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Research article

On the complete moment convergence of moving average processes generated by negatively dependent random variables under sub-linear expectations

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Abstract: The moving average processes $X_k = \sum_{i=-\infty}^{\infty} a_{i+k} Y_i$ are studied, where $\{Y_i, -\infty < i < \infty\}$ is a double infinite sequence of negatively dependent random variables under sub-linear expectations, and $\{a_i, -\infty < i < \infty\}$ is an absolutely summable sequence of real numbers. We establish the complete moment convergence of a moving average process under proper conditions, extending the corresponding results in classic probability space to those in sub-linear expectation space.

Keywords: moving average process; complete moment convergence; weighted sums; negatively dependent random variables; sub-linear expectations
Mathematics Subject Classification: 60F05, 60F15

1. Introduction

Since Peng [1, 2] initiated the concept of the sub-linear expectations space to study the uncertainty in probability, many scholars try to investigate the limit theorems under sub-linear expectations. Zhang [3–5] studied the famous exponential inequalities, Rosenthal's inequalities, and Donsker's invariance principle under sub-linear expectations. Chen and Wu [6] investigated complete convergence theorems for a moving average process generated by independent random variables under sub-linear expectations. Under sub-linear expectations, Xu et al. [7], Xu and Kong [8] obtained complete convergence and complete moment convergence of weighted sums of negatively dependent random variables under sub-linear expectations. For more limit theorems under sub-linear expectations, the readers could refer to Zhang [9], Xu and Zhang [10, 11], Wu and Jiang [12], Zhang and Lin [13], Zhong and Wu [14], Hu et al. [15], Gao and Xu [16], Kuczmaszewska [17], Zhang [5], Chen [18], Zhang [19], Chen and Wu [20], Xu and Cheng [21, 22], Xu et al. [23], Xu [24, 25], and references therein.

Guo et al. [26] studied the complete moment convergence of moving average processes under negative association assumptions. For more results about complete moment convergence of moving average processes, the interested reader could refer to Hossenni and Nezakati [27] and references therein. Motivated by the work of Guo et al. [26], Chen and Wu [6], and Xu et al. [23], we try to prove complete moment convergence of moving average processes generated by negatively dependent random variables under sub-linear expectations, complementing the corresponding results obtained in Guo et al. [26]. The differences between the works of Xu et al. [7], Xu and Kong [8], Xu [24,25], and the results in this article are that under sub-linear expectations the comlete convergence of weighted sums of negatively dependent or extended negatively dependent random variables are studied in Xu et al. [7], Xu and Kong [8], and Xu [25], the complete convergence of moving average processes produced by negatively dependent random variables is studied in Xu [24], and the complete moment convergence of moving average processes generated by negatively dependent random variables is investigated here. The novelty here is that the results in this paper could imply those in Xu and Kong [8] and Xu [24] in some sense, and the results here extend the corresponding ones in probability space.

The rest of this paper is organized as follows. We present some necessary basic notions, concepts and corresponding properties, and give necessary lemmas under sublinear expectations in the next section. In Section 3, we present our results, Theorems 3.1–3.3, and the proofs of which are given in Section 4.

2. Preliminary

Hereafter, we use notions similar to that in the works by Peng [2], Zhang [4]. Assume that $(\Omega, \mathcal{F})$ is a given measurable space. Suppose that $\mathcal{H}$ is a set of all random variables on $(\Omega, \mathcal{F})$ fulfilling $\varphi(X_1, \cdots, X_n) \in \mathcal{H}$ for $X_1, \cdots, X_n \in \mathcal{H}$, and each $\varphi \in C_{l,Lip}(\mathbb{R}^n)$, where $C_{l,Lip}(\mathbb{R}^n)$ is the set of φ fulfilling

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq C(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)(|\mathbf{x} - \mathbf{y}|), \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

for $C > 0$, $m \in \mathbb{N}$ relying on φ.

Definition 2.1. *A sub-linear expectation $\mathbb{E}$ on $\mathcal{H}$ is a functional $\mathbb{E} : \mathcal{H} \mapsto \bar{\mathbb{R}} := [-\infty, \infty]$ fulfilling the following: for every $X, Y \in \mathcal{H}$,*

(a) $X \geq Y$ *implies* $\mathbb{E}[X] \geq \mathbb{E}[Y]$*;*

(b) $\mathbb{E}[c] = c$, $\forall c \in \mathbb{R}$*;*

(c) $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$, $\forall \lambda \geq 0$*;*

(d) $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$ *whenever* $\mathbb{E}[X] + \mathbb{E}[Y]$ *is not of the form* $\infty - \infty$ *or* $-\infty + \infty$.

Definition 2.2. *We say that $\{X_n; n \geq 1\}$ is stochastically dominated by a random variable X in $(\Omega, \mathcal{H}, \mathbb{E})$, if there exists a constant C such that $\forall n \geq 1$, for all non-negative $h \in C_{l,Lip}(\mathbb{R})$, $\mathbb{E}(h(X_n)) \leq C\mathbb{E}(h(X))$.*

$V : \mathcal{F} \mapsto [0, 1]$ is named to be a capacity if

(a) $V(\emptyset) = 0$, $V(\Omega) = 1$;

(b) $V(A) \le V(B)$, $A \subset B$, $A, B \in \mathcal{F}$.
Furthermore, if V is continuous, then V obeys

(c) $A_n \uparrow A$ yields $V(A_n) \uparrow V(A)$.

(d) $A_n \downarrow A$ yields $V(A_n) \downarrow V(A)$.

V is said to be sub-additive when $V(A \bigcup B) \le V(A) + V(B)$, $A, B \in \mathcal{F}$.

In $(\Omega, \mathcal{H}, \mathbb{E})$, set $\mathbb{V}(A) := \inf\{\mathbb{E}[\xi] : I_A \le \xi, \xi \in \mathcal{H}\}$, $\forall A \in \mathcal{F}$ (cf. Zhang [3]). $\mathbb{V}$ is a sub-additive capacity. Write

$$C_{\mathbb{V}}(X) := \int_0^{\infty} \mathbb{V}(X > x)\mathrm{d}x + \int_{-\infty}^{0} (\mathbb{V}(X > x) - 1)\mathrm{d}x.$$

As in 4.3 of Zhang [3], throughout this paper, define an extension of $\mathbb{E}$ on the space of all random variables by

$$\mathbb{E}^*(X) = \inf\{\mathbb{E}[Y] : X \le Y, Y \in \mathcal{H}\}.$$

Then $\mathbb{E}^*$ is a sublinear expectation on the space of all random variables, $\mathbb{E}[X] = \mathbb{E}^*[X]$, $\forall X \in \mathcal{H}$, and $\mathbb{V}(A) = \mathbb{E}^*(I_A)$, $\forall A \in \mathcal{F}$.

Suppose $\mathbf{X} = (X_1, \cdots, X_m)$, $X_i \in \mathcal{H}$ and $\mathbf{Y} = (Y_1, \cdots, Y_n)$, $Y_i \in \mathcal{H}$ are two random vectors on $(\Omega, \mathcal{H}, \mathbb{E})$. $\mathbf{Y}$ is named to be negatively dependent to $\mathbf{X}$, if for ψ_1 on $C_{l,Lip}(\mathbb{R}^m)$, ψ_2 on $C_{l,Lip}(\mathbb{R}^n)$, $\mathbb{E}[\psi_1(\mathbf{X})\psi_2(\mathbf{Y})] \le \mathbb{E}[\psi_1(\mathbf{X})]\mathbb{E}[\psi_2(\mathbf{Y})]$ whenever $\psi_1(\mathbf{X}) \ge 0$, $\mathbb{E}[\psi_2(\mathbf{Y})] \ge 0$, $\mathbb{E}[|\psi_1(\mathbf{X})\psi_2(\mathbf{Y})|] < \infty$, $\mathbb{E}[|\psi_1(\mathbf{X})|] < \infty$, $\mathbb{E}[|\psi_2(\mathbf{Y})|] < \infty$, and either ψ_1 and ψ_2 are coordinatewise nondecreasing or ψ_1 and ψ_2 are coordinatewise nonincreasing (see Definition 2.3 of Zhang [3], Definition 1.5 of Zhang [4]). $\{X_n\}_{n=-\infty}^{\infty}$ is said to be negatively dependent, if X_{n+l} is negatively dependent to $(X_l, X_{l+1}, \cdots, X_{l+n-1})$ for each $n \ge 1$, $-\infty < l < \infty$.

Suppose $\mathbf{X}_1$ and $\mathbf{X}_2$ are two n-dimensional random vectors in $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ respectively. They are said to be identically distributed if for every $\psi \in C_{l,Lip}(\mathbb{R}^n)$,

$$\mathbb{E}_1[\psi(\mathbf{X}_1)] = \mathbb{E}_2[\psi(\mathbf{X}_2)].$$

$\{X_n; n \ge 1\}$ is called to be identically distributed if for every $i \ge 1$, X_i and X_1 are identically distributed.

Throughout this paper, we suppose that $\mathbb{E}$ is countably sub-additive, i.e., $\mathbb{E}(X) \le \sum_{n=1}^{\infty} \mathbb{E}(X_n)$ could be implied by $X \le \sum_{n=1}^{\infty} X_n$, $X, X_n \in \mathcal{H}$, and $X \ge 0$, $X_n \ge 0$, $n = 1, 2, \ldots$. Therefore $\mathbb{E}^*$ is also countably sub-additive. Moreover $\mathbb{V}$ is also countably sub-additive (cf. Zhang [3]). Let C denote a positive constant which may change from line to line. $I(A)$ or I_A is the indicator function of A. The symbol $a_x \approx b_x$ means that there exists two positive constants C_1, C_2 fulfilling $C_1|b_x| \le |a_x| \le C_2|b_x|$, x^+ stands for $\max\{x, 0\}$, $x^- = (-x)^+$, for $x \in \mathbb{R}$, $a \bigvee b = \max\{a, b\}$, for $a, b \in \mathbb{R}$.

As in Zhang [4], if $X_1, X_2, \ldots, X_n$ are negatively dependent random variables and $f_1(x), f_2(x), \ldots, f_n(x) \in C_{l,Lip}(\mathbb{R})$ are all non increasing (or non decreasing) functions, then $f_1(X_1)$, $f_2(X_2), \ldots, f_n(X_n)$ are negatively dependent random variables.

We cite the following under sub-linear expectations.

Lemma 2.1. *(cf. Lemma 4.5 (iii) of Zhang [3]) If* $\mathbb{E}$ *is countably sub-additive under* $(\Omega, \mathcal{H}, \mathbb{E})$*, then for* $X \in \mathcal{H}$*,*

$$\mathbb{E}|X| \le C_{\mathbb{V}}(|X|).$$

Lemma 2.2. *(cf. Theorem 2.1 in Zhang [4]) Write $S_k = Y_1 + \cdots + Y_k$, $S_0 = 0$. Suppose that Y_{k+1} is negatively dependent to $(Y_1, \ldots, Y_k)$ for $k = 1, 2, \ldots, n-1$, or Y_k is negatively dependent to $(Y_{k+1}, \ldots, Y_n)$ for $k = 0, \ldots, n-1$ in sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Then for $p \geq 2$,*

$$\mathbb{E}\left[\max_{k\leq n} |S_k|^p\right] \leq C_p \left\{ \sum_{k=1}^{n} \mathbb{E}[|Y_k|^p] + \left(\sum_{k=1}^{n} \mathbb{E}[|Y_k|^2] \right)^{p/2} + \left(\sum_{k=1}^{n} [|\mathbb{E}(Y_k)| + |\mathbb{E}(-Y_k)|] \right)^p \right\}. \tag{2.1}$$

By Lemma 2.2 of Zhong and Wu [14], the following lemma holds.

Lemma 2.3. *Suppose $Y \in \mathcal{H}$, $r > 0$, $p > 0$, and $l(x)$ is a slowly varying function. (i) Then for any $c > 0$,*

$$C_{\mathbb{V}}\{|Y|^r l(|Y|^p)\} < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^{r/p-1} l(n) \mathbb{V}\left(|Y| > cn^{1/p}\right) < \infty.$$

(ii) Suppose $C_{\mathbb{V}}\{|Y|^r l(|Y|^p)\} < \infty$. Then for any $\theta > 1$ and $c > 0$,

$$\sum_{k=1}^{\infty} \theta^{kr/p} l(\theta^k) \mathbb{V}\left(|Y| > c\theta^{k/p}\right) < \infty.$$

3. Main results

Theorem 3.1. *Assume that $X_n = \sum_{i=-\infty}^{\infty} a_{i+n} Y_i$, $n \geq 1$, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers fulfilling $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, $\{Y_i, -\infty < i < \infty\}$ is a sequence of negatively dependent random variables, and $\{Y_i, -\infty < i < \infty\}$ is stochastically dominated by Y in sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Let $l(x)$ be a slowly varying function and $1 \leq p < 2$, $r \geq 1 + p/2$. Suppose that $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$ for all $-\infty < i < \infty$, and $C_{\mathbb{V}}(|Y|^r (1 \bigvee l(|Y|^p))) < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) C_{\mathbb{V}} \left\{ \left[\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_i \right|^{1/t} - \epsilon n^{1/(pt)} \right]^+ \right\} < \infty, \text{ for all } \epsilon > 0 \text{ and } t > \tfrac{1}{r}, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} n^{r/p-2} l(n) C_{\mathbb{V}} \left\{ \left[\sup_{k\geq n} \left| k^{-1/p} \sum_{i=1}^{k} X_i \right|^{1/t} - \epsilon \right]^+ \right\} < \infty, \text{ for all } \epsilon > 0 \text{ and } t > \tfrac{1}{r}. \tag{3.2}$$

Theorem 3.2. *Suppose that $X_n = \sum_{i=-\infty}^{\infty} a_{i+n} Y_i$, $n \geq 1$, where $\{a_i, -\infty < i < \infty\}$ is a sequence of real numbers fulfilling $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, $\{Y_i, -\infty < i < \infty\}$ is a sequence of negatively dependent random variables, and $\{Y_i, -\infty < i < \infty\}$ is stochastically dominated by Y in sub-linear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$. Let $l(x)$ be a non-decreasing and slowly varying function. Assume $1 \leq p < 2$, $r > 1 + p/2$. Suppose that $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$, for all $-\infty < i < \infty$ and $C_{\mathbb{V}}\left(|Y|^{1/t} (1 \bigvee l(|Y|^p))\right) < \infty$. Then*

$$\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) C_{\mathbb{V}} \left\{ \left[\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_i \right|^{1/t} - \epsilon n^{1/(pt)} \right]^+ \right\} < \infty, \text{ for all } \epsilon > 0 \text{ and } t < \tfrac{1}{r}, \tag{3.3}$$

and

$$\sum_{n=1}^{\infty} n^{r/p-2} l(n) C_{\mathbb{V}} \left\{ \left[\sup_{k\geq n} \left| k^{-1/p} \sum_{i=1}^{k} X_i \right|^{1/t} - \epsilon \right]^+ \right\} < \infty, \text{ for all } \epsilon > 0 \text{ and } t < \tfrac{1}{r}. \tag{3.4}$$

Theorem 3.3. *Assume that* $X_n = \sum_{i=-\infty}^{\infty} a_{i+n}Y_i$, $n \geq 1$, *where* $\{a_i, -\infty < i < \infty\}$ *is a sequence of real numbers fulfilling* $\sum_{i=-\infty}^{\infty} |a_i| < \infty$, $\{Y_i, -\infty < i < \infty\}$ *is a sequence of negatively dependent random variables, and* $\{Y_i, -\infty < i < \infty\}$ *is stochastically dominated by* Y *in sub-linear expectation space* $(\Omega, \mathcal{H}, \mathbb{E})$. *Assume that* $l(x)$ *is a slowly varying function and* $1 < p < 2$. *Suppose* $\mathbb{E}(Y_i) = \mathbb{E}(-Y_i) = 0$ *for* $-\infty < i < \infty$ *and* $C_{\mathbb{V}}(|Y|^p(1 \bigvee l(|Y|^p))) < \infty$. *Then*

$$\sum_{n=1}^{\infty} n^{-1-1/(pt)} l(n) C_{\mathbb{V}} \left\{ \left[\max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_i \right|^{1/t} - \epsilon n^{1/(pt)} \right]^{+} \right\} < \infty, \text{ for all } \epsilon > 0 \text{ and } t > \tfrac{1}{p}. \tag{3.5}$$

As in Remark 2.3 of Guo et al. [26] and Remark 1.2 of Li and Zhang [28], by Theorems 3.1, 3.2, we could obtain the following corollaries.

Corollary 3.1. *Under the assumptions of Theorem 3.1, and assume that* $C_{\mathbb{V}}(|Y|^r(1 \bigvee l(|Y|^p))) < \infty$. *Then*

$$\sum_{n=1}^{\infty} n^{r/p-2} l(n) \mathbb{V} \left\{ \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_i \right| > \epsilon n^{1/p} \right\} < \infty \text{ for } \epsilon > 0;$$

$$\sum_{n=1}^{\infty} n^{r/p-2} l(n) \mathbb{V} \left\{ \sup_{k\geq n} \left| k^{-1/p} \sum_{i=1}^{k} X_i \right| > \epsilon \right\} < \infty \text{ for } \epsilon > 0.$$

Corollary 3.2. *Under the assumptions of Theorem 3.3, and assume that* $C_{\mathbb{V}}(|Y|^p(1 \bigvee l(|Y|^p))) < \infty$. *Then*

$$\sum_{n=1}^{\infty} n^{-1} l(n) \mathbb{V} \left\{ \max_{1\leq k\leq n} \left| \sum_{i=1}^{k} X_i \right| > \epsilon n^{1/p} \right\} < \infty \text{ for } \epsilon > 0.$$

Remark 3.1. *In Theorems 3.1, 3.2, 3.3, Corollaries 3.1, 3.2, we all assume that* $\mathbb{E}(Y_j) = \mathbb{E}(-Y_j) = 0$, $\forall j \geq 1$. *Readers may wonder what the intrinsic difference between the sub-linear expectation and linear expectation in probability space is? The following example heuristically implies the diffenrence in some extent. Suppose that* Y_1 *is G-normally distributed, i.e., for* $a, b > 0$, $aY_1 + b\bar{Y}_1$ *and* $\sqrt{a^2 + b^2}Y_1$ *are identically distributed, where* $\bar{Y}_1$ *and* Y_1 *are independent and identically distributed (cf. Definition 2.2.8 and Remark 2.2.9 of Peng [2]). We know that* $\mathbb{E}(Y_1) = \mathbb{E}(-Y_1) = 0$ *(cf. Remark 2.2.5 of Peng [2]). Assume that* $\mathbb{E}(Y_1^2) = 1 > -\mathbb{E}(-Y_1^2) > 0$. *Then by the Remarks 3 and 14 of Hu [29], we know that* $\mathbb{E}(Y_1^{2n+1}) = \mathbb{E}(-Y_1^{2n+1}) > 0$, $\forall n \geq 1$. *Hence, for any* $n \geq 2$, $\mathbb{E}(Y_1^n) \neq -\mathbb{E}(-Y_1^n)$ *(cf. Proposition 2.2.15 of Peng [2]).*

4. Proofs of the main results

Hereafter, as in Chen and Wu [6], we define some useful functions. Assume that $2^{-1/p} < \mu < 1$, $g(y) \in C_{l,Lip}(\mathbb{R})$ is a decreasing function for $y \geq 0$, $0 \leq g(y) \leq 1$ for all y and $g(y) = 1$ if $|y| \leq \mu$, $g(y) = 0$ if $|y| > 1$. We see that

$$I(|y| \leq \mu) \leq g(|y|) \leq I(|y| \leq 1), \quad I(|y| > 1) \leq 1 - g(|y|) \leq I(|y| > \mu). \tag{4.1}$$

Define $g_j(y) \in C_{l,Lip}(\mathbb{R})$, $j \ge 1$ such that $0 \le g_j(y) \le 1$ for all y and $g_j\left(\frac{|y|}{2^{j/p}}\right) = 1$ if $2^{(j-1)/p} < |y| \le 2^{j/p}$, $g_j\left(\frac{|y|}{2^{j/p}}\right) = 0$ if $|y| \le \mu 2^{(j-1)/p}$ or $|y| > (1+\mu)2^{j/p}$. We see that

$$I\left(2^{(j-1)/p} < |Y| \le 2^{j/p}\right) \le g_j\left(\frac{|Y|}{2^{j/p}}\right) \le I\left(\mu 2^{(j-1)/p} < |Y| \le (1+\mu)2^{j/p}\right), \tag{4.2}$$

$$|Y|^\alpha g\left(\frac{|Y|}{2^{k/p}}\right) \le 1 + \sum_{j=1}^{k} |Y|^\alpha g_j\left(\frac{|Y|}{2^{j/p}}\right), \quad \forall \alpha > 0, \tag{4.3}$$

$$|Y|^\alpha \left(1 - g\left(\frac{|Y|}{2^{k/p}}\right)\right) \le \sum_{j=k}^{\infty} |Y|^\alpha g_j\left(\frac{|Y|}{2^{j/p}}\right), \quad \forall \alpha > 0. \tag{4.4}$$

Proof of Theorem 3.1. Here we adopt some ideas from the proofs of Theorem 2.1 in Guo et al. [26]. Write $Y_{xi}^{(1)} = Y_i I(|Y_i| < x) - xI(Y_i \le -x) + xI(Y_i \ge x)$, $Y_{xi}^{(2)} = Y_i - Y_{xi}^{(1)}$, $Y_x^{(1)} = YI(|Y| < x) - xI(Y \le -x) + xI(Y \ge x)$, $Y_x^{(2)} = Y - Y_x^{(1)}$ for any $x \ge 0$ and $-\infty < i < \infty$. Note that

$$\sum_{k=1}^{n} X_k = \sum_{k=1}^{n} \sum_{i=-\infty}^{\infty} a_{i+k} Y_i = \sum_{i=-\infty}^{\infty} a_i \sum_{k=1}^{n} Y_{i-k} = \sum_{i=-\infty}^{\infty} a_i \sum_{j=i-n}^{i-1} Y_j.$$

We see that

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) C_{\mathbb{V}} \left\{ \left[\max_{1\le k\le n} \left| \sum_{i=1}^{k} X_i \right|^{1/t} - \epsilon n^{1/(pt)} \right]^+ \right\} \\
&= \sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{\epsilon n^{1/(pt)}}^{\infty} \mathbb{V}\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} X_i \right| > x^t \right\} \mathrm{d}x \qquad (\text{letting } y = (x/\epsilon)^t) \\
&= \sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} \mathbb{V}\left\{ \max_{1\le k\le n} \left| \sum_{i=1}^{k} X_i \right| > \epsilon^t y \right\} \frac{\epsilon}{t} y^{\frac{1}{t}-1} \mathrm{d}y \\
&\le C \sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1} \mathbb{V}\left\{ \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i-k}^{i-1} Y_{xj}^{(2)} \right| \ge x \frac{\epsilon^t}{2} \right\} \mathrm{d}x \\
&\quad + C \sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1} \mathbb{V}\left\{ \max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i-k}^{i-1} Y_{xj}^{(1)} \right| \ge x \frac{\epsilon^t}{2} \right\} \mathrm{d}x \\
&:= I_1 + I_2.
\end{aligned} \tag{4.5}$$

For I_1, observe that $r/p - 1 - 1/(pt) > -1$ and $C_{\mathbb{V}}(|Y|^r l(|Y|^p)) < \infty$, by Lemmas 2.2 and 2.3, Markov inequality under sub-linear expectations, (4.1), (4.4), we get

$$\begin{aligned}
I_1 &\le C \sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-2} \mathbb{E}^*\left[\max_{1\le k\le n} \left| \sum_{i=-\infty}^{\infty} a_i \sum_{j=i-k}^{i-1} Y_{xj}^{(2)} \right| \right] \mathrm{d}x \\
&\le C \sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-2} \max_{-\infty<i<\infty} \mathbb{E}^*\left[\left| \sum_{j=i-n}^{i-1} |Y_j| \left(1 - g\left(\frac{|Y_j|}{x}\right)\right) \right| \right] \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-2} \max_{-\infty<i<\infty} \mathbb{E}\left[\left|\sum_{j=i-n}^{i-1} |Y_j|\left(1-g\left(\frac{|Y_j|}{x}\right)\right)\right|\right] \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} n^{r/p-1-1/(pt)} l(n) \sum_{k=n}^{\infty} \int_{k^{1/p}}^{(k+1)^{1/p}} x^{\frac{1}{t}-2} \mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{x}\right)\right)\right) \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} n^{r/p-1-1/(pt)} l(n) \sum_{k=n}^{\infty} k^{1/(pt)-1/p-1} \mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right) \\
&\leq C\sum_{k=1}^{\infty} k^{1/(pt)-1/p-1} \mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right) \sum_{n=1}^{k} n^{r/p-1-1/(pt)} l(n) \\
&\leq C\sum_{k=1}^{\infty} k^{r/p-1-1/p} l(k) \mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right) \\
&= C\sum_{n=0}^{\infty} \sum_{k=2^n}^{2^{n+1}-1} k^{r/p-1-1/p} l(k) \mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right) \\
&\leq C\sum_{n=1}^{\infty} 2^{n(r/p-1/p)} l(2^n) \mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{2^{n/p}}\right)\right)\right) \\
&\leq C\sum_{n=1}^{\infty} 2^{n(r/p-1/p)} l(2^n) \mathbb{E}^*\left(\sum_{j=n}^{\infty} |Y| g_j\left(\frac{|Y|}{2^{j/p}}\right)\right) \\
&\leq C\sum_{n=1}^{\infty} 2^{n(r/p-1/p)} l(2^n) \sum_{j=n}^{\infty} \mathbb{E}^*\left(|Y| g_j\left(\frac{|Y|}{2^{j/p}}\right)\right) \\
&= C\sum_{j=1}^{\infty} \mathbb{E}\left(|Y| g_j\left(\frac{|Y|}{2^{j/p}}\right)\right) \sum_{n=1}^{j} 2^{n(r/p-1/p)} l(2^n) \\
&\leq C\sum_{j=1}^{\infty} 2^{jr/p} l(2^j) \mathbb{V}\left\{|Y| > \mu 2^{(j-1)/p}\right\} < \infty. \qquad (4.6)
\end{aligned}
$$

Next we establish I_2. By Lemma 2.2, Markov's inequality under sub-linear expectations, Hölder inequality, we see that for $q \geq 2$,

$$
\begin{aligned}
I_2 &\leq C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1} x^{-q} \mathbb{E}^*\left[\max_{1\leq k\leq n}\left|\sum_{i=-\infty}^{\infty} a_i \sum_{j=i-k}^{i-1} Y_{xj}^{(1)}\right|^q\right] \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1-q} \mathbb{E}^*\left[\max_{1\leq k\leq n} \sum_{i=-\infty}^{\infty} (|a_i|^{1-1/q})(|a_i|^{1/q}) \left|\sum_{j=i-k}^{i-1} Y_{xj}^{(1)}\right|^q\right] \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1-q} \left(\sum_{i=-\infty}^{\infty} |a_i|\right)^{q-1} \left(\sum_{i=-\infty}^{\infty} |a_i| \mathbb{E}^*\left(\max_{1\leq k\leq n}\left|\sum_{j=i-k}^{i-1} Y_{xj}^{(1)}\right|^q\right)\right) \mathrm{d}x \\
&\leq C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)} l(n) \int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1-q} \max_{-\infty<i<\infty} \mathbb{E}\left(\max_{1\leq k\leq n}\left|\sum_{j=i-k}^{i-1} Y_{xj}^{(1)}\right|^q\right) \mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1-q}\max_{-\infty<i<\infty}\left(\sum_{j=i-n}^{i-1}\mathbb{E}\left|Y_{xj}^{(1)}\right|^{q}\right)\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1-q}\max_{-\infty<i<\infty}\left(\sum_{j=i-n}^{i-1}\mathbb{E}\left(\left|Y_{xj}^{(1)}\right|^{2}\right)\right)^{q/2}\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1-q}\max_{-\infty<i<\infty}\left(\sum_{j=i-n}^{i-1}\left[\left|\mathbb{E}\left(Y_{xj}^{(1)}\right)\right|+\left|\mathbb{E}\left(-Y_{xj}^{(1)}\right)\right|\right]\right)^{q}\mathrm{d}x\\
&:= I_{21}+I_{22}+I_{23}.
\end{aligned}
$$

For I_{21}, take $q>\max\{r,2\}$, by Lemma 2.3, (4.1), (4.2) and (4.3), and $\forall x>0$, $f(\cdot):=|\cdot|^{q}I(|\cdot|\le x)+x^{q}I(|\cdot|>x)\in C_{l,Lip}(\mathbb{R})$, we see that

$$
\begin{aligned}
I_{21} &\le C\sum_{n=1}^{\infty} n^{r/p-2-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1-q}\left(n\mathbb{E}\left|Y_{x}^{(1)}\right|^{q}\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{r/p-1-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1-q}\left[x^{q}\mathbb{E}\left(1-g\left(\frac{|Y|}{x}\right)\right)+\mathbb{E}\left(|Y|^{q}g\left(\frac{\mu|Y|}{x}\right)\right)\right]\mathrm{d}x\\
&= C\sum_{n=1}^{\infty} n^{r/p-1-1/(pt)}l(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{\frac{1}{t}-1}\mathbb{E}\left(1-g\left(\frac{|Y|}{x}\right)\right)\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty} n^{r/p-1-1/(pt)}l(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{\frac{1}{t}-1-q}\mathbb{E}\left(|Y|^{q}g\left(\frac{\mu|Y|}{x}\right)\right)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty} n^{r/p-1-1/(pt)}l(n)\sum_{m=n}^{\infty} m^{\frac{1}{tp}-1}\mathbb{V}\left\{|Y|>\mu m^{1/p}\right\}\\
&\quad +C\sum_{n=1}^{\infty} n^{r/p-1-1/(pt)}l(n)\sum_{m=n}^{\infty} m^{\frac{1}{tp}-1-q/p}\mathbb{E}\left(|Y|^{q}g\left(\frac{\mu|Y|}{(m+1)^{1/p}}\right)\right)\\
&\le \sum_{m=1}^{\infty} m^{\frac{r}{p}-1}l(m)\mathbb{V}\left\{|Y|>\mu m^{1/p}\right\}\\
&\quad +C\sum_{m=1}^{\infty} m^{\frac{1}{tp}-1-q/p}\mathbb{E}\left(|Y|^{q}g\left(\frac{\mu|Y|}{(m+1)^{1/p}}\right)\right)\sum_{n=1}^{m} n^{r/p-1-1/(pt)}l(n)\\
&\le C\sum_{k=0}^{\infty}\sum_{m=2^{k}}^{2^{k+1}-1} m^{\frac{r}{p}-1-q/p}l(m)\mathbb{E}\left(|Y|^{q}g\left(\frac{\mu|Y|}{(m+1)^{1/p}}\right)\right)\\
&\le C\sum_{k=1}^{\infty} 2^{\frac{kr}{p}-kq/p}l(2^{k})\mathbb{E}\left(|Y|^{q}g\left(\frac{\mu|Y|}{2^{(k+1)/p}}\right)\right)\\
&\le C\sum_{k=1}^{\infty} 2^{\frac{kr}{p}-kq/p}l(2^{k})\mathbb{E}\left(1+\sum_{j=1}^{k}|Y|^{q}g_{j}\left(\frac{\mu|Y|}{2^{(j+1)/p}}\right)\right)\\
&\le C\sum_{k=1}^{\infty} 2^{k(r/p-q/p)}l(2^{k})+C\sum_{k=1}^{\infty} 2^{\frac{kr}{p}-kq/p}l(2^{k})\sum_{j=1}^{k}\mathbb{E}\left(|Y|^{q}g_{j}\left(\frac{\mu|Y|}{2^{(j+1)/p}}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\le C\sum_{j=1}^{\infty}2^{jq/p}\mathbb{V}\left\{|Y|>2^{j/p}\right\}\sum_{k=j}^{\infty}2^{k(r/p-q/p)}l(2^k)\\
&\le C\sum_{j=1}^{\infty}2^{jr/p}l(2^j)\mathbb{V}\left\{|Y|>2^{j/p}\right\}<\infty. \qquad (4.7)
\end{aligned}
$$

For I_{22}, we study the following two cases. If $r\le 2$, we take $q>2$. Note that $r/p-(r/p-1)q/2<1$ and $r/p-2-1/(pt)+q/2>-1$. We get

$$
\begin{aligned}
I_{22} &\le C\sum_{n=1}^{\infty}n^{r/p-2-1/(pt)+q/2}l(n)\int_{n^{1/p}}^{\infty}x^{\frac{1}{t}-1-q}\left(\mathbb{E}|Y_x^{(1)}|^2\right)^{q/2}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}n^{r/p-2-1/(pt)+q/2}l(n)\int_{n^{1/p}}^{\infty}x^{\frac{1}{t}-1-q}x^{(2-r)q/2}\left(\mathbb{E}|Y_x^{(1)}|^r\right)^{q/2}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}n^{r/p-(r/p-1)q/2-2}\left(\mathbb{E}|Y|^r\right)^{q/2}\\
&\le C\sum_{n=1}^{\infty}n^{r/p-(r/p-1)q/2-2}\left(C_{\mathbb{V}}\left(|Y|^r\right)\right)^{q/2}<\infty. \qquad (4.8)
\end{aligned}
$$

If $r>2$, we take $q>\max\left\{2p(r/p-1)/(2-p),t^{-1}\right\}$, then $r/p-q/p+q/2<1$. Note that $\mathbb{E}(Y^2)<C_{\mathbb{V}}(Y^2)\le CC_{\mathbb{V}}\left(|Y|^r l(|Y|^p)\right)<\infty$ in this case. Therefore, we get

$$
\begin{aligned}
I_{22} &\le C\sum_{n=1}^{\infty}n^{r/p-2-1/(pt)+q/2}l(n)\int_{n^{1/p}}^{\infty}x^{\frac{1}{t}-1-q}\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}n^{r/p-2-q/p+q/2}l(n)<\infty. \qquad (4.9)
\end{aligned}
$$

Combining (4.8) and (4.9) results in $I_{22}<\infty$.

For I_{23}, we take $q>2$. Observe that $r\ge 1+p/2>p$. By $\mathbb{E}(Y_i)=\mathbb{E}(-Y_i)=0$, Proposition 1.3.7 of Peng (2019), and Lemma 2.1, we see that

$$
\begin{aligned}
I_{23} &\le C\sum_{n=1}^{\infty}n^{r/p-2-1/(pt)}l(n)\sum_{k=n}^{\infty}\int_{k^{1/p}}^{(k+1)^{1/p}}x^{\frac{1}{t}-1-q}\max_{-\infty<i<\infty}\left(\sum_{j=i-n}^{i-1}\left[\mathbb{E}\left|Y_{xj}^{(1)}-Y_j\right|+\mathbb{E}\left|-Y_{xj}^{(1)}+Y_j\right|\right]\right)^q\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}n^{r/p-2-1/(pt)}l(n)\sum_{k=n}^{\infty}\int_{k^{1/p}}^{(k+1)^{1/p}}x^{\frac{1}{t}-1-q}\max_{-\infty<i<\infty}\left(\sum_{j=i-n}^{i-1}\mathbb{E}\left|Y_{xj}^{(1)}-Y_j\right|\right)^q\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}n^{r/p-2-1/(pt)+q}l(n)\sum_{k=n}^{\infty}\int_{k^{1/p}}^{(k+1)^{1/p}}x^{\frac{1}{t}-1-q}\left(\mathbb{E}|Y|\left(1-g\left(\frac{|Y|}{x}\right)\right)\right)^q\mathrm{d}x\\
&\le C\sum_{k=1}^{\infty}k^{\frac{1}{tp}-1-q/p}\left(\mathbb{E}|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right)^q\sum_{n=1}^{k}n^{r/p-2-1/(pt)+q}l(n)\\
&\le C\sum_{k=1}^{\infty}k^{1/(pt)-1-q/p}\left(\mathbb{E}|Y|^r l(|Y|^p)/\left(k^{(r-1)/p}l(k)\right)\right)^q k^{r/p-1-1/(pt)+q}l(k)
\end{aligned}
$$

$$\leq \quad C\sum_{k=1}^{\infty} k^{-(r/p-1)(q-1)-1}/l(k)^{q-1}\left(C_{\mathbb{V}}\{|Y|^r l(|Y|^p)\}\right)^q < \infty. \tag{4.10}$$

Hence, by (4.5) and (4.6)–(4.10), we establish (3.1).

Now we prove (3.2). By $r/p > 1$ and the countable sub-additivity of $\mathbb{V}$, we obtain

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r/p-2} l(n) C_{\mathbb{V}}\left\{\left[\sup_{k\geq n}\left|k^{-1/p}\sum_{i=1}^{k} X_i\right|^{1/t} - \epsilon\right]^+\right\}\\
&= \sum_{n=1}^{\infty} n^{r/p-2} l(n)\int_{\epsilon}^{\infty}\mathbb{V}\left\{\sup_{k\geq n}\left|k^{-1/p}\sum_{i=1}^{k} X_i\right|^{1/t} > x\right\}\mathrm{d}x\\
&= \sum_{j=0}^{\infty}\sum_{n=2^j}^{2^{j+1}-1} n^{r/p-2} l(n)\int_{\epsilon}^{\infty}\mathbb{V}\left\{\sup_{k\geq n}\left|k^{-1/p}\sum_{i=1}^{k} X_i\right|^{1/t} > x\right\}\mathrm{d}x\\
&\leq C\sum_{j=0}^{\infty} 2^{j(r/p-1)} l(2^j)\int_{\epsilon}^{\infty}\mathbb{V}\left\{\sup_{k\geq 2^j}\left|k^{-1/p}\sum_{i=1}^{k} X_i\right|^{1/t} > x\right\}\mathrm{d}x\\
&\leq C\sum_{j=0}^{\infty} 2^{j(r/p-1)} l(2^j)\sum_{\ell=j}^{\infty}\int_{\epsilon}^{\infty}\mathbb{V}\left\{\sup_{2^\ell\leq k\leq 2^{\ell+1}}\left|\sum_{i=1}^{k} X_i\right|^{1/t} > x2^{\ell/(pt)}\right\}\mathrm{d}x\\
&\leq C\sum_{\ell=0}^{\infty} 2^{\ell(r/p-1)} l(2^\ell)\int_{\epsilon}^{\infty}\mathbb{V}\left\{\sup_{2^\ell\leq k\leq 2^{\ell+1}}\left|\sum_{i=1}^{k} X_i\right|^{1/t} > x2^{\ell/(pt)}\right\}\mathrm{d}x\\
&\leq C\sum_{n=0}^{\infty} n^{r/p-2} l(n)\int_{\epsilon'}^{\infty}\mathbb{V}\left\{\sup_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^{1/t} > xn^{1/(pt)}\right\}\mathrm{d}x \qquad (\text{letting } \epsilon' = \epsilon 2^{-1/(pt)})\\
&\leq C\sum_{n=0}^{\infty} n^{r/p-2-1/(pt)} l(n)\int_{\epsilon' n^{1/(pt)}}^{\infty}\mathbb{V}\left\{\sup_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^{1/t} > x\right\}\mathrm{d}x\\
&\leq C\sum_{n=0}^{\infty} n^{r/p-2-1/(pt)} l(n) C_{\mathbb{V}}\left\{\left[\max_{1\leq k\leq n}\left|\sum_{i=1}^{k} X_i\right|^{1/t} - \epsilon' n^{1/(pt)}\right]^+\right\} < \infty.
\end{aligned} \tag{4.11}$$

Hence (3.2) is proved. □

Proof of Theorem 3.2. As in the proof of Theorem 3.1, it is sufficient to prove that $I_1 < \infty$, $I_{21} < \infty$, $I_{22} < \infty$, $I_{23} < \infty$. Indeed, observe that $r/p - 1 - 1/(pt) < -1$ yields $\sum_{n=1}^{\infty} n^{r/p-1-1/(pt)} < \infty$. Therefore, by the proofs of (4.6) and (4.4), and Lemma 2.3, we get

$$\begin{aligned}
I_1 \quad &\leq \quad C\sum_{k=1}^{\infty} k^{1/(pt)-1/p-1}\mathbb{E}\left[|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right]\sum_{n=1}^{k} n^{r/p-1-1/(pt)} l(n)\\
&\leq \quad C\sum_{k=1}^{\infty} k^{1/(pt)-1/p-1}\mathbb{E}\left[|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right]
\end{aligned}$$

$$
\begin{aligned}
&= C\sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}-1} k^{1/(pt)-1/p-1}\mathbb{E}\left[|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right]\\
&\le C\sum_{n=1}^{\infty} 2^{n(1/(pt)-1/p)}\mathbb{E}\left[|Y|\left(1-g\left(\frac{|Y|}{2^{n/p}}\right)\right)\right]\\
&\le C\sum_{n=1}^{\infty} 2^{n(1/(pt)-1/p)}\mathbb{E}^*\left[\sum_{j=n}^{\infty}|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\right] \le C\sum_{j=1}^{\infty}\mathbb{E}^*\left[|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\right]\sum_{n=1}^{j}2^{n(1/(pt)-1/p)}\\
&\le C\sum_{j=1}^{\infty}2^{j/(pt)}\mathbb{E}\left[g_j\left(\frac{|Y|}{2^{j/p}}\right)\right] \le C\sum_{j=1}^{\infty}2^{j/(pt)}\mathbb{V}\left\{|Y|>\mu 2^{(j-1)/p}\right\}<\infty.
\end{aligned}
$$

For I_{22}, I_{23}, we take $q > \max\{t^{-1}, 2(r-p)/(2-p), 2+2(1/t-r)/p\}$. By the proofs of (4.8), (4.9) and (4.10), we can obtain $I_{22} < \infty$, $I_{23} < \infty$.

For I_{21}, take $q > \max\{2, t^{-1}\}$, by the proof of (4.7), and (4.3), we see that

$$
\begin{aligned}
I_{21} &\le C\sum_{m=1}^{\infty} m^{1/(pt)-q/p-1}\mathbb{E}\left[|Y|^q g\left(\frac{\mu|Y|}{(m+1)^{1/p}}\right)\right]\sum_{n=1}^{m} n^{r/p-1-1/(pt)}l(n)\\
&\le C\sum_{k=0}^{\infty}\sum_{m=2^k}^{2^{k+1}-1} m^{1/(pt)-q/p-1}\mathbb{E}\left[|Y|^q g\left(\frac{\mu|Y|}{(m+1)^{1/p}}\right)\right]\\
&\le C\sum_{k=1}^{\infty} 2^{k(1/(pt)-q/p)}\mathbb{E}\left[|Y|^q g\left(\frac{\mu|Y|}{2^{(k+1)/p}}\right)\right]\\
&\le C\sum_{k=1}^{\infty} 2^{k(1/(pt)-q/p)}\mathbb{E}\left[1+\sum_{j=1}^{k}|Y|^q g_j\left(\frac{\mu|Y|}{2^{(j+1)/p}}\right)\right]\\
&\le C\sum_{k=1}^{\infty} 2^{k(1/(pt)-q/p)} + C\sum_{k=1}^{\infty} 2^{k(1/(pt)-q/p)}\sum_{j=1}^{k}\mathbb{E}\left[|Y|^q g_j\left(\frac{\mu|Y|}{2^{(j+1)/p}}\right)\right]\\
&\le C\sum_{j=1}^{\infty} 2^{jq/p}\mathbb{V}\left\{|Y|>2^{j/p}\right\}\sum_{k=j}^{\infty}2^{k(1/(pt)-q/p)}\\
&\le C\sum_{j=1}^{\infty} 2^{j/(tp)}\mathbb{V}\left\{|Y|>2^{j/p}\right\}<\infty.
\end{aligned}
$$

□

Proof of Theorem 3.3. By the proof of (4.5), we get

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{-1-1/(pt)}l(n)C_{\mathbb{V}}\left\{\left[\max_{1\le k\le n}\left|\sum_{i=1}^{k}X_i\right|^{1/t}-\epsilon n^{1/(pt)}\right]^+\right\}\\
&\le C\sum_{n=1}^{\infty} n^{-1-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-1}\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i-k}^{i-1}Y_{xj}^{(2)}\right|\ge x\frac{\epsilon^t}{2}\right\}\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&+C\sum_{n=1}^{\infty}n^{-1-1/(pt)}l(n)\int_{n^{1/p}}^{\infty}x^{\frac{1}{t}-1}\mathbb{V}\left\{\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i-k}^{i-1}Y_{xj}^{(1)}\right|\ge x\frac{\epsilon^t}{2}\right\}\mathrm{d}x\\
&:= J_1+J_2. \qquad (4.12)
\end{aligned}
$$

Observe that $pt>1$ and $C_{\mathbb{V}}\left(|Y|^{1/t}l(|Y|^p)\right)<\infty$, by Markov's inequality under sub-linear expectations and Lemmas 2.2, 2.3, (4.4), we have

$$
\begin{aligned}
J_1 &\le C\sum_{n=1}^{\infty}n^{-1-1/(pt)}l(n)\int_{n^{1/p}}^{\infty}x^{\frac{1}{t}-2}\mathbb{E}^*\max_{1\le k\le n}\left|\sum_{i=-\infty}^{\infty}a_i\sum_{j=i-k}^{i-1}Y_{xj}^{(2)}\right|\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}n^{-1/(pt)}l(n)\sum_{k=n}^{\infty}\int_{k^{1/p}}^{(k+1)^{1/p}}x^{\frac{1}{t}-2}\mathbb{E}(|Y_x^{(2)}|)\mathrm{d}x\\
&\le C\sum_{n=1}^{\infty}n^{-1/(pt)}l(n)\sum_{k=n}^{\infty}\int_{k^{1/p}}^{(k+1)^{1/p}}x^{\frac{1}{t}-2}\mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{x}\right)\right)\right)\mathrm{d}x\\
&\le C\sum_{k=1}^{\infty}k^{1/(tp)-1/p-1}\mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right)\sum_{n=1}^{k}n^{-1/(pt)}l(n)\\
&\le C\sum_{k=1}^{\infty}k^{-1/p}l(k)\mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right)\\
&= C\sum_{n=0}^{\infty}\sum_{k=2^n}^{2^{n+1}-1}k^{-1/p}l(k)\mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right)\\
&\le C\sum_{n=1}^{\infty}2^{(1-1/p)n}l(2^n)\mathbb{E}\left(|Y|\left(1-g\left(\frac{|Y|}{2^{n/p}}\right)\right)\right)\\
&\le C\sum_{n=1}^{\infty}2^{(1-1/p)n}l(2^n)\mathbb{E}^*\left(\sum_{j=n}^{\infty}|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\right)\\
&\le C\sum_{j=1}^{\infty}\mathbb{E}^*\left(|Y|g_j\left(\frac{|Y|}{2^{j/p}}\right)\right)\sum_{n=1}^{j}2^{(1-1/p)n}l(2^n)\\
&\le C\sum_{j=1}^{\infty}2^jl(2^j)\mathbb{V}\left\{|Y|>\mu 2^{-1/p}2^{j/p}\right\}<\infty. \qquad (4.13)
\end{aligned}
$$

For J_2, as in the proof of I_2, choose $q=2$, by (2.1), we get

$$
\begin{aligned}
J_2 &\le C\sum_{n=1}^{\infty}n^{-1-1/(pt)}l(n)\int_{n^{1/p}}^{\infty}x^{\frac{1}{t}-3}\max_{-\infty<i<\infty}\left(\sum_{j=i-n}^{i-1}\mathbb{E}(|Y_{xj}^{(1)}|^2)\right)\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty}n^{-1-1/(pt)}l(n)\int_{n^{1/p}}^{\infty}x^{\frac{1}{t}-3}\max_{-\infty<i<\infty}\left(\sum_{j=i-n}^{i-1}\left[|\mathbb{E}Y_{xj}^{(1)}|+|\mathbb{E}(-Y_{xj}^{(1)})|\right]\right)^2\mathrm{d}x\\
&= :J_{21}+J_{22}.
\end{aligned}
$$

By Lemma 2.3, (4.1), (4.3), we conclude that

$$
\begin{aligned}
J_{21} &= C\sum_{n=1}^{\infty} n^{-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-3}\mathbb{E}\left(|Y_x^{(1)}|^2\right)\mathrm{d}x\\
&= C\sum_{n=1}^{\infty} n^{-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-3}\left[x^2\mathbb{E}\left(1-g\left(\frac{|Y|}{x}\right)\right)+\mathbb{E}|Y|^2 g\left(\frac{\mu|Y|}{x}\right)\right]\mathrm{d}x\\
&= C\sum_{n=1}^{\infty} n^{-1/(pt)}l(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{\frac{1}{t}-1}\mathbb{E}\left(1-g\left(\frac{|Y|}{x}\right)\right)\mathrm{d}x\\
&\quad +C\sum_{n=1}^{\infty} n^{-1/(pt)}l(n)\sum_{m=n}^{\infty}\int_{m^{1/p}}^{(m+1)^{1/p}} x^{\frac{1}{t}-3}\mathbb{E}|Y|^2 g\left(\frac{\mu|Y|}{x}\right)\mathrm{d}x\\
&\leq C\sum_{m=1}^{\infty} m^{\frac{1}{tp}-1}\mathbb{E}\left(1-g\left(\frac{|Y|}{m^{1/p}}\right)\right)\sum_{n=1}^{m} n^{-1/(pt)}l(n)\\
&\quad +C\sum_{m=1}^{\infty} m^{\frac{1}{tp}-\frac{2}{p}-1}\mathbb{E}|Y|^2 g\left(\frac{\mu|Y|}{(m+1)^{1/p}}\right)\sum_{n=1}^{m} n^{-1/(pt)}l(n)\\
&\leq C\sum_{m=1}^{\infty} l(m)\mathbb{V}\left\{|Y|>\mu m^{1/p}\right\}+C\sum_{m=1}^{\infty} m^{-\frac{2}{p}}l(m)\mathbb{E}|Y|^2 g\left(\frac{\mu|Y|}{(m+1)^{1/p}}\right)\\
&= C\sum_{n=0}^{\infty}\sum_{m=2^n}^{2^{n+1}-1} m^{-\frac{2}{p}}l(m)\mathbb{E}|Y|^2 g\left(\frac{\mu|Y|}{(m+1)^{1/p}}\right)\\
&\leq C\sum_{n=1}^{\infty} 2^{(1-2/p)n}l(2^n)\mathbb{E}|Y|^2 g\left(\frac{\mu|Y|}{(2)^{(n+1)/p}}\right)\\
&\leq C\sum_{n=1}^{\infty} 2^{(1-2/p)n}l(2^n)\mathbb{E}\left[1+\sum_{j=1}^{n}|Y|^2 g_j\left(\frac{\mu|Y|}{(2)^{(j+1)/p}}\right)\right]\\
&\leq C\sum_{n=1}^{\infty} 2^{(1-2/p)n}l(2^n)+C\sum_{n=1}^{\infty} 2^{(1-2/p)n}l(2^n)\sum_{j=1}^{n}\mathbb{E}\left[|Y|^2 g_j\left(\frac{\mu|Y|}{(2)^{(j+1)/p}}\right)\right]\\
&\leq C\sum_{j=1}^{\infty} 2^{2j/p}\mathbb{V}\left\{|Y|>2^{j/p}\right\}\sum_{n=j}^{\infty} 2^{(1-2/p)n}l(2^n)\\
&\leq C\sum_{j=1}^{\infty} 2^{j}l(2^j)\mathbb{V}\left\{|Y|>2^{j/p}\right\}<\infty.
\end{aligned}
$$

By $\mathbb{E}(-Y_i)=\mathbb{E}(Y_i)=0$, Proposition 1.3.7 of Peng (2019), (4.1), and Lemma 2.1, we see that

$$
\begin{aligned}
J_{22} &\leq C\sum_{n=1}^{\infty} n^{-1-1/(pt)}l(n)\int_{n^{1/p}}^{\infty} x^{\frac{1}{t}-3}\left[n\mathbb{E}|Y|\left(1-g\left(\frac{|Y|}{x}\right)\right)\right]^2\mathrm{d}x\\
&= C\sum_{n=1}^{\infty} n^{1-1/(pt)}l(n)\sum_{k=n}^{\infty}\int_{k^{1/p}}^{(k+1)^{1/p}} x^{(1/t)-3}\left[\mathbb{E}|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right]^2\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&\leq C\sum_{k=1}^{\infty}k^{1-2/p}l(k)\left[\mathbb{E}|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right]^2\leq C\sum_{k=1}^{\infty}k^{1-2/p}l(k)\left[C_{\mathbb{V}}\left(|Y|\left(1-g\left(\frac{|Y|}{k^{1/p}}\right)\right)\right)\right]^2\\
&\leq C\sum_{k=1}^{\infty}k^{1-2/p}l(k)\left[C_{\mathbb{V}}\left(|Y|I\left(|Y|>\mu k^{1/p}\right)\right)\right]^2\\
&\leq C\sum_{k=1}^{\infty}k^{1-2/p}l(k)\left[\int_0^{\mu k^{1/p}}\mathbb{V}\left(|Y|>\mu k^{1/p}\right)\mathrm{d}y+\int_{\mu k^{1/p}}^{\infty}\mathbb{V}\left(|Y|>y\right)\mathrm{d}y\right]^2\\
&\leq C\sum_{k=1}^{\infty}kl(k)\left[\mathbb{V}\left\{|Y|>\mu k^{1/p}\right\}\right]^2+C\sum_{k=1}^{\infty}k^{1-2/p}l(k)\left[\int_{\mu k^{1/p}}^{\infty}\mathbb{V}\left\{|Y|>y\right\}\mathrm{d}y\right]^2\\
&\leq C\int_1^{\infty}xl(x)\mathbb{V}^2\left\{|Y|>\mu x^{1/p}\right\}\mathrm{d}x\\
&\quad +C\int_1^{\infty}x^{1-2/p}l(x)\mathrm{d}x\int_{\mu x^{1/p}}^{\infty}\mathbb{V}\left\{|Y|>y\right\}\mathrm{d}y\int_{\mu x^{1/p}}^{y}\mathbb{V}\left\{|Y|>z\right\}\mathrm{d}z\\
&\leq C\int_1^{\infty}\left(xl(x)\mathbb{V}\left\{|Y|^pl(|Y|^p)>Cxl(x)\right\}\right)\mathbb{V}\left\{|Y|^p>Cx\right\}\mathrm{d}x\\
&\quad +C\int_{\mu}^{\infty}\mathbb{V}\left\{|Y|>y\right\}\mathrm{d}y\int_{\mu}^{y}\mathbb{V}\left\{|Y|>z\right\}\mathrm{d}z\int_1^{(z/\mu)^p}x^{1-2/p}l(x)\mathrm{d}x\\
&\leq C\int_1^{\infty}\mathbb{V}\left\{|Y|^p>Cx\right\}\mathrm{d}x\\
&\quad +CC\int_{\mu}^{\infty}\mathbb{V}\left\{|Y|>y\right\}\mathrm{d}y\int_{\mu}^{y}\mathbb{V}\left\{|Y|>z\right\}z^{2p-2}l(z^p)\mathrm{d}z\\
&\leq CC_{\mathbb{V}}\left\{|Y|^p\right\}+CC\int_{\mu}^{\infty}\mathbb{V}\left\{|Y|>y\right\}\mathrm{d}y\int_{\mu}^{y}\frac{\mathbb{E}(|Y|^p)}{z^p}z^pz^{p-2}l(z^p)\mathrm{d}z\\
&\leq C+C\int_{\mu}^{\infty}\mathbb{V}\left\{|Y|>y\right\}C_{\mathbb{V}}\{|Y|^p\}y^{p-1}l(y^p)\mathrm{d}y\\
&\leq CC_{\mathbb{V}}\left\{|Y|^pl(|Y|^p)\right\}<\infty.
\end{aligned}
$$

Hence, (3.5) is proved. □

5. Conclusions

We have obtained new results about complete moment convergence for maximal partial sums of moving average processes produced by negatively dependent random variables under sub-linear expectations. Results obtained in our article generalize those for negatively dependent random variables in probability space, and Theorems 3.1–3.3 complement the results of Xu et al. [7, 23], Xu and Kong [8], and Xu [24] in some sense.

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Use of AI tools declaration

Artificial Intelligence tools were not used.

Conflict of interest

The author declares that there are no conflicts of interest.

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