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Research article

The hybrid power mean of the generalized Gauss sums and the generalized two-term exponential sums

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Abstract: This article applied the properties of character sums, quadratic character, and classical Gauss sums to study the calculations of the hybrid power mean of the generalized Gauss sums and the generalized two-term exponential sums. It also provided exact formulas for calculating these hybrid power means.

Keywords: two-term exponential sums; character sums; the generalized Gauss sums
Mathematics Subject Classification: 11L03, 11L05

1. Introduction

The concept of exponential sums was first introduced to solve the Waring problem. The estimation of exponential sums has consistently been a fundamental focus in analytic number theory. Throughout the history of research on exponential sums, researchers have observed that individual values of exponential sums show highly irregular behavior, while their higher power mean and hybrid power mean exhibit regular patterns. Investigating two-term exponential sums is vital for various areas, including integer factorization, prime factorization, and the study of number theory functions. Therefore, in order to promote the mutual advancement of analytic number theory and its related fields, it is essential to explore the higher power mean and hybrid power mean of two-term exponential sums.

Let $q > 3$ be a positive integer and m and n be integers satisfying $(mn, q) = 1$. For any Dirichlet character χ, the generalized Gauss sums $G(m, n, \chi; q)$ and the generalized two-term exponential sums $C(m, k, h, \chi; q)$ are defined as follows:

$$G(m, n, \chi; q) = \sum_{a=0}^{q-1} \chi(a) e\left(\frac{ma^n}{q}\right),$$

$$C(m, k, h, \chi; q) = \sum_{a=1}^{q-1} \chi(a) e\left(\frac{ma^k + a^h}{q}\right),$$

where $e(y) = e^{2\pi iy}$, $k \geq 1$, $n \geq 1$, and $h > 1$ are integers.

Due to the importance of the generalized Gauss sum in analytic number theory, it has attracted the attention of many experts and resulted in some significant findings. For example, according to the results of A. Weil [1], an upper bound estimate can be obtained as $\left|\sum_{x=1}^{p-1} \chi(x) e(\frac{ax^n+bx}{p})\right| \leq n\sqrt{p}$, where p is an odd prime.

Therefore, we can speculate that

$$\left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2}{p}\right)\right| \leq 2\sqrt{p}.$$

Zhang Wenpeng and Lv Xingxing [2] studied the following identity that holds when p is an odd prime satisfying $p \equiv 3 \bmod 4$ and for any integer m satisfying $(m, p) = 1$,

$$\sum_{\chi \bmod p} \left|\sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2}{p}\right)\right|^2 \left|\sum_{b=1}^{p-1} \chi(b+\overline{b})\right|^2 = (p-1)(3p^2-6p-1).$$

Goran Djanković et al. [3] discussed the following equation that holds when p is an odd prime and n is an integer satisfying $(n, p) = 1$,

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left|\sum_{a=1}^{p-1} \chi^2(a) e\left(\frac{ma^2}{p}\right)\right|^2 \left|\sum_{b=1}^{p-1} \chi(b) e\left(\frac{mb+n\overline{b}}{p}\right)\right|^2$$
$$= \begin{cases} p(p-1)\left(2p^2-5p-2\left(\dfrac{n}{p}\right)+5\right), & \text{if } p \equiv 1 \bmod 4, \\ p(p-1)(2p^2-5p+5), & \text{if } p \equiv 3 \bmod 4, \end{cases}$$

where $\left(\frac{*}{p}\right)$ denotes the Legendre symbol and $\overline{b}$ denotes the multiplicative inverse of b mod p.

Li Xiaoxue and Wu Chengjing [4] discussed the following statement that holds for any integer m and n satisfying $(mn, p) = 1$,

$$\sum_{\chi \bmod p} \left|\sum_{a=1}^{p-1} \chi^2(a) e\left(\frac{ma^2}{p}\right)\right|^2 \left|\sum_{b=1}^{p-1} \chi(b) e\left(\frac{nb+\overline{b}}{p}\right)\right|^2$$
$$= p(p-1)^2 + p(p-1)\sum_{b=1}^{p-1} e\left(\frac{2nb+2\bar{b}}{p}\right) - (p-1)\left(\sum_{b=1}^{p-1} e\left(\frac{2nb+2\bar{b}}{p}\right)\right)^2,$$

where p is an odd prime satisfying $p \equiv 3 \bmod 4$.

Liu Xiaoge and Meng Yuanyuan [5] obtained the following result when p is an odd prime with $p \equiv 1 \bmod 12$,

$$\sum_{m=1}^{p-1} \left|\sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^4}{p}\right)\right|^2 \left|\sum_{b=1}^{p-1} \chi_4(b) e\left(\frac{ma^3}{p}\right)\right|^2 = 12p^2(p-1),$$

for any cubic character λ and quartic character χ_4 modulo p.

Consulting references [6–11] reveals additional significant results related to two-term exponential sums; these findings will not be restated in this paper.

The main focus of this paper is to study the hybrid power mean of the generalized Gauss sums $G(m, n, \chi; p)$ and the generalized two-term exponential sums $C(m, k, h, \chi; p)$ given by

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \lambda(a) e\left(\frac{ma^n}{p}\right) \right|^{2l} \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^h + mb^k}{p}\right) \right|^2, \tag{1.1}$$

where $\sum\limits_{\chi \bmod p}$ denotes the sum over all characters χ modulo p and $\lambda = \chi^2$ or $\lambda = \chi$.

This paper considers the computational problem posed by Eq (1.1), and it seems that no one has studied this specific topic before, at least not in the existing literature. Generally, summing over $\chi \bmod p$ does not yield ideal results. However, through calculations, we can gain a more accurate understanding of the relationship between the hybrid power mean of the generalized Gauss sums and the generalized two-term exponential sums.

In this paper, we utilize elementary and analytic methods, as well as the properties of character sums and classical quadratic Gauss sums, to investigate the computational problems presented in Eq (1.1). Here, we will discuss the hybrid power mean of several kinds of generalized two-term sums with generalized Gauss sums under the conditions of $n = 2, 4; l = 1, 2; h = 3, 4; k = 1, 2$. As a result, we derive several meaningful computational formulas. In other words, we can obtain the following:

Theorem 1.1. *When p is an odd prime satisfying $(3, p-1) = 1$ and χ is any character modulo p, we have*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi^2(a) e\left(\frac{ma^2}{p}\right) \right|^2 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^3 + mb^2}{p}\right) \right|^2$$
$$= \begin{cases} p(p-1)(2p^2 - 5p + 7), & \text{if } p \equiv 5 \bmod 24, \\ p(p-1)(2p^2 - 5p + 5), & \text{if } p \equiv 3, 11 \bmod 24, \\ p(p-1)(2p^2 - 5p + 11), & \text{if } p \equiv 17 \bmod 24, \\ p(p-1)(2p^2 - 5p + 9), & \text{if } p \equiv 23 \bmod 24. \end{cases}$$

Theorem 1.2. *When p is an odd prime satisfying $(3, p-1) = 1$ and χ is any character modulo p, we can derive the following identity*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi^2(a) e\left(\frac{ma^2}{p}\right) \right|^2 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^3 + mb}{p}\right) \right|^2$$
$$= \begin{cases} p(p-1)(2p^2 - 5p - 2\sqrt{p} + 5), & \text{if } p \equiv 5 \bmod 12, \\ p(p-1)(2p^2 - 5p + 5), & \text{if } p \equiv 11 \bmod 12 \text{ or } p = 3. \end{cases}$$

Theorem 1.3. *Let p be an odd prime satisfying $(3, p-1) = 1$ and χ is any character modulo p, and we can obtain*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2}{p}\right) \right|^2 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^3 + mb^2}{p}\right) \right|^2 = p(p-1)(p^2 - 5p + 8).$$

Theorem 1.4. *Let p be an odd prime satisfying* $(3, p-1) = 1$ *and* χ *is any character modulo p, and we will obtain the identity*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2}{p}\right) \right|^2 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^3+mb}{p}\right) \right|^2$$
$$= \begin{cases} p(p-1)(p^2-3p-2\tau(\chi_2)+4+\chi_4(3)\tau(\overline{\chi}_4)^2+\overline{\chi}_4(3)\tau(\chi_4)^2), & \text{if } p \equiv 5 \bmod 12, \\ p(p-1)(p^2-3p+4), & \text{if } p \equiv 11 \bmod 12 \text{ or } p=3. \end{cases}$$

Theorem 1.5. *Let p be an odd prime satisfying* $p > 5$ *and* χ *is any character modulo p, and we can conclude that*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^4}{p}\right) \right|^2 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{mb^4+b}{p}\right) \right|^2 = \begin{cases} p(p-1)(p^2-5p+8), & \text{if } 4 \nmid p-1, \\ p(p-1)(p^2-9p+24), & \text{if } 4 \mid p-1. \end{cases}$$

Theorem 1.6. *Let p be an odd prime satisfying* $(3, p-1) = 1$ *and* χ *is any character modulo p, and we can conclude that*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2}{p}\right) \right|^4 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^3+mb^2}{p}\right) \right|^2$$
$$= p(p-1)\left(3p^3-18p^2+27p+14+6\left(\frac{-1}{p}\right)\right).$$

Theorem 1.7. *When* $p > 5$ *is an odd prime satisfying* $p \equiv 3 \bmod 4$ *and* χ *is any character modulo p, we can conclude that*

$$\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^4}{p}\right) \right|^4 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{mb^4+b}{p}\right) \right|^2 = p(p-1)(3p^3-18p^2+27p+8).$$

2. Several lemmas

In this section, we will provide some important lemmas that are necessary for proving the theorems. In the following, we will apply the properties of characters modulo p and quadratic Gauss sums, and relevant content can be referenced from literature [12–15]. Therefore, we will not elaborate on it here. First, we introduce the relevant properties of quadratic character in [13]:

$$\sum_{\chi \bmod p} \left(\frac{x(ax+b)}{p}\right) = \sum_{\chi \bmod p} \left(\frac{ax+b}{p}\right) = -\left(\frac{a}{p}\right),$$

where $(ab, p) = 1$.

Next, we present the following lemmas:

Lemma 2.1. *Let p be an odd prime satisfying* $(3, p-1) = 1$, *then we can obtain the following equation:*

$$\sum_{\substack{a=1 \\ a^2+c^2 \equiv b^2+1 \bmod p \\ a^2 c \equiv b^2 \bmod p \\ c^3 \equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} 1 = 2p-2.$$

Proof. Through applying the properties of exponential sums and residue systems, we can conclude that

$$\sum_{\substack{a=1 \\ a^2+c^2\equiv b^2+1 \bmod p \\ a^2c\equiv b^2 \bmod p \\ c^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a^2+c^2\equiv b^2+1 \bmod p \\ a^2c\equiv b^2 \bmod p \\ c\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a^2+1\equiv b^2+1 \bmod p \\ a^2\equiv b^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1} 1 = 2p-2.$$

This result verifies Lemma 2.1. □

Lemma 2.2. *Let p be an odd prime satisfying $(3, p-1) = 1$, and we will obtain the identity*

$$\sum_{\substack{a=1 \\ a^2+c^2\equiv b^2+1 \bmod p \\ ac\equiv b \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 = 4p-8.$$

Proof. By utilizing the solutions to congruence equations and the computational method described in Lemma 2.1, we can easily obtain the following result:

$$\sum_{\substack{a=1 \\ a^2+c^2\equiv b^2+1 \bmod p \\ ac\equiv b \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a^2+c^2\equiv a^2c^2+1 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 = \sum_{\substack{a=1 \\ (a^2-1)(c^2-1)\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 = 4(p-1)-4 = 4p-8.$$

This proves Lemma 2.2. □

Lemma 2.3. *Let p be a prime satisfying $p > 5$, and we will obtain the identity*

$$\sum_{\substack{a=1 \\ a^4+c^4\equiv b^4+1 \bmod p \\ ac\equiv b \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 = \begin{cases} 4p-8, & \text{if } 4 \nmid p-1, \\ 8p-24, & \text{if } 4 \mid p-1. \end{cases}$$

Proof. By utilizing the solutions to congruence equations, we obtain the following result:

$$\begin{aligned}\sum_{\substack{a=1 \\ a^4+c^4\equiv b^4+1 \bmod p \\ ac\equiv b \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 &= \sum_{\substack{a=1 \\ a^4+c^4\equiv a^4c^4+1 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 \\ &= \sum_{\substack{a=1 \\ (a^4-1)(c^4-1)\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 = \begin{cases} 4p-8, & \text{if } 4 \nmid p-1, \\ 8p-24, & \text{if } 4 \mid p-1. \end{cases}\end{aligned}$$

This proves Lemma 2.3. □

Lemma 2.4. *Let p be an odd prime with* $(3, p-1) = 1$*, then we have*

$$\sum_{\substack{a=1 \\ a^2+c^2\equiv b^2+1 \bmod p \\ a^2c\equiv b^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 = \begin{cases} 3p-7, & \textit{if } p \equiv 5 \bmod 24, \\ 3p-5, & \textit{if } p \equiv 3, 11 \bmod 24, \\ 3p-11, & \textit{if } p \equiv 17 \bmod 24, \\ 3p-9, & \textit{if } p \equiv 23 \bmod 24. \end{cases}$$

Proof. Through utilizing the properties of character sums and residue systems, we can derive the following precise calculation formula:

$$\sum_{\substack{a=1 \\ a^2+c^2\equiv b^2+1 \bmod p \\ a^2c\equiv b^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 = \sum_{\substack{a=1 \\ a^2+c^2\equiv b+1 \bmod p \\ a^2c\equiv b \bmod p}}^{p-1}\sum_{c=1}^{p-1}\sum_{b=1}^{p-1} (1+\chi_2(b))$$

$$= \sum_{\substack{a=1 \\ a^2+c^2\equiv a^2c+1 \bmod p}}^{p-1}\sum_{c=1}^{p-1} (1+\chi_2(a^2c)) = \sum_{\substack{a=1 \\ a^2+c^2\equiv a^2c+1 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 + \sum_{\substack{a=1 \\ a^2+c^2\equiv a^2c+1 \bmod p}}^{p-1}\sum_{c=1}^{p-1} \chi_2(c)$$

$$= \sum_{\substack{a=1 \\ (c-1)(c+1-a^2)\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 + \sum_{\substack{a=1 \\ (c-1)(c+1-a^2)\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} \chi_2(c)$$

$$= \sum_{\substack{a=1 \\ c-1\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 + \sum_{\substack{a=1 \\ c+1-a^2\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 - \sum_{\substack{a=1 \\ (1+1-a^2)\equiv 0 \bmod p}}^{p-1} 1$$

$$+ \sum_{\substack{a=1 \\ c-1\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} \chi_2(c) + \sum_{\substack{a=1 \\ c+1-a^2\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} \chi_2(c) - \sum_{\substack{a=1 \\ (1+1-a^2)\equiv 0 \bmod p}}^{p-1} 1$$

$$= p-1-(1+\chi_2(2)) + \sum_{\substack{a=1 \\ c+1-a\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} (1+\chi_2(a)) + p-1-(1+\chi_2(2))$$

$$+ \sum_{\substack{a=1 \\ c+1-a\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} (1+\chi_2(a))\chi_2(c)$$

$$= 2(p-1)-2(1+\chi_2(2)) + \sum_{\substack{a=1 \\ c+1-a\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} 1 + \sum_{\substack{a=1 \\ c+1-a\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} \chi_2(a)$$

$$+ \sum_{\substack{a=1 \\ c+1-a\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} \chi_2(c) + \sum_{\substack{a=1 \\ c+1-a\equiv 0 \bmod p}}^{p-1}\sum_{c=1}^{p-1} \chi_2(ac)$$

$$= 2(p-1)-2(1+\chi_2(2)) + p-2-1-1-\chi_2(-1)$$

$$= 3p-8-2\chi_2(2)-\chi_2(-1)$$

$$= \begin{cases} 3p-7, & \text{if } p \equiv 5 \bmod 24, \\ 3p-5, & \text{if } p \equiv 3, 11 \bmod 24, \\ 3p-11, & \text{if } p \equiv 17 \bmod 24, \\ 3p-9, & \text{if } p \equiv 23 \bmod 24. \end{cases}$$

This proves Lemma 2.4. □

Lemma 2.5. *When p is an odd prime that satisfies $(3, p-1) = 1$, we can prove the following:*

$$\sum_{\substack{a=1 \\ a^2 \equiv d \bmod p \\ b^2 \equiv c \bmod p \\ c \not\equiv d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) = \begin{cases} -p+3-2\sqrt{p}, & \text{if } p \equiv 5 \bmod 12, \\ -p+3, & \text{if } p \equiv 11 \bmod 12 \text{ or } p = 3. \end{cases}$$

Proof. For $p > 3$, by solving congruence equations and utilizing the properties of classical Gauss sums, we can obtain

$$\sum_{\substack{a=1 \\ a^2 \equiv d \bmod p \\ b^2 \equiv c \bmod p \\ c \not\equiv d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) = \sum_{\substack{a=1 \\ a^2 \equiv d \bmod p \\ b^2 \equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) - \sum_{\substack{a=1 \\ a^2 \equiv d \bmod p \\ b^2 \equiv c \bmod p \\ c \equiv d \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)$$

$$= \sum_{\substack{a=1 \\ a \equiv d \bmod p \\ b \equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} (1+\chi_2(a))(1+\chi_2(b)) e\left(\frac{c^3-d^3}{p}\right) - 2(p-1)$$

$$= \sum_{\substack{a=1 \\ a \equiv d \bmod p \\ b \equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) + \sum_{\substack{a=1 \\ a \equiv d \bmod p \\ b \equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi_2(a) e\left(\frac{c^3-d^3}{p}\right)$$

$$+ \sum_{\substack{a=1 \\ a \equiv d \bmod p \\ b \equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi_2(b) e\left(\frac{c^3-d^3}{p}\right) + \sum_{\substack{a=1 \\ a \equiv d \bmod p \\ b \equiv c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi_2(ab) e\left(\frac{c^3-d^3}{p}\right) - 2(p-1)$$

$$= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} e\left(\frac{b^3-a^3}{p}\right) + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(a) e\left(\frac{b^3-a^3}{p}\right)$$

$$+ \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b^3-a^3}{p}\right) + \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \chi_2(ab) e\left(\frac{b^3-a^3}{p}\right) - 2(p-1)$$

$$= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} e\left(\frac{a^3(b^3-1)}{p}\right) + \tau(\chi_2) \sum_{b=1}^{p-1} \chi_2(b)\chi_2(b^3-1)$$

$$+\tau(\chi_2) \sum_{b=1}^{p-1} \chi_2(b^3-1) + p \sum_{\substack{b=1 \\ b^3 \equiv 1 \bmod p}}^{p-1} \chi_2(b) - \sum_{b=1}^{p-1} \chi_2(b) - 2(p-1)$$

$$= 1 + \tau(\chi_2)\sum_{b=1}^{p-1}\chi_2(b)\chi_2(b^3-1) + \tau(\chi_2)\sum_{b=1}^{p-1}\chi_2(b^3-1) + p\sum_{\substack{b=1\\ b^3\equiv 1 \bmod p}}^{p-1}\chi_2(b) - \sum_{b=1}^{p-1}\chi_2(b) - 2(p-1)$$

$$= p + 1 + \tau(\chi_2)\sum_{b=1}^{p-1}(1+\chi_2(b))\chi_2(b^3-1) - 2(p-1)$$

$$= p + 1 + \tau(\chi_2)\sum_{b=1}^{p-1}(1+\chi_2(b))\chi_2(b-1) - 2(p-1)$$

$$= \begin{cases} -p+3-2\sqrt{p}, & \text{if } p \equiv 5 \bmod 12,\\ -p+3, & \text{if } p \equiv 11 \bmod 12.\end{cases}$$

Specifically, when $p = 3$, and we can calculate the sums:

$$\sum_{\substack{a=1\\ a^2\equiv d \bmod 3\\ b^2\equiv c \bmod 3\\ c\not\equiv d \bmod 3}}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2} e\left(\frac{c^3-d^3}{3}\right) = \sum_{\substack{a=1\\ a^2\equiv d \bmod 3\\ b^2\equiv c \bmod 3}}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2} e\left(\frac{c^3-d^3}{3}\right) - \sum_{\substack{a=1\\ a^2\equiv d \bmod 3\\ b^2\equiv c \bmod 3\\ c\equiv d \bmod 3}}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2} e\left(\frac{c^3-d^3}{3}\right)$$

$$= \sum_{\substack{a=1\\ a^2\equiv d \bmod 3\\ b^2\equiv c \bmod 3}}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2}\sum_{d=1}^{2} e\left(\frac{c^3-d^3}{3}\right) - 4 = \sum_{\substack{a=1\\ a^2\equiv 1 \bmod 3\\ b^2\equiv 1 \bmod 3}}^{2}\sum_{b=1}^{2} 1 - 4 = 0,$$

which equals $-p + 3$.

This proves Lemma 2.5. □

Lemma 2.6. *Let p be an odd prime that satisfies* $(3, p-1) = 1$. *We can prove the following:*

$$\sum_{\substack{a=1\\ a^2-b^2+c-d\equiv 0 \bmod p\\ ac\equiv bd \bmod p\\ d\not\equiv \pm c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) = \begin{cases} \chi_4(3)\tau(\overline{\chi}_4)^2 + \overline{\chi}_4(3)\tau(\chi_4)^2 - 2\tau(\chi_2) - p + 3, & \text{if } p \equiv 5 \bmod 12,\\ -p+3, & \text{if } p \equiv 11 \bmod 12 \text{ or } p = 3.\end{cases}$$

Proof. Since the case when $p = 3$ is trivial, we will not discuss it further here. By solving congruence equations and reducing the residue classes modulo p, we can deduce that $a^2 - b^2 + c - d \equiv 0 \bmod p$ and $ac \equiv bd \bmod p$ are equivalent to $b^2(\bar{c}^2d^2 - 1) \equiv d - c \bmod p$ and $a \equiv b\bar{c}d \bmod p$, or $b^2 \equiv c^2\overline{(d+c)} \bmod p$ and $a \equiv b\bar{c}d \bmod p$. The number of solutions for this equation can be represented as $1 + \chi_2(c^2\overline{(c+d)})$=$1 + \chi_2(c+d)$, where, in the case of $p \equiv 5 \bmod 12$, $\chi_2 = \chi_4^2$, $\chi_2(*)$ represents the Legendre symbol and $\chi_4(*)$ is any fourth character modulo p. Therefore, we can prove

$$\sum_{\substack{a=1\\ a^2-b^2+c-d\equiv 0 \bmod p\\ ac\equiv bd \bmod p\\ d\not\equiv \pm c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) = \sum_{\substack{a=1\\ b^2(\bar{c}^2d^2-1)\equiv d-c \bmod p\\ a\equiv b\bar{c}d \bmod p\\ d\not\equiv \pm c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)$$

$$
\begin{aligned}
&= \sum_{\substack{c=1 \\ d\not\equiv \pm c \bmod p}}^{p-1} \sum_{d=1}^{p-1} (1+\chi_2(c+d)) e\left(\frac{c^3-d^3}{p}\right) \\
&= \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} (1+\chi_2(c+d)) e\left(\frac{c^3-d^3}{p}\right) - \sum_{c=1}^{p-1} (1+\chi_2(2c)) - \sum_{c=1}^{p-1} e\left(\frac{2c^3}{p}\right) \\
&= \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) + \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi_2(c+d) e\left(\frac{c^3-d^3}{p}\right) - (p-1) + 1 \\
&= 1 + \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi_2(c+d) e\left(\frac{c^3-d^3}{p}\right) - (p-1) + 1 \\
&= \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi_2(c+1)\chi_2(d) e\left(\frac{d^3(c^3-1)}{p}\right) - p + 3 \\
&= \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} \chi_2(c+1)\chi_2(d) e\left(\frac{d(c^3-1)}{p}\right) - p + 3 \\
&= \tau(\chi_2) \sum_{c=1}^{p-1} \chi_2(c+1)\chi_2(c^3-1) - p + 3 \\
&= \tau(\chi_2) \sum_{c=0}^{p-1} \chi_2(c+1)\chi_2(c^3-1) - \chi_2(-1)\tau(\chi_2) - p + 3 \\
&= \tau(\chi_2) \sum_{c=0}^{p-1} \chi_2(c+2)\chi_2(c^3+3c^2+3c) - \tau(\chi_2) - p + 3 \\
&= \tau(\chi_2) \sum_{c=1}^{p-1} \chi_2(2c+1)\chi_2(3c^2+3c+1) - \tau(\chi_2) - p + 3 \\
&= \tau(\chi_2) \sum_{c=0}^{p-1} \chi_2(2c+1)\chi_2(3c^2+3c+1) - 2\tau(\chi_2) - p + 3 \\
&= \tau(\chi_2) \sum_{c=1}^{p-1} \chi_2(c)\chi_2(3c^2+1) - 2\tau(\chi_2) - p + 3 \\
&= \sum_{b=1}^{p-1} \chi_2(b) e\left(\frac{b}{p}\right) \sum_{c=1}^{p-1} \chi_2(c) e\left(\frac{3bc^2}{p}\right) - 2\tau(\chi_2) - p + 3 \\
&= \sum_{b=1}^{p-1} \chi_4(b^2) e\left(\frac{b}{p}\right) \sum_{c=1}^{p-1} \chi_4(c^2) e\left(\frac{3bc^2}{p}\right) - 2\tau(\chi_2) - p + 3 \\
&= \sum_{b=1}^{p-1} \chi_4(b^2) e\left(\frac{b}{p}\right) \sum_{c=1}^{p-1} \chi_4(c)(1+\chi_2(c)) e\left(\frac{3bc}{p}\right) - 2\tau(\chi_2) - p + 3
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{b=1}^{p-1}\chi_4(b^2)e\left(\frac{b}{p}\right)\sum_{c=1}^{p-1}(\chi_4(c)+\overline{\chi_4}(c))e\left(\frac{3bc}{p}\right)-2\tau(\chi_2)-p+3\\
&= \sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b}{p}\right)\sum_{c=1}^{p-1}\chi_4(c)e\left(\frac{3bc}{p}\right)+\sum_{b=1}^{p-1}\chi_2(b)e\left(\frac{b}{p}\right)\sum_{c=1}^{p-1}\overline{\chi_4}(c)e\left(\frac{3bc}{p}\right)-2\tau(\chi_2)-p+3\\
&= \chi_4(3)\tau(\overline{\chi_4})^2+\overline{\chi_4}(3)\tau(\chi_4)^2-2\tau(\chi_2)-p+3,
\end{aligned}
$$

when $p \equiv 11 \bmod 12$, and we have $\chi_2(a) = -\chi_2(-a)$; thus,

$$
\begin{aligned}
&\sum_{\substack{a=1\\a^2-b^2+c-d\equiv 0 \bmod p\\ac\equiv bd \bmod p\\d\not\equiv \pm c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) = \sum_{\substack{a=1\\b^2(\bar{c}^2d^2-1)\equiv d-c \bmod p\\a\equiv b\bar{c}d \bmod p\\d\not\equiv \pm c \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)\\
&= \sum_{\substack{c=1\\d\not\equiv \pm c \bmod p}}^{p-1}\sum_{d=1}^{p-1}(1+\chi_2(c+d))e\left(\frac{c^3-d^3}{p}\right)\\
&= \sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi_2(c+1)\chi_2(d)e\left(\frac{d(c^3-1)}{p}\right)-p+3\\
&= \tau(\chi_2)\sum_{c=1}^{p-1}\chi_2(c+1)\chi_2(c^3-1)-p+3\\
&= \tau(\chi_2)\sum_{c=0}^{p-1}\chi_2(c+1)\chi_2(c^3-1)-\chi_2(-1)\tau(\chi_2)-p+3\\
&= \tau(\chi_2)\sum_{c=1}^{p-1}\chi_2(c)\chi_2(3c^2+1)+\tau(\chi_2)-p+3\\
&= -p+3.
\end{aligned}
$$

This proves Lemma 2.6. □

Lemma 2.7. *Let p be an odd prime, and we can establish the identity*

$$
\sum_{\substack{a=1\\a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = p^3+6p^2-19p-14-6\left(\frac{-1}{p}\right).
$$

Proof. Through utilizing the properties of the reduced residue system modulo p, we know that if d and e traverse reduced residue system modulo p, then da and eb will also traverse this system. We first clarify the results based on Lemma 1 presented in article [16], then improve the results for two specific parts

$$
\sum_{\substack{a=1\\a^2+b^2+d^2e^2\equiv a^2d^2+b^2e^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1
$$

$$
\begin{aligned}
&= \sum_{\substack{a=1 \\ a+b+de\equiv ad+be+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} (1+2\chi_2(d)+2\chi_2(a)+\chi_2(de)+\chi_2(ab)) \\
&+ \sum_{\substack{a=1 \\ a+b+de\equiv ad+be+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} (2\chi_2(ad)+2\chi_2(bd)+2\chi_2(ade)+2\chi_2(dab)+\chi_2(abde) \\
&= W_1+2W_2+2W_3+W_4+W_5+2W_6+2W_7+2W_8+2W_9+W_{10},
\end{aligned} \tag{2.1}
$$

where $\chi_2(*)$ denotes the Legendre symbol modulo p.

$$
\begin{aligned}
W_2 = &\sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} \chi_2(d) \\
=&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1} 1 + \sum_{\substack{a=1 \\ b(e-1)\equiv e-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{e=2}^{p-1} 1 + \sum_{\substack{a=1 \\ a(d-1)\equiv d-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1} \chi_2(d) + \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1} \chi_2(d) \\
=&(p-1)^2+(p-1)(p-2)+(p-1)\sum_{d=2}^{p-1}\chi_2(d) + \sum_{\substack{a=0 \\ a+b\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\chi_2(d) - \sum_{\substack{b=1 \\ b\equiv de-1 \bmod p}}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1}\chi_2(d) \\
=&(p-1)(2p-4)-(p-1)(p-2)+p-3 = p^2-2p-1.
\end{aligned} \tag{2.2}
$$

$$
\begin{aligned}
W_7 = &\sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} \chi_2(bd) \\
=&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\chi_2(b) + \sum_{\substack{a=1 \\ b(e-1)\equiv e-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{e=2}^{p-1} \chi_2(b) + \sum_{\substack{a=1 \\ a(d-1)\equiv d-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1} \chi_2(bd) + \sum_{\substack{a=1 \\ a(d-1)+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1} \chi_2(bd) \\
=& \sum_{\substack{a=1 \\ b(e-1)\equiv e-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{e=2}^{p-1} \chi_2(b) + \sum_{\substack{a=1 \\ a+b(e-1)\equiv de-1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=2}^{p-1}\sum_{e=2}^{p-1} \chi_2(bd) \\
=&(p-1)(p-2)+(p-3) = p^2-2p-1.
\end{aligned} \tag{2.3}
$$

Consequently, by combining Eqs (2.1)–(2.3) and Lemma 1 of article [16], we can deduce the following result:

$$
\sum_{\substack{a=1 \\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = p^3+6p^2-19p-14-6\left(\frac{-1}{p}\right).
$$

□

Lemma 2.8. *Let p be an odd prime that satisfies* $p \equiv 3 \bmod 4$*, and we have the following identity:*

$$\sum_{\substack{a=1 \\ a^4+b^4+c^4\equiv d^4+e^4+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 = p^3+6p^2-19p-8.$$

Proof. When p satisfies $p \equiv 3 \bmod 4$ and is an odd prime, then $\chi_2(abde) = -\chi_2(-abde)$. By utilizing the properties of the reduced residue system modulo p and the solutions to congruence equations and combining them with the results presented in Lemma 2.7, we can obtain

$$\begin{aligned}
\sum_{\substack{a=1 \\ a^4+b^4+c^4\equiv d^4+e^4+1 \bmod p \\ abc\equiv de \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 &= \sum_{\substack{a=1 \\ a^4+b^4+d^4e^4\equiv a^4d^4+b^4e^4+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= \sum_{\substack{a=1 \\ a^2+b^2+d^2e^2\equiv a^2d^2+b^2e^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} (1+\chi_2(a))(1+\chi_2(b))(1+\chi_2(d))(1+\chi_2(e)) \\
&= \sum_{\substack{a=1 \\ a^2+b^2+d^2e^2\equiv a^2d^2+b^2e^2+1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= p^3+6p^2-19p-8.
\end{aligned}$$

□

3. Proof of theorems

We know the trigonometric identity:

$$\sum_{a=0}^{p-1} e\left(\frac{ma}{p}\right) = \begin{cases} p, & \text{if } p|m, \\ 0, & \text{if } p \nmid m. \end{cases} \tag{3.1}$$

First, we give the proofs for Theorems 1.1–1.4. Let p be an odd prime and suppose $(3, p-1) = 1$, then by Lemmas 2.1 and 2.4, we can obtain

$$\begin{aligned}
&\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi^2(a)e\left(\frac{ma^2}{p}\right)\right|^2\left|\sum_{b=1}^{p-1}\chi(b)e\left(\frac{b^3+mb^2}{p}\right)\right|^2 \\
=& \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)\sum_{\chi \bmod p}\chi(a^2c)\overline{\chi}(b^2d)\sum_{m=0}^{p-1} e\left(\frac{m(a^2-b^2+c^2-d^2)}{p}\right) \\
=& \; p(p-1)\sum_{\substack{a=1 \\ a^2-b^2+c^2-d^2\equiv 0 \bmod p \\ a^2c\equiv b^2d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)
\end{aligned}$$

$$= p^2(p-1)\sum_{\substack{a=1\\ a^2-b^2+c^2-1\equiv 0 \bmod p\\ a^2c\equiv b^2 \bmod p\\ c^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 - p(p-1)\sum_{\substack{a=1\\ a^2-b^2+c^2-1\equiv 0 \bmod p\\ a^2c\equiv b^2 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1$$

$$= p(p-1)(2p^2-2p) - p(p-1)(3p-8-2\chi_2(2)-\chi_2(-1))$$

$$= \begin{cases} p(p-1)(2p^2-5p+7), & \text{if } p\equiv 5 \bmod 24,\\ p(p-1)(2p^2-5p+5), & \text{if } p\equiv 3, 11 \bmod 24,\\ p(p-1)(2p^2-5p+11), & \text{if } p\equiv 17 \bmod 24,\\ p(p-1)(2p^2-5p+9), & \text{if } p\equiv 23 \bmod 24, \end{cases}$$

and by combining Lemma 2.5, we have

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi^2(a)e\left(\frac{ma^2}{p}\right)\right|^2\left|\sum_{b=1}^{p-1}\chi(b)e\left(\frac{b^3+mb}{p}\right)\right|^2$$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)\sum_{\chi \bmod p}\chi(a^2c)\overline{\chi}(b^2d)\sum_{m=0}^{p-1} e\left(\frac{m(a^2-b^2+c-d)}{p}\right)$$

$$= p(p-1)\sum_{\substack{a=1\\ a^2-b^2+c-d\equiv 0 \bmod p\\ a^2c\equiv b^2d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)$$

$$= p(p-1)\sum_{\substack{a=1\\ a^2-b^2+c-d\equiv 0 \bmod p\\ a^2c\equiv b^2d \bmod p\\ c\equiv d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) + p(p-1)\sum_{\substack{a=1\\ a^2-b^2+c-d\equiv 0 \bmod p\\ a^2c\equiv b^2d \bmod p\\ c\not\equiv d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)$$

$$= 2p(p-1)(p-1)^2 + p(p-1)\sum_{\substack{a=1\\ a^2-b^2+c-d\equiv 0 \bmod p\\ a^2c\equiv b^2d \bmod p\\ c\not\equiv d \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)$$

$$= \begin{cases} p(p-1)(2p^2-5p-2\sqrt{p}+5), & \text{if } p\equiv 5 \bmod 12,\\ p(p-1)(2p^2-5p+5), & \text{if } p\equiv 11 \bmod 12 \text{ or } p=3, \end{cases}$$

therefore, according to Lemma 2.2, we can obtain the identity

$$\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^2\left|\sum_{b=1}^{p-1}\chi(b)e\left(\frac{b^3+mb^2}{p}\right)\right|^2$$

$$= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right)\sum_{\chi \bmod p}\chi(ac)\overline{\chi}(bd)\sum_{m=0}^{p-1} e\left(\frac{m(a^2-b^2+c^2-d^2)}{p}\right)$$

$$
\begin{aligned}
&= p(p-1) \sum_{\substack{a=1 \\ a^2-b^2+c^2-d^2\equiv 0 \bmod p \\ ac\equiv bd \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) \\
&= p^2(p-1) \sum_{\substack{a=1 \\ a^2-b^2+c^2-1\equiv 0 \bmod p \\ ac\equiv b \bmod p \\ c^3\equiv 1 \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} 1 - p(p-1) \sum_{\substack{a=1 \\ a^2-b^2+c^2-1\equiv 0 \bmod p \\ ac\equiv b \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} 1 \\
&= p(p-1)(p^2-p) - p(p-1)(4p-8) \\
&= p(p-1)(p^2-5p+8).
\end{aligned}
$$

We combine Lemma 2.6 to provide the proof of Theorem 1.4:

$$
\begin{aligned}
&\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^2}{p}\right) \right|^2 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{b^3+mb}{p}\right) \right|^2 \\
&= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) \sum_{\chi \bmod p} \chi(ac)\overline{\chi}(bd) \sum_{m=0}^{p-1} e\left(\frac{m(a^2-b^2+c-d)}{p}\right) \\
&= p(p-1) \sum_{\substack{a=1 \\ a^2-b^2+c-d\equiv 0 \bmod p \\ ac\equiv bd \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) \\
&= p(p-1) \sum_{\substack{a=1 \\ a^2-b^2+c-d\equiv 0 \bmod p \\ a^2c\equiv b^2d \bmod p \\ d\equiv \pm c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) + p(p-1) \sum_{\substack{a=1 \\ a^2-b^2+c-d\equiv 0 \bmod p \\ ac\equiv bd \bmod p \\ d\not\equiv \pm c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) \\
&= p(p-1)(p-1)^2 + p(p-1) \sum_{\substack{a=1 \\ a^2-b^2+c-d\equiv 0 \bmod p \\ ac\equiv bd \bmod p \\ d\not\equiv \pm c \bmod p}}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c^3-d^3}{p}\right) \\
&= \begin{cases} p(p-1)(p^2-3p-2\tau(\chi_2)+4+\chi_4(3)\tau(\overline{\chi}_4)^2+\overline{\chi}_4(3)\tau(\chi_4)^2), & \text{if } p\equiv 5 \bmod 12, \\ p(p-1)(p^2-3p+4), & \text{if } p\equiv 11 \bmod 12 \text{ or } p=3. \end{cases}
\end{aligned}
$$

Here, according to Lemma 2.3, we can obtain

$$
\begin{aligned}
&\sum_{\chi \bmod p} \sum_{m=0}^{p-1} \left| \sum_{a=1}^{p-1} \chi(a) e\left(\frac{ma^4}{p}\right) \right|^2 \left| \sum_{b=1}^{p-1} \chi(b) e\left(\frac{mb^4+b}{p}\right) \right|^2 \\
&= \sum_{a=1}^{p-1} \sum_{b=1}^{p-1} \sum_{c=1}^{p-1} \sum_{d=1}^{p-1} e\left(\frac{c-d}{p}\right) \sum_{\chi \bmod p} \chi(ac)\overline{\chi}(bd) \sum_{m=0}^{p-1} e\left(\frac{m(a^4-b^4+c^4-d^4)}{p}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= p(p-1)\sum_{\substack{a=1 \\ a^4-b^4+c^4-d^4\equiv 0 \bmod p \\ ac\equiv bd \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} e\left(\frac{c-d}{p}\right) \\
&= p^2(p-1)\sum_{\substack{a=1 \\ a^4-b^4+c^4-1\equiv 0 \bmod p \\ ac\equiv b \bmod p \\ c\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 - p(p-1)\sum_{\substack{a=1 \\ a^4-b^4+c^4-1\equiv 0 \bmod p \\ ac\equiv b \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 \\
&= \begin{cases} p(p-1)(p^2-5p+8), & \text{if } 4\nmid p-1, \\ p(p-1)(p^2-9p+24), & \text{if } 4\mid p-1. \end{cases}
\end{aligned}
$$

This proves Theorem 1.5.

Now, combining Lemmas 2.2 and 2.7 with odd prime p and satisfying the condition $(3, p-1) = 1$, we can get the identity

$$
\begin{aligned}
&\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^2}{p}\right)\right|^4\left|\sum_{b=1}^{p-1}\chi(b)e\left(\frac{b^3+mb^2}{p}\right)\right|^2 \\
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{e^3-f^3}{p}\right)\sum_{\chi \bmod p}\chi(ace)\overline{\chi}(bdf)\sum_{m=0}^{p-1} e\left(\frac{m(a^2+c^2+e^2-b^2-d^2-f^2)}{p}\right) \\
&= p(p-1)\sum_{\substack{a=1 \\ a^2+c^2+e^2\equiv b^2+d^2+f^2 \bmod p \\ ace\equiv bd \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{e^3-f^3}{p}\right) \\
&= p^2(p-1)\sum_{\substack{a=1 \\ a^2+c^2+e^2\equiv b^2+d^2+1 \bmod p \\ ace\equiv bd \bmod p \\ e^3\equiv 1 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 - p(p-1)\sum_{\substack{a=1 \\ a^2+c^2+e^2\equiv b^2+d^2+1 \bmod p \\ ace\equiv bd \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= p^2(p-1)^2\sum_{\substack{a=1 \\ a^2+c^2\equiv b^2+1 \bmod p \\ ac\equiv b \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1} 1 - p(p-1)\sum_{\substack{a=1 \\ a^2+c^2+e^2\equiv b^2+d^2+1 \bmod p \\ ace\equiv bd \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1} 1 \\
&= p(p-1)(4p^3-12p^2+8p) - p(p-1)\left(p^3+6p^2-19p-14-6\left(\frac{-1}{p}\right)\right) \\
&= p(p-1)\left(3p^3-18p^2+27p+14+6\left(\frac{-1}{p}\right)\right).
\end{aligned}
$$

This proves Theorem 1.6.

Finally, according to Lemmas 2.3 and 2.8, when $p\equiv 3 \bmod 4$, we can prove Theorem 1.7:

$$
\sum_{\chi \bmod p}\sum_{m=0}^{p-1}\left|\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^4}{p}\right)\right|^4\left|\sum_{b=1}^{p-1}\chi(b)e\left(\frac{mb^4+b}{p}\right)\right|^2
$$

$$
\begin{aligned}
&= \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} e\left(\frac{e-f}{p}\right)\sum_{\chi \bmod p}\chi(ace)\overline{\chi}(bdf)\sum_{m=0}^{p-1} e\left(\frac{m(a^4+c^4+e^4-b^4-d^4-f^4)}{p}\right)\\
&= p(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1\\ a^4+c^4+e^4\equiv b^4+d^4+f^4 \bmod p\\ ace\equiv bd \bmod p}}^{p-1} e\left(\frac{e-f}{p}\right)\\
&= p^2(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a^4+c^4+e^4\equiv b^4+d^4+1 \bmod p\\ ace\equiv bd \bmod p\\ e\equiv 1 \bmod p}}^{p-1} 1 - p(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a^4+c^4+e^4\equiv b^4+d^4+1 \bmod p\\ ace\equiv bd \bmod p}}^{p-1} 1\\
&= p^2(p-1)^2\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ a^4+c^4\equiv b^4+1 \bmod p\\ ac\equiv b \bmod p}}^{p-1} 1 - p(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a^4+c^4+e^4\equiv b^4+d^4+1 \bmod p\\ ace\equiv bd \bmod p}}^{p-1} 1\\
&= p^2(p-1)^2\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{\substack{c=1\\ a^2+c^2\equiv b^2+1 \bmod p\\ ac\equiv b \bmod p}}^{p-1} 1 - p(p-1)\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ a^2+c^2+b^2d^2\equiv a^2b^2+c^2d^2+1 \bmod p}}^{p-1} 1\\
&= p(p-1)(4p^3-12p^2+8p) - p(p-1)\left(p^3+6p^2-19p-8\right)\\
&= p(p-1)(3p^3-18p^2+27p+8).
\end{aligned}
$$

This completes the proofs of our results.

4. Conclusions

The main result of this paper was to investigate the computational problem involving the hybrid power mean of the generalized Gauss sums with the generalized two-term exponential sums. We also obtained exact computational formulas. The main results were obtained by using Lemma 2.7 to complete the proof. At this point, we let

$$
h(p) = \sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{\substack{e=1\\ a^2+b^2+c^2\equiv d^2+e^2+1 \bmod p\\ abc\equiv de \bmod p}}^{p-1} 1.
$$

Here, we provided an example to calculate the exact results for prime numbers p. The precise calculation results were summarized in Table 1. Additionally, our results also offered effective solutions to the problem of calculating the higher power mean of two-term exponential sums. We believe that these works will play a positive role in advancing the research on related problems.

Table 1. The calculation of $h(p)$.

p	$h(p)$	p	$h(p)$
101	$h(101)=1089568$	103	$h(103)=1154416$
109	$h(109)=1364227$	107	$h(107)=1291696$
113	$h(113)=1517344$	127	$h(127)=2142736$
137	$h(137)=2681344$	131	$h(131)=2348560$
149	$h(149)=3438304$	139	$h(139)=2798868$
173	$h(173)=5353984$	151	$h(151)=3576880$
181	$h(181)=6122848$	157	$h(157)=4014796$
197	$h(197)=7874464$	163	$h(163)=4487056$
229	$h(229)=12319264$	167	$h(167)=4821616$
233	$h(233)=12970624$	191	$h(191)=7183120$

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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