
				0.6590	0.4960	0.5710	0.4040	0.3900	0.5800	0.4440	0.4480
			10	0.8220	0.7010	0.5360	0.4540	0.4000	0.7520	0.6500	0.5400
				0.4780	0.3510	0.3330	0.2350	0.2100	0.3930	0.3110	0.2490
				0.6890	0.5090	0.4470	0.3700	0.3530	0.5790	0.4480	0.3670
			16	0.9170	0.8620	0.6360	0.5560	0.5010	0.9110	0.8130	0.7080
				0.5760	0.4880	0.4470	0.2660	0.2100	0.5350	0.4420	0.3500
				0.8550	0.7440	0.4680	0.3600	0.3110	0.8350	0.6570	0.4940

Table 2. Monte Carlo point results of ξ_1 and ξ_2 when $(\xi_1, \xi_2) = (0.75, 1.5)$.

Par.	T	(m,n)	r	MLE	SEL	LL1	LL2	LL3	GEL1	GEL2	GEL3
ξ_1	1	(10,10)	6	1.0470	0.8470	0.8950	0.6470	0.5300	0.9550	0.7380	0.5090
				0.8700	0.3500	0.5550	0.3030	0.3050	0.4110	0.3180	0.3650
				0.7680	0.3620	0.4890	0.3320	0.3520	0.4190	0.3390	0.4180
			10	1.0850	0.9600	0.8460	0.6520	0.5480	1.0260	0.8930	0.7500
				0.7140	0.4460	0.4660	0.2960	0.2930	0.4920	0.4070	0.3560
				0.6490	0.4390	0.4260	0.3230	0.3380	0.4810	0.3990	0.3720
			16	1.0540	0.9080	0.8010	0.6670	0.5810	0.9890	0.8270	0.6490
				0.6950	0.3790	0.3960	0.2970	0.2860	0.4340	0.3370	0.3120
				0.6310	0.3810	0.3830	0.3200	0.3210	0.4370	0.3430	0.3400
	2	(10,10)	6	1.0370	0.8600	0.8970	0.6470	0.5300	0.9590	0.7600	0.5440
				0.8020	0.3560	0.5620	0.3020	0.3040	0.4140	0.3200	0.3490
				0.7040	0.3610	0.4870	0.3310	0.3520	0.4170	0.3350	0.3960
			10	1.0580	0.9110	0.8320	0.6460	0.5440	0.9920	0.8300	0.6520
				0.6930	0.3780	0.4580	0.2940	0.2930	0.4320	0.3350	0.3080
				0.6270	0.3780	0.4130	0.3200	0.3390	0.4350	0.3390	0.3360
			16	1.2880	1.1110	0.8410	0.7150	0.6310	1.1740	1.0480	0.9140
				0.8000	0.5070	0.3870	0.2680	0.2420	0.5640	0.4530	0.3490
				0.7630	0.5190	0.3550	0.2800	0.2710	0.5910	0.4530	0.3400
ξ_2	1	(10,10)	6	2.2270	1.8980	2.0410	1.3700	1.0910	1.9830	1.8140	1.6360
				1.3500	0.6990	1.1900	0.4620	0.5080	0.7630	0.6400	0.5400
				0.5990	0.3500	0.5060	0.2520	0.2960	0.3860	0.3190	0.2730
			10	1.9730	1.7600	1.9900	1.4200	1.1500	1.8530	1.6670	1.4700
				1.2120	0.5240	1.0600	0.4600	0.4710	0.5890	0.4680	0.4040
				0.5340	0.2680	0.4690	0.2470	0.2690	0.3040	0.2410	0.2150
			16	2.0230	1.6520	2.0800	1.5300	1.2600	1.7690	1.2850	1.5360
				1.0410	0.4570	1.2200	0.5450	0.4570	0.5250	0.4280	0.4120
				0.4820	0.2370	0.5180	0.2780	0.2550	0.2720	0.2360	0.2170
	2	(10,10)	6	2.0280	1.6760	2.0510	1.3800	1.1000	1.7850	1.5680	1.3350
				1.0390	0.4680	1.2100	0.4670	0.5070	0.5360	0.4200	0.4110
				0.4800	0.2420	0.5140	0.2530	0.2960	0.2770	0.2200	0.2240
			10	1.9660	1.7630	2.0200	1.4400	1.1710	1.8560	1.6710	1.4740
				1.1220	0.5220	1.0610	0.4670	0.4660	0.5880	0.4660	0.4000
				0.5020	0.2660	0.4860	0.2500	0.2660	0.3030	0.2390	0.2130
			16	2.5790	2.1040	2.3600	1.7200	1.4100	2.1890	2.0200	1.8420
				1.6640	0.8380	1.4800	0.6210	0.4120	0.9140	0.7650	0.6220
				0.7580	0.4330	0.6470	0.3050	0.2240	0.4810	0.3880	0.3080

Table 3. Monte Carlo interval results of ξ_1 and ξ_2.

(ξ_1, ξ_2)	T	(m, n)	r	ξ_1		ξ_2		ξ_1		ξ_2	
				ACL	CP	ACL	CP	ACL	CP	ACL	CP
(0.5,0.5)	1	(10,10)	6	1.411	0.875	1.246	0.906	1.531	0.892	1.324	0.928
			10	1.204	0.893	1.174	0.918	1.368	0.902	1.229	0.938
			16	1.156	0.903	1.050	0.928	1.213	0.913	1.156	0.949
	2	(10,10)	6	1.328	0.882	1.221	0.917	1.461	0.901	1.259	0.931
			10	1.166	0.908	1.142	0.924	1.268	0.911	1.145	0.940
			16	1.109	0.917	1.040	0.933	1.170	0.918	1.049	0.947
(0.75,1.5)	1	(10,10)	6	2.154	0.804	1.671	0.844	1.897	0.881	1.696	0.907
			10	2.061	0.815	1.542	0.856	1.696	0.904	1.545	0.918
			16	1.893	0.837	1.473	0.879	1.679	0.908	1.457	0.922
	2	(10,10)	6	2.128	0.817	1.616	0.858	1.786	0.900	1.627	0.918
			10	2.008	0.828	1.520	0.869	1.656	0.916	1.520	0.934
			16	1.779	0.854	1.397	0.897	1.618	0.925	1.379	0.944

From Figures 2–5, we present the following statements:

- The Bayes' estimation method is better than the frequentist (maximum likelihood) estimation method, and both are applicable and feasible. It can also be seen that the proposed estimation methodologies of ξ_1 and ξ_2 work satisfactorily.
- As n and m increase, the RMSEs and ARABs of all point estimates decrease as expected. Obviously, to improve numerical results, try increasing the r size.
- As T increases, it is noted that
  - The RMSEs and ARABs of all estimates of ξ_1 and ξ_2 increase.
  - The ACLs of 95% ACIs (or BCIs) of ξ_1 and ξ_2 narrowed down while their CPs grew.
- As $(\frac{r}{N})100\%$ increases, the RMSEs and ARABs of ξ_1 and ξ_2 decrease at $T = 1$ while these increase at $T = 2$.
- Bayes estimates, due to additional information provided by gamma prior, performed better than others.
- When the true value of ξ_1 and ξ_2 decreases, the proposed point estimates become even better.
- It is also clear that the proposed Bayes estimates using the LL (or GEL) function of ξ_1 and ξ_2 are overestimates when $\upsilon(\text{or } \tau) < 0$ while these are underestimates when $\upsilon(\text{or } \tau) > 0$.
- This result is due to fact that the Bayes estimates under asymmetric type loss have additional flexibility, due to the shape parameter loss, compared to those obtained based on symmetric type loss.
- When n and m increase, the ACLs of 95% ACIs/BCIs tend to decrease.
- When ξ_1 and ξ_2 increase, the ACLs of ACIs/BCIs increase while the corresponding CPs decrease. Similar behavior is observed when T increases.
- As the failure percentage increases, for fixed n and m, the ACLs and CPs decrease for ACIs/BCIs of ξ_1 and ξ_2.

- Finally, employing the Bayes paradigm to offer point (or interval) estimates of the Rayleigh population parameters is recommended.

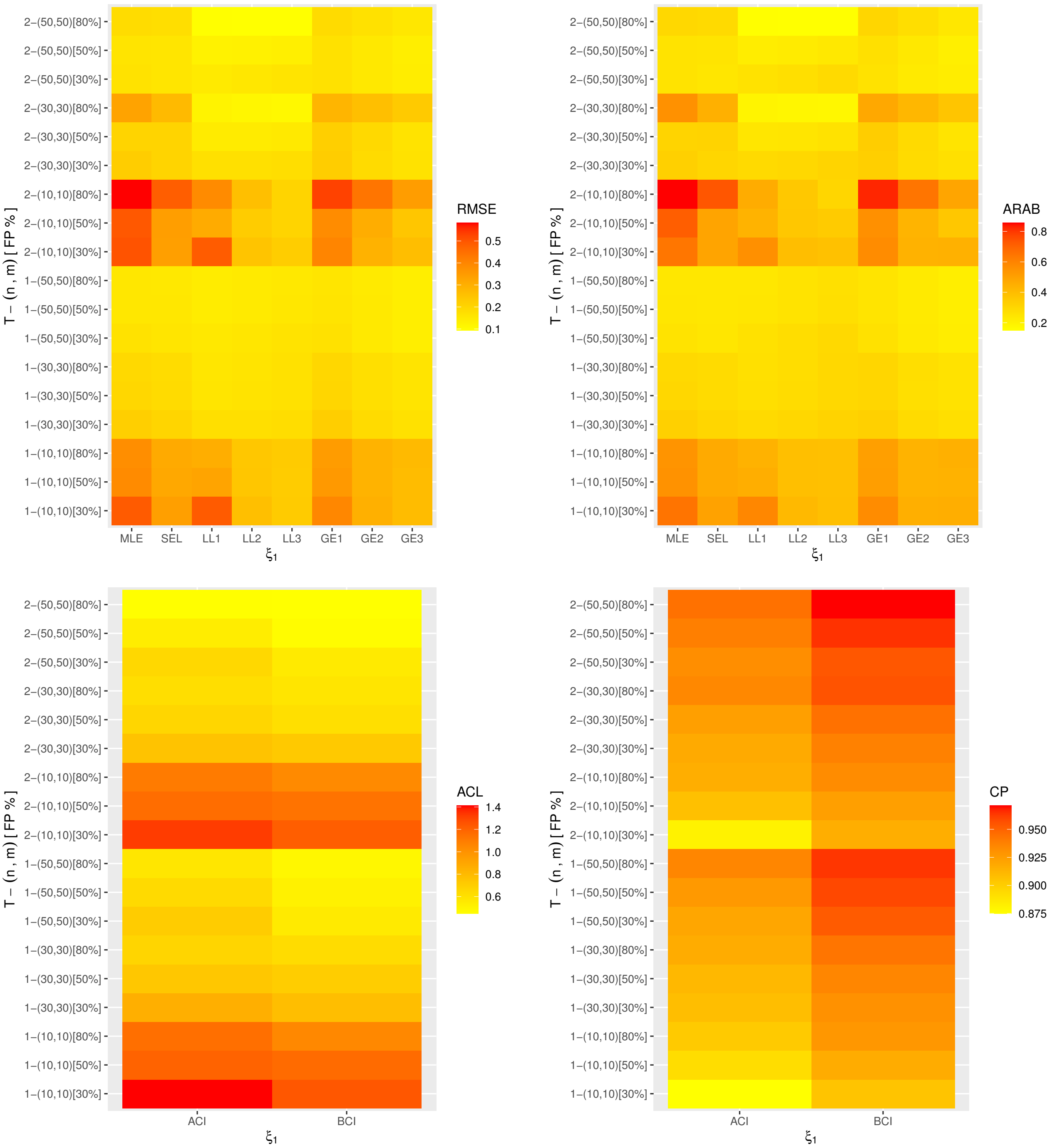

Figure 2. Heat-Map for outcomes of ξ_1 at $(\xi_1, \xi_2) = (0.5, 0.5)$.

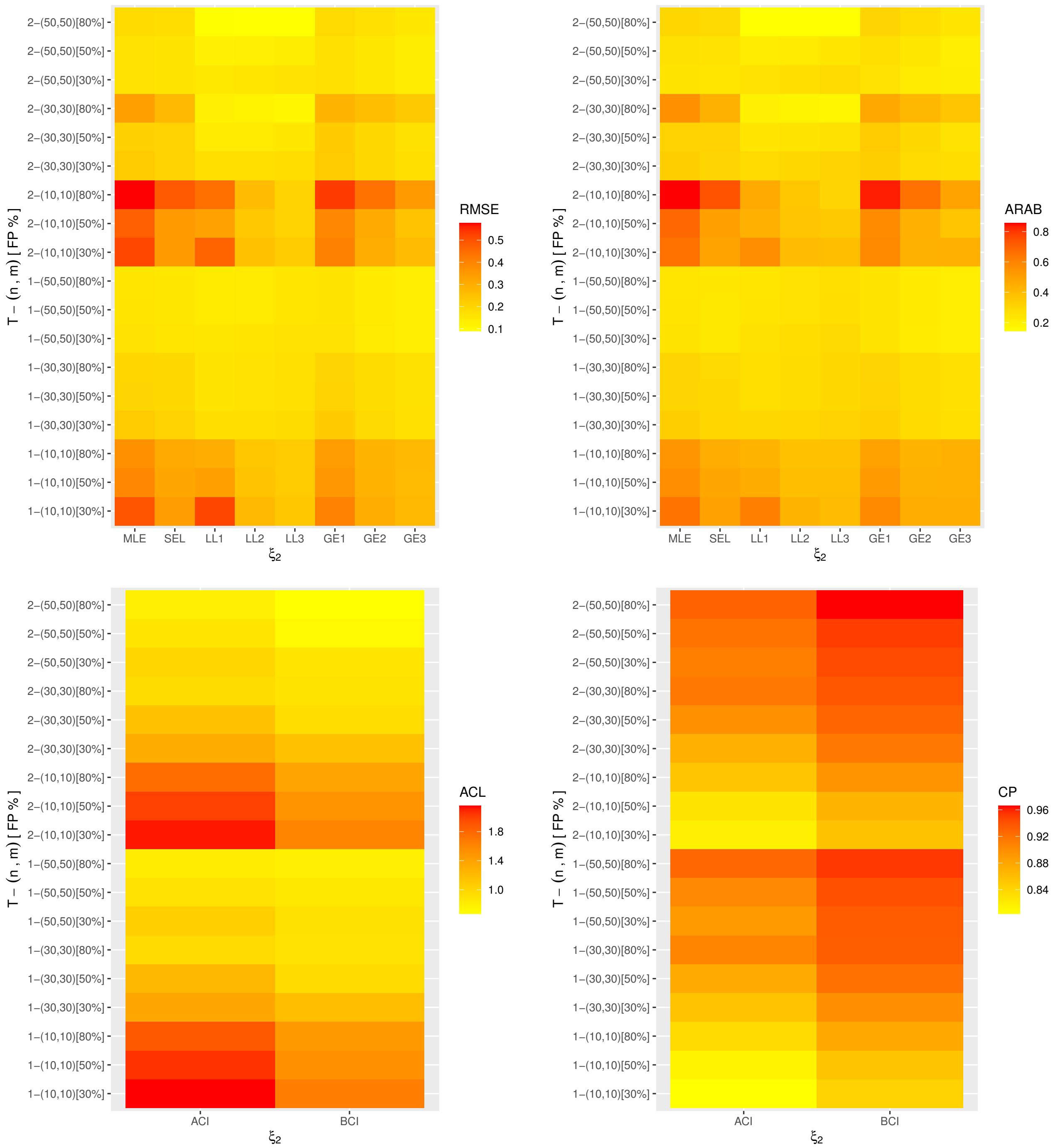

Figure 3. Heat-Map for outcomes of ξ_2 at $(\xi_1, \xi_2) = (0.5, 0.5)$.

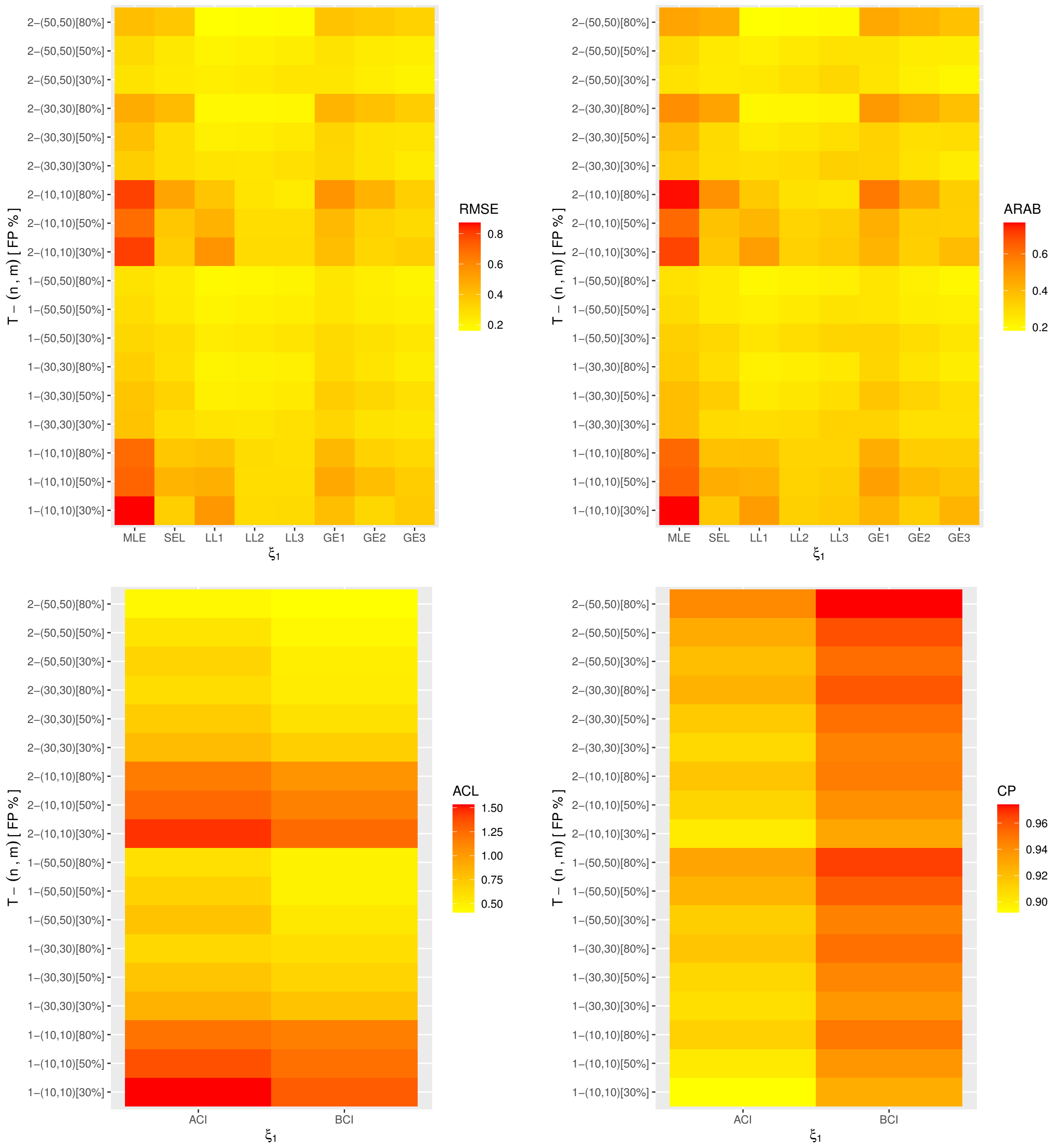

Figure 4. Heat-Map for outcomes of ξ_1 at $(\xi_1, \xi_2) = (0.75, 1.5)$.

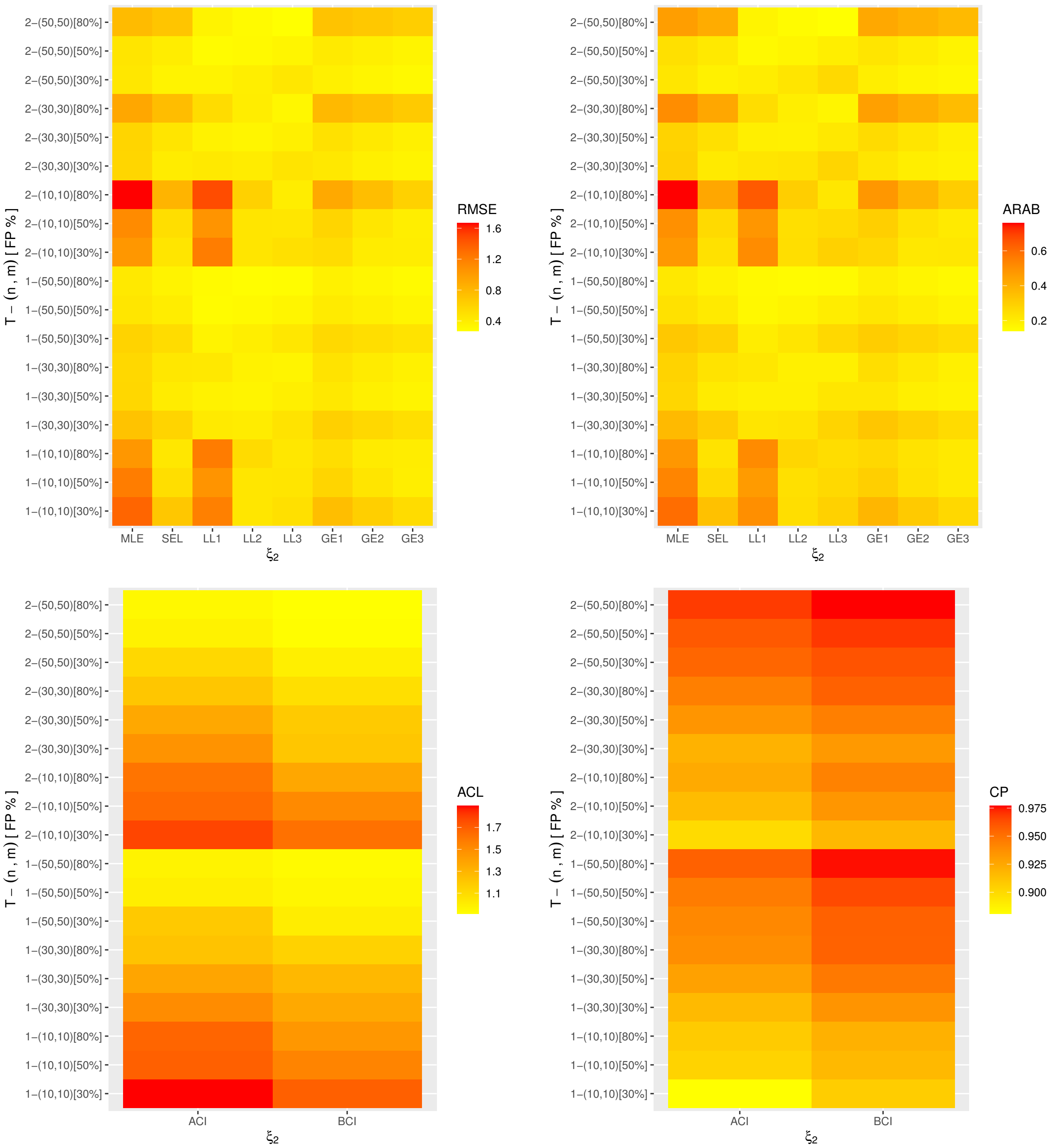

Figure 5. Heat-Map for outcomes of ξ_2 at $(\xi_1, \xi_2) = (0.75, 1.5)$.

5. Engineering applications

Two separate engineering scenarios are analyzed to demonstrate how to make use of the suggested inferential methodologies.

5.1. Insulating fluid

In this application, we shall consider two samples (groups 3 and 6) of size $m = n = 10$ each from Nelson [18]. These samples indicate the time required for an insulating liquid exposed to extreme voltage stresses to break down (in minutes). The failure times of groups 3 (say **X**) are 1.99, 0.64, 2.15, 1.08, 2.57, 0.93, 4.75, 0.82, 2.06, 0.49, as well as the failure times of groups 6 (say **Y**) are: 2.12, 3.97, 1.56, 1.34, 1.49, 8.71, 2.10, 7.21, 3.83, 5.13.

To highlight the validity of the Rayleigh population for given datasets, the Kolmogorov-Smirnov (K-S) statistic (with its P-value) is computed. First, the MLEs with their standard errors (in parentheses) of the Rayleigh parameters ξ_i, $i = 1, 2$ using **X** and **Y** datasets are 0.2205(0.0697) and 0.0500(0.0158), respectively. Thus, the K-S distances with their P-values (in parentheses) from **X** and **Y** datasets become 0.2732(0.375) and 0.2987(0.275), respectively. Hence, we can conclude that the real times of **X** and **Y** come from the Rayleigh populations. For more illustration, using **X** and **Y** datasets, the fitted/empirical distribution functions of Rayleigh populations, probability-probability (PP), and quantile-quantile (QQ) are displayed in Figure 6. It further confirms the same numerical findings of K-S statistics.

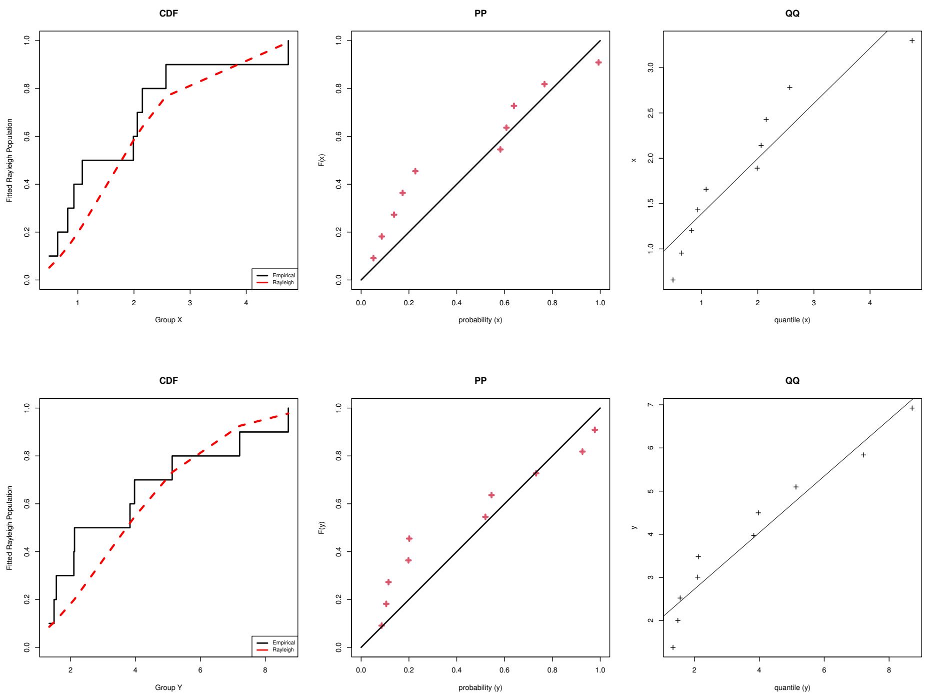

Figure 6. Plots of CDF, PP, and QQ using **X** (top) and **Y** (bottom) in the insulating liquid data.

Moreover, before proceeding, to highlight the superiority of the proposed Rayleigh population among others, we reexamine it with two well-known models in the literature called Maxwell-Boltzmann (MB(ξ)) and Muth(ξ) lifetime models. Recently, these competitive models have been analyzed by Elshahhat et al. [19] and Alotaibi et al. [20], respectively. As a result, Table 4 indicates that the Rayleigh population is the best from the insulating liquid data compared to others.

Table 4. Summary fit of the Rayleigh population and its competitors using insulating liquid groups.

Population	MLE(Standard–Error)	K-S(P-value)
	Group **X**	
Rayleigh	0.2205(0.0697)	0.2732(0.375)
MB	3.0231(0.7805)	0.3562(0.122)
Muth	0.0678(0.1551)	0.3718(0.095)
	Group **Y**	
Rayleigh	0.0500(0.0158)	0.2987(0.275)
MB	13.329(3.4417)	0.3792(0.084)
Muth	0.2015(0.1357)	0.7641(0.001)

Using various choices of r, at $T = 2.5$ (in minutes), three different JHC-T1 samples are generated and reported in Table 5. Since we lack any prior knowledge about ξ_1 and ξ_2, the Bayes estimates using improper gamma priors, i.e., $a_i = b_i = 0$, for $i = 1, 2$, are developed. The Bayes estimates of ξ_1 or ξ_2 are acquired based on the SEL, LL($\upsilon(= -2, -0.02, +2)$), and GEL($\tau(= -1, -0.01, +1)$) functions. For calculation convenience, we have set all hyperparameter values to 0.001.

From Table 5, the suggested point and 95% interval estimates of ξ_1 and ξ_2 are computed; see Tables 6 and 7, respectively. From Table 6, we can decide that the offered classical and Bayes estimates of ξ_1 and ξ_2 are quite similar, as expected. In addition, from Table 7, the 95% credible intervals of ξ_1 and ξ_2 are also close to those acquired by asymptotic intervals.

To estimate the mean lifetime of the Rayleigh population, the corresponding MLEs of ξ_1 and ξ_2 based on samples 1–3 are used. Using the invariance property $\hat{\xi}_1$ and $\hat{\xi}_2$, the estimated mean lifetimes $\sqrt{\frac{\pi}{4\hat{\xi}_1}}$ and $\sqrt{\frac{\pi}{4\hat{\xi}_2}}$ relative to groups **X** and **Y** become (1.5068, 1.3434, 1.2476) and (2.6410, 2.7886, 2.2156) from samples (1, 2, 3) respectively. To sum up, the group **X** has a higher lifetime compared to the others. Using the offered Bayes' estimates of ξ_1 and ξ_2, the same result may likewise be reached for the mean lifetime of each group.

Table 5. Three JHC-T1 samples from insulating fluid datasets.

Sample	r	$w(z)$	(m^*, n^*)
1	7	0.49(1), 0.64(1), 0.82(1), 0.93(1), 1.08(1), 1.34(0), 1.49(0)	(5,8)
2	10	0.49(1), 0.64(1), 0.82(1), 0.93(1), 1.08(1), 1.34(0), 1.49(0), 1.56(0), 1.99(1), 2.06(1)	(3,7)
3	15	0.49(1), 0.64(1), 0.82(1), 0.93(1), 1.08(1), 1.34(0), 1.49(0), 1.56(0), 1.99(1), 2.06(1), 2.10(0), 2.12(0), 2.15(1)	(2,5)

Table 6. Point estimates of ξ_1 and ξ_2 from insulating fluid datasets.

Sample	Par.	MLE	SEL	LL			GEL		
				$\upsilon = -2$	$\upsilon = -0.02$	$\upsilon = +2$	$\tau = -1$	$\tau = -0.01$	$\tau = +1$
1	ξ_1	0.3459	0.3459	0.3723	0.3462	0.3240	0.3459	0.3123	0.2767
	ξ_2	0.1126	0.1126	0.1195	0.1127	0.1067	0.1126	0.0862	0.0563
2	ξ_1	0.4352	0.4352	0.4647	0.4355	0.4102	0.4352	0.4048	0.3730
	ξ_2	0.1010	0.1010	0.1045	0.1010	0.0977	0.1010	0.0849	0.0673
3	ξ_1	0.5046	0.5046	0.5394	0.5049	0.4752	0.5654	0.4744	0.3749
	ξ_2	0.1600	0.1600	0.1653	0.1601	0.1551	0.1902	0.1448	0.0923

Table 7. Interval estimates of ξ_1 and ξ_2 from insulating fluid datasets.

Sample	Par.	ACI	BCI
1	ξ_1	(0.0427,0.8253)	(0.1123,0.7086)
	ξ_2	(0.0139,0.2687)	(0.0136,0.3137)
2	ξ_1	(0.1128,0.9277)	(0.1750,0.8119)
	ξ_2	(0.0262,0.2153)	(0.0208,0.2432)
3	ξ_1	(0.1549,0.9469)	(0.2179,0.9098)
	ξ_2	(0.0491,0.3002)	(0.0520,0.3277)

5.2. Mechanical equipment

In this application, from Murthy et al. [21], we consider the time between thirty failures for repairable mechanical equipment. To illustrate the findings of the paper, we partitioned (randomly) the complete data into two samples of equal sizes denoted here as groups **X** and **Y**, see Table 8. Using **X** and **Y** datasets, the MLEs of ξ_1 and ξ_2 with their (standard errors) are 0.3487(0.0900) and 0.2299(0.0593), respectively.

To see the validity of the proposed model, the K-S(P-value) for both **X** and **Y** samples are 0.2132(0.442) and 0.2003(0.520), respectively. These results showed that the Rayleigh populations fit the repairable mechanical equipment datasets quite well. In Figure 7, the fitted/empirical distribution functions of Rayleigh populations as well as PP and QQ plots are displayed based on **X** and **Y**. It also supports the findings of the goodness statistics.

Table 8. Two groups for mechanical equipment dataset.

Group	Failure times/hour
X	0.11, 0.45, 0.63, 0.70, 0.71, 0.77, 1.17, 1.23, 1.24, 1.43, 1.74, 1.86, 1.97, 2.23, 4.36
Y	0.30, 0.40, 0.59, 0.74, 0.94, 1.06, 1.23, 1.46, 1.49, 1.82, 2.37, 2.46, 2.63, 3.46, 4.73

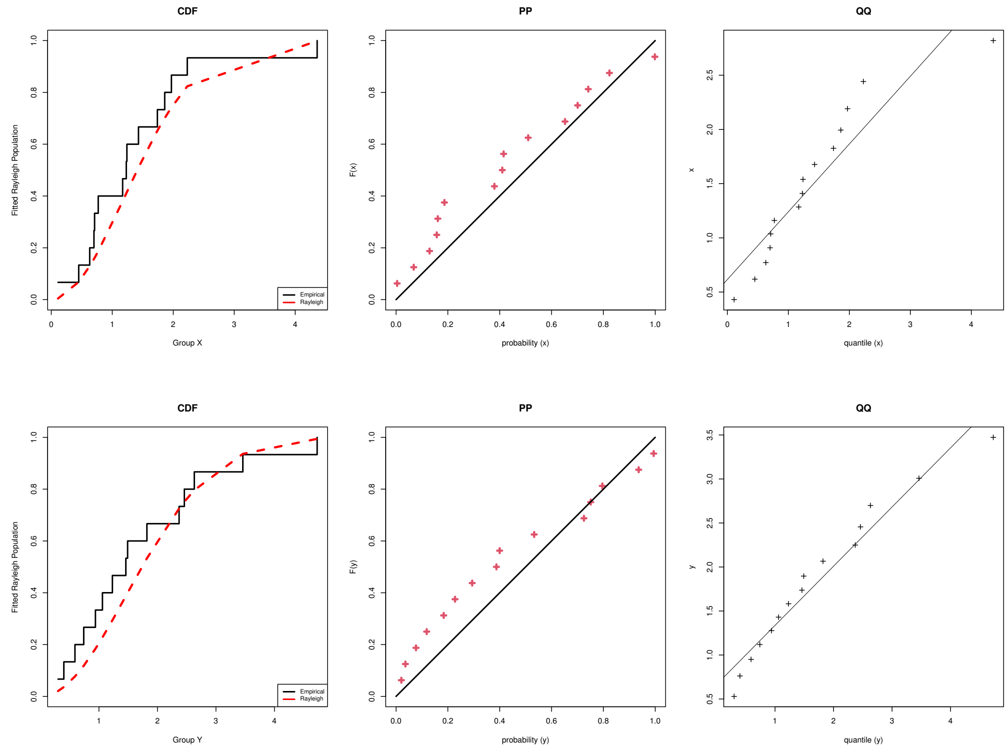

Figure 7. Plots of CDF, PP, and QQ using **X** (top) and **Y** (bottom) in the mechanical equipments data.

Again, before continuing, we compare the utility of the proposed Rayleigh ensemble with its competitors Maxwell-Boltzmann (MB(ξ)) and Muth(ξ) lifetime models. Consequently, Table 9 points out that the suggested Rayleigh population is the best compared to others based on the mechanical equipment data.

Table 9. Summary fit of the Rayleigh population and its competitors using mechanical equipment groups.

Population	MLE(Standard–Error)	K-S(P-value)
	Group **X**	
Rayleigh	0.3487(0.0900)	0.2132(0.442)
MB	1.9117(0.4030)	0.2917(0.126)
Muth	0.1635(0.1496)	0.2957(0.117)
	Group **Y**	
Rayleigh	0.2299(0.0593)	0.2003(0.520)
MB	2.9004(0.6114)	0.2751(0.170)
Muth	0.0817(0.1308)	0.3268(0.063)

Using both **X** and **Y** datasets, when $T = 1.5$(hours) and $r(= 10, 18, 25)$, three different JHC-T1 samples are created, see Table 10. Using each JHC-T1 sample, the maximum likelihood and Bayes estimates of ξ_1 and ξ_2, for $a_i = b_i = 0.001,\ i = 1, 2$, are computed and listed in Table 11. Under SEL, LL (for $\upsilon(= -2, -0.02, +2)$), and GEL (for $\tau(= -3, -0.03, +3)$), the Bayes estimates of ξ_1 and ξ_2 are developed. Further, the 95% ACIs/BCIs of ξ_1 and ξ_2 are calculated and provided in Table 12.

As expected, Tables 11 and 12 showed that the acquired point and 95% interval estimates of ξ_1 and ξ_2 are similar to each other, as expected. Using the fitted values of $\hat{\xi}_1$ and $\hat{\xi}_2$, the estimated mean lifetimes of the Rayleigh populations from the groups **X** and **Y** become (0.9174, 1.1446, 1.1646) and (1.1749, 1.2655, 1.2219) under the samples (1, 2, 3), respectively. It can also be seen that the mean lifetimes of the group **Y** have a larger lifetime than the others. Using the Bayesian estimates, the same conclusion about the mean lifetimes can be easily drawn. Since the jointly Type-I hybrid censoring made a balance between reducing duration tests and observing extreme failures, we therefore recommend terminating the experiment as soon as the first of $g_{(r)}$ (or T) reaches.

Table 10. Three JHC-T1 samples from mechanical equipment dataset.

Sample	r	$w(z)$	(m^*, n^*)
1	10	0.11(1), 0.30(0), 0.40(0), 0.45(1), 0.59(0), 0.63(1), 0.70(1), 0.71(1), 0.74(0), 0.77(1)	(9,11)
2	18	0.11(1), 0.30(0), 0.40(0), 0.45(1), 0.59(0), 0.63(1), 0.70(1), 0.71(1), 0.74(0), 0.77(1), 0.94(0), 1.06(0), 1.17(1), 1.23(1), 1.23(0), 1.24(1), 1.43(1), 1.46(0)	(5,7)
3	25	0.11(1), 0.30(0), 0.40(0), 0.45(1), 0.59(0), 0.63(1), 0.70(1), 0.71(1), 0.74(0), 0.77(1), 0.94(0), 1.06(0), 1.17(1), 1.23(1), 1.23(0), 1.24(1), 1.43(1), 1.46(0), 1.49(0)	(5,6)

Table 11. Point estimates of ξ_1 and ξ_2 from mechanical equipment dataset.

Sample	Par.	MLE	SEL	LL			GEL		
				$\upsilon = -2$	$\upsilon = -0.02$	$\upsilon = +2$	$\tau = -3$	$\tau = -0.03$	$\tau = +3$
1	ξ_1	0.9330	0.9330	1.1175	0.9344	0.8123	1.0810	0.8587	0.6087
	ξ_2	0.5690	0.5690	0.6695	0.5698	0.5007	0.7016	0.5017	0.2585
2	ξ_1	0.5995	0.5995	0.6386	0.5999	0.5662	0.6577	0.5707	0.4771
	ξ_2	0.4904	0.4904	0.5232	0.4907	0.4626	0.5494	0.4610	0.3644
3	ξ_1	0.5790	0.5790	0.6153	0.5793	0.5748	0.6351	0.5512	0.4608
	ξ_2	0.5260	0.5260	0.5593	0.5263	0.4974	0.5824	0.4979	0.4063

Table 12. Interval estimates of ξ_1 and ξ_2 from mechanical equipment dataset.

Sample	Par.	ACI	BCI
1	ξ_1	(0.1865,1.8473)	(0.3424,1.8144)
	ξ_2	(0.1137,1.1266)	(0.1550,1.2471)
2	ξ_1	(0.2279,1.0150)	(0.2875,1.0243)
	ξ_2	(0.1865,0.8302)	(0.2117,0.8841)
3	ξ_1	(0.2201,0.9572)	(0.2776,0.9892)
	ξ_2	(0.1999,0.8696)	(0.2405,0.9212)

6. Conclusions

A new type of Type-II hybrid censoring, which enables the investigation of the lifespan of two (or more) competing goods in the same facility, has been investigated in this study. Various estimation procedures in the presence of such censoring have been proposed when the lifetimes of experimental items from two populations have been assumed to follow Rayleigh distributions with different scale parameters. Besides the maximum likelihood approach, Bayes' inferential approach has also been considered. Monte Carlo computations have been run to assess the efficiency of the suggested methods. The computational results indicate that the acquired estimators of the Rayleigh population parameters derived from the Bayes method perform quite satisfactorily compared to those derived by the classical method. Two actual data sets from engineering-related fields have been shown to test the usefulness of the suggested procedures. To sum up, the use of the asymmetric Bayes paradigm to estimate the Rayleigh population parameters under jointly Type-I hybrid censoring is recommended. The conclusions and approach described here may easily be extended to other lifetime groups as a future study, e.g., gamma, Weibull, or generalized exponential distribution, among others. It is better to extend the generalized Rayleigh results in Maiti and Kayal [22] to a joint progressive Type-II strategy. We also hope that the methodologies discussed here will be beneficial to data analysts and reliability managers.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [GRANT No. 5422].

Conflict of interest

The authors declare no conflicts of interest.

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