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Research article

Approximation of functions in a certain Banach space by some generalized singular integrals

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Abstract: We introduced the q-Picard, the q-Picard-Cauchy, the q-Gauss-Weierstrass, and the q-truncated Picard singular integrals. Using the last three mentioned integrals, the orders of approximation for functions from a generalized Hölder space were determined, both in the L^p-norm and in the generalized Hölder-norm.

Keywords: degree of approximation; q-Singular integrals; modulus of continuity; generalized Hölder class; generalized Minkowski inequality
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1. Introduction

Already we know well that the approximation theory is concerned with how functions can best be approximated with simpler functions and with quantitatively characterizing the errors introduced. A closely related topic is the approximation of functions by Fourier series. Staying in this particular topic, the approximation of 2π-periodic and integrable functions by their Fourier series in the Hölder metric has been studied regularly by lots of researchers. Das et al. studied the degree of approximation of functions by matrix means of their Fourier series in the generalized Hölder metric [6], generalizing some well-known previous results. Again, Das et al. [7] studied the rate of the convergence problem of the Fourier series in a new Banach space of functions conceived as a generalization of the spaces introduced by Prössdorf [26] and Leindler [20]. Later on, Nayak et al. [24, 25] studied the rate of the convergence problem of the Fourier series by delayed arithmetic mean in the generalized Hölder metric space, which was earlier introduced in [7]. This obtained a sharper estimate of Jackson's order and was the main objective of their results. In [18], Kim treated the degree of approximation of functions in the same generalized Hölder metric by using the so-called even-type delayed arithmetic mean of Fourier series. Intentionally, we do not want to mention all published results here because they are somehow beyond the topic treated here. However, for the sake of the interested reader, recent results on that topic can be found in references [14–18, 24, 25] and the references therein.

We return back again to the paper of Prössdorf, referenced by [26], who studied the degree of approximation problems of Fourier series of functions from H_α $(0 < \alpha \leq 1)$ space in the Hölder metric. Let $C_{2\pi}$ be the Banach space of 2π-periodic continuous functions defined in $[-\pi, \pi]$ under the sup-norm. For $0 < \alpha \leq 1$ and some positive constant K, the function space H_α is given by

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x - y|^\alpha\}.$$

The space H_α is a Banach space with the norm $\|f\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_C + \sup_{x,y}\{\Delta^\alpha f(x, y)\},$$

where

$$\|f\|_C = \sup_{-\pi \leq x \leq \pi} |f(x)|$$

and

$$\Delta^\alpha f(x, y) := |f(x) - f(y)||x - y|^{-\alpha}, \quad x \neq y.$$

At this stage, we agree to write (by convention) $\Delta^0 f(x, y) := 0$.

Now, let us recall the well-known Picard, Picard-Cauchy and Gauss-Weierstrass singular integrals given by

$$P_\xi(f; x) = \frac{1}{2\xi}\int_{-\infty}^{\infty} f(x + t)e^{-\frac{|t|}{\xi}}dt, \tag{1.1}$$

$$Q_\xi(f; x) = \frac{\xi}{\pi}\int_{-\pi}^{\pi} \frac{f(x + t)}{t^2 + \xi^2}dt \tag{1.2}$$

and

$$W_\xi(f; x) = \frac{1}{\sqrt{\pi\xi}}\int_{-\pi}^{\pi} f(x + t)e^{-\frac{t^2}{\xi}}dt, \tag{1.3}$$

respectively, where ξ is a positive parameter that tends to zero.

Everywhere in this paper, we write

$$\varphi_x(t) := \frac{1}{2}[f(x + t) + f(x - t) - 2f(x)]$$

and $u = O(v)$, whenever there exists a positive constant K, not necessarily the same at each occurrence, such that $u \leq Kv$.

Let f be a bounded real valued function defined on the real line $\mathbb{R}$ or $(-\pi, \pi)$. By $\mathbb{B}$ we denote the Banach space of such functions under the sup-norm.

Mohapatra and Rodrigez [23] yielded the error bound of $f \in H_\alpha$ in the norm $\|\cdot\|_\beta$ for $(0 \leq \beta < \alpha \leq 1)$. Among others, they proved the following theorems.

Theorem 1.1. *Let $f \in \mathbb{B}$ and*

$$\omega(\delta) = \sup_{|t| \leq \delta}|f(x + t) - f(x)|, \quad (\delta > 0),$$

such that $\omega(t)/t$ is a nonincreasing function of t, then as $\xi \to 0+$, the following hold:

$$\|f - P_\xi(f;\cdot)\|_C = O(\omega(\xi)), \tag{1.4}$$
$$\|f - Q_\xi(f;\cdot)\|_C = O(\omega(\xi)|\ln(1/\xi)|), \tag{1.5}$$
$$\|f - W_\xi(f;\cdot)\|_C = O\left(\omega(\xi)\xi^{-\frac{1}{2}}\right). \tag{1.6}$$

Theorem 1.2. *Let* $0 \le \beta < \alpha \le 1$ *and* $f \in H_\alpha$*, then as* $\xi \to 0+$*,*

$$\|f - P_\xi(f;\cdot)\|_\beta = O\left(\xi^{\alpha-\beta}\right),$$
$$\|f - Q_\xi(f;\cdot)\|_\beta = O\left(\xi^{\alpha-\beta}|\ln(1/\xi)|\right),$$
$$\|f - W_\xi(f;\cdot)\|_\beta = O\left(\xi^{\alpha-\beta-\frac{1}{2}}\right).$$

Seemingly wishing to generalize the Hölder metric (see [20]), Leindler introduced the function space H^ω (a truly generalization) given by

$$H^\omega := \{f \in C[-\pi,\pi] : |f(x+t) - f(x)| = O(\omega(|t|))\},$$

where $\omega(\delta, f)$ is the modulus of continuity of f and ω is a modulus of continuity; that is, ω is a positive nondecreasing continuous function on $[0, 2\pi]$ having the properties $\omega(0) = 0$ and $\omega(\delta_1 + \delta_2) \le \omega(\delta_1) + \omega(\delta_2)$ for $0 \le \delta_1 \le \delta_2 \le 2\pi$.

He also introduced the norm $\|\cdot\|_\omega$ on the space H^ω by

$$\|f\|_\omega = \|f\|_C + \sup_{t\neq 0} \frac{|f(x+t) - f(x)|}{\omega(|t|)}.$$

In the case when $\omega(\delta) = \delta^\alpha$, $0 < \alpha \le 1$ the space H^ω reduces to H_α space and the norm $\|\cdot\|_\omega$ clearly becomes $\|\cdot\|_\alpha$–norm, which, as we mentioned above, was introduced by Prössdorf. It is known (see [26]) that

$$H_\alpha \subseteq H_\beta \subseteq C_{2\pi}, \quad 0 \le \beta < \alpha \le 1.$$

Das, Nath and Ray [7] generalized further the space H^ω by

$$H_p^{(\omega)} := \left\{ f \in L^p[0,2\pi] : \sup_{t\neq 0} \frac{\|f(\cdot + t) - f(\cdot)\|_p}{\omega(|t|)} < \infty \right\}$$

and

$$\|f\|_p^{(\omega)} := \|f\|_p + \sup_{t\neq 0} \frac{\|f(\cdot + t) - f(\cdot)\|_p}{\omega(|t|)},$$

where

$$\|f\|_p := \left(\frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

for $1 \le p < \infty$, while for $p = \infty$,

$$\|f\|_\infty := \operatorname{ess} \sup_{x\in(0,2\pi)} \{|f(x)|\}.$$

Choosing $\omega(t)$ and $v(t)$ so that $\omega(t)/v(t)$ is nondecreasing, then

$$\|f\|_p^{(v)} \le \max(1, \omega(2\pi)/v(2\pi)) \|f\|_p^{(\omega)},$$

which shows the relations

$$H_p^{(\omega)} \subseteq H_p^{(v)} \subseteq L^p, \ \ p \ge 1.$$

The singular integrals $P_\xi(f;x)$, $Q_\xi(f;x)$, $W_\xi(f;x)$, as well as their generalizations, are widely used in the problems of approximation of certain class functions. The approximation of functions, belonging to a certain class of functions, have been studied by Khan and Umar [12], Gal [9,10], Deeba et al. [5], Mezei [22], Anastassiou and Mezei [1], Anastassiou and Gal [2], Anastassiou and Aral [3], Rempulska and Tomczak [27], Firlejy and Rempulska [8], Bogalska et al. [4], Leśniewicz et al. [19], Khan and Ram [13] and, of course, there are many other results established by numerous researchers.

Following from [3], for $q > 0$ and for all nonnegative real ξ, the q-ξ real number $[\xi]_q$ is defined by

$$[\xi]_q := \frac{1-q^\xi}{1-q} \quad \text{for} \quad q \neq 1 \quad \text{and} \quad [\xi]_q := \xi \quad \text{for} \quad q = 1.$$

Now, we are in able to introduce the following generalizations of the integrals (1.1)–(1.3):

$$P_{\xi;q}(f;x) := \frac{1}{2[\xi]_q} \int_{-\infty}^{\infty} f(x+t) e^{-\frac{|t|}{[\xi]_q}} dt,$$

$$Q_{\xi;q}(f;x) := \frac{[\xi]_q}{\pi} \int_{-\pi}^{\pi} \frac{f(x+t)}{t^2 + [\xi]_q^2} dt$$

and

$$W_{\xi;q}(f;x) := \frac{1}{\sqrt{\pi[\xi]_q}} \int_{-\pi}^{\pi} f(x+t) e^{-\frac{t^2}{[\xi]_q}} dt.$$

Since, for $q = 1$, these integrals reduce to the singular integrals $P_\xi(f;x)$, $Q_\xi(f;x)$ and $W_\xi(f;x)$, respectively, it make sense to name the integrals $P_{\xi;q}(f;x)$, $Q_{\xi;q}(f;x)$ and $W_{\xi;q}(f;x)$ the q-Picard singular integrals, the q-Picard-Cauchy singular integrals and q-Gauss-Weierstrass singular integrals, respectively.

Before we pass to the aim of this paper, we introduce the integral

$$\overline{P}_{\xi;q}(f;x) := \frac{1}{2(1 - e^{-\frac{\pi}{[\xi]_q}})[\xi]_q} \int_{-\pi}^{\pi} f(x+t) e^{-\frac{|t|}{[\xi]_q}} dt,$$

which we name it “the q-truncated Picard singular integral”. The idea of introducing such an integral is that it enables Lemma 2.1 (see section two) to be applied in the proof of the main results, and in this way it might enlarge its applicability and usefulness in approximations problems. As we will see, the application of the q-truncated Picard singular integral, in approximation of a function f, provides the same degree of approximation, which we are going to show in section three.

The purpose of this paper is to prove the homologous of Theorems 1.1 and 1.2 in the metric $\|\cdot\|_p^{(\cdot)}$ of the space $H_p^{(\cdot)}$. To my best knowledge and the accessible literature, such results are not reported so far. For the proofs of the main results, we use the same lines of the arguments of [23,26].

2. Helpful lemmas

The generalized Minkowski's inequality for integrals states that the norm of an integral is less or equal to the integral of the corresponding norm. For the L^p spaces, it can be formulated as follows.

Lemma 2.1. *(Generalized Minkowski inequality [11]) If $z(x,t)$ is a function in two variables defined for $c \leq t \leq d$, $a \leq x \leq b$, then*

$$\left\{\int_a^b \left|\int_c^d z(x,t)dt\right|^p dx\right\}^{\frac{1}{p}} \le \int_c^d \left\{\int_a^b |z(x,t)|^p dx\right\}^{\frac{1}{p}} dt, \quad p \ge 1.$$

Lemma 2.2. *Let $f \in H_p^{(\omega)}$ $(p \ge 1)$ and ω and v be two moduli of continuity, such that ω/v is a nondecreasing function of t, then*

(i) $\|\varphi_\cdot(t)\|_p = O(\omega(t))$;

(ii) $\|\varphi_{\cdot+h}(t) - \varphi_\cdot(t)\|_p = O(\omega(t))$;

(iii) $\|\varphi_{\cdot+h}(t) - \varphi_\cdot(t)\|_p = v(|h|)O(\omega(t)/v(t))$.

Proof. The proof can be done in the same way as Lemma 1 in [7]. □

3. Main results

First we report the following first main result.

Theorem 3.1. *Let $q_\xi \in (0,1)$ such that $q_\xi \to 1$ as $\xi \to 0+$. Let $f \in H_p^{(\omega)}$ with $p \ge 1$ and $\omega(t)/t$ be a nonincreasing function of t, then*

$$\|\overline{P}_{\xi;q_\xi}(f;\cdot) - f\|_p = O\left(\omega([\xi]_{q_\xi})\right), \tag{3.1}$$

$$\|Q_{\xi;q_\xi}(f;\cdot) - f\|_p = O\left(\omega([\xi]_{q_\xi})|\ln(1/[\xi]_{q_\xi})|\right), \tag{3.2}$$

$$\|W_{\xi;q_\xi}(f;\cdot) - f\|_p = O\left(\omega([\xi]_{q_\xi})[\xi]_{q_\xi}^{-\frac{1}{2}}\right). \tag{3.3}$$

Proof. Throughout the proof (for simplicity of notation) we simply write q instead of q_ξ. Taking into account the equality

$$\int_{-\pi}^{\pi} e^{-\frac{|t|}{[\xi]_q}} dt = 2(1 - e^{-\frac{\pi}{[\xi]_q}})[\xi]_q$$

and the truncated Picard singular integral $\overline{P}_{\xi;q}(f;x)$ we can write

$$\overline{P}_{\xi;q}(f;x) - f(x) = \frac{1}{(1 - e^{-\frac{\pi}{[\xi]_q}})\xi} \int_0^\pi \varphi_x(t) e^{-\frac{|t|}{[\xi]_q}} dt.$$

Therefore, using Lemma 2.1, we have

$$\begin{aligned}
\|\overline{P}_{\xi;q}(f;\cdot) - f\|_p &= \left\{\frac{1}{2\pi}\int_0^{2\pi} \left|\frac{1}{(1 - e^{-\frac{\pi}{[\xi]_q}})[\xi]_q} \int_0^\pi \varphi_x(t) e^{-\frac{|t|}{[\xi]_q}} dt\right|^p dx\right\}^{\frac{1}{p}} \\
&\le \frac{1}{(1 - e^{-\frac{\pi}{[\xi]_q}})[\xi]_q} \int_0^\pi \left\{\frac{1}{2\pi}\int_0^{2\pi} |\varphi_x(t)|^p dx\right\}^{\frac{1}{p}} e^{-\frac{|t|}{[\xi]_q}} dt \\
&= \underbrace{\frac{1}{(1 - e^{-\frac{\pi}{[\xi]_q}})[\xi]_q} \int_0^{[\xi]_q} \|\varphi_\cdot(t)\|_p e^{-\frac{|t|}{[\xi]_q}} dt}_{:=\mathbb{P}_1} \\
&\quad + \underbrace{\frac{1}{(1 - e^{-\frac{\pi}{[\xi]_q}})[\xi]_q} \int_{[\xi]_q}^{\pi} \|\varphi_\cdot(t)\|_p e^{-\frac{|t|}{[\xi]_q}} dt}_{:=\mathbb{P}_2},
\end{aligned} \tag{3.4}$$

where $0 < [\xi]_q < \pi$.

Since $\omega(t)$ is monotonically increasing, Lemma 2.1 (i) implies

$$\begin{aligned}\mathbb{P}_1 &= O(1)\frac{1}{(1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q}\int_0^{[\xi]_q}\omega(t)e^{-\frac{|t|}{[\xi]_q}}dt \\ &= O(1)\frac{\omega([\xi]_q)}{(1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q}\int_0^{\xi}e^{-\frac{|t|}{[\xi]_q}}dt \\ &= O(1)\frac{\omega([\xi]_q)}{(1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q}\cdot\frac{e-1}{e}[\xi]_q \\ &= O(\omega([\xi]_q)). \end{aligned} \tag{3.5}$$

Moreover, using the condition that $\omega(t)/t$ is nonincreasing and integrated by parts, we also obtain

$$\begin{aligned}\mathbb{P}_2 =&O(1)\frac{1}{[\xi]_q}\int_{[\xi]_q}^{\pi}\omega(t)e^{-\frac{|t|}{[\xi]_q}}dt \\ =&O(1)\frac{1}{[\xi]_q}\int_{[\xi]_q}^{\pi}\frac{\omega(t)}{t}te^{-\frac{|t|}{[\xi]_q}}dt \\ =&O(1)\frac{\omega([\xi]_q)}{[\xi]_q^2}\int_{[\xi]_q}^{\pi}te^{-\frac{|t|}{[\xi]_q}}dt \\ =&O(1)\frac{\omega([\xi]_q)}{[\xi]_q^2}[\xi]_q\left(2\xi e^{-1}-e^{-\frac{\pi}{[\xi]_q}}([\xi]_q+\pi)\right) \\ =&O(\omega([\xi]_q)). \end{aligned} \tag{3.6}$$

Consequently, (3.4) along with (3.5) and (3.6) imply (3.1).

By (1.2) and some appropriate operations, we arrive at

$$Q_{\xi;q}(f;x)-f(x)=\frac{2[\xi]_q}{\pi}\int_0^{\pi}\frac{\varphi_x(t)}{t^2+[\xi]_q^2}dt-f(x)L([\xi]_q), \tag{3.7}$$

where

$$L([\xi]_q):=-\frac{2[\xi]_q}{\pi}\int_0^{\pi}\frac{dt}{t^2+[\xi]_q^2}+1.$$

The L'hospital's rule gives us

$$\lim_{[\xi]_q\to0+}\frac{L([\xi]_q)}{[\xi]_q}=\lim_{[\xi]_q\to0+}\left(\frac{1}{[\xi]_q}-\frac{2}{\pi}\frac{1}{[\xi]_q}\arctan\left(\frac{\pi}{[\xi]_q}\right)\right)=\frac{2}{\pi^2}$$

and therefore $(0<[\xi]_q\le\pi)$

$$\begin{aligned}L([\xi]_q) =&O([\xi]_q)=O\left(\frac{[\xi]_q}{\omega([\xi]_q)}\omega([\xi]_q)\right) \\ =&O\left(\frac{\pi}{\omega(\pi)}\omega([\xi]_q)\right)=O\left(\omega([\xi]_q)\right),\end{aligned} \tag{3.8}$$

since $t/\omega(t)$ is an increasing function of t.

Now, we need to estimate from upper the integral appearing in (3.7). Indeed, first we write

$$
\begin{aligned}
q :=& \frac{2[\xi]_q}{\pi}\int_0^{\pi}\frac{\varphi_x(t)}{t^2+[\xi]_q^2}dt\\
=& \frac{2[\xi]_q}{\pi}\Bigg(\underbrace{\int_0^{[\xi]_q}\frac{\varphi_x(t)}{t^2+[\xi]_q^2}dt}_{:=Q_1}+\underbrace{\int_{[\xi]_q}^{\pi}\frac{\varphi_x(t)}{t^2+[\xi]_q^2}dt}_{:=Q_2}\Bigg),
\end{aligned}
$$

then, using Lemma 2.1 and $f\in H_p^{(\omega)}$, we obtain

$$
\begin{aligned}
\|Q_1\|_p =& O(1)\int_0^{[\xi]_q}\frac{\|\varphi_{\cdot}(t)\|_p}{t^2+[\xi]_q^2}dt = O(1)\int_0^{[\xi]_q}\frac{\omega(t)}{t^2+[\xi]_q^2}dt\\
=& O(\omega([\xi]_q))\int_0^{[\xi]_q}\frac{dt}{t^2+[\xi]_q^2} = O(\omega([\xi]_q)/[\xi]_q).
\end{aligned}
$$

Once more, using Lemma 2.1, $f\in H_p^{(\omega)}$ and the assumption, we get

$$
\begin{aligned}
\|Q_2\|_p =& O(1)\int_{[\xi]_q}^{\pi}\frac{\|\varphi_{\cdot}(t)\|_p}{t^2+[\xi]_q^2}dt = O(1)\int_{[\xi]_q}^{\pi}\frac{\omega(t)t}{t(t^2+[\xi]_q^2)}dt\\
=& O\left(\omega([\xi]_q)/[\xi]_q\right)\int_{[\xi]_q}^{\pi}\frac{tdt}{t^2+[\xi]_q^2} = O(\omega([\xi]_q)/[\xi]_q)\ln\frac{\pi^2+[\xi]_q^2}{2[\xi]_q^2}.
\end{aligned}
$$

Besides that, the limit

$$
\lim_{[\xi]_q\to 0+}\frac{\ln\frac{\pi^2+[\xi]_q^2}{2[\xi]_q^2}}{\ln\left(\frac{1}{[\xi]_q}\right)} = \lim_{[\xi]_q\to 0+}\frac{2\pi^2}{\pi^2+[\xi]_q^2} = 2
$$

implies

$$
\|Q_2\|_p = O\left((\omega([\xi]_q)/[\xi]_q)|\ln(1/[\xi]_q)|\right).
$$

Thus,

$$
\|q\|_p = O([\xi]_q)\left(\|Q_1\|_p+\|Q_2\|_p\right) = O\left(\omega([\xi]_q)|\ln(1/[\xi]_q)|\right). \tag{3.9}
$$

Consequently, (3.7)–(3.9) imply

$$
\|Q_{\xi;q}(f;\cdot)-f\|_p \le \|q\|_p + \|f\|_p|L([\xi]_q)| = O\left(\omega([\xi]_q)\ln\left|1/[\xi]_q\right|\right),
$$

which is (3.2).

When all was said and done, we can write

$$
W_{\xi;q}(f;x) = \frac{2}{\sqrt{\pi[\xi]_q}}\int_0^{\pi}\varphi_x(t)e^{-\frac{t^2}{[\xi]_q}}dt + \frac{2f(x)}{\sqrt{\pi[\xi]_q}}\int_0^{\pi}e^{-\frac{t^2}{[\xi]_q}}dt,
$$

and taking into account that

$$
\int_{-\infty}^{\infty}e^{-\frac{t^2}{[\xi]_q}}dt = \sqrt{\pi[\xi]_q}\quad ([\xi]_q>0), \tag{3.10}
$$

we get

$$\begin{aligned} &W_{\xi;q}(f;x) - f(x) \\ &= \underbrace{\frac{2}{\sqrt{\pi[\xi]_q}} \int_0^{\pi} \varphi_x(t) e^{-\frac{t^2}{[\xi]_q}} dt}_{:=\mathcal{W}_1} - \underbrace{\frac{1}{\sqrt{\pi[\xi]_q}} \left(\int_{-\infty}^{\infty} - \int_{-\pi}^{\pi} \right) f(x) e^{-\frac{t^2}{[\xi]_q}} dt}_{:=\mathcal{W}_2} . \end{aligned} \tag{3.11}$$

Using Lemma 2.1 and $f \in H_p^{(\omega)}$, we have

$$\begin{aligned} \|\mathcal{W}_1\|_p =& O\left(\frac{1}{\sqrt{[\xi]_q}}\right) \int_0^{\pi} \|\varphi_{\cdot}(t)\|_p e^{-\frac{t^2}{[\xi]_q}} dt \\ =& O\left(\frac{1}{\sqrt{[\xi]_q}}\right) \Bigg(\underbrace{\int_0^{[\xi]_q} \omega(t) e^{-\frac{t^2}{[\xi]_q}} dt}_{:=\mathcal{W}_{11}} + \underbrace{\int_{[\xi]_q}^{\pi} \omega(t) e^{-\frac{t^2}{[\xi]_q}} dt}_{:=\mathcal{W}_{12}} \Bigg). \end{aligned} \tag{3.12}$$

For $\mathcal{W}_{11}$ (using (3.10)), we have

$$\begin{aligned} \mathcal{W}_{11} &\leq \omega([\xi]_q) \int_0^{[\xi]_q} e^{-\frac{t^2}{[\xi]_q}} dt \\ &\leq \omega([\xi]_q) \int_0^{\infty} e^{-\frac{t^2}{[\xi]_q}} dt = \frac{\sqrt{\pi}}{2} [\xi]_q^{\frac{1}{2}} \omega([\xi]_q) \leq \frac{\pi}{2} \omega([\xi]_q), \end{aligned} \tag{3.13}$$

while for $\mathcal{W}_{12}$, we obtain

$$\begin{aligned} \mathcal{W}_{12} &\leq \omega([\xi]_q) \int_{[\xi]_q}^{\pi} \frac{t}{[\xi]_q} e^{-\frac{t^2}{[\xi]_q}} dt \\ &\leq \omega([\xi]_q) \int_{[\xi]_q}^{\infty} \frac{t}{[\xi]_q} e^{-\frac{t^2}{[\xi]_q}} dt = \frac{1}{2e^{[\xi]_q}} \omega([\xi]_q) \leq \frac{1}{2} \omega([\xi]_q). \end{aligned} \tag{3.14}$$

Thus, from (3.12)–(3.14), we get

$$\|\mathcal{W}_1\|_p = O\left(\frac{1}{\sqrt{[\xi]_q}}\right) (\mathcal{W}_{11} + \mathcal{W}_{12}) = O\left(\frac{\omega([\xi]_q)}{\sqrt{[\xi]_q}}\right). \tag{3.15}$$

For $\mathcal{W}_2$, we can write

$$\mathcal{W}_2 = \frac{2f(x)}{\sqrt{\pi[\xi]_q}} \int_{\pi}^{\infty} e^{-\frac{t^2}{[\xi]_q}} dt$$

and, therefore, using the assumption that $t/\omega(t)$ is increasing with respect to t, we find that

$$\begin{aligned} \|\mathcal{W}_2\|_p &\leq \frac{2\|f\|_p}{\pi\sqrt{\pi[\xi]_q}} \int_{\pi}^{\infty} t e^{-\frac{t^2}{[\xi]_q}} dt = \frac{\|f\|_p}{\pi\sqrt{\pi[\xi]_q}} \cdot \frac{[\xi]_q}{e^{\frac{\pi^2}{[\xi]_q}}} \cdot \frac{\omega([\xi]_q)}{[\xi]_q} \cdot \frac{[\xi]_q}{\omega([\xi]_q)} \\ &\leq \frac{\|f\|_p}{\omega(\pi)\sqrt{\pi} e^{\frac{\pi^2}{[\xi]_q}}} \cdot \frac{\omega([\xi]_q)}{\sqrt{[\xi]_q}} = O\left(\frac{\omega([\xi]_q)}{\sqrt{[\xi]_q}}\right) \quad \text{for} \quad 0 < [\xi]_q < \pi. \end{aligned} \tag{3.16}$$

Combining (3.11), (3.15) and (3.16), we obtain

$$\|W_{\xi;q}(f;\cdot)-f\|_p \le \|\mathcal{W}_1\|_p + \|\mathcal{W}_2\|_p = O\left(\frac{\omega([\xi]_q)}{\sqrt{[\xi]_q}}\right),$$

which is (3.3).

The proof is completed. □

Remark 3.1. *The assumption in Theorem 3.1,* $q_\xi \in (0,1)$ *such that* $q_\xi \to 1$ *as* $\xi \to 0+$ *is significant to the rate of convergence in the metric* $\|\cdot\|_p$*. Otherwise, if* $q \in (0,1)$*, then*

$$\lim_{\xi\to 0}[\xi]_q = \frac{1}{1-q} \neq 0.$$

To overcome this inconvenience, for example, let us choose q_ξ *so that*

$$\frac{1}{2} \le 1-\xi \le q_\xi < 1$$

for some $\xi > 0$*, then*

$$[\xi]_{q_\xi} = \frac{1-q_\xi^\xi}{1-q_\xi} \le 2(1-q_\xi) \le 2\xi,$$

which shows that $[\xi]_{q_\xi} \to 0$ *as* $\xi \to 0$*.*

If we replace the function $\omega(t)$ by t^α $(0 < \alpha \le 1)$ in definition of the $H_p^{(\omega)}$ space, then it reduces to the Banach space $H(\alpha, p)$ equipped with the norm $\|\cdot\|_{(\alpha,p)}$ (for more details, see [6], p. 140). In this case, the condition that $\omega(t)/t$ has to be a nonincreasing function of t is satisfied automatically. For this reason, Theorem 3.1 implies the following corollary.

Corollary 3.1. *Let* $q_\xi \in (0,1)$ *such that* $q_\xi \to 1$ *as* $\xi \to 0+$*. Moreover, let* $f \in H(\alpha, p)$ *with* $p \ge 1$ *and* $0 < \alpha \le 1$*, then*

$$\begin{aligned}
\|\overline{P}_{\xi;q_\xi}(f;\cdot)-f\|_{(\alpha,p)} &= O\left([\xi]_{q_\xi}^{\alpha}\right),\\
\|Q_{\xi;q_\xi}(f;\cdot)-f\|_{(\alpha,p)} &= O\left([\xi]_{q_\xi}^{\alpha}|\ln(1/[\xi]_{q_\xi})|\right),\\
\|W_{\xi;q_\xi}(f;\cdot)-f\|_{(\alpha,p)} &= O\left([\xi]_{q_\xi}^{\alpha-\frac{1}{2}}\right).
\end{aligned}$$

Note here that $H(\alpha,\infty)$ is the familiar H_α space introduced earlier by Prössdorf [26]. Therefore, from Theorem 3.1 ($q = 1$), we also derive the following.

Corollary 3.2. *[23] Let* $f \in H_\alpha$ *with* $0 < \alpha \le 1$*, then as* $\xi \to 0+$*,*

$$\begin{aligned}
\|Q_\xi(f;\cdot)-f\|_C &= O\left(\xi^\alpha|\ln(1/\xi)|\right),\\
\|W_\xi(f;\cdot)-f\|_C &= O\left(\xi^{\alpha-\frac{1}{2}}\right).
\end{aligned}$$

Remark 3.2. *Note that Theorem 1.1 (recalled in the introduction of this article) can be implied from Theorem 3.1 as a special case (except relation (3.14)).*

Now, we prove the homologous statement of Theorem 1.2.

Theorem 3.2. *Let $q_\xi \in (0,1)$ such that $q_\xi \to 1$ as $\xi \to 0+$. Moreover, let $f \in H_p^{(\omega)}$ with $p \geq 1$, $\frac{\omega(t)}{v(t)}$ be a nondecreasing function, $\omega(t)/t$ be a nonincreasing function and $h \in [[\xi]_{q_\xi}, \pi]$, then*

$$\|f - \overline{P}_{\xi;q_\xi}(f;\cdot)\|_p^{(v)} = O\left(\frac{\omega([\xi]_{q_\xi})}{v([\xi]_{q_\xi})}\right), \tag{3.17}$$

$$\|f - Q_{\xi;q_\xi}(f;\cdot)\|_p^{(v)} = O\left(\frac{\omega([\xi]_{q_\xi})}{v([\xi]_{q_\xi})}|\ln(1/[\xi]_{q_\xi})|\right), \tag{3.18}$$

$$\|f - W_{\xi;q_\xi}(f;\cdot)\|_p^{(v)} = O\left(\frac{\omega([\xi]_{q_\xi})}{v([\xi]_{q_\xi})\sqrt{[\xi]_{q_\xi}}}\right). \tag{3.19}$$

Proof. For simplicity (only in the proof), we write q instead of q_ξ. If we put

$$D_{\xi;q}(f;x) := \overline{P}_{\xi;q}(f;x) - f(x),$$

then we can write

$$\begin{aligned}
&D_{\xi;q}(f;x+h) - D_{\xi;q}(f;x)\\
&= \frac{1}{(1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q}\int_0^\pi [\varphi_{x+h}(t) - \varphi_x(t)]e^{-\frac{|t|}{[\xi]_q}}dt\\
&= \frac{1}{(1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q}\Bigg(\underbrace{\int_0^{[\xi]_q}[\varphi_{x+h}(t) - \varphi_x(t)]e^{-\frac{|t|}{[\xi]_q}}dt}_{:=\mathcal{D}_1}\\
&\quad + \underbrace{\int_{[\xi]_q}^\pi [\varphi_{x+h}(t) - \varphi_x(t)]e^{-\frac{|t|}{[\xi]_q}}dt}_{:=\mathcal{D}_2}\Bigg),
\end{aligned} \tag{3.20}$$

where $0 < [\xi]_q \leq h \leq \pi$.

Using Lemma 2.1 and Lemma 2.2 (iii), we find that

$$\begin{aligned}
\|\mathcal{D}_1\|_p =& O(1)\int_0^{[\xi]_q}\|\varphi_{\cdot+h}(t) - \varphi_\cdot(t)\|_p e^{-\frac{|t|}{[\xi]_q}}dt = O(v(|h|))\int_0^{[\xi]_q}\frac{\omega(t)}{v(t)}e^{-\frac{|t|}{[\xi]_q}}dt\\
=& O(v(|h|))\frac{\omega([\xi]_q)}{v([\xi]_q)}\int_0^\pi e^{-\frac{|t|}{[\xi]_q}}dt = O\Big((1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q v(|h|)\frac{\omega([\xi]_q)}{v([\xi]_q)}\Big).
\end{aligned} \tag{3.21}$$

Analogously, using Lemma 2.1, Lemma 2.2 (ii), and the assumption that $\omega(t)/t$ is a nonincreasing function of t, we have

$$\begin{aligned}
\|\mathcal{D}_2\|_p =& O(1)\int_{[\xi]_q}^\pi \|\varphi_{\cdot+h}(t) - \varphi_\cdot(t)\|_p e^{-\frac{|t|}{[\xi]_q}}dt = O(1)\int_{[\xi]_q}^\pi \frac{\omega(t)}{t}te^{-\frac{|t|}{[\xi]_q}}dt\\
=& O\Big(\frac{\omega([\xi]_q)}{[\xi]_q}\Big)\int_{[\xi]_q}^\pi te^{-\frac{|t|}{[\xi]_q}}dt = O\Big(\frac{\omega([\xi]_q)}{[\xi]_q}\Big)\pi\int_0^\pi e^{-\frac{|t|}{[\xi]_q}}dt\\
=& O\Big((1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q\frac{\omega([\xi]_q)}{v([\xi]_q)}v([\xi]_q)\Big) = O\Big((1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q v(|h|)\frac{\omega([\xi]_q)}{v([\xi]_q)}\Big)
\end{aligned} \tag{3.22}$$

for $0 < [\xi]_q \le h \le \pi$.

Using Minkowski's inequality in (3.20), and keeping in mind (3.21) and (3.22), we obtain

$$\begin{aligned}\frac{\|D_{\xi;q}(f;\cdot+h)-D_{\xi;q}(f;\cdot)\|_p}{v(|h|)} =& O\left(\frac{1}{(1-e^{-\frac{\pi}{[\xi]_q}})[\xi]_q}\right)\left(\|\mathcal{D}_1\|_p+\|\mathcal{D}_2\|_p\right)\\ =& O\left(\frac{\omega([\xi]_q)}{v([\xi]_q)}\right).\end{aligned} \tag{3.23}$$

Based on relation (3.1) of Theorem 3.1, we can write

$$\|\overline{P}_{\xi;q}(f;\cdot)-f\|_p = O\left(\omega([\xi]_q)\right) = O\left(\frac{\omega([\xi]_q)}{v([\xi]_q)}v([\xi]_q)\right) = O\left(\frac{\omega([\xi]_q)}{v([\xi]_q)}\right) \tag{3.24}$$

because of $v([\xi]_q) \le v(\pi)$.

Thus, using (3.23) and (3.24), we have

$$\|f-\overline{P}_{\xi;q}(f;\cdot)\|_p^{(v)} = O\left(\frac{\omega([\xi]_q)}{v([\xi]_q)}\right),$$

which is (3.17) as requested.

The proofs of (3.18) and (3.19) can be done in a similar way as (3.17). Therefore, we have intentionally skipped them.

The proof is completed. □

Let $\omega(t) = t^\alpha$, $v(t) = t^\beta$, $0 \le \beta < \alpha \le 1$. These conditions enable us (from Theorem 1.2) to extract the following.

Corollary 3.3. *Let $q_\xi \in (0,1)$ such that $q_\xi \to 1$ as $\xi \to 0+$. Moreover, let $f \in H(\alpha, p)$ with $p \ge 1$, $0 \le \beta < \alpha \le 1$ and $h \in [[\xi]_{q_\xi}, \pi]$, then*

$$\begin{aligned}\|f-\overline{P}_{\xi;q_\xi}(f;\cdot)\|_{(\beta,p)} &= O\left([\xi]_{q_\xi}^{\alpha-\beta}\right),\\ \|f-Q_{\xi;q_\xi}(f;\cdot)\|_{(\beta,p)} &= O\left([\xi]_{q_\xi}^{\alpha-\beta}|\ln(1/[\xi]_{q_\xi})|\right),\\ \|f-W_{\xi;q_\xi}(f;\cdot)\|_{(\beta,p)} &= O\left([\xi]_{q_\xi}^{\alpha-\beta-1/2}\right).\end{aligned}$$

Remark 3.3. *Let $\omega(t) = t^\alpha$, $v(t) = t^\beta$, $0 \le \beta < \alpha \le 1$ and $f \in H_\alpha$, then Theorem 1.2 can be implied by Theorem 3.2 (except relation (3.17)).*

Remark 3.4. *We note that, even the singular integrals $P_\xi(f;x)$ and $\overline{P}_{\xi;1}(f;x)$ differ in their limits of integration and they give the same order of approximation in various metrics.*

4. Conclusions

In this paper we have introduced the q-Picard, the q-Picard-Cauchy, the q-Gauss-Weierstrass, and the q-truncated Picard singular integrals in a simple form. The deviations, in the L^p-norm and in the generalized Hölder-norm, between these integrals and the functions from a generalized Hölder space has been obtained. We have demonstrated that the obtained degrees of approximation are of Jackson's order, with exception in the case when the q-Picard-Cauchy singular integral has been used. To achieve these degrees of approximation, we have shown that the values of the number $q > 0$ have to be restricted. These results, in the point of view for future research, open new perspectives for further generalizations.

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author has no conflict of interest to declare.

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